ON THE PERIODICITY OF COXETER TRANSFORMATIONS AND THE NON-NEGATIVITY OF THEIR EULER FORMS

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Abstract. We show that for piecewise hereditary algebras, the periodicity of the Coxeter transformation implies the non-negativity of the Euler form. Contrary to previous assumptions, the condition of piecewise heredity cannot be omitted, even for triangular algebras, as demonstrated by incidence algebras of posets.

We also give a simple, direct proof, that certain products of reflections, defined for any square matrix $A$ with 2 on its main diagonal, and in particular the Coxeter transformation corresponding to a generalized Cartan matrix, can be expressed as $-A^{-1}_{+}A^{-1}_{-}$, where $A_{+}, A_{-}$ are closely associated with the upper and lower triangular parts of $A$.

1. Introduction

Let $V$ be a free abelian group of finite rank and let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Z}$ be a non-degenerate $\mathbb{Z}$-bilinear form on $V$. The Coxeter transformation $\Phi : V \to V$ corresponding to $\langle \cdot, \cdot \rangle$ is defined via the equation $\langle x, y \rangle = - \langle y, \Phi x \rangle$ for $x, y \in V$ [14].

The purpose of this paper is to study the relations between positivity properties of the form $\langle \cdot, \cdot \rangle$ and periodicity properties of its Coxeter transformation $\Phi$. Recall that $\langle \cdot, \cdot \rangle$ is positive if $\langle x, x \rangle > 0$ for all $0 \neq x \in V$, non-negative if $\langle x, x \rangle \geq 0$ for all $x \in V$ and indefinite otherwise. The transformation $\Phi$ is periodic if $\Phi^m$ equals the identity $I$ for some integer $m \geq 1$ and weakly periodic [20] if $(\Phi^m - I)^n = 0$ for some integers $m, n \geq 1$.

Implications in one direction are given in the paper [20], where linear algebra techniques are used to show that the Coxeter matrix $\Phi$ is periodic if $\langle \cdot, \cdot \rangle$ is positive and weakly periodic if $\langle \cdot, \cdot \rangle$ is non-negative. It is much harder to establish implications in the other direction. As already noted in [20], even if $\Phi$ is periodic, $\langle \cdot, \cdot \rangle$ may be indefinite, so additional constraints are needed.

An alternative definition of the Coxeter matrix is as a certain product of reflections defined by a generalized Cartan matrix [1, 19], whereas the definition given above is $-C^{-1}C^t$ where $C$ is the matrix of the bilinear form.

We claim similarly to [6], and give a simple, direct proof, that for any square matrix $A$ with 2 on its main diagonal, the product of the $n$ reflections it defines can be expressed as $-A_{+}^{-1}A_{-}^{-1}$, where $A_{+}, A_{-}$ are closely associated

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with the upper and lower triangular parts of $A$, see Section 2. This claim can be generalized to products in arbitrary order, and no other conditions on $A$, such as being generalized Cartan, bipartite [1] or symmetric [12], are needed. In particular, when $\langle\cdot,\cdot\rangle$ is triangular, $\Phi$ can be written as a product of the reflections defined by the symmetrization of $\langle\cdot,\cdot\rangle$.

Further connections between periodicity and non-negativity are achieved when we restrict ourselves to pairs $(V,\langle\cdot,\cdot\rangle)$ for which there exists a finite dimensional $k$-algebra $A$ over an algebraically closed field $k$, having finite global dimension, such that $V \cong K_0(\text{mod } A)$ and $\langle\cdot,\cdot\rangle$ coincides, under that isomorphism, with the Euler form $\langle\cdot,\cdot\rangle_A$ of $A$. Here $\text{mod } A$ denotes the category of finite dimensional right $A$-modules. Since $\text{gl.dim } A < \infty$, the form $\langle\cdot,\cdot\rangle_A$ is non-degenerate, hence its Coxeter transformation $\Phi_A$ is well-defined and coincides with the image in $K_0(\text{mod } A)$ of the Auslander-Reiten translation on the bounded derived category $D^b(\text{mod } A)$.

In Section 3 of the paper, we show that if $A$ is piecewise hereditary, i.e. its bounded derived category $D^b(\text{mod } A)$ is equivalent as a triangulated category to $D^b(H)$ for a hereditary abelian category $H$, then the periodicity of $\Phi_A$ implies the non-negativity of $\langle\cdot,\cdot\rangle_A$.

In that Section, we also show that when $A$ is an incidence algebra of a poset $X$, the Euler form $\langle\cdot,\cdot\rangle_A$ and its Coxeter transformation $\Phi_A$ can be explicitly described in terms of the combinatorics of $X$.

Previously, [7] claimed that the condition of $\langle\cdot,\cdot\rangle$ being triangular, that is, its matrix with respect to some basis of $V$ is upper triangular with ones on the main diagonal, is enough for the periodicity of $\Phi$ to imply the non-negativity of $\langle\cdot,\cdot\rangle$. We find however examples of incidence algebras of posets negating this claim, see Section 4.

2. Coxeter transformations of bilinear forms

2.1. The definition of the Coxeter matrix. Let $V$ be a free abelian group of finite rank and let $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{Z}$ be a non-degenerate $\mathbb{Z}$-bilinear form on $V$. Recall that $\langle\cdot,\cdot\rangle$ is positive if $\langle v, v \rangle > 0$ for all $0 \neq v \in V$, non-negative if $\langle v, v \rangle \geq 0$ for all $v \in V$ and indefinite otherwise. The Coxeter transformation $\Phi : V \to V$ corresponding to $\langle\cdot,\cdot\rangle$ is defined via the equation $\langle v, w \rangle = -\langle w, \Phi v \rangle$ for all $v, w \in V$ [14].

We consider the elements of $\mathbb{Z}^n$ as column vectors, and denote by $M^t$ the transpose of a matrix $M$. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{Z}^n$. By choosing a $\mathbb{Z}$-basis $v_1, \ldots, v_n$ of $V$, we may identify $V$ with $\mathbb{Z}^n$ and $\langle\cdot,\cdot\rangle$ with the form $\langle\cdot,\cdot\rangle_C$ defined by $\langle x, y \rangle_C = x^t C y$ where $C \in \text{GL}_n(\mathbb{Z})$ is the matrix whose entries are $C_{ij} = \langle v_i, v_j \rangle$ for $1 \leq i, j \leq n$. In other words, $\langle v_i, v_j \rangle = \langle e_i, e_j \rangle_C$. Under this identification, the matrix of $\Phi$ is $-C^{-1}C^t$, hence we define the Coxeter matrix $\Phi_C$ of a matrix $C \in \text{GL}_n(\mathbb{Z})$ to be $\Phi_C = -C^{-1}C^t$.

Note that $v = -C^{-1}C^t v$ if and only if $(C + C^t)v = 0$, hence the geometric multiplicity of the eigenvalue 1 in $\Phi_C$ equals the dimension of the radical of the symmetrized bilinear form $C + C^t$.

Definition 2.1. A matrix $\Phi \in \text{GL}_n(\mathbb{Z})$ is periodic if $\Phi^m = I$ for some $m \geq 1$. $\Phi$ is weakly periodic if for some $m \geq 1$, $\Phi^m - I$ is nilpotent.
Definition 2.2. A matrix \( C \in \text{GL}_n(\mathbb{Z}) \) is unitriangular if \( C \) is upper triangular and \( C_{ii} = 1 \) for \( 1 \leq i \leq n \).

Relations between the positivity of the bilinear form \( \langle \cdot, \cdot \rangle_C \) and the periodicity of \( \Phi_C \) have been studied in [7, 20] and are summarized as follows:

Theorem 2.3. Let \( C \in \text{GL}_n(\mathbb{Z}) \). Then:

1. [20, (2.8)] \( \Phi_C \) is periodic if \( \langle \cdot, \cdot \rangle_C \) is positive.
2. [20, (3.4)] \( \Phi_C \) is weakly periodic if \( \langle \cdot, \cdot \rangle_C \) is non-negative.

However, [20, (3.8)] is an example of a matrix whose Coxeter matrix is periodic but the corresponding bilinear form is indefinite.

2.2. Alternative definition as a product of reflections. Following [1, 2, 19], we review an alternative definition of the Coxeter matrix as a product of reflections.

Let \( A \) be an \( n \times n \) matrix with integer entries satisfying

1. \( A_{ii} = 2 \quad 1 \leq i \leq n \) \((A1)\)
2. \( A_{ij} = 0 \quad \text{if and only if } A_{ij} = 0, 1 \leq i, j \leq n \) \((A2)\)

The primitive graph of \( A \) (cf. [2]) is an undirected graph with \( n \) vertices, where two vertices \( i \neq j \) are connected by an edge if \( A_{ij} \neq 0 \). The matrix \( A \) is indecomposable if its primitive graph is connected.

Define reflections \( r_1, \ldots, r_n \) by

\[
(2.1) \quad r_i(e_j) = e_j - A_{ij}e_i \quad 1 \leq j \leq n
\]

In other words, \( r_i \) is the matrix obtained from the identity matrix by subtracting the \( i \)-th row of \( A \). Denote by \( I \) the \( n \times n \) identity matrix.

Lemma 2.4. Let \( A \) be a matrix satisfying \((A1)\).

(a) \( r_i^2 = I \) for \( 1 \leq i \leq n \).
(b) If \( A \) satisfies also \((A2)\), then \( r_ir_j = r_jr_i \) for any two non-adjacent vertices \( i, j \) on the primitive graph of \( A \).

Proof. Since \( A_{ii} = 2 \), we have \( r_i(e_i) = -e_i \), thus

\[
r_i(e_j) = e_j - A_{ij}e_i = e_t - A_{it}e_i - A_{it}r_i(e_i) = e_t
\]

for all \( 1 \leq t \leq n \), and the first assertion is proved.

If \( A_{ij} = 0 \) then \( r_i(e_j) = e_j \). The assumptions on \( A \) imply that if \( i, j \) are not adjacent, then \( r_i(e_j) = e_j \) and \( r_j(e_i) = e_i \). Therefore, if \( 1 \leq t \leq n \),

\[
r_ir_j(e_t) = r_i(e_t - A_{jt}e_j) = e_t - A_{it}e_i - A_{it}r_i(e_i) = e_t - A_{it}e_i - A_{jt}e_j
\]

is symmetric in \( i \) and \( j \), hence \( r_ir_j = r_jr_i \). \( \Box \)

Consider the following two additional properties:

1. \( A_{ij} \leq 0 \quad \text{for all } i \neq j \) \((A3)\)
2. The primitive graph of \( A \) is bipartite \((A4)\).

Definition 2.5. A matrix \( A \) is a generalized Cartan matrix if it satisfies \((A1)\), \((A2)\) and \((A3)\). A matrix \( A \) is bipartite if it satisfies \((A1)\), \((A2)\) and \((A4)\).
For a generalized Cartan matrix $A$ and a permutation $\pi$ of $\{1, 2, \ldots, n\}$, a Coxeter transformation is defined in [19] by $\Phi(A, \pi) = r_{\pi(1)}r_{\pi(2)} \cdots r_{\pi(n)}$. For a bipartite matrix $A$, let $\Sigma_1 \sqcup \Sigma_2$ be a corresponding partition of $\{1, 2, \ldots, n\}$ and consider $R_A = R_1R_2$ where $R_k = \prod_{i \in \Sigma_k} r_i$, $k = 1, 2$, see [2]. Note that by Lemma 2.4, the matrices $R_k$ do not depend on the order of reflections within each product. Note also that $R_A$ equals $\Phi(A, \pi)$ for a suitable $\pi$.

Recall that the spectrum of a square matrix $\Phi$ with complex entries, denoted $\text{spec}(\Phi)$, is the set of (complex) roots of the characteristic polynomial of $\Phi$. Let $\rho(\Phi) = \max \{|\lambda| : \lambda \in \text{spec}(\Phi)\}$ be the spectral radius of $\Phi$.

We recall two results on the spectrum of Coxeter transformations corresponding to generalized Cartan and bipartite matrices.

**Theorem 2.6** ([19]). Let $A$ be an indecomposable generalized Cartan matrix, $\pi \in S_n$. If $A$ is not of finite or affine type, then $\rho(\Phi(A, \pi)) > 1$.

**Theorem 2.7** ([1, p. 63],[2, p. 344]). Let $A$ be a bipartite matrix.

(a) $\lambda^2 \in \text{spec}(R_A)$ if and only if $\lambda + 2 + \lambda^{-1} \in \text{spec}(A)$.

(b) If $A$ is also symmetric, then $\text{spec}(R_A) \subset S^1 \cup \mathbb{R}$.

### 2.3. Linking the two definitions.

Let $R$ be any commutative ring with 1 and let $e_1, \ldots, e_n$ be a basis of a free $R$-module of rank $n$. Let $A$ be an $n \times n$ matrix with entries in $R$ satisfying (A1) (where 2 means 1 + 1), and define the reflections $r_1, \ldots, r_n$ as in (2.1). When we want to stress the dependence of the reflections on $A$, we shall use the notation $r_1^{A}, \ldots, r_n^{A}$.

**Lemma 2.8.** Let $1 \leq s \leq n$. Then for every $1 \leq t \leq n$,

$$
(r_1 \cdots r_s)(e_t) = e_t + \sum_{k=1}^{s} (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq s} A_{i_1i_2} \cdots A_{i_{k-1}i_k} e_{i_k} t e_{i_1}
$$

**Proof.** By induction on $s$, the case $s = 1$ being just the definition of $r_1$, and for the induction step, expand $r_{s+1}(e_t)$ as $e_t - A_{s+1,t} e_{s+1}$ and use the hypothesis for $s$.

$$
(r_1 \cdots r_s r_{s+1})(e_t) = (r_1 \cdots r_s)(e_t) - A_{s+1,t} (r_1 \cdots r_s)(e_{s+1})
$$

$$
= e_t - A_{s+1,t} e_{s+1} + \sum_{k=1}^{s} (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq s} A_{i_1i_2} \cdots A_{i_{k-1}i_k} e_{i_k} t e_{i_1}
$$

$$
+ \sum_{k=1}^{s+1} (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq s} A_{i_1i_2} \cdots A_{i_{k-1}i_k} A_{i_k,s+1} e_{s+1} t e_{i_1}
$$

$$
= e_t - \sum_{k=1}^{s+1} (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq s+1} A_{i_1i_2} \cdots A_{i_{k-1}i_k} e_{i_k} t e_{i_1}
$$

\[\square\]

Define two $n \times n$ matrices $A_+$ and $A_-$ by

$$
(A_+)_ij = \begin{cases} 
A_{ij} & i < j \\
1 & i = j \\
0 & i > j 
\end{cases} \quad (A_-)_ij = \begin{cases} 
A_{ij} & i < j \\
1 & i = j \\
0 & i > j 
\end{cases}
$$
Then $A = A_+ + A_-^t$, and one can think of $A_+$, $A_-$ as the upper and lower triangular parts of $A$. The matrices $A_+$ and $A_-$ are invertible since $A_+ - I$ and $A_+ - I$ are nilpotent. Note that $A$ is symmetric if and only if $A_+ = A_-^t$.

**Theorem 2.9.** If $A$ satisfies (A1), then $r_1^A r_2^A \cdots r_n^A = -A_+^{-1} A_-^t$.

*Proof.* By Lemma 2.8 with Theorem 2.9.

If $r_i$ and $A$ satisfy (A2), then $A$ is triangular with respect to the original one using permutation matrices.

**Remark 2.10.** Theorem 2.9 is still true when we drop the condition (A1) and slightly change the definition of $A_-$, by $(A_-)_{ii} = A_{ii} - 1$ for $1 \leq i \leq n$. However, in that case the matrices $r_i$ are no longer reflections.

Theorem 2.9 provides a link between the definition of the Coxeter matrix as a specific automorphism of the bilinear form and its definition as a product of $n$ reflections, as shown by the following corollary.

**Corollary 2.11.** Let $C \in \text{GL}_n(\mathbb{Z})$ be a unitriangular matrix. Then $\Phi_C = \Phi(C + C^t, id)$, that is, $\Phi_C = r_1^A r_2^A \cdots r_n^A$ for $A = C + C^t$.

In fact, this corollary is proved in [12] for the case where $\Phi_C$ is a Coxeter element in an arbitrary Coxeter group of finite rank represented as a group of linear transformations on a real inner product space, so that the Cartan matrix $A$ is symmetric.

*Proof.* Apply Theorem 2.9 for the matrix $A = C + C^t$, which satisfies (A1), (A2) and $A_+ = A_- = C$.

Denote by $S_n$ the group of permutations on $\{1, 2, \ldots, n\}$ and let $\pi \in S_n$. One could deduce a generalized version of Theorem 2.9 for the product of the $n$ reflections in an arbitrary order by proving an analogue of Lemma 2.8 for arbitrary $\pi$. Instead, we will derive the generalized version from the original one using permutation matrices.

Define the permutation matrix $P_\pi$ by $P_\pi(e_i) = e_{\pi(i)}$ for all $1 \leq i \leq n$. Note that $P_\pi^{-1} = P_{\pi}^t$. Given a matrix $A$, let $A_\pi$ denote the matrix $P_\pi^{-1} A P_\pi$, so that $(A_\pi)_{ij} = A_{\pi(i)\pi(j)}$. Obviously, if $A$ satisfies (A1), so does $A_\pi$.

**Lemma 2.12.** Let $1 \leq i \leq n$. Then $r_i^{A_\pi} = P_\pi^{-1} r_i^A P_\pi$.

*Proof.* For all $1 \leq t \leq n$,

$$
\left(P_\pi^{-1} r_i^A P_\pi\right)(e_t) = P_\pi^{-1} \left(e_{\pi(t)} - A_{\pi(i)\pi(t)} e_{\pi(i)}\right) = e_t - A_{\pi(i)\pi(t)} e_t = r_i^{A_\pi} (e_t)
$$

□
Define two $n \times n$ matrices $A_{\pi,+}$ and $A_{\pi,-}$ by

$$(A_{\pi,+})_{ij} = \begin{cases} A_{ij} & \pi^{-1}(i) < \pi^{-1}(j) \\ 1 & i = j \\ 0 & \text{otherwise} \end{cases} \quad (A_{\pi,-})_{ij} = \begin{cases} A_{ji} & \pi^{-1}(i) < \pi^{-1}(j) \\ 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

Direct calculation shows that $A_{\pi,+} = P_\pi(A_\pi)_+ P_\pi^{-1}, A_{\pi,-} = P_\pi(A_\pi)_- P_\pi^t$ and $A = A_{\pi,+} + A_{\pi,-}^t$.

**Corollary 2.13.** Let $A$ satisfy (A1) and let $\pi \in S_n$. Then

$$r^A_{\pi(1)} r^A_{\pi(2)} \cdots r^A_{\pi(n)} = -A_{\pi,+}^{-1} A_{\pi,-}^t$$

**Proof.** By Lemma 2.12 and Theorem 2.9 applied for $A_\pi$,

$$r^A_{\pi(1)} r^A_{\pi(2)} \cdots r^A_{\pi(n)} = P_\pi \left( r^A_1 r^A_2 \cdots r^A_n \right) P_\pi^{-1} = -P_\pi (A_\pi)_+^{-1} (A_\pi)_-^t P_\pi^t$$

$$= - (P_\pi (A_\pi)_+^{-1} P_\pi^{-1}) \left( P_\pi (A_\pi)_-^t P_\pi^t \right) = -A_{\pi,+}^{-1} A_{\pi,-}^t \quad \square$$

### 3. Periodicity and Non-negativity for Piecewise Hereditary Algebras and Posets

Let $k$ be a field, and let $\mathcal{A}$ be an abelian $k$-category of finite global dimension with finite dimensional Ext-spaces. Denote by $D^b(\mathcal{A})$ its bounded derived category and by $K_0(\mathcal{A})$ its Grothendieck group. The expression

$$\langle X, Y \rangle_{\mathcal{A}} = \sum_{i \in \mathbb{Z}} (-1)^i \dim_k \text{Hom}_{D^b(\mathcal{A})}(X, Y[i])$$

is well-defined for $X, Y \in D^b(\mathcal{A})$ and induces a $\mathbb{Z}$-bilinear form on $K_0(\mathcal{A})$, known as the *Euler form*. When $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ is non-degenerate, the unique transformation $\Phi_{\mathcal{A}} : K_0(\mathcal{A}) \to K_0(\mathcal{A})$ satisfying $\langle x, y \rangle_{\mathcal{A}} = -\langle y, \Phi_{\mathcal{A}} x \rangle_{\mathcal{A}}$ for all $x, y \in K_0(\mathcal{A})$ is called the *Coxeter transformation* of $\mathcal{A}$. For more details we refer the reader to [15].

Two such abelian $k$-categories $\mathcal{A}$ and $\mathcal{B}$ are said to be *derived equivalent* if there exists a triangulated equivalence $F : D^b(\mathcal{A}) \simeq D^b(\mathcal{B})$. In this case, the forms $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ are equivalent over $\mathbb{Z}$, hence the positivity properties of the Euler form and the periodicity properties of the Coxeter transformation are invariants of derived equivalence.

Let $\Lambda$ be a finite dimensional algebra of finite global dimension over an algebraically closed field $k$, and consider the $k$-category mod $\Lambda$ of finitely generated right modules over $\Lambda$. Denote by $D^b(\Lambda)$ its bounded derived category, by $K_0(\Lambda)$ its Grothendieck group and by $\langle \cdot, \cdot \rangle_{\Lambda}$ the Euler form. Then $K_0(\Lambda)$ is free of finite rank, with a $\mathbb{Z}$-basis consisting of the representatives of the isomorphism classes of simple modules in mod $\Lambda$. The form $\langle \cdot, \cdot \rangle_{\Lambda}$ is non-degenerate, and its Coxeter transformation $\Phi_{\Lambda}$ coincides with the linear map on $K_0(\Lambda)$ induced by the Auslander-Reiten translation on $D^b(\Lambda)$. For more details see [9, (III.1)], [18, (2.4)] or [15].
3.1. Path algebras of quivers without oriented cycles. The first example of algebras $\Lambda$ for which the connection between the positivity of $\langle \cdot, \cdot \rangle_\Lambda$ and the periodicity of $\Phi_\Lambda$ is completely understood is the class of path algebras of quivers without oriented cycles, or more generally hereditary algebras, see [15, Theorem 18.5]. We briefly review the main results.

A (finite) quiver $Q$ is a directed graph with a finite number of vertices and edges. The underlying graph of $Q$ is the undirected graph obtained from $Q$ by forgetting the orientations of the edges. An oriented cycle is a nontrivial path in $Q$ starting and ending at the same vertex. The path algebra $kQ$ is the algebra over $k$ having as a $k$-basis the set of all (oriented) paths in $Q$; the product of two paths is their composition, if defined, and zero otherwise. When $Q$ has no oriented cycles, the path algebra $kQ$ is hereditary and finite-dimensional. Denote by $\langle \cdot, \cdot \rangle_Q$ its Euler form and by $\Phi_Q$ its Coxeter transformation. The matrix of $\langle \cdot, \cdot \rangle_Q$ with respect to the basis of simple modules is unitriangular, and its symmetrization is generalized Cartan. The relations between the periodicity of $\Phi_Q$ and the positivity of $\langle \cdot, \cdot \rangle_Q$ are summarized in the following well-known proposition, see [1, 3, 5, 19] and [18, (1.2)].

Proposition 3.1. Let $Q$ be a connected quiver without oriented cycles. Then:

(a) $\Phi_Q$ is periodic if and only if $\langle \cdot, \cdot \rangle_Q$ is positive, equivalently the underlying graph of $Q$ is a Dynkin diagram of type $A$, $D$ or $E$.

(b) $\Phi_Q$ is weakly periodic if and only if $\langle \cdot, \cdot \rangle_Q$ is non-negative, equivalently the underlying graph of $Q$ is a Dynkin diagram or an extended Dynkin diagram of type $\tilde{A}$, $\tilde{D}$ or $\tilde{E}$.

3.2. Canonical algebras. Another interesting class of algebras for which the connection between non-negativity and periodicity is established are the canonical algebras, introduced in [18].

The Grothendieck group and the Euler form of canonical algebras were thoroughly studied in [14]. If $\Lambda$ is canonical of type $(p, \lambda)$ where $p = (p_1, \ldots, p_t)$ and $\lambda = (\lambda_3, \ldots, \lambda_t)$ is a sequence of pairwise distinct elements of $k \setminus \{0\}$, then the rank of $K_0(\Lambda)$ is $\sum_{i=1}^t p_i - (t - 2)$ and the characteristic polynomial of the Coxeter transformation $\Phi_\Lambda$ equals $(T - 1)^2 \prod_{i=1}^t \frac{T^{p_i} - 1}{T - 1}$ [14, Prop. 7.8]). In particular, $\rho(\Phi_\Lambda) = 1$ and the eigenvalues of $\Phi_\Lambda$ are roots of unity, hence $\Phi_\Lambda$ is weakly periodic.

The following proposition follows from [14, Prop. 10.3], see also [16].

Proposition 3.2. Let $\Lambda$ be a canonical algebra of type $(p, \lambda)$. If $\Phi_\Lambda$ is periodic then $p$ is one of $(2, 3, 6)$, $(2, 4, 4)$, $(3, 3, 3)$ or $(2, 2, 2, 2)$. In any of these cases, $\langle \cdot, \cdot \rangle_\Lambda$ is non-negative.

3.3. Extending to piecewise hereditary algebras. We extend the results of the previous sections to the class of all piecewise hereditary algebras.

Definition 3.3. An algebra $\Lambda$ over $k$ is piecewise hereditary if there exist a hereditary abelian category $\mathcal{H}$ and a triangulated equivalence $\mathcal{D}^b(\Lambda) \simeq \mathcal{D}^b(\mathcal{H})$. 


Theorem 3.4. Let $k$ be an algebraically closed field and let $\Lambda$ be a finite dimensional piecewise hereditary $k$-algebra. If $\Phi_\Lambda$ is periodic, then $\langle \cdot, \cdot \rangle_\Lambda$ is non-negative.

Proof. By definition, there exists a hereditary category $\mathcal{H}$ and an equivalence of triangulated categories $F : \mathcal{D}^b(\Lambda) \simeq \mathcal{D}^b(\mathcal{H})$. By the invariance under derived equivalence, it is enough to prove the theorem for $\Phi_\mathcal{H}$ and $\langle \cdot, \cdot \rangle_\mathcal{H}$. Moreover, we can assume that $\mathcal{H}$ is connected.

Now $\mathcal{H}$ is an Ext-finite $k$-category and $F(\Lambda_\Lambda)$ is a tilting object in $\mathcal{D}^b(\mathcal{H})$, so by [11, Theorem 1.7], $\mathcal{H}$ admits a tilting object, that is, an object $T$ with $\text{Ext}^1_T(T, T) = 0$ such that for any object $X$ of $\mathcal{H}$, the condition $\text{Hom}_\mathcal{H}(T, X) = 0 = \text{Ext}^1_\mathcal{H}(T, X)$ implies that $X = 0$.

By the classification of hereditary connected Ext-finite $k$-categories with tilting object up to derived equivalence over an algebraically closed field [10], $\mathcal{H}$ is derived equivalent to mod $H$ for a finite dimensional hereditary algebra $H$ or to mod $\Lambda$ for a canonical algebra $\Lambda$. Again by invariance under derived equivalence we may assume that $\mathcal{H} = \text{mod } H$ or $\mathcal{H} = \text{mod } \Lambda$.

For $\mathcal{H} = \text{mod } H$, we can replace $H$ by a path algebra of a finite connected quiver without oriented cycles, and then use Proposition 3.1. For $\mathcal{H} = \text{mod } \Lambda$, the result follows from Proposition 3.2. □

3.4. Incidence algebras of posets. Let $X$ be a finite partially ordered set (poset) and let $k$ be a field. The incidence algebra $kX$ is the $k$-algebra spanned by elements $e_{xy}$ for the pairs $x \leq y$ in $X$, with multiplication defined by $e_{xy}e_{zw} = \delta_{yz}e_{xw}$. Finite dimensional right modules over $kX$ can be identified with commutative diagrams of finite dimensional $k$-vector spaces over the Hasse diagram of $X$ which is the directed graph whose vertices are the points of $X$, with an arrow from $x$ to $y$ if $x < y$ and there is no $z \in X$ with $x < z < y$.

We recollect the basic facts on the Euler form of posets and refer the reader to [13] for details. The algebra $kX$ is of finite global dimension, hence its Euler form, denoted $\langle \cdot, \cdot \rangle_X$, is well-defined and non-degenerate. Denote by $C_X$, $\Phi_X$ the matrices of $\langle \cdot, \cdot \rangle_X$ and its Coxeter transformation with respect to the basis of simple $kX$-modules.

The incidence matrix of $X$, denoted $1_X$, is the $X \times X$ matrix defined by

$$
(1_X)_{xy} = \begin{cases} 
1 & x \leq y \\
0 & \text{otherwise}
\end{cases}
$$

By extending the partial order on $X$ to a linear order, we can always arrange the elements of $X$ such that the incidence matrix is unitriangular. In particular, $1_X$ is invertible over $\mathbb{Z}$. Recall that the Möbius function $\mu_X : X \times X \to \mathbb{Z}$ is defined by $\mu_X(x, y) = (1_X)^{-1}xy$.

Lemma 3.5 ([13, Prop. 3.11]). $C_X = 1_X^{-1}$.

Lemma 3.6. Let $x, y \in X$. Then $(\Phi_X)_{xy} = -\sum_{z : z \geq x} \mu_X(y, z)$.

Proof. Since $\Phi_X = -C_X^{-1}C_X^{-1} = -1_X1_X^{-1}$,

$$(\Phi_X)_{xy} = -\sum_{z \in X} (1_X)_{xz}(1_X^{-1})_{yz} = -\sum_{z : z \geq x} \mu_X(y, z)$$
When the Hasse diagram of $X$ has the property that any two vertices $x, y$ are connected by at most one directed path, the Möbius function takes a very simple form, namely

$$
\mu_X(x, y) = \begin{cases} 
1 & y = x \\
-1 & x \to y \text{ is an edge in the Hasse diagram} \\
0 & \text{otherwise}
\end{cases}
$$

In this case, Lemma 3.6 coincides with Proposition 3.1 of [4], taking the Hasse diagram as the quiver.

**Lemma 3.7.** If $X$ and $Y$ are posets, then

$$
C_{X \times Y} = C_X \otimes C_Y \quad \Phi_{X \times Y} = -\Phi_X \otimes \Phi_Y
$$

**Proof.** Observe that $1_{X \times Y} = 1_X \otimes 1_Y$. □

**Corollary 3.8.** Let $X, Y$ be posets with periodic Coxeter matrices. Then $X \times Y$ has also periodic Coxeter matrix.

Since non-negativity of forms is not preserved under tensor products, Corollary 3.8 can be used to construct posets with periodic Coxeter matrix but with indefinite Euler form, see Example 4.4.

4. Examples

For a poset $X$, let $C_X, \Phi_X$ be as in the previous section. In particular we may assume that $C_X$ is unitriangular. The symmetrization $A_X = C_X + C_X^t$ satisfies (A1) and (A2), but in general it is not bipartite nor generalized Cartan.

4.1. Spectral properties of $\Phi_X$. 

**Example 4.1.** The spectrum of $\Phi_X$ does not determine that of $A_X$ (Compare with Theorem 2.7a).

The four posets whose Hasse diagrams are depicted in Figure 1 are derived equivalent (as they are all piecewise hereditary of type $D_5$), hence their Coxeter matrices are similar and have the same spectrum, namely the roots of the characteristic polynomial $x^5 + x^4 + x + 1$. However, the spectra of the corresponding symmetrized forms are different. Figure 1 also shows for each poset $X$ the characteristic polynomial of the matrix of its symmetrized form.

**Example 4.2.** A poset $X$ with $\text{spec } \Phi_X \not\subseteq S^1 \cup \mathbb{R}$ (Compare with Theorem 2.7b).

Let $X$ be the following poset.
The characteristic polynomial of $\Phi_X$ is $(x + 1)^4(x^4 - 2x^3 + 6x^2 - 2x + 1)$, whose roots, besides $-1$, are $z, \bar{z}, z^{-1}, \bar{z}^{-1}$ with $\Re z = \frac{1 + \sqrt{2\sqrt{3} - 3}}{2 - \sqrt{3}}$ and $|z|^2 = \frac{1 + \sqrt{2\sqrt{3} - 3}}{2 - \sqrt{3}} - 1$. These four roots are neither real nor on the unit circle.

An example of similar spectral behavior for path algebra of a quiver is given in [15, Example 18.1].

Note that for all posets $X$ with 7 elements or less, $\text{spec}(\Phi_X) \subseteq S^1 \cup \mathbb{R}$. This was verified using the database [8] and the Magma software package.

4.2. Counterexamples to [7, Prop. 1.2]. We give two examples of posets showing that in general, for triangular algebras, the periodicity of the Coxeter transformation (and even of the Auslander-Reiten translation up to a shift) does not imply the non-negativity of the Euler form.

Example 4.3. Consider the poset $X$ with the following Hasse diagram.

Then $\Phi_X^{d_X} = I$ but $\nu^t C_X \nu = -1$ for $\nu = (1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0)^t$ (the vertices are ordered in layers from top to bottom).

Example 4.4. Let $X = A_3 \times D_4$ with the following orientations:
The Hasse diagram of $X$ is given by

```
1,1 → 1,2 ← 2,1 → 2,2
\vdash \downarrow \downarrow \uparrow \uparrow \uparrow
1,3 \quad 3,1 → 3,2 \quad 2,3
\downarrow \downarrow \downarrow \downarrow \downarrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow
1,4 \quad 3,3 \quad 2,4
\downarrow
3,4
```

so that $X$ contains the following wild quiver as a subposet.

```
1,3 \quad 3,1 ← 3,2 \quad 2,3
\downarrow
3,3
\downarrow
3,4
```

It follows [17] that $kX$ is not of finite representation type, hence by [21, Theorem 6] the form $\langle \cdot, \cdot \rangle_X$ is not weakly positive, that is, there exists a vector $v \neq 0$ with non-negative coordinates such that $\langle v, v \rangle_X \leq 0$.

Moreover, we can exhibit a non-negative vector $v$ such that $v^tC_Xv = -1$, namely $v = (v_x)_{x \in X}$ where the integers $v_x$ are placed at the vertices as in the following picture:

```
0 \quad 0 \quad 0 \quad 0
\downarrow \downarrow \downarrow \downarrow
1 \quad 1 \quad 1 \quad 1
\downarrow \downarrow \downarrow \downarrow
1 \quad 1 
```

On the other hand, the Coxeter matrices of the quivers $A_3$ and $D_4$ are periodic, their orders are 4 and 6 respectively. By Corollary 3.8, the Coxeter matrix of $X$ is periodic of order 12.

Contrary to Example 4.3, one can show that not only the image $\Phi_X$ of the Auslander-Reiten translation $\tau_X : D^b(X) \to D^b(X)$ in the Grothendieck group is periodic, but also that actually $\tau_X^d \simeq [d]$ for some integers $d, e \geq 1$.

References