Seminar on Steenrod Operations and Homotopy Groups of Spheres

Time and venue: Tuesdays, 14:15-15:45, SR 1.007
Prerequisites: Topologie I, Topologie II, Algebraic Topology I

In this seminar, we will study stable operations on mod p singular cohomology, i.e. natural transformations

$$\theta\colon H^*(X;\mathbb{F}_p) \to H^{*+i}(X;\mathbb{F}_p)$$

that respect the suspension isomorphisms. The $\mathbb{F}_p$-algebra $\mathcal{A}_p$ of all such operations is called the mod p *Steenrod algebra* and acts in a natural way on the cohomology of any space. The resulting $\mathcal{A}_p$-module structure on $H^*(X;\mathbb{F}_p)$ provides additional information about the space X; e.g., $\Sigma\mathbb{C}P^2$ and $S^3 \vee S^5$ have cohomology rings that agree as $\mathbb{F}_p$-algebras, but are different as modules over the Steenrod algebra.

A choice of generators for the mod 2 Steenrod algebra $\mathcal{A}_2$ is given by the *Steenrod squares*

$$Sq^i\colon H^n(X;\mathbb{F}_2) \to H^{n+i}(X;\mathbb{F}_2);\ i \geq 0.$$

They are uniquely determined by the following axioms:

(1) They are natural with respect to continuous maps of topological spaces.

(2) $Sq^0 = \mathrm{id}$

(3) $Sq^n(x) = x^2$ for $x \in H^n(X;\mathbb{F}_2)$

(4) $Sq^n(x) = 0$ for $n > deg(x)$

(5) The *Cartan formula* holds: $Sq^n(xy) = \sum_{i+j=n} Sq^i(x)Sq^j(y)$.

(The mod p Steenrod algebra for odd p is generated by the Bockstein homomorphism and the *Steenrod powers* P^i that can be characterized in a similar way. We will mostly focus on the case $p = 2$.)

Famous applications include:

- detecting non-trivial elements in the stable homotopy groups of spheres, e.g. the generator $\Sigma\eta \in \pi_4(S^3) \cong \mathbb{Z}/2$
- a map $f\colon S^{2n-1} \to S^n$ cannot have Hopf invariant one if n is not a power of 2
- the non-existence of 2^m linearly independent vector fields on S^{n-1} if $n = 2^m(2s+1)$.

A theorem of Milnor exhibits the dual Steenrod algebra $\mathcal{A}_p^*$ as a free graded commutative algebra on a certain set of generators. In the final talks, we will study projective resolutions over $\mathcal{A}_2^*$ to set up the *Adams spectral sequence*, a powerful tool that computes the 2-torsion in the stable homotopy groups of spheres $\pi_i^s = \mathrm{colim}_k\, \pi_{i+k}(S^k)$.

Overview of talks

It is highly recommended to consult with the organizers 1-2 weeks before your talk.

1. **Axiomatic description and first applications** (Piet Glas) .... April 2
(Stable) cohomology operations [Ha, p. 488] [Mo-Ta, Ch. 1, pp. 2-4]. Axiomatic description of Steenrod squares and powers [St-Ep, Ch. I, §1; Ch. VI, §1].
Easy applications: calculation of the Steenrod action on $\mathbb{R}P^\infty$; non-triviality of suspensions of Hopf maps, non-existence of stable splittings of projective space [Ha, pp. 489-492] [St-Ep, Ch. I, §2].
If time permits: proof that the axioms determine the Sq^is uniquely [St-Ep, Ch. VII, §3].

2. **Construction of the Steenrod squares** (Dimitris Oikomonou) .... April 9
Failure of (signed) commutativity of the cup-product on cochain level [Br, VI, §16].
Construction of Sq^is via $\cup_i$-products [Mo-Ta, Ch. 2, pp. 14-20].

3. **Properties of the Steenrod squares** (Solomiya Mizyuk) .... April 16
Verification of the axioms; $\mathbb{F}_2$-cohomology of $K(\mathbb{Z}/n; n)$ [Mo-Ta, Ch. 3, pp. 22-28]

4. **The Adem relations** (Domenico Marasco) .... April 30
Proof of the relations; examples.
[Mo-Ta, Ch. 3, pp. 29-31], [Bu-MacD]

5. **The Hopf invariant** (Marco La Vecchia) .... May 7
Definition, properties, application: non-existence of maps with Hopf invariant one.
[Mo-Ta, Ch. 4, pp. 33-38]

6. **Vector fields on spheres** (Jerónimo García) .... May 14
Stiefel manifolds, cell decomposition, number of vector fields.
[Mo-Ta, Ch. 5, pp. 39-44]

7. **The Steenrod algebra** (Miguel Barrero) .... May 21
Steenrod algebra, decomposable elements, Hopf algebras.
[Mo-Ta, Ch. 5, pp. 45-50]

8. **The dual of the Steenrod algebra** (Sil Linskens) .... May 28
The dual algebra, Milnor basis, theorem of Milnor-Moore.
[Mo-Ta, Ch. 5, pp. 50-57], [Mi]

9. **Some homotopy groups of spheres I** (N.N.) .... June 4
An approximation to S^n, first part of the calculation of π_i^s for $i \leq 7$ [Mo-Ta, Ch. 12, pp. 112-118].

10. **Some homotopy groups of spheres II** (Vincent Grande) .... June 25
Second part of the calculation of π_i^s for $i \leq 7$ [Mo-Ta, Ch. 12, pp. 118-122]

11. **The Adams spectral sequence I** (Avital Berry) . . . . July 2
Setting up the spectral sequence [Rav, Ch. 2, §1] (or alternatively [Mo-Ta, Ch. 18, pp. 185-199])

12. **The Adams spectral sequence II** (Mihail Arabadji) . . . . July 9
Computation of π_i^s for $i \leq 14$ [Mo-Ta, Ch. 18, pp. 199-205].

References

[Bu-MacD] S.R. Bullet, I.G. Macdonald: *On the Adem relations.* Topology 21 (1982), 329-332.

[Br] G.E. Bredon: *Topology and Geometry.* Graduate Texts in Mathematics, vol. 139, Springer-Verlag (1993, corrected 3rd print 1997).

[Ha] A. Hatcher: *Algebraic Topology.* Cambridge University Press (2002).

[Mi] J. Milnor: *The Steenrod algebra and its dual.* Ann. Math. (2) 67 (1958), 150-171.

[Mo-Ta] R.E. Mosher, M.C. Tangora: *Cohomology Operations and Applications in Homotopy Theory.* Harper & Row Publishers (1968).

[Rav] D.C. Ravenel: *Complex cobordism and stable homotopy groups of spheres.* Pure and Applied Mathematics 121 (1986).

[St-Ep] N.E. Steenrod. D.B.A. Epstein: *Cohomology Operations.* Princeton University Press (1962).