Generalized reflections and derived equivalences of posets

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Posets, diagrams and sheaves

$X$ – *poset* (finite partially ordered set)

$\mathcal{A}$ – abelian category

$\mathcal{A}^X$ – the category of *diagrams* over $X$ with values in $\mathcal{A}$, or *functors* $F : X \to \mathcal{A}$, consisting of:

- An *object* $F_x$ of $\mathcal{A}$ for each $x \in X$.
- A *morphism* $r_{xx'} \in \text{Hom}_{\mathcal{A}}(F_x, F_{x'})$ for each $x < x'$.

such that $r_{xx''} = r_{x'x''}r_{xx'}$ for all $x < x' < x''$ (*commutativity*).

Natural *topology* on $X$: $U \subseteq X$ is *open* if $x \in U$, $x \leq x' \Rightarrow x' \in U$

Diagrams can be identified with *sheaves* over $X$ with values in $\mathcal{A}$. 
Posets, diagrams and sheaves – Example

Let $X = \{1, 2, 3, 4\}$ with $1 < 2$, $1 < 3$, $1 < 4$, $2 < 4$, $3 < 4$.

A *diagram* over $X$ is shown on the right:

$$
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
4 \\
\downarrow \\
3 \\
\end{array}
\quad
\begin{array}{c}
F_1 \\
\downarrow r_{12} \\
F_2 \\
\downarrow r_{24} \\
F_4 \\
\downarrow r_{13} \\
F_3 \\
\end{array}

r_{24}r_{12} = r_{14} = r_{34}r_{13}

The *open sets* are

$\phi, \{4\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}$. 

Derived categories

$\mathcal{B}$ – abelian category, $\mathcal{C}^b(\mathcal{B})$ – the category of bounded complexes

$$K^\bullet = \ldots \xrightarrow{d} K^{-1} \xrightarrow{d} K^0 \xrightarrow{d} K^1 \xrightarrow{d} \ldots$$

with $K^i \in \mathcal{B}$, $d^2 = 0$ and $K^i = 0$ for $|i| \gg 0$.

A morphism $f : K^\bullet \to L^\bullet$ is a quasi-isomorphism if

$$H^i f : H^i K^\bullet \to H^i L^\bullet$$

are isomorphisms for all $i \in \mathbb{Z}$.

The bounded derived category $\mathcal{D}^b(\mathcal{B})$ is obtained from $\mathcal{C}^b(\mathcal{B})$ by localization with respect to the quasi-isomorphisms (that is, we formally invert all quasi-isomorphisms).
Universal derived equivalence

Two posets $X$ and $Y$ are *universally derived equivalent* ($X \overset{u}{\sim} Y$) if
\[ D^b(A^X) \simeq D^b(A^Y) \]
for any abelian category $A$.

Fix a field $k$, and specialize:
mod $k$ – the category of finite dimensional vector spaces over $k$.

$(\text{mod } k)^X$ can be identified with the category of finitely generated *right modules* over the *incidence algebra* of $X$ over $k$.

$X$ and $Y$ are *derived equivalent* ($X \sim Y$) if
\[ D^b(\text{mod } kX) \simeq D^b(\text{mod } kY) \]
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Comments on derived equivalence

No known \textit{algorithm} that decides if \( X \sim Y \) (or \( X^u \sim Y \)).

However, one can use:

- \textit{Invariants} of the derived category;
  If \( X \sim Y \) then \( X \) and \( Y \) must have the same invariants.

Examples of invariants are:

- The \textit{number of points} of \( X \).
- The \textit{Euler bilinear form} on \( X \), closely related to the \textit{Möbius function} of \( X \).

- \textit{Constructions}
  Start with some “nice” \( X \) and get many \( Y \)-s with \( X \sim Y \).
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Known constructions

• **BGP Reflection**
  When $X$ is a tree and $s \in X$ is a *source* (or a *sink*), invert all arrows from (to) $s$ and get a new tree $X'$ with $X' \sim X$.

**Example.**

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• **The square and $D_4$**

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and

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•

are equivalent.
Constructions of derived equivalent posets

**Common theme:** structured reversal of order relations.

- Generalized reflections (universal derived equivalences)
  - *Flip-Flops*, with application to posets of cluster tilting objects
  - *Generalized BGP reflections*
  - Hybrid construction

- Mirroring with respect to a *bipartite* structure
  - *Mates* of triangular matrix algebras
Generalized reflections and derived equivalences of posets

Flip-Flops

Let \((X, \leq_X), (Y, \leq_Y)\) be posets, \(f : X \rightarrow Y\) order-preserving.

Define two partial orders \(\leq^+_f, \leq^-_f\) on \(X \sqcup Y\) as follows:

- Keep the original partial orders inside \(X\) and \(Y\).
- Add the relations

\[
\begin{align*}
  x \leq^+_f y & \iff f(x) \leq_Y y \\
  y \leq^-_f x & \iff y \leq_Y f(x)
\end{align*}
\]

for \(x \in X, y \in Y\).

**Theorem.** \((X \sqcup Y, \leq^+_f) \sim (X \sqcup Y, \leq^-_f)\).
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Flip-Flop – Example

\[(X \sqcup Y, \leq_f)\]

\[(X \sqcup Y, \leq_f^-)\]
Application – Posets of cluster tilting objects

$Q$ – quiver without oriented cycles, $k$ – field

$\mathcal{T}_Q$ – poset of cluster tilting objects in the cluster category of $kQ$ [BMRRT, CCS, FZ]

When $Q$ is a Dynkin diagram of type $A$, $D$, or $E$, the poset $\mathcal{T}_Q$ is known as a Cambrian lattice [Reading], which is a quotient of the weak order on the corresponding Coxeter group.

In type $A$ with the linear orientation, we get the Tamari lattices. Their Hasse diagrams are the 1-skeletons of polytopes known as the Stasheff Associahedra.
Tamari Lattices for $A_1$ and $A_2$

$A_1$:

$T \bullet \quad (ab)c \rightarrow a(bc)$

$A_2$:

$T \bullet \rightarrow \bullet \quad (((ab)c)d \quad ((ab)c)d \quad (ab)(cd) \quad (a(bc))d \quad (a(bc))d \quad a((bc)d) \quad a((bc)d) \quad a(b(cd)) \quad a(b(cd))$
Tamari Lattice for $A_3$
Flip-flops on posets of cluster tilting objects

$Q$ – quiver without oriented cycles.

$x$ – a sink in $Q$.

$Q'$ – the *BGP reflection* with respect to $x$.

**Theorem.** $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$ are related via a flip-flop.

$$
\mathcal{T}_Q \simeq (T_Q^x \sqcup T_Q \setminus T_Q^x, \leq^f \cup) \hspace{1cm} \mathcal{T}_{Q'} \simeq (T_{Q'}^x \sqcup T_{Q'} \setminus T_{Q'}^x, \leq^{f'})
$$

**Corollary.** If $Q_1 \sim Q_2$ then $\mathcal{T}_{Q_1} \sim \mathcal{T}_{Q_2}$. 
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**Generalized BGP reflections**

Let \((Y, \leq)\) be a poset, \(Y_0 \subseteq Y\) a subset with the property

\[
[y, \cdot] \cap [y', \cdot] = \phi = [\cdot, y] \cap [\cdot, y'] \quad \text{for all } y \neq y' \in Y_0
\]

Define two partial orders \(\leq_{Y_0}^+, \leq_{Y_0}^-\) on \(\{\ast\} \cup Y\) as follows:

- Keep the original partial order inside \(Y\).
- Add the relations

\[
\ast <_{Y_0}^+ y \iff \exists y_0 \in Y_0 \text{ with } y_0 \leq y \\
y <_{Y_0}^- \ast \iff \exists y_0 \in Y_0 \text{ with } y \leq y_0
\]

for \(y \in Y\).
The vertex \(*\) is a \textit{source} in the Hasse diagram of \(\leq_{Y_0}^+\), with arrows ending at the vertices of \(Y_0\).

The Hasse diagram of \(\leq_{Y_0}^-\) is obtained by reverting the orientations of the arrows from \(*\), making it into a \textit{sink}.

**Theorem.** \(\left({\ast}\cup Y, \leq_{Y_0}^+\right) \sim \left({\ast}\cup Y, \leq_{Y_0}^-\right)\).

**Example.**

\[
\begin{array}{c}
\text{The vertex } \ast \text{ is a source in the Hasse diagram of } \leq_{Y_0}^+, \text{ with arrows ending at the vertices of } Y_0. \\
\text{The Hasse diagram of } \leq_{Y_0}^- \text{ is obtained by reverting the orientations of the arrows from } \ast, \text{ making it into a sink.} \\
\textbf{Theorem.} \left({\ast}\cup Y, \leq_{Y_0}^+\right) \sim \left({\ast}\cup Y, \leq_{Y_0}^-\right). \\
\textbf{Example.}
\end{array}
\]
Hybrid construction – setup

$(X, \leq_X), (Y, \leq_Y)$ – posets, $\{Y_x\}_{x \in X}$ – collection of subsets $Y_x \subseteq Y$, with the properties:

- For all $x \in X$,
  \[
  [y, \cdot] \cap [y', \cdot] = \phi = [\cdot, y] \cap [\cdot, y'] \quad \text{for all } y \neq y' \text{ in } Y_x
  \]

- For all $x \leq x'$, there exists an isomorphism $\varphi_{x,x'} : Y_x \xrightarrow{\sim} Y_{x'}$ with
  \[
  y \leq_Y \varphi_{x,x'}(y) \quad \text{for all } y \in Y_x
  \]

It follows that $\{Y_x\}_{x \in X}$ is a local system of subsets of $Y$:

\[
\varphi_{x,x''} = \varphi_{x',x''} \varphi_{x,x'} \quad \text{for all } x \leq x' \leq x''.
\]
Hybrid construction – result

Define two partial orders on $\leq_+$, $\leq_-$ on $X \sqcup Y$ as follows:

- Keep the original partial orders inside $X$ and $Y$.
- Add the relations

\[
\begin{align*}
x \leq_+ y & \iff \exists y_x \in Y_x \text{ with } y_x \leq_Y y \\
y \leq_- x & \iff \exists y_x \in Y_x \text{ with } y \leq_Y y_x
\end{align*}
\]

for $x \in X$, $y \in Y$.

**Theorem.** $(X \sqcup Y, \leq_+) \sim (X \sqcup Y, \leq_-)$.

**Remarks.**

- When $X = \{\ast\}$, we recover the generalized BGP reflection.
- When $Y_x = \{\ast\}$ for all $x \in X$, we recover the flip-flop.
Mirroring with respect to a bipartite structure

Let $S$ be bipartite. ($S = S_0 \sqcup S_1$ with $s < s' \Rightarrow s \in S_0$ and $s' \in S_1$)

Let $\mathcal{X} = \{X_s\}_{s \in S}$ be a collection of posets indexed by $S$.

Define two partial orders $\leq_+$ and $\leq_-$ on $\bigcup_{s \in S} X_s$ as follows:

- Keep the original partial order inside each $X_s$.
- Add the relations

  $$x_s <_+ x_t \iff s < t \iff x_t <_-_ x_s$$

  for $x_s \in X_s$, $x_t \in X_t$.

**Theorem.** $(\bigcup_{s \in S} X_s, \leq_+) \overset{u}{\sim} (\bigcup_{s \in S} X_s, \leq_-).$
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Bipartite structure – example

\[ S = \]

\[ \mathcal{X} = \]

\[ (\bigcup_{s \in S} X_s, \leq_{\scriptscriptstyle +}) \]

\[ (\bigcup_{s \in S} X_s, \leq_{\scriptscriptstyle -}) \]
Mates of triangular matrix algebras

Let \( k \) be a field, \( R \) and \( S \) \( k \)-algebras and \( R^M S \) bimodule. Consider the triangular matrix algebras

\[
\Lambda = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \quad \text{and} \quad \tilde{\Lambda} = \begin{pmatrix} S & DM \\ 0 & R \end{pmatrix}
\]

where \( DM = \text{Hom}_k(M, k) \).

**Theorem.** \( \mathcal{D}^b(\text{Mod} \Lambda) \cong \mathcal{D}^b(\text{Mod} \tilde{\Lambda}) \), under the assumptions:

- \( \dim_k M < \infty \)
- \( \dim_k S < \infty, \text{gl.dim} S < \infty \)