Universal derived equivalences of posets and applications to cluster-tilting objects

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Posets, diagrams and sheaves

$X$ – *poset* (finite partially ordered set)

$\mathcal{A}$ – abelian category

$\mathcal{A}^X$ – the category of *diagrams* over $X$ with values in $\mathcal{A}$, or *functors* $F : X \to \mathcal{A}$, consisting of:

- An *object* $F_x$ of $\mathcal{A}$ for each $x \in X$.
- A *morphism* $r_{xx'} \in \text{Hom}_\mathcal{A}(F_x, F_{x'})$ for each $x < x'$.

such that $r_{xx''} = r_{x'x''}r_{xx'}$ for all $x < x' < x''$ (*commutativity*).

Natural *topology* on $X$: $U \subseteq X$ is *open* if $x \in U$, $x \leq x' \Rightarrow x' \in U$

Diagrams can be identified with *sheaves* over $X$ with values in $\mathcal{A}$.
Posets, diagrams and sheaves – Example

Let $X = \{1, 2, 3, 4\}$ with $1 < 2$, $1 < 3$, $1 < 4$, $2 < 4$, $3 < 4$.

A *diagram* over $X$ is shown on the right:

The *open sets* are

$$\emptyset, \{4\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}.$$
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Derived categories

$\mathcal{B}$ – abelian category, $\mathcal{C}^b(\mathcal{B})$ – the category of *bounded complexes*

$$K^\bullet = \ldots \xrightarrow{d} K^{-1} \xrightarrow{d} K^0 \xrightarrow{d} K^1 \xrightarrow{d} \ldots$$

with $K^i \in \mathcal{B}$, $d^2 = 0$ and $K^i = 0$ for $|i| \gg 0$.

A morphism $f : K^\bullet \to L^\bullet$ is a *quasi-isomorphism* if

$$H^i f : H^i K^\bullet \to H^i L^\bullet$$

are isomorphisms for all $i \in \mathbb{Z}$.

The *bounded derived category* $\mathcal{D}^b(\mathcal{B})$ is obtained from $\mathcal{C}^b(\mathcal{B})$ by *localization* with respect to the quasi-isomorphisms (that is, we formally invert all quasi-isomorphisms).
Universal derived equivalence

Two posets $X$ and $Y$ are *universally derived equivalent* ($X \overset{u}{\sim} Y$) if

$$\mathcal{D}^b(A^X) \cong \mathcal{D}^b(A^Y)$$

for any abelian category $A$.

Fix a field $k$, and specialize:

mod $k$ – the category of finite dimensional vector spaces over $k$.

$(\text{mod } k)^X$ can be identified with the category of finitely generated *right modules* over the *incidence algebra* of $X$ over $k$.

$X$ and $Y$ are *derived equivalent* ($X \sim Y$) if

$$\mathcal{D}^b(\text{mod } kX) \cong \mathcal{D}^b(\text{mod } kY)$$
Comments on derived equivalence

No known *algorithm* that decides if $X \sim Y$ (or $X^u \sim Y$).

However, one can use:

- *Invariants* of the derived category;
  If $X \sim Y$ then $X$ and $Y$ must have the same invariants.

Examples of invariants are:
- The *number of points* of $X$.
- The *Euler bilinear form* on $X$, closely related to the *Möbius function* of $X$.

- *Constructions*
  Start with some “nice” $X$ and get many $Y$-s with $X \sim Y$. 
Known constructions

- **BGP Reflection**
  When $X$ is a tree and $s \in X$ is a source (or a sink), invert all arrows from (to) $s$ and get a new tree $X'$ with $X' \sim X$.

**Example.**

- $\bullet \leftrightarrow \bullet \rightarrow \bullet$
- $\bullet \rightarrow \bullet \rightarrow \bullet$
- $\bullet \rightarrow \bullet \leftrightarrow \bullet$

- **The square and $D_4$**

The square and $D_4$ are equivalent.
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New construction – Flip-flops

Let \((X, \leq_X), (Y, \leq_Y)\) be posets, \(f : X \to Y\) order-preserving.

Define two partial orders \(\leq^+_f, \leq^-_f\) on \(X \sqcup Y\) as follows:

- Keep the original partial orders inside \(X\) and \(Y\).
- Add the relations

\[
\begin{align*}
x \leq^+_f y & \iff f(x) \leq_Y y \\
y \leq^-_f x & \iff y \leq_Y f(x)
\end{align*}
\]

for \(x \in X, y \in Y\).

**Theorem [L1].** \((X \sqcup Y, \leq^+_f) \sim (X \sqcup Y, \leq^-_f)\).
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Flip-flop – An example

\[ f : \quad 8 \leftrightarrow 1 \quad 9 \leftrightarrow 2 \quad 11 \leftrightarrow 10 \quad 12 \leftrightarrow 6 \quad 14 \leftrightarrow 13 \]

\((X \sqcup Y, \leq^f)\)

\((X \sqcup Y, \leq^f)\)
Cluster tilting objects

$Q$ – quiver without oriented cycles with $n$ vertices; $k$ – field

The **cluster category** associated with $Q$ [BMRRT, CCS, FZ] is defined as the orbit category

$$
\mathcal{C}_Q = \mathcal{D}^b(\text{mod } kQ)/\nu \cdot [-2]
$$

where $\nu : \mathcal{D}^b(\text{mod } kQ) \to \mathcal{D}^b(\text{mod } kQ)$ is the **Serre functor**.

Indecomposables: $\text{ind } \mathcal{C}_Q = \text{ind } kQ \cup \{P_x[1] : x \text{ is a vertex of } Q\}$.

A (basic) object $T$ of $\mathcal{C}_Q$ is **cluster tilting** if $\text{Hom}_{\mathcal{C}_Q}(T, T[1]) = 0$ and $T$ has $n$ indecomposable summands.

$\mathcal{T}_Q$ – the set of cluster tilting objects in $\mathcal{C}_Q$. 
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Partial order on the set of cluster tilting objects

For $T = \bigoplus_{i=1}^{n} T_i \in \mathcal{T}_Q$, let $\hat{T} \in \text{mod } kQ$ be the sum of the $T_i$ which are in $\text{ind } kQ$. Consider the torsion class

$$\text{fac } \hat{T} = \left\{ M \in \text{mod } kQ : M \text{ is a quotient of } \hat{T}^m \text{ for some } m \geq 1 \right\}$$

and define [IT] a partial order on $\mathcal{T}_Q$ by setting $T \leq T'$ if $\text{fac } \hat{T} \supset \text{fac } \hat{T}'$.

When $Q$ is a Dynkin diagram of type $A$, $D$, or $E$, the poset $\mathcal{T}_Q$ is known also as a Cambrian lattice [R], which is a quotient of the weak order on the corresponding Coxeter group.

In type $A$ with the linear orientation, we get the Tamari lattices. Their Hasse diagrams are the 1-skeletons of polytopes known as the Stasheff Associahedra.
Tamari Lattices for $A_1$ and $A_2$

$A_1$: 

$\mathcal{T}_\bullet \rightarrow (ab)c \rightarrow a(bc)$

$A_2$: 

$\mathcal{T}_\bullet \rightarrow ((ab)c)d \rightarrow ((a(bc))d \rightarrow (a((bc)d) \rightarrow a((bc)d) \rightarrow a(b(cd)) \rightarrow a(b(cd))$
Tamari Lattice for $A_3$
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Flip-flops on posets of cluster tilting objects

$Q$ – quiver without oriented cycles; $x$ – a sink in $Q$.
$Q'$ – the BGP reflection with respect to $x$.

**Theorem [L2].** $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$ are related via a flip-flop.

$$
\mathcal{T}_Q \simeq (\mathcal{T}_Q^x \sqcup \mathcal{T}_Q \setminus \mathcal{T}_Q^x, \leq f) \quad \mathcal{T}_{Q'} \simeq (\mathcal{T}_{Q'}^{x[1]} \sqcup \mathcal{T}_{Q'} \setminus \mathcal{T}_{Q'}^{x[1]}, \leq f')
$$

$\mathcal{T}_Q^x$ – cluster tilting objects in $\mathcal{T}_Q$ containing $P_x$ as summand.
$\mathcal{T}_{Q'}^{x[1]}$ – cluster tilting objects in $\mathcal{T}_{Q'}$ containing $P_x[1]$ as summand.

$f$ and $f'$ are defined via cluster mutation.

**Corollary.** If $\mathcal{D}^b(\text{mod } kQ_1) \simeq \mathcal{D}^b(\text{mod } kQ_2)$ then $\mathcal{T}_{Q_1} \overset{u}{\sim} \mathcal{T}_{Q_2}$. 
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References


