SPACES VERSUS SIMPLICIAL SETS

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Abstract. This documents contains lecture notes on the relationship between simplicial sets and topological spaces. I make no claim to originality.

Contents

1. Notation, terminology and recollections

The category Δ has the objects $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$, and as morphisms all weakly monotone maps. A *simplicial set* is a contravariant functor from Δ to the category of sets; a *morphism* of simplicial sets is a natural transformation of functors. We write **sset** for the category of simplicial sets.

If $X : \Delta^{\mathrm{op}} \longrightarrow$ (sets) is a simplicial set, we write

$$X_n = X([n])$$

and for a morphism $\alpha : [m] \longrightarrow [n]$ we write

$$\alpha^* = X(\alpha) : X_n \longrightarrow X_m .$$

Elements of X_n are called *n-simplices* of X.

An n-simplex x of a simplicial set X is *degenerate* if $x = s_i^*(y)$ for some $0 \leq i \leq n-1$ and some $y \in X_{n-1}$. And x is *non-degenerate* if it is not degenerate. If x is an arbitrary n-simplex of X, then there is a unique pair (σ, z) consisting of a surjective morphism $\sigma : [n] \longrightarrow [l]$ in Δ and a non-degenerate simplex $z \in X_l$ such that $x = \sigma^*(z)$.

We recall the Yoneda lemma for simplicial sets. The *simplicial n-simplex* is the functor

$$\Delta^n = \Delta(-, [n]) : \Delta^{\mathrm{op}} \longrightarrow (sets)$$

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represented by the object $[n]$ of Δ. For every simplicial set X, all $n \geq 0$ and every n-simplex $x \in X_n$, there is a unique morphism

$$x^\flat \ : \ \Delta^n \ \longrightarrow \ X$$

such that $x^\flat_n(\mathrm{Id}_{[n]}) = x$. We refer to $x^\flat$ as the *characteristic morphism* of the n-simplex x; it is given in dimensions m by

$$x^\flat_m \ : \ (\Delta^n)_m \ = \ \Delta([m],[n]) \ \longrightarrow \ X_m \ , \quad x^\flat_m(\alpha) \ = \ \alpha^*(x) \ .$$

The *boundary* $\partial\Delta^n$ is the simplicial subset of Δ^n with

$$(\partial\Delta^n)_m \ = \ \{\alpha \in \Delta([m],[n]) \ : \ \ \alpha \text{ is not surjective}\} \ .$$

Example 1.1. The k-simplices of Δ^n are all weakly monotone maps from $[k]$ to $[n]$. Such a map $\alpha : [k] \longrightarrow [n]$ is non-degenerate as a simplex of Δ^n if and only if it is injective.

2. Minimal representatives for geometric realization

The *topological n-simplex* is

$$\nabla^n \ = \ \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \ : \ t_i \geq 0, \ t_0 + \cdots + t_n = 1\} \ .$$

This space is the convex hull of the standard basis vectors $e_0, e_1, \dots, e_n$ of $\mathbb{R}^{n+1}$, where $e_i = (0, \dots, 0, 1, 0, \dots)$ with 1 in the $(i+1)$st coordinate. The topological simplices assemble into a covariant functor $\nabla^\bullet : \Delta \longrightarrow \mathbf{Top}$ by sending a morphism $\alpha : [m] \longrightarrow [n]$ in Δ to the continuous map

$$\alpha_* \ : \ \nabla^m \ \longrightarrow \nabla^n \ , \quad (\alpha(t_0, \dots, t_m))_j \ = \ \sum_{\alpha(i)=j} t_i \ .$$

The map α_* can be characterized as the unique affine linear map whose behavior on the vertices is given by $\alpha_*(e_i) = e_{\alpha(i)}$.

The *geometric realization* of a simplicial set X is the space

$$|X| \ = \ \Big(\coprod\nolimits_{n\geq 0} X_n \times \nabla^n\Big) / \sim \ ;$$

here the set X_n is given the discrete topology, so that $X_n \times \nabla^n$ is a topological disjoint union of copies of ∇^n indexed by X_n. The equivalence relation '$\sim$' is generated by the relation

$$(x, \alpha_*(t)) \ \sim \ (\alpha^*(x), t) \tag{2.1}$$

for all $x \in X_n$, $t \in \nabla^m$ and $\alpha : [m] \longrightarrow [n]$ a morphism in Δ. In more abstract categorical terms, the realization is the *coend* of the functor

$$\Delta^{\mathrm{op}} \times \Delta \ \longrightarrow \ \mathbf{Top} \ , \quad ([m],[n]) \ \longmapsto \ X_m \times \nabla^n \ .$$

Because the generating relation (2.1) for the geometric realization is not symmetric, it is not immediately obvious when exactly points in the disjoint union of the spaces $X_n \times \nabla^n$ are equivalent. We will now argue that the equivalence classes have unique *minimal representatives*, which are moreover easy to characterize.

Proposition 2.2. *Let X be a simplicial set.*

(i) *Every equivalence class under the equivalence relation generated by* (2.1) *has a unique representative $(x, t) \in X_l \times \nabla^l$ of minimal dimension l.*

(ii) *An element* (y,s) *of* $X_n \times \nabla^n$ *is the minimal representative in its equivalence class if and only if* y *is non-degenerate and* s *belongs to the interior of* ∇^n*.*

(iii) *If* $(x,t) \in X_l \times \nabla^l$ *is the minimal representative in the equivalence class of* $(y,s) \in X_n \times \nabla^n$*, then there is a unique triple*

$$(\delta, \sigma, u)$$

consisting of an injective morphism $\delta : [k] \longrightarrow [n]$*, a surjective morphism* $\sigma : [k] \longrightarrow [l]$*, and an interior point* u *of* Δ^k *such that*

$$\delta^*(y) \;=\; \sigma^*(x) \ , \quad s \;=\; \delta_*(u) \quad \textit{and} \quad t = \sigma_*(u) \ .$$

Proof. We write X_l^{nd} for the set of non-degenerate l-simplices of X, and we write

$$\mathrm{int}(\nabla^l) \;=\; \{(t_0, \ldots, t_l) \in \nabla^l \ : \ t_0 > 0, \ldots, t_l > 0\}$$

for the interior of the topological l-simplex. We define a map

$$\rho \ : \ \coprod_{n \geq 0} X_n \times \nabla^n \ \longrightarrow \ \coprod_{l \geq 0} X_l^{\mathrm{nd}} \times \mathrm{int}(\nabla^l) \tag{2.3}$$

such that for $\rho(y,s)$ is equivalent to (y,s).

We consider any pair $(y,s) \in X_n \times \nabla^n$. We suppose that $s = (s_0, \ldots, s_n)$. Since the numbers $s_0, \ldots, s_n$ are non-negative and sum up to 1, at least one of them must be positive. We suppose that $k+1$ of the real numbers $s_0, \ldots, s_n$ are positive. We let $u = (u_0, \ldots, u_k)$ be the sequence obtained from $(s_0, \ldots, s_n)$ by deleting the 0's and keeping the other entries in their order. Then there is a unique injective monotone map $\delta : [k] \longrightarrow [n]$ such that

$$s = (s_0, \ldots, s_n) \;=\; \delta_*(u_0, \ldots, u_k) \;=\; \delta_*(u) \ .$$

Then

$$(y,s) = (y, \delta_*(u)) \ \sim \ (\delta^*(y), u) \ .$$

The k-simplex $\delta^*(y)$ is of the form

$$\delta^*(y) \;=\; \sigma^*(x)$$

for a unique surjective morphism $\sigma : [k] \longrightarrow [l]$ in Δ and a unique non-degenerate simplex $x \in X_l^{\mathrm{nd}}$. Thus

$$(\delta^*(y), u) = (\sigma^*(x), u) \ \sim \ (x, \sigma_*(u)) \ .$$

Since all coordinates of u are positive and $\sigma_* : \nabla^k \longrightarrow \nabla^l$ is summing up coordinates, all coordinates of $\sigma_*(u)$ are again positive, so $\sigma_*(u)$ is an interior point of ∇^l. We can thus define

$$\rho(y,s) \;=\; (x, \sigma_*(u)) \ \in \ X_l^{\mathrm{nd}} \times \mathrm{int}(\nabla^l) \ .$$

Claim: If $(y,s) \in X_n \times \nabla^n$ and $(\bar{y}, \bar{s}) \in X_{\bar{n}} \times \nabla^{\bar{n}}$ are equivalent, then $\rho(y,s) = \rho(\bar{y}, \bar{s})$. It suffices to show the claim whenever (y,s) and $(\bar{y}, \bar{s})$ are related by a generating relation (2.1) i.e., we can assume that $y = \alpha^*(\bar{y})$ and $\bar{s} = \alpha_*(s)$ for some morphism $\alpha : [n] \longrightarrow [\bar{n}]$. We let (δ, u, σ, x) be as in the construction of $\rho(y,s)$. We choose a factorization (necessarily unique)

$$\alpha \circ \delta \;=\; \bar{\delta} \circ \bar{\sigma} \ : \ [k] \ \longrightarrow \ [\bar{n}] \tag{2.4}$$

as a surjective morphism $\bar{\sigma} : [k] \longrightarrow [\bar{k}]$ followed by an injective morphism $\bar{\delta} : [\bar{k}] \longrightarrow [\bar{n}]$. Then

$$\bar{s} \;=\; \alpha_*(s) \;=\; \alpha_*(\delta_*(u)) \;=\; \bar{\delta}_*(\bar{\sigma}_*(u)) \ .$$

Since u is an interior point of ∇^k and $\bar{\sigma}$ is injective, $\bar{\sigma}_*(u)$ is an interior point of $\nabla^{\bar{k}}$. This shows that $(\bar{\delta}, \bar{\sigma}_*(u))$ is the data in the first step of the construction of $\rho(\bar{y}, \bar{s})$.

Then we write

$$\bar{\delta}^*(\bar{y}) \ = \ \hat{\sigma}^*(\hat{x})$$

for a surjective morphism $\hat{\sigma} : [\bar{k}] \longrightarrow [\hat{l}]$ and a non-degenerate element $\hat{x} \in X^{\text{nd}}_{\hat{l}}$ (necessarily unique). Then

$$\begin{aligned} \sigma^*(x) \ &= \ \delta^*(y) \ = \ \delta^*(\alpha^*(\bar{y})) \\ \overset{(2.4)}{=} \ &\bar{\sigma}^*(\bar{\delta}^*(\bar{y})) \ = \ \bar{\sigma}^*(\hat{\sigma}^*(\hat{x})) \ = \ (\hat{\sigma}\bar{\sigma})^*(\hat{x}) \ . \end{aligned}$$

Since x and $\hat{x}$ are non-degenerate and σ and $\hat{\sigma}\bar{\sigma}$ are surjective, the uniqueness shows that $l = \hat{l}$, $\sigma = \hat{\sigma}\bar{\sigma}$ and $x = \hat{x}$. We conclude that

$$\bar{\delta}^*(\bar{y}) \ = \ \hat{\sigma}^*(\hat{x}) \ = \ \hat{\sigma}^*(x) \ .$$

Altogether this shows that $(\bar{\delta}, \bar{\sigma}_*(u), \hat{\sigma}, x)$ is the data in the construction of $\rho(\bar{y}, \bar{s})$. So

$$\rho(\bar{y}, \bar{s}) \ = \ [x, \hat{\sigma}_*(\bar{\sigma}_*(u))] \ = \ (x, \sigma_*(u)) \ = \ \rho(y, s) \ .$$

This proves the claim.

Now we can prove the proposition.

(i) Suppose that (y, s) is of minimal dimension among all pairs in its equivalence class. In the construction of $\rho(y, s)$ the map $\delta : [k] \longrightarrow [n]$ is injective and $\sigma : [k] \longrightarrow [l]$ is surjective, so we have $n \geq k \geq l$. Since (y, s) is of minimal dimension in its equivalence class and equivalent to $\rho(y, s)$, we must have $n = k = l$. But then $\delta = \sigma = \mathrm{Id}_{[n]}$, and so $(y, s) = \rho(y, s)$. If (y', s') is another representative of minimal dimension in the same equivalence class, then

$$(y, s) \ = \ \rho(y, s) \ = \ \rho(y', s') \ = \ (y', s') \ ,$$

where the second equation is the above claim. We record for later use that we have also shown that the unique minimal representative in the equivalence class of (y, s) is $\rho(y, s)$.

(ii) We showed in the proof of (i) that the minimal representative in every equivalence class of (y, s) is $\rho(y, s)$; this consists, by construction, of a non-degenerate simplex and interior point. Now suppose conversely that (y, s) is a pair of a non-degenerate simplex and an interior point. In the construction of $\rho(y, s)$ we must then have $\delta = \mathrm{Id}_{[n]}$ (because s is an interior point); because $y = \delta^*(y) = \sigma^*(x)$ is non-degenerate, we must have $\sigma = \mathrm{Id}_{[n]}$. So $(y, s) = \rho(y, s)$; we showed in the proof of (i) that therefore, (y, s) is the minimal representative in its equivalence class.

(iii) We showed in the proof of part (i) that $(x, t) = \rho(y, s)$ is the minimal representative in the class of (y, s), and the morphisms σ and δ and the point u were constructed in the definition of $\rho(y, s)$. To prove the uniqueness clause we observe that we had no other choice in the definition of $\rho(y, s)$: because u is non-degenerate, all of its coordinates are non-zero. So the relation $s = \delta_*(u)$ forces k to be one less than the number of non-zero coordinates of s, and $\delta : [k] \longrightarrow [n]$ is unique determined by the positions of the 0s in s. The pair (σ, x) is then determined because $\delta^*(y)$ is uniquely a degeneracy of some non-degenerate simplex. □

Corollary 2.5. *Let $f : X \longrightarrow Y$ be a morphism of simplicial sets such that $f_n : X_n \longrightarrow Y_n$ is injective for every $n \geq 0$.*

(i) *For every non-degenerate n-simplex x of X, the simplex $f_n(x)$ is non-degenerate.*
(ii) *The continuous map $|f| : |X| \longrightarrow |Y|$ is injective.*

Proof. (i) We let $x \in X_n$ be non-degenerate. We argue by contradiction and suppose that $f_n(x) = s_i^*(y)$ for some $0 \leq i \leq n-1$ and $y \in Y_{n-1}$. Then

$$f_n(s_i^*(d_i^*(x))) \;=\; s_i^*(d_i^*(f_n(x))) \;=\; s_i^*(d_i^*(s_i^*(y))) \;=\; s_i^*(y) \;=\; f_n^*(x)\ .$$

Because f_n is injective, we conclude that $x = s_i^*(d_i^*(x))$. This contradicts the hypothesis that x is non-degenerate.

(ii) We let $(x,t) \in X_n \times \nabla^n$ and $(x',t') \in X_m \times \nabla^m$ be the minimal representatives of two points in $|X|$ such that $|f|[x,t] = |f|[x',t']$. Then

$$[f_n(x),t] \;=\; |f|[x,t] \;=\; |f|[x',t'] \;=\; [f_m(x'),t']\ .$$

As minimal representatives, the simplices x and x' are non-degenerate and the points t and t' are interior points in the respective simplices. By part (i) the simplices $f_n(x)$ and $f_m(x')$ are again non-degenerate. So $(f_n(x),t)$ and $(f_m(x'),t')$ are minimal representatives of the same equivalence class in $|Y|$. Uniqueness of minimal representatives thus forces

$$n = m\ , \quad f_n(x) = f_m(x') \qquad \text{and} \qquad t = t'\ .$$

Because $f_n = f_m$ is injective, also $x = x'$. So $[x,t] = [x',t']$, and we have shown that $|f|$ is injective. □

Remark 2.6. Something stronger than stated in Corollary 2.5 is actually true. For every morphism of simplicial sets $f : X \longrightarrow Y$ that is dimensionwise injective, the continuous map $|f| : |X| \longrightarrow |Y|$ is closed and a homeomorphism onto its image. Even better: with respect to the preferred CW-structures, the map $|f|$ is a cellular homeomorphism onto a CW-subcomplex of $|Y|$. We will show these facts in Theorem 3.8 below.

Corollary 2.7. *For every simplicial set X, the composite*

$$\coprod\nolimits_{n\geq 0} X_n^{\mathrm{nd}} \times \nabla^n \xrightarrow{\text{inclusion}} \coprod\nolimits_{n\geq 0} X_n \times \nabla^n \xrightarrow{\text{quotient}} |X|$$

is surjective.

For example, if X has only finitely many non-degenerate simplices altogether, then the source of the above composite is compact, and hence the realization $|X|$ is quasi-compact. We will see below that much more is true: if X has only finitely many non-degenerate simplices, $|X|$ is a finite CW-complex.

Similarly, the existence, characterization and uniqueness of minimal representations are equivalent to the fact that the continuous composite

$$\coprod\nolimits_{n\geq 0} X_n^{\mathrm{nd}} \times \mathrm{int}(\nabla^n) \xrightarrow{\text{inclusion}} \coprod\nolimits_{n\geq 0} X_n \times \nabla^n \xrightarrow{\text{quotient}} |X|$$

is bijective, where the source is now the disjoint union of the *open* simplices. This continuous bijection is typically *not* a homeomorphism. We will later recognize the images of the sets $\{x\} \times \mathrm{int}(\nabla^n)$ as the open cells in the preferred CW-structure on $|X|$, see Theorem 3.8 below.

The two maps

$$\begin{aligned} |\Delta^m| &\longrightarrow \nabla^m\ , & [\alpha,t] &\longmapsto \alpha_*(t) \\ \nabla^m &\longrightarrow |\Delta^m|\ , & t &\longmapsto [\mathrm{Id}_{[m]},t] \end{aligned} \tag{2.8}$$

are mutually inverse homeomorphisms. The next proposition shows that these homeomorphisms also identify the realization of $|\partial\Delta^m|$ with the boundary

$$\partial\nabla^m \ = \ \{(t_0,\dots,t_m)\in\nabla^m \ : \ t_i=0 \text{ for some } 0\leq i\leq m\}$$

of the topological m-simplex.

Proposition 2.9. *The composite*

$$|\partial\Delta^m| \ \xrightarrow{|\text{inclusion}|} \ |\Delta^m| \ \xrightarrow[\cong]{[\alpha,t]\mapsto\alpha_*(t)} \ \nabla^m$$

is a homeomorphism onto the closed subspace $\partial\nabla^m$ of ∇^m.

Proof. The first map induced by the inclusion $\partial\Delta^m\longrightarrow\Delta^m$ is injective by Corollary 2.5. The non-degenerate simplices of $\partial\Delta^m$ are all injective morphisms $\alpha:[k]\longrightarrow[m]$ *except* $\mathrm{Id}_{[m]}$. Since $\partial\Delta^m$ has only finitely many non-degenerate simplices, the realization $|\partial\Delta^m|$ is quasi-compact by Corollary 2.7. Since ∇^m is a Hausdorff space, the composite is a closed map, and so an homeomorphism onto its image. Again by Corollary 2.7, the image of the composite is the union of the images of the maps

$$\alpha_* \ :\nabla^k \ \longrightarrow \ \nabla^m$$

for all injective morphisms $\alpha:[k]\longrightarrow[m]$ other than $\mathrm{Id}_{[m]}$. This shows that the image of the composite coincides with the boundary of ∇^m. □

3. The preferred CW-structure on a geometric realization

In this section we will show that the geometric realization $|X|$ of a simplicial set X comes with a preferred CW-structure. The strategy to construct it is to observe that the skeleton filtration and the 'cell attachments' already exist in the world of simplicial sets; because geometric realization is left adjoint to the singular complex functor, it preserves colimits such as the simplicial cell attachments.

We will also see that the CW-structure on $|X|$ construction does not involve any choices; as a result, morphisms of simplicial sets will realize to cellular maps for the preferred CW-structures.

Construction 3.1 (Simplicial skeleta)**.** We let X be a simplicial set, and we consider a natural number $m\geq 0$. The *m-skeleton* is the simplicial subset $\mathrm{sk}^m X$ of X defined by

$$(\mathrm{sk}^m X)_n \ = \ \{x\in X_n\colon x=\alpha^*(y) \text{ for some } y\in X_m \text{ and } \alpha:[n]\longrightarrow[m]\} \ .$$

The sets $(\mathrm{sk}^m X)_n$ are clearly closed under the simplicial structure maps of X, so they indeed form a simplicial subset. By definition, $\mathrm{sk}^m X$ is the smallest simplicial subset of X that contains all m-simplices of X.

Example 3.2. Every constant simplicial set is 0-dimensional, i.e., it coincides with its 0-skeleton. Conversely, every simplicial set X with $X=\mathrm{sk}^0 X$ is isomorphic to the constant simplicial set with value X_0.

Example 3.3. The m-simplex Δ^m is 'm-dimensional' in the sense that $\mathrm{sk}^m(\Delta^m)=\Delta^m$. The $(m-1)$-skeleton of the simplicial m-simplex is its boundary:

$$\mathrm{sk}^{m-1}(\Delta^m) \ = \ \partial\Delta^m \ .$$

Proposition 3.4. *Let X be a simplicial set and $m\geq 0$.*

(i) *For* $n \leq m$*, we have* $(\mathrm{sk}^m X)_n = X_n$*.*
(ii) *For* $n > m$*, every* n*-simplex of* $\mathrm{sk}^m X$ *is degenerate.*
(iii) *The simplicial set* $\mathrm{sk}^m X$ *is contained in* $\mathrm{sk}^{m+1} X$*.*
(iv) *The simplicial set* X *is a colimit of the sequence of simplicial sets*

$$\mathrm{sk}^0 X \ \subseteq \ \mathrm{sk}^1 X \ \subseteq \ \mathrm{sk}^2 X \ \subseteq \ \dots \ \subseteq \ \mathrm{sk}^m X \ \subseteq \ \dots \ .$$

(v) *Every morphism* $f : X \longrightarrow Y$ *of simplicial sets sends* $\mathrm{sk}^m X$ *to* $\mathrm{sk}^m Y$*.*

Proof. (i) For $n \leq m$ we can choose an injective morphism $\alpha : [n] \longrightarrow [m]$ and a surjective morphism $\sigma : [m] \longrightarrow [n]$ in the category Δ such that $\sigma \circ \alpha = \mathrm{Id}_{[n]}$. Then every $x \in X_n$ satisfies

$$x \ = \ (\sigma \circ \alpha)^*(x) \ = \ \alpha^*(\sigma^*(x)) \ ,$$

so x belongs to $(\mathrm{sk}^m X)_n$.

(ii) We suppose that $n > m$, and that $x = \alpha^*(y)$ for some $y \in X_m$ and some morphism $\alpha : [n] \longrightarrow [m]$. Since n is larger than m, the morphism α cannot be injective, so there is an $i \in \{0, \dots, n-\}$ such that $\alpha(i) = \alpha(i+1)$. Then $\alpha = \beta \circ s_i$ for some morphism $\beta : [n-1] \longrightarrow [m]$, and hence

$$x \ = \ \alpha^*(y) \ = \ s_i^*(\beta^*(y))$$

is degenerate.

(iii) We consider an n-simplex of the form $x = \alpha^*(y)$ for some $\alpha : [n] \longrightarrow [m]$ and $y \in X_m$. Then

$$x \ = \ \alpha^*(y) \ = \ (s_0 \circ d_0 \circ \alpha)^*(y) \ = \ (d_0 \circ \alpha)^*(s_0^*(y)) \ ,$$

which shows that $x \in (\mathrm{sk}^{m+1} X)_n$.

(iv) Because the category of simplicial sets is a functor category, limits and colimits of simplicial sets are calculated objectwise. So it suffices to show that for every $n \geq 0$, the set X_n is a colimit of the sequence of sets

$$(\mathrm{sk}^0 X)_n \ \subseteq \ (\mathrm{sk}^1 X)_n \ \subseteq \ (\mathrm{sk}^2 X)_n \ \subseteq \ \dots \ \subseteq \ (\mathrm{sk}^m X)_n \ \subseteq \ \dots \ .$$

However, by part (i) this sequence stabilizes from the n-skeleton onward, i.e.,

$$X_n \ = \ (\mathrm{sk}^n X)_n \ = \ (\mathrm{sk}^{n+1} X)_n \ = \ (\mathrm{sk}^{n+2} X)_n \ = \ \dots \ ,$$

so X_n is a colimit of this sequence.

(v) If $x = \alpha^*(y) \in X_n$ for some $\alpha : [n] \longrightarrow [m]$ and $y \in X_m$, then

$$f_n(x) \ = \ f_n(\alpha^*(y)) \ = \ \alpha^*(f_m(y)) \ ,$$

so $f_n(x)$ belongs to $(\mathrm{sk}^m Y)_n$. □

Remark 3.5. We let $f : X \longrightarrow Y$ be any morphism of simplicial sets. By part (v) of the previous proposition, f restricts to a morphism $\mathrm{sk}^m f : \mathrm{sk}^m X \longrightarrow \mathrm{sk}^m Y$ between the m-skeleta. This extends the m-skeleton construction to a functor

$$\mathrm{sk}^m \ : \ \mathbf{sset} \ \longrightarrow \ \mathbf{sset} \ .$$

Moreover, for varying simplicial sets X, the inclusions $\mathrm{sk}^m X \longrightarrow \mathrm{sk}^{m+1} X$ and $\mathrm{sk}^m X \longrightarrow X$ are natural transformations $\mathrm{sk}^m \longrightarrow \mathrm{sk}^{m+1}$ and $\mathrm{sk}^m \longrightarrow \mathrm{Id}$ of endofunctors on the category of simplicial sets.

For the following proposition we record an observation. We let X be a simplicial set and $m \geq 0$. Let Y be a simplicial subset of X such that $X_{m-1} = Y_{m-1}$. Equivalently, $\mathrm{sk}^{m-1} Y = \mathrm{sk}^{m-1} X$. Then the following square commutes:

$$\begin{array}{ccccc} \partial\Delta^m & = & \mathrm{sk}^{m-1}\Delta^m & \longrightarrow & \Delta^m \\ \downarrow{\scriptstyle x^\flat|_{\partial\Delta^m}} & & \downarrow{\scriptstyle \mathrm{sk}^{m-1}(x^\flat)} & & \downarrow{\scriptstyle x^\flat} \\ \mathrm{sk}^{m-1} Y & = & \mathrm{sk}^{m-1} X & \longrightarrow & X \end{array}$$

In other words: the characteristic morphism of every m-simplex of X sends $\partial\Delta^m$ into the simplicial subset Y.

Proposition 3.6. *Let X be a simplicial set and $m \geq 0$. Let Y be a simplicial subset of X such that $X_{m-1} = Y_{m-1}$. Suppose moreover that for $n > m$, every simplex in $X_n \setminus Y_n$ is degenerate.*

(i) *The commutative square*

$$\begin{array}{ccc} \coprod_{x\in X_m\setminus Y_m} \partial\Delta^m & \xrightarrow{\mathrm{incl}} & \coprod_{x\in X_m\setminus Y_m} \Delta^m \\ \downarrow{\scriptstyle \coprod x^\flat|_{\partial\Delta^m}} & & \downarrow{\scriptstyle \coprod x^\flat} \\ Y & \xrightarrow[\mathrm{incl}]{} & X \end{array}$$

is a pushout square of simplicial sets.

(ii) *The commutative square*

$$\begin{array}{ccc} \coprod_{x\in X_m\setminus Y_m} |\partial\Delta^m| & \xrightarrow{\mathrm{incl}} & \coprod_{x\in X_m\setminus Y_m} |\Delta^m| \\ \downarrow{\scriptstyle \coprod |x^\flat||_{\partial\Delta^m}} & & \downarrow{\scriptstyle \coprod x^\flat} \\ |Y| & \xrightarrow[\mathrm{incl}]{} & |X| \end{array}$$

is a pushout square of topological spaces.

(iii) *The geometric realization $|X|$ can be obtained from $|Y|$ by attaching m-cells indexed by the set $X_m \setminus Y_m$.*

Proof. (i) Because the category of simplicial sets is a functor category, colimits of simplicial sets are calculated objectwise. So it suffices to show that for every $k \geq 0$, the following square is a pushout in the category of sets:

$$\begin{array}{ccc} \coprod_{x\in X_m\setminus Y_m} (\partial\Delta^m)_k & \xrightarrow{\mathrm{incl}} & \coprod_{x\in X_m\setminus Y_m} (\Delta^m)_k \\ \downarrow{\scriptstyle \coprod x^\flat_k|_{\partial\Delta^m}} & & \downarrow{\scriptstyle \coprod x^\flat_k} \\ Y_k & \xrightarrow[\mathrm{incl}]{} & X_k \end{array} \qquad (3.7)$$

Because the two horizontal maps are inclusions, the pushout property is equivalent to the property that the right vertical map restricts to a bijection

$$(X_m \setminus Y_m) \times ((\Delta^m)_k \setminus (\partial\Delta^m)_k) \;\longrightarrow\; X_k \setminus Y_k \ , \quad (x, \alpha) \longmapsto \alpha^*(x) \ .$$

The k-simplices of $\partial\Delta^m$ are all morphisms $\alpha : [k] \longrightarrow [m]$ that are not surjective. So we must show that the map

$$(X_m \setminus Y_m) \times \{\alpha : [k] \longrightarrow [m] \colon \ \alpha \text{ surjective}\} \ \longrightarrow \ X_k \setminus Y_k \ , \quad (x, \alpha) \longmapsto \alpha^*(x)$$

is bijective. Injectivity is the fact, proven earlier, that the representation of a simplex as a degeneracy of some non-degenerate simplex is unique. Surjectivity is trivial for $k < m$ (because then $X_k = Y_k$) and obvious for $k = m$. For $k > m$ we consider any simplex $x \in X_k \setminus Y_k$ we write $x = \alpha^*(\bar{x})$ for some surjective morphism $\alpha : [k] \longrightarrow [n]$ and some non-degenerate simplex $\bar{x} \in X_n$. Because x does not belong to Y_k, the simplex $\bar{x}$ does not belong to Y_n. So we must have $n \geq m$, because otherwise $X_n = Y_n$. But we must also have $n \leq m$, because otherwise all simplices in $X_n \setminus Y_n$ are degenerate by hypothesis. Hence $n = m$, and so x is in the image of the map.

(ii) The geometric realization functor $|-| : \textbf{sset} \longrightarrow \textbf{Top}$ is left adjoint to the singular complex functor $\mathcal{S} : \textbf{Top} \longrightarrow \textbf{sset}$. So geometric realization preserves colimits, in particular coproducts and pushouts.

(iii) Part (ii) shows that $|X|$ can be obtained from $|Y|$ by attaching copies of $|\Delta^m|$ along $|\partial\Delta^m|$ indexed by the set $X_m \setminus Y_m$. Proposition 2.9 shows that the pair $(|\Delta^m|, |\partial\Delta^m|)$ is homeomorphic to the pair $(\nabla^m, \partial\nabla^m)$, and hence also to the pair (D^m, S^{m-1}). This proves the claim. □

The following theorem is the main results of this section, and it summarizes all the key properties of the preferred CW-structure on $|X|$.

Theorem 3.8. *Let X be a simplicial set.*

(i) *The subspaces $|\operatorname{sk}^m X|$ for $m \geq 0$ form a CW-structure on the geometric realization $|X|$, the* preferred CW-structure*.*
(ii) *The m-cells of the preferred CW-structure biject with the set of non-degenerate m-simplices of X.*
(iii) *Suppose that for $n > m$, every n-simplex of X is degenerate. Then the preferred CW-structure on $|X|$ is m-dimensional.*
(iv) *Suppose that the total number of non-degenerate simplices of X is finite. Then the preferred CW-structure on $|X|$ is finite. In particular, the space $|X|$ is compact.*
(v) *For every morphism $f : Y \longrightarrow X$ of simplicial sets, the continuous map $|f| : |Y| \longrightarrow |X|$ is cellular with respect to the preferred CW-structures.*
(vi) *If Y is a simplicial subset of X, then the preferred CW-structure on $|Y|$ is a subcomplex of the preferred CW-structure on $|X|$.*

Proof. We prove parts (i) and (ii) together. We have $(\operatorname{sk}^{m-1} X)_{m-1} = X_{m-1} = (\operatorname{sk}^m X)_{m-1}$, and for $n > m$, every n-simplex of $\operatorname{sk}^m X$ is degenerate. So Proposition 3.6 applies to the pair $(\operatorname{sk}^m X, \operatorname{sk}^{m-1} X)$. Since

$$(\operatorname{sk}^m X)_m \setminus (\operatorname{sk}^{m-1} X)_m$$

is precisely the set of non-degenerate m-simplices of X, we conclude that $|\operatorname{sk}^m X|$ can be obtained from $|\operatorname{sk}^{m-1} X|$ by attaching m-cells indexed by the non-degenerate m-simplices of X.

The simplicial set X is the colimit of the sequence of its skeleta $\operatorname{sk}^m X$. The geometric realization functor is a left adjoint, so it preserves colimits. Hence $|X|$ is a colimit, in the category of topological spaces and continuous maps, of the sequence

$$|\operatorname{sk}^0 X| \ \longrightarrow \ |\operatorname{sk}^1 X| \ \longrightarrow \ |\operatorname{sk}^2 X| \ \longrightarrow \ \ldots \ \longrightarrow \ |\operatorname{sk}^m X| \ \longrightarrow \ \ldots \ .$$

Each of these maps is a cell attachment, and hence in particular a closed embedding. So a colimit of the sequence is the union with the weak topology.

Parts (iii) through (vi) are immediate consequences of parts (i) and (ii). □

Example 3.9 (Preferred CW-structure on $|\Delta^m|$)**.** We have seen an explicit homeomorphism

$$|\Delta^m| \cong \nabla^m , \quad [\alpha, t] \longmapsto \alpha_*(t)$$

between the realization of the simplicial m-simplex and the topological m-simplex. This homeomorphism identifies the preferred CW-structure on $|\Delta^m|$ with the 'linear' CW-structure on ∇^m, i.e., by the linear dimension of the faces. The k-skeleton of this CW-structure is

$$(\nabla^m)_k = \{(t_0, \dots, t_m) \in \nabla^m : \text{ at least } m-k \text{ of the coordinates } t_0, \dots t_m \text{ are } 0\} .$$

Example 3.10 (Preferred CW-structure on $|\Delta^m/\partial\Delta^m|$)**.** We write $\Delta^m/\partial\Delta^m$ for the quotient simplicial set where the simplicial subset $\partial\Delta^m$ of Δ^m is collapsed. In other words, in every dimension n, all the n-simplices of $\partial\Delta^m$ are identified to a single n-simplex. This simplicial set participates in a pushout square

$$\begin{array}{ccc} \partial\Delta^m & \xrightarrow{\text{inclusion}} & \Delta^m \\ \downarrow & & \downarrow{\scriptstyle \text{projection}} \\ \Delta^0 & \longrightarrow & \Delta^m/\partial\Delta^m \end{array}$$

Since geometric realization preserves colimits (such as pushouts), it produces a pushout square of topological spaces

$$\begin{array}{ccc} |\partial\Delta^m| & \xrightarrow{|\text{inclusion}|} & |\Delta^m| \\ \downarrow & & \downarrow{\scriptstyle |\text{projection}|} \\ |\Delta^0| & \longrightarrow & |\Delta^m/\partial\Delta^m| \end{array}$$

The space $|\Delta^0| \cong \nabla^0$ is a single point, so this pushout says that the continuous map $|\text{projection}| : |\Delta^m| \longrightarrow |\Delta^m/\partial\Delta^m|$ factors through a homeomorphism

$$|\Delta^m|/|\partial\Delta^m| \cong |\Delta^m/\partial\Delta^m| .$$

By Proposition 2.9, the pair $(|\Delta^m|, |\partial\Delta^m|)$ is homeomorphic to the pair $(\nabla^m, \partial\nabla^m)$ and so the realization of $\Delta^m/\partial\Delta^m$ is homeomorphic to

$$\nabla^m/\partial\nabla^m \cong S^m .$$

We invite the reader to check that $\Delta^m/\partial\Delta^m$ has precisely two non-degenerate simplices: the 0-simplex consisting of the identified $(\partial\Delta^m)_0$, and (the image of) the m-simplex $\mathrm{Id}_{[m]}$. So the preferred CW-structure on $|\Delta^m/\partial\Delta^m|$ corresponds to the minimal CW-structure on S^m, with one 0-cell and one m-cell (or with two 0-cells if $m = 0$).

4. The singular complex functor is homotopical

In this section we show that the singular complex functor takes weak homotopy equivalences of topological spaces to homotopy equivalences of simplicial sets.

The functors of geometric realization and singular complex form an adjoint pair

$$\mathbf{sset} \underset{\mathcal{S}}{\overset{|-|}{\rightleftarrows}} \mathbf{Top} \ .$$

The data of this adjunction is a bijection of morphisms sets

$$\mathrm{Hom}_{\mathbf{Top}}(|X|, A) \ \cong \ \mathrm{Hom}_{\mathbf{sset}}(X, \mathcal{S}(A))$$

that is natural in the simplicial set X and the topological space A. The data of the adjunction can alternatively be specified by the unit or the counit of the adjunction. In our case, the unit is the morphism of simplicial sets $\eta_X : X \longrightarrow \mathcal{S}|X|$ with components

$$(\eta_X)_n \ : \ X_n \ \longrightarrow \ (\mathcal{S}|X|)_n = \mathrm{Hom}_{\mathbf{Top}}(\nabla^n, |X|) \ , \quad (\eta_X)_n(x)(t) \ = \ [x,t] \ .$$

The counit is the continuous map

$$\epsilon_Z \ : \ |\mathcal{S}(Z)| \ \longrightarrow \ Z \ , \quad \epsilon_Z[f,t] \ = \ f(t) \ .$$

Here $f : \nabla^n \longrightarrow Z$ is a singular n-simplex in the space Z, and $t \in \nabla^n$.

We will now argue that the adjunction bijection passes to a similar bijection of homotopy classes. In the context of simplicial sets, 'homotopy' now refers to the equivalence relation generated by elementary homotopies, i.e., morphisms of simplicial sets $X \times \Delta^1 \longrightarrow \mathcal{S}(A)$. As a matter of fact, for the target $\mathcal{S}(A)$, this elementary homotopy relation is already symmetric and transitive, because $\mathcal{S}(A)$ is an example of a so-called *Kan complex*, see also Remark 7.21; however, we will not use this fact.

Proposition 4.1. *Let A be a topological space and X a simplicial set. Then two continuous maps $f, g : |X| \longrightarrow A$ are homotopic if and only if the adjoint morphisms of simplicial sets $f^\flat, g^\flat : X \longrightarrow \mathcal{S}(A)$ are simplicially homotopic.*

Proof. In one direction we let $H : |X| \times \nabla^1 \longrightarrow A$ by a homotopy from f to g, i.e., such that

$$H(-,(0,1)) \ = \ f \qquad \text{and} \qquad H(-,(1,0)) \ = \ g \ .$$

We define a morphism of simplicial sets

$$\kappa \ : \ X \times \Delta^1 \ \longrightarrow \ \mathcal{S}(|X| \times \nabla^1)$$

in simplicial dimension n as the map

$$\kappa_n \ : \ X_n \times \Delta([n],[1]) \ \longrightarrow \ \mathcal{S}_n(|X| \times \nabla^1)$$
$$\kappa_n(x,\alpha)(t) \quad = \ ([x,t], \alpha_*(t)) \ \in \ |X| \times \nabla^1 \ .$$

Then the composite

$$X \times \Delta^1 \ \xrightarrow{\ \kappa\ } \ \mathcal{S}(|X| \times \nabla^1) \ \xrightarrow{\ \mathcal{S}(H)\ } \ \mathcal{S}(A)$$

is a simplicial homotopy from $f^\flat$ to $g^\flat$.

For the converse implication we suppose that $f^\flat$ and $g^\flat$ are simplicially homotopic. Because 'homotopy' of continuous maps is an equivalence relation, we may assume without loss of generality that there is an elementary homotopy

$$K \ : \ X \times \Delta^1 \ \longrightarrow \ \mathcal{S}(A)$$

from $f^\flat$ to $g^\flat$. Then the composite

$$|X| \times \nabla^1 \ \cong \ |X \times \Delta^1| \ \xrightarrow{|\kappa|} \ |\mathcal{S}(A)| \ \xrightarrow{\epsilon_A} \ A$$

is a homotopy from f to g, where the first map is the inverse of the homeomorphism

$$|X \times \Delta^1| \ \xrightarrow[\cong]{(|p_1|,|p_2|)} \ |X| \times |\Delta^1| \cong |X| \times \nabla^1 \ ,$$

compare Exercise 11.2. □

Theorem 4.2. *Let $f : A \longrightarrow B$ be a weak homotopy equivalence of topological spaces. Then $\mathcal{S}(f) : \mathcal{S}(A) \longrightarrow \mathcal{S}(B)$ is homotopy equivalence of simplicial sets.*

Proof. Since f is a weak homotopy equivalence, the induced map of homotopy classes

$$f_* \ = \ [K, f] \ : \ [K, A] \ \longrightarrow \ [K, B]$$

is bijective for every CW-complex K, compare Exercise 10.2. In particular, the map

$$f_* \ : \ [|\mathcal{S}(B)|, A] \ \longrightarrow \ [|\mathcal{S}(B)|, B]$$

is bijective. So there is a continuous map $\lambda : |\mathcal{S}(B)| \longrightarrow A$, unique up to homotopy, such that the composite $f \circ \lambda : |\mathcal{S}(B)| \longrightarrow B$ is homotopic to the counit ϵ_B of the adjunction between geometric realization and singular complex. The adjoint

$$\lambda^\sharp \ : \ \mathcal{S}(B) \ \longrightarrow \ \mathcal{S}(A)$$

is then a morphism of simplicial sets, and Proposition 4.1 shows that $\mathcal{S}(f) \circ \lambda^\sharp : \mathcal{S}(B) \longrightarrow \mathcal{S}(B)$ is simplicially homotopic to the identity of $\mathcal{S}(B)$.

Now we will argue that the other composite $\lambda^\sharp \circ \mathcal{S}(f) : \mathcal{S}(A) \longrightarrow \mathcal{S}(A)$ is also simplicially homotopic to the identity. To see that, we exploit that the composite

$$f \circ \lambda \circ |\mathcal{S}(f)| \ : \ |\mathcal{S}(A)| \ \longrightarrow \ B$$

is homotopic to

$$\epsilon_B \circ |\mathcal{S}(f)| \ = \ f \circ \epsilon_A \ .$$

Since the map

$$f_* \ : \ [|\mathcal{S}(A)|, A] \ \longrightarrow \ [|\mathcal{S}(A)|, B]$$

is bijective, we conclude that $\lambda \circ |\mathcal{S}(f)| : |\mathcal{S}(A)| \longrightarrow A$ is homotopic to the counit ϵ_A. Passing to adjoints shows that $\lambda^\sharp \circ f$ is simplicially homotopic to the identity of $\mathcal{S}(A)$, by another application of Proposition 4.1. □

5. Homology of the geometric realization

In this section we prove that for every simplicial set X and all coefficient groups A, the unit $\eta_X : X \longrightarrow \mathcal{S}|X|$ of the adjunction induces an isomorphism

$$H_n(\eta_X) \ : \ H_n(X;A) \ \longrightarrow \ H_n(|X|;A)$$

from the homology of the simplicial set to the singular homology of its geometric realization. As a corollary we will deduce that for every space Z, the adjunction counit $\epsilon_Z : |\mathcal{S}(Z)| \longrightarrow Z$ induces isomorphisms of all singular homology groups.

Construction 5.1. We let (K,L) be a pair of simplicial sets, i.e., L is a simplicial subset of K. The adjunction units and inclusions give rise to a commutative square of chain complexes:

$$\begin{array}{ccc} C_*(L;A) & \xrightarrow{\text{incl}} & C_*(K;A) \\ {\scriptstyle C_*(\eta_L;A)}\downarrow & & \downarrow{\scriptstyle C_*(\eta_K;A)} \\ C_*(\mathcal{S}|L|;A) & \xrightarrow[\text{incl}]{} & C_*(\mathcal{S}|K|;A) \end{array}$$

We write

$$\eta_{K,L} \ : \ C_*(K;A)/C_*(L;A) \ \longrightarrow \ C_*(\mathcal{S}|K|;A)/C_*(\mathcal{S}|L|;A) \tag{5.2}$$

for the chain map induced on quotient complexes. The homology of the complex $C_*(K;A)/C_*(L;A)$ is the relative homology $H_*(K,L;A)$ of the pair of simplicial sets (K,L); the homology of the complex $C_*(\mathcal{S}|K|;A)/C_*(\mathcal{S}|L|;A)$ is the relative homology $H_*(|K|,|L|;A)$ of the pair of spaces $(|K|,|L|)$.

Theorem 5.3. *For every pair of simplicial sets (K,L) and all abelian groups A, the chain map $\eta_{K,L}$ from* (5.2) *is a quasi-isomorphism. Hence the induced maps*

$$H_n(\eta_{K,L}) \ : \ H_n(K,L;A) \ \longrightarrow \ H_n(|K|,|L|;A)$$

are isomorphisms of homology groups.

Proof. Claim 1: We consider two pairs (M,N) and (K,L) of simplicial sets that participate in a pushout square:

$$\begin{array}{ccc} N & \xrightarrow{\text{incl}} & M \\ {\scriptstyle f|_N}\downarrow & & \downarrow{\scriptstyle f} \\ L & \xrightarrow[\text{incl}]{} & K \end{array}$$

If the theorem holds for the pair (M,N), then it also holds for the pair (K,L). The pushout square gives rise to a commutative square of chain complexes

$$\begin{array}{ccc} C_*(M;A)/C_*(N;A) & \xrightarrow[\simeq]{\eta_{M,N}} & C_*(\mathcal{S}|M|;A)/C_*(\mathcal{S}|N|;A) \\ {\scriptstyle f_*}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathcal{S}|f|_*} \\ C_*(K;A)/C_*(L;A) & \xrightarrow[\eta_{K,L}]{} & C_*(\mathcal{S}|K|;A)/C_*(\mathcal{S}|L|;A) \end{array} \tag{5.4}$$

Because the square is a pushout, the right vertical morphism $M \longrightarrow K$ restricts to bijections from $M_n \setminus N_n$ onto $K_n \setminus L_n$ for every $n \geq 0$. So the left vertical chain map in (5.4) is an isomorphism of chain complexes. The upper horizontal morphism $\eta_{M,N}$ is a quasi-isomorphism by hypothesis. The effect of the right vertical morphism on homology groups is

$$|f|_* \ : \ H_n(|M|,|N|;A) \ \longrightarrow \ H_n(|K|,|L|;A) \ ,$$

the induced homomorphism of the relative homology groups of the geometric realizations. Since the square

$$\begin{array}{ccc} |N| & \xrightarrow{|\mathrm{incl}|} & |M| \\ {\scriptstyle |f|_{|N|}}\downarrow & & \downarrow{\scriptstyle |f|} \\ |L| & \xrightarrow[|\mathrm{incl}|]{} & |K| \end{array}$$

is a pushout of spaces and the horizontal maps are inclusions of CW-subcomplexes, excision implies that the map of relative homology groups is an isomorphism. This means that also the right vertical chain map in (5.4) is also a quasi-isomorphism. Since the other three chain maps in (5.4) are quasi-isomorphisms, so is $\eta_{K,L}$, and this proves Claim 1.

Claim 2: We consider a triple of simplicial sets (K,L,M). If the theorem holds for two of the three pairs (K,L), (L,M) and (K,M), then it also holds for the third pair. The triple of simplicial sets yields two triples of chain complexes

$$\begin{array}{rcl} C_*(M;A) \ \subset C_*(L;A) & \subset & C_*(K;A) \qquad \text{and} \\ C_*(\mathcal{S}|M|;A) \ \subset C_*(\mathcal{S}|L|;A) & \subset & C_*(\mathcal{S}|K|;A) \ . \end{array}$$

From these we form two short exact sequences of chain complexes, related by the η-morphisms in a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_*(L;A)/C_*(M;A) & \longrightarrow & C_*(K;A)/(M;A) & \longrightarrow & C_*(K;A)/C_*(L;A) & \longrightarrow & 0 \\ & & {\scriptstyle \eta_{L,M}}\downarrow & & {\scriptstyle \eta_{K,M}}\downarrow & & \downarrow{\scriptstyle \eta_{K,L}} & & \\ 0 & \longrightarrow & C_*(\mathcal{S}|L|;A)/C_*(\mathcal{S}|M|;A) & \longrightarrow & C_*(\mathcal{S}|K|;A)/(\mathcal{S}|M|;A) & \longrightarrow & C_*(\mathcal{S}|K|;A)/C_*(\mathcal{S}|L|;A) & \longrightarrow & 0 \end{array}$$

We apply the 5-lemma to the resulting commutative diagram of long exact homology sequences: since the theorem holds for two of the three pairs, every two out of three vertical morphisms of homology groups are isomorphisms, hence so are the remaining ones. So the theorem holds for the third pair, which proves Claim 2.

Claim 3: The theorem holds for the pair $(\Delta^m, \partial\Delta^m)$ for all $m \geq 0$. We argue by induction on m. The induction starts with $m = 0$; then Δ^0 is a constant simplicial set with one vertex and $\partial\Delta^0 = \emptyset$; also, the space $|\Delta^0|$ consists of a single point. So in this case we have individually calculated $H_*(\Delta^0, \partial\Delta^0; A)$ and $H_*(|\Delta^0|, |\partial\Delta^0|; A)$, and both consists of a copy of A concentrated in dimension 0. Moreover, the map $\eta_{\Delta^0,\partial\Delta^0}$ is a quasi-isomorphism by explicit verification.

For $m \geq 1$ we let Λ^m_0 denote the '0-th horn' of Δ^m, i.e., the simplicial subset generated by $d_1, \dots, d_m : [m-1] \longrightarrow [m]$. Then both Δ^m and Λ^m_0 are simplicially contractible, and their geometric realizations $|\Delta^m|$ and $|\Lambda^m_0|$ are contractible. So the relative homology groups $H_*(\Delta^m, \Lambda^m_0; A)$ and $H_*(|\Delta^m|, |\Lambda^m_0|; A)$ are all trivial, and $\eta_{\Delta^m,\Lambda^m_0}$ is a quasi-isomorphism. The simplex $d_0 : [m-1] \longrightarrow [m]$ is the unique non-degenerate simplex of $\partial\Delta^m$ that does not belong to Λ_0^m. So the following square is a pushout of simplicial sets:

$$\begin{array}{ccc} \partial\Delta^{m-1} & \xrightarrow{\text{incl}} & \Delta^{m-1} \\ {\scriptstyle (d_0)_*}\downarrow & & \downarrow{\scriptstyle (d_0)_*} \\ \Lambda_0^m & \xrightarrow[\text{incl}]{} & \partial\Delta^m \end{array}$$

Since the theorem holds for the pair $(\Delta^{m-1}, \partial\Delta^{m-1})$ by induction, it holds for the pair $(\partial\Delta^m, \Lambda_0^m)$ by Claim 1. Since the theorem holds for the pairs (Δ^m, Λ_0^m) and $(\partial\Delta^m, \Lambda_0^m)$, it holds for the pair $(\Delta^m, \partial\Delta^m)$ by Claim 2. This completes the inductive step, and hence the proof of Claim 3.

Claim 4: Suppose that the theorem holds for a family $\{(K_i, L_i)\}_{i\in I}$ of pairs of simplicial sets. Then the theorem also holds for the pair $(\amalg_{i\in I} K_i, \amalg_{i\in I} L_i)$ of disjoint unions. The functors of geometric realization and singular complex both preserves disjoint unions, and the functor $C_*(-;A)$ takes disjoint unions of simplicial sets to direct sums of chain complexes. So in the commutative square

$$\begin{array}{ccc} \bigoplus_{i\in I} C_*(K_i;A)/C_*(L_i;A) & \xrightarrow{\bigoplus \eta_{K_i,L_i}} & \bigoplus_{i\in I} C_*(\mathcal{S}|K_i|;A)/C_*(\mathcal{S}|L_i|;A) \\ \cong\downarrow & & \downarrow\cong \\ C_*(\amalg K_i;A)/C_*(\amalg L_i;A) & \xrightarrow[\eta_{\amalg K_i,\amalg L_i}]{} & C_*(\mathcal{S}|\amalg K_i|;A)/C_*(\mathcal{S}|\amalg L_i|;A) \end{array}$$

the canonical vertical morphisms are isomorphisms. Any direct sum of quasi-isomorphisms is a quasi-isomorphism, so the upper horizontal morphism is a quasi-isomorphism since all pairs (K_i, L_i) satisfy the hypothesis of the theorem. Hence the lower horizontal map is a quasi-isomorphism, which proves Claim 4.

Claim 5: The theorem holds for the pair $((\mathrm{sk}^m K)\cup L, L)$ for all $m \geq -1$. We argue by induction on m; because $\mathrm{sk}^{-1} K$ is empty, there is nothing to show for $m = -1$. Now we suppose that $m \geq 0$, and we assume the theorem for the pair $((\mathrm{sk}^{m-1} K)\cup L, L)$. We write

$$N \ = \ K_m^{n.d.} \setminus L_m$$

for the set of non-degenerate m-simplices in $K \setminus L$. Proposition 3.6 (i) provides a pushout in the category of simplicial sets:

$$\begin{array}{ccc} \coprod_N \partial\Delta^m & \longrightarrow & \coprod_N \Delta^m \\ \downarrow & & \downarrow \\ (\mathrm{sk}^{m-1} K)\cup L & \longrightarrow & (\mathrm{sk}^m K)\cup L \end{array}$$

The theorem holds for the pair $(\Delta^m, \partial\Delta^m)$ by Claim 3, and hence also for the disjoint union $(\coprod_N \Delta^m, \coprod_N \partial\Delta^m)$ by Claim 4. So the theorem holds for the pair $((\mathrm{sk}^m K)\cup L, (\mathrm{sk}^{m-1} K)\cup L)$ by Claim 1. Since the theorem holds for the pair $((\mathrm{sk}^{m-1} K)\cup L, L)$ by induction, Claim 2 shows that theorem for the pair $((\mathrm{sk}^m K)\cup L, L)$.

Claim 6: The theorem holds in general. To show this, we fix a homology dimension n. Then all simplices in $K \setminus ((\mathrm{sk}^{n+1} K)\cup L)$, and all cells in $|K| \setminus |(\mathrm{sk}^{n+1} K)\cup L|$ have dimension at least

$n+2$, so the inclusion $(\mathrm{sk}^{n+1} K) \cup L \longrightarrow K$ induces isomorphisms

$$H_n((\mathrm{sk}^{n+1} K) \cup L, L; A) \xrightarrow{\cong} H_n(K, L; A) \quad \text{and}$$
$$H_n(|(\mathrm{sk}^{n+1} K) \cup L|, |L|; A) \xrightarrow{\cong} H_n(|K|, |L|; A) \ .$$

The theorem holds for the pair $((\mathrm{sk}^{n+1} K) \cup L, L)$ by Claim 5, so naturality of the η-maps establishes the theorem for the pair (K, L). □

Corollary 5.5. *For every space Z, the counit of the adjunction $\epsilon_Z : |\mathcal{S}(Z)| \longrightarrow Z$ induces isomorphisms of all singular homology groups.*

Proof. The composite

$$\mathcal{S}(Z) \xrightarrow{\eta_{\mathcal{S}(Z)}} \mathcal{S}|\mathcal{S}(Z)| \xrightarrow{\mathcal{S}(\epsilon_Z)} \mathcal{S}(Z)$$

is the identity, hence so is the composite

$$H_*(Z; A) \xrightarrow{H_*(\eta_{\mathcal{S}(Z)})} H_*(|\mathcal{S}(Z)|; A) \xrightarrow{H_*(\epsilon_Z)} H_*(Z; A) \ .$$

The first map is an isomorphism by Theorem 5.3, applied to the pair $(K, L) = (\mathcal{S}(Z), \emptyset)$. So the map $H_*(\epsilon_Z)$ is an isomorphism. □

6. The counit weak equivalence

In this section we will show that for every space Z, the counit of the adjunction $\epsilon_Z : |\mathcal{S}(Z)| \longrightarrow Z$ is a weak homotopy equivalence. This result was originally proved by John Milnor [4, Theorem 4], who credits the result to Giever [2]. Since the space $|\mathcal{S}(Z)|$ comes with a preferred CW-structure, this also yields a functorial and natural CW-approximation for any space Z.

Proposition 6.1. *Let X be a non-empty simplicial set with the following properties:*

- *for all vertices $x, y \in X_0$ there is a 1-simplex $w \in X_1$ with $d_1^*(w) = x$ and $d_0^*(w) = y$;*
- *for all 1-simplices $u, v, w \in X_1$ such that*

$$d_0^*(u) = d_1^*(v) \ , \quad d_0^*(v) = d_0^*(w) \quad \text{and} \quad d_1^*(u) = d_1^*(w)$$

there is a 2-simplex $z \in X_2$ such that

$$d_0^*(z) = v \ , \quad d_1^*(z) = w \quad \text{and} \quad d_2^*(w) = u \ .$$

Then the geometric realization $|X|$ is simply connected.

Proof. We choose a vertex $x \in X_0$; we abuse notation and also denote the corresponding 0-cell of $|X|$ by x, which we use as the basepoint. For every $y \in X_0$ with $y \neq x$ we choose a 1-simplex $s(y) \in X_1$ such that $d_0^*(s(y)) = y$ and $d_1^*(s(y)) = x$; such 1-simplices exist by the first hypothesis. We write T for the 1-dimensional simplicial subset of X generated by X_0 and the 1-simplices $s(y)$ for all $y \in X_0 \setminus \{x\}$. Then $|T|$ is a 1-dimensional CW-complex with X_0 as its set of 0-cells, and such that every $y \neq x$ is connected by a unique 1-cell to x. In particular, $|T|$ is contractible. Since $|T|$ is a CW-subcomplex of $|X|$, the inclusion $|T| \longrightarrow |X|$ has the homotopy extension property, and so the quotient map

$$|X| \longrightarrow |X|/|T| \cong |X/T|$$

is a homotopy equivalence. We may thus show that the geometric realization of X/T is simply connected. Because $T_0 = X_0$, the simplicial set X/T has a unique vertex t, so its realization is path connected.

The geometric realization of $\mathrm{sk}^1(X/T) = (\mathrm{sk}^1 X)/T$ is a 1-dimensional CW-complex with a single 0-cell, i.e., a wedge of circles indexed by the non-degenerated 1-simplices

$$\left(\mathrm{sk}^1(X/T)\right)^{\mathrm{n.d.}} \;=\; X_1 \setminus T_1 \ .$$

By covering space theory, or an iterated application of the van Kampen theorem, the fundamental group $\pi_1(|\,\mathrm{sk}^1(X/T)|, t)$ is a free group, generated by the loops represented by the non-degenerate 1-simplices of X/T. By cellular approximation, the inclusion $\mathrm{sk}^1(X/T) \longrightarrow X/T$ induces an epimorphism of fundamental groups after geometric realization. So the fundamental group $\pi_1(|X/T|, t)$ is generated by the homotopy classes of the loops

$$v^\flat \ : \ [0,1] \ \longrightarrow \ |X/T| \ , s \ \longmapsto \ [v,(s,1-s)]$$

for all $v \in X_1 \setminus T_1$.

We will now use the second hypothesis to show that all the loops $v^\flat$ are nullhomotopic. The 1-simplices $u = s(d_1^*(y)), v$ and $w = s(d_0^*(y))$ satisfy the relations

$$d_0^*(s(d_1^*(v)))) = d_1^*(v) \ , \quad d_0^*(s(d_0^*(v))) = d_0^*(v) \quad \text{and} \quad d_1^*(s(d_1^*(v))) = d_1^*(s(d_0^*(v))) = x \ .$$

These are precisely the hypotheses of the second assumption, so there exists a 2-simplex $z \in X_2$ such that

$$d_0^*(z) = v \ , \quad d_1^*(z) = w \quad \text{and} \quad d_2^*(w) = u \ .$$

The composite

$$[0,1]\times[0,1] \xrightarrow{(s,t)\mapsto(t,(1-t)(1-s),(1-t)s)} \nabla^2 \xrightarrow{t\mapsto[z,t]} |X| \xrightarrow{\simeq} |X/T|$$

then provides a homotopy, relative endpoints, from the loop $v^\flat$ to the constant loop at the basepoint t. □

Remark 6.2. Proposition 6.1 is a special case of a more general fact. Indeed, the two hypotheses on the simplicial set in the previous proposition can equivalently be stated as extension properties: the first condition is equivalent to requiring that every morphism of simplicial sets $\partial\Delta^1 \longrightarrow X$ admits an extension to Δ^1; the second condition is equivalent to requiring that every morphism of simplicial sets $\partial\Delta^2 \longrightarrow X$ admits an extension to Δ^2.

The generalization of Proposition 6.1 is as follows: let X be a simplicial set with the property that for all $0 \leq k \leq n$, every morphism of simplicial sets $\partial\Delta^k \longrightarrow X$ admits an extension to Δ^k. Then the geometric realization $|X|$ is $(n-1)$-connected.

Corollary 6.3. *Let Z be a simply connected space.*

(i) *The singular complex $\mathcal{S}(Z)$ satisfies the hypotheses of Proposition 6.1.*
(ii) *The space $|\mathcal{S}(Z)|$ is simply connected.*

Proof. (i) We let x, y be vertices of $\mathcal{S}(Z)$, i.e., maps $x, y : \nabla^0 \longrightarrow Z$ (automatically continuous). Since Z is path connected, we can choose a path from $x(1)$ to $y(1)$, which we can parameterize as a continuous map $w : \nabla^1 \longrightarrow Z$ such that $\nabla(1,0) = x(1)$ and $\nabla(0,1) = y(1)$. Then w is a 1-simplex of $\mathcal{S}(Z)$ that satisfies

$$d_1^*(w)(1) \ = \ w(d_1(1)) \ = \ w(1,0) \ = \ x(1) \ ;$$

so $d_1^*(w) = x$, and similarly $d_0^*(w) = y$.

Now we let $u, v, w \in \mathcal{S}(Z)$ be singular 1-simplices, i.e., continuous maps $u, v, w : \nabla^1 \longrightarrow Z$. The hypotheses $d_0^*(u) = d_1^*(v)$ $d_0^*(v) = d_0^*(w)$ and $d_1^*(u) = d_1^*(w)$ then translate into the relations

$$u(0,1) = v(1,0) \ , \quad v(0,1) = w(0,1) \qquad \text{and} \qquad u(1,0) = w(1,0) \ .$$

We can thus define a continuous map

$$\langle u, v, w \rangle \ : \ \partial\nabla^2 \ \longrightarrow \ Z$$

by

$$\langle u, v, w \rangle(t_0, t_1, t_2) \ = \ \begin{cases} v(t_1, t_2) & \text{if } t_0 = 0, \\ w(t_0, t_2) & \text{if } t_1 = 0, \text{ and} \\ u(t_0, t_1) & \text{if } t_2 = 0. \end{cases}$$

Because Z is simply-connected, the map $\langle u, v, w \rangle$ admits a continuous extension $h : \nabla^2 \longrightarrow Z$. This extension is a 2-simplex of $\mathcal{S}(Z)$ that satisfies $d_0^*(z) = v$, $d_1^*(z) = w$ and $d_2^*(z) = u$. Part (ii) is now an application of Proposition 6.1. □

Theorem 6.4. *For every topological space Z, the continuous map $\epsilon_Z : |\mathcal{S}(Z)| \longrightarrow Z$ is a weak homotopy equivalence.*

Proof. We prove the result for successively more general classes of spaces.

Case 1: The space Z is simply connected and underlies a CW-complex. By Corollary 5.5 and Corollary 6.3 (ii), the map ϵ_Z is a homology isomorphism between simply connected spaces, and hence a weak homotopy equivalence by the corollary to the Hurewicz theorem.

Case 2: The space Z is path connected and underlies a CW-complex. Every CW-complex is locally path connected and semi-locally simply connected. So there exists a universal cover $p : \tilde{Z} \longrightarrow Z$, and $\tilde{Z}$ is a simply connected space that admits a CW-structure. We let G be the group of deck transformations of the universal cover p. We consider a singular simplex $f : \nabla^n \longrightarrow Z$. Because simplices are simply connected and locally path connected, the deck transformation group G acts freely and transitively on the set of lifts of f to $\tilde{Z}$. Hence the action of G on $\mathcal{S}(\tilde{Z})$ is free and the morphism of simplicial sets

$$\mathcal{S}(p) \ : \ \mathcal{S}(\tilde{Z}) \ \longrightarrow \ \mathcal{S}(Z)$$

factors through an isomorphism of simplicial sets

$$\mathcal{S}(\tilde{Z})/G \ \cong \ \mathcal{S}(Z) \ .$$

Since the action of G on $\mathcal{S}(\tilde{Z})$ is free, the induced action on $|\mathcal{S}(\tilde{Z})|$ is free and properly discontinuous, compare Exercise 11.3. So the quotient map

$$|\mathcal{S}(\tilde{Z})| \ \longrightarrow \ |\mathcal{S}(\tilde{Z})|/G$$

is a covering map. Since $\tilde{Z}$ is simply connected, so is $|\mathcal{S}(\tilde{Z})|$ by Corollary 6.3 (ii). So this quotient map exhibits $|\mathcal{S}(\tilde{Z})|$ as a universal cover of $|\mathcal{S}(\tilde{Z})|/G$.

The geometric realization functor is a left adjoint, so it preserves colimits, such as orbits by a group action. So the continuous map

$$|\mathcal{S}(p)| \ : \ |\mathcal{S}(\tilde{Z})| \ \longrightarrow \ |\mathcal{S}(Z)|$$

factors through a homeomorphism

$$|\mathcal{S}(\tilde{Z})|/G \;\cong\; \mathcal{S}(Z)\ .$$

Altogether we conclude that the map $|\mathcal{S}(p)|$ is a universal covering.

In the commutative square

$$\begin{array}{ccc} |\mathcal{S}(\tilde{Z})| & \xrightarrow{\epsilon_{\tilde{Z}}} & \tilde{Z} \\ {\scriptstyle |\mathcal{S}(p)|}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ |\mathcal{S}(Z)| & \xrightarrow[\epsilon_Z]{} & Z \end{array}$$

both vertical maps are universal coverings, with isomorphic deck transformation groups. So the map $\epsilon_Z : |\mathcal{S}(Z)| \longrightarrow Z$ induces isomorphisms of fundamental groups for arbitrary basepoints. Since $\tilde{Z}$ is simply connected and admits a CW-structure, the map $\epsilon_{\tilde{Z}} : |\mathcal{S}(\tilde{Z})| \longrightarrow \tilde{Z}$ is a weak homotopy equivalence by Case 1. So it induces isomorphisms of all homotopy groups, and for all choices of base points. Since covering maps induce isomorphisms of homotopy groups on π_n for $n \geq 2$, we conclude that the map $\epsilon_Z : |\mathcal{S}(Z)| \longrightarrow Z$ also induces isomorphisms of higher homotopy groups.

Case 3: The space Z is path connected. We choose a CW-approximation, i.e., a weak homotopy equivalence $f : Y \longrightarrow Z$ whose source admits a CW-structure. Because Z is path connected, so is Y. We contemplate the commutative square:

$$\begin{array}{ccc} |\mathcal{S}(Y)| & \xrightarrow{\epsilon_Y} & Y \\ {\scriptstyle |\mathcal{S}(f)|}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ |\mathcal{S}(Z)| & \xrightarrow[\epsilon_Z]{} & Z \end{array}$$

The map ϵ_Y is a weak homotopy equivalence by Case 2. The morphism $\mathcal{S}(f) : \mathcal{S}(Y) \longrightarrow \mathcal{S}(Z)$ is a homotopy equivalence of simplicial sets by Theorem 4.2. So the geometric realization $|\mathcal{S}(f)| : |\mathcal{S}(Y)| \longrightarrow |\mathcal{S}(Z)|$ is a homotopy equivalence of spaces. So the other three of the four maps in the commutative square are weak homotopy equivalences; hence the map ϵ_Z is a weak homotopy equivalence, too.

Case 4: The space Z is arbitrary. We show that the map $\pi_0(\epsilon_Z) : \pi_0(|\mathcal{S}(Z)|) \longrightarrow \pi_0(Z)$ is bijective. Every point $z \in Z$ provides a singular 0-simplex $\hat{z} : \nabla^0 \longrightarrow Z$ with image z. The point $[\hat{z}, 1] \in |\mathcal{S}(Z)|$ than maps to z under ϵ_Z. This shows in particular that $\pi_0(\epsilon_Z)$ is surjective.

Every path component of $|\mathcal{S}(Z)|$ contains a 0-cell in the preferred CW-structure, which are indexed by vertices of $\mathcal{S}(Z)$. For injectivity we can thus consider two 0-cells $[f,1], [g,1] \in |\mathcal{S}(Z)|$ corresponding to singular 0-simplices $f, g : \nabla^0 \longrightarrow |\mathcal{S}(Z)|$, such that $\epsilon_Z[f,1]$ and $\epsilon_Z[g,1]$ belong to the same path component of Z. We can then choose a path connecting these two points, and reparameterize it as a continuous map

$$h \ :\ \nabla^1 \ \longrightarrow\ Z$$

such that $h(0,1) = \epsilon_Z[f,1]$ and $h(1,0) = \epsilon_Z[g,1]$. The map h is then a singular 1-simplex, and the path

$$[0,1] \ \longrightarrow\ |\mathcal{S}(Z)|\ , \quad s \ \longmapsto\ [h,(s,1-s)]$$

connects the points $[f,1]$ and $[g,1]$. This shows that the map $\pi_0(\epsilon_Z)$ is also injective.

Now we show that for every point $x \in |\mathcal{S}(Z)|$ and all $n \geq 1$, the map

$$(\epsilon_Z)_* \ : \ \pi_n(|\mathcal{S}(Z)|, x) \ \longrightarrow \ \pi_n(Z, \epsilon_Z(x))$$

is bijective. We let $Z\langle x \rangle$ denote the path component of Z that contains the point $\epsilon(x)$, endowed with the subspace topology. We consider the commutative square:

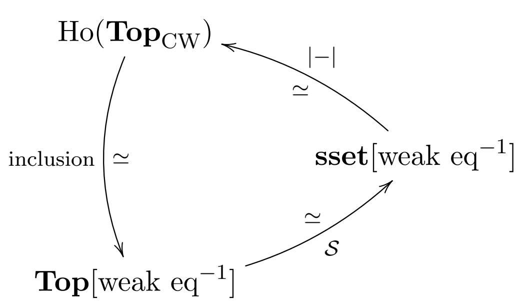
$$\begin{array}{ccc} |\mathcal{S}(Z_x)| & \xrightarrow{\epsilon_{Z_x}} & Z_x \\ {\scriptstyle |\mathcal{S}(\text{inclusion})|}\downarrow & & \downarrow{\scriptstyle \text{inclusion}} \\ |\mathcal{S}(Z)| & \xrightarrow[\epsilon_Z]{} & Z \end{array}$$

Because Z_x is path connected, the upper horizontal map is a weak homotopy equivalence by Case 3.

Since the simplices ∇^n are all path connected, every singular simplex $\nabla^n \longrightarrow Z$ has image in a unique path component of Z. So the singular complex $\mathcal{S}(Z)$ is the disjoint union of the singular complexes of the path components Z_x of Z, for $x \in \pi_0(Z)$, where Z_x is endowed with the subspace topology. Geometric realization preserves coproducts, so it takes disjoint unions of simplicial sets to topological disjoint unions of spaces. Hence the left vertical map is the inclusion of the path component of $|\mathcal{S}(Z)|$ that contains the point x. Higher homotopy groups only depend the path component of the base point. So both vertical maps induce isomorphisms of higher homotopy groups. Hence the lower map ϵ_Z also induces isomorphisms of all higher homotopy groups. So the map ϵ_Z is a weak homotopy equivalence. □

7. Equivalences of homotopy categories

In this section we prove that the functors of singular complex and geometric realization descend to equivalences of homotopy categories of topological spaces and simplicial sets. The upshot of this section is a diagram of equivalences of categories:



Here, $\mathrm{Ho}(\mathbf{Top}_{\mathrm{CW}})$ is the *homotopy category of CW-complexes*. Its objects are all spaces that admit the structure of a CW-complex; its morphisms are homotopy classes of continuous maps. The two other categories are certain *localizations*, a notion that we will introduce momentarily.

Many arguments in this section will be formal and categorical; all the hard work was done in the previous sections in proving that the realization of every simplicial set admits a CW-structure (Theorem 3.8), that the singular complex functor takes weak equivalences to homotopy equivalences (Theorem 4.2), and that the adjunction counit $\epsilon_Z : |\mathcal{S}(Z)| \longrightarrow Z$ is a weak homotopy equivalence (Theorem 6.4).

Definition 7.1. Let $\mathcal{C}$ be a category, and let $\mathcal{W}$ be a class of morphisms of $\mathcal{C}$. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is *$\mathcal{W}$-inverting* if it sends all morphisms in $\mathcal{W}$ to isomorphisms in $\mathcal{D}$. A *localization* of $\mathcal{C}$ at $\mathcal{W}$ is a functor $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ that is initial among $\mathcal{W}$-inverting functors.

Let us take the time to spell out the previous definition in more detail: a functor $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ is a localization at $\mathcal{W}$ if and only if:

- the functor γ is $\mathcal{W}$-inverting, and
- for every $\mathcal{W}$-inverting functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ there exists a unique functor $G : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{D}$ such that $G\gamma = F$.

We offer some comments to put localizations of categories into perspective.

(a) As always for objects with universal properties, localizations are unique up to preferred isomorphism of categories. Indeed, suppose that $\gamma : \mathcal{C} \longrightarrow \mathcal{D}$ and $\mu : \mathcal{C} \longrightarrow \mathcal{E}$ are two localizations at the same class $\mathcal{W}$ of $\mathcal{C}$-morphisms. Since both functors take $\mathcal{W}$ to isomorphisms, the universal properties provide two functors $G : \mathcal{D} \longrightarrow \mathcal{E}$ and $H : \mathcal{E} \longrightarrow \mathcal{D}$ such that
$$G \circ \gamma \ = \ \mu \qquad \text{and} \qquad H \circ \mu \ = \ \gamma \ .$$
Then
$$H \circ G \circ \gamma \ = \ \gamma \ = \ \mathrm{Id}_{\mathcal{D}} \circ \gamma \ ,$$
so the uniqueness part of the universal property forces $H \circ G = \mathrm{Id}_{\mathcal{D}}$. Reversing the roles of γ and μ shows that $G \circ H = \mathrm{Id}_{\mathcal{E}}$. So G and H are mutually inverse isomorphisms of categories.

(b) Localizations of categories $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ are bijective on objects. To see that, we let X be any set, and we let EX denote the category with object set X, and with a unique morphism $(y, x) : x \longrightarrow y$ between any pair objects. The morphism (y, x) is then an isomorphism with inverse $(x, y) : y \longrightarrow x$. The category EX is sometimes called the *indiscrete category* with object set X. Because EX is a groupoid, every functor from $\mathcal{C}$ to EX is $\mathcal{W}$-inverting, so every functor $\mathcal{C} \longrightarrow EX$ extends uniquely over γ to a functor $\mathcal{C}[\mathcal{W}^{-1}] \longrightarrow EX$. However, if $\mathcal{D}$ is any category, then every map from the objects of $\mathcal{D}$ to X can be uniquely extended to a functor $\mathcal{D} \longrightarrow EX$. So altogether we conclude that every map from the object set of $\mathcal{C}$ to X extends uniquely over γ to a map from the object set of $\mathcal{C}[\mathcal{W}^{-1}]$ to X. So γ is bijective on objects.

(c) If a localization of $\mathcal{C}$ at $\mathcal{W}$ exists, then we can choose a localization $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ with the special property that $\mathcal{C}$ and $\mathcal{C}[\mathcal{W}^{-1}]$ have the same objects, and γ is the identity on objects. Indeed, by the previous item, the localization functor is bijective on objects, so we can just rename to object set of $\mathcal{C}[\mathcal{W}^{-1}]$ by $\mathcal{C}$ using γ.

(d) Localizations of categories are analogous to localizations in ring theory. A localization of a ring R at a subset S is a ring homomorphism $R \longrightarrow R[S^{-1}]$ that takes all elements of S to units, and that is initial with this property. In ring theory, rings are usually first encountered for *commutative* rings, and typically under the assumption that the set S is multiplicatively closed and contains 1. In that case, a localization $R[S^{-1}]$ can be constructed as *fractions*, i.e., equivalence class of pairs $(r, s) \in R \times S$, where $(r, s) \sim (r', s')$ if and only if there is a $t \in S$ such that
$$rs't \ = \ r'st \ .$$
The equivalence class of (r, s) is then denoted as a fraction r/s, a ring structure on the set $R[S^{-1}]$ of fraction is defined by the rules from high school, and the map
$$R \ \longrightarrow \ R[S^{-1}] \ , \quad r \ \longmapsto \ r/1$$

is a localization.

(e) Localizations exist also for non-commutative rings, but in that generality the ring $R[S^{-1}]$ might not have any explicit description that resembles 'fractions'. There are sufficient conditions, known as *calculus of fractions*, or the *Ore condition*, compare
`https://en.wikipedia.org/wiki/Ore_condition`
With respect to localizations, categories behave very much like non-commutative rings, in that in complete generality, it might be hard to get one's hands on the localization $\mathcal{C}[\mathcal{W}^{-1}]$.

(f) When interested in general localizations of categories, one needs to confront a certain amount of set theory. The naive approach to assume a 'class of objects' and, for each pair of objects, a 'set of morphisms' becomes problematic. When pressed for a commitment on the set-theoretic details, many people would probably choose to work in ZFC (Zermelo-Fraenkel axioms, plus the axiom of choice), and also assume Grothendieck's axiom of existence of universes, compare
`https://en.wikipedia.org/wiki/Grothendieck_universe`
For the localizations that we need, we can give concrete constructions, so we don't worry about the set theory foundations and continue to employ the naive approach to categories.

(g) In complete generality, one has to worry whether a localization $\mathcal{C}[\mathcal{W}^{-1}]$ might fail to exist for set-theoretic size reasons. As explained in (c), one can always take $\mathcal{C}[\mathcal{W}^{-1}]$ to have the same objects as $\mathcal{C}$, and let $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ be the identity on objects. Every morphism $X \longrightarrow Y$ in $\mathcal{C}[\mathcal{W}^{-1}]$ is then a finite composite of the form
$$X = A_1 \xrightarrow{\gamma(f_1)} B_1 \xrightarrow{\gamma(w_1)^{-1}} A_2 \xrightarrow{\gamma(f_1)} B_2 \xrightarrow{\gamma(w_n)^{-1}} \dots \ B_{n-1} \xrightarrow{\gamma(w_{n-1})^{-1}} A_n = Y \ .$$
But without further conditions on $(\mathcal{C}, \mathcal{W})$, there need not be a bound on the length of such a sequence, and there is no control on which intermediate objects occur. In particular, there is no guarantee that one can do with a *set* worth of intermediate objects, and that the morphisms in $\mathcal{C}[\mathcal{W}^{-1}]$ form a set (as opposed to a class). If one works in ZFC with Grothendieck universes, the localization will always exist if one is willing to pass to a larger universe.

(h) Gabriel and Zisman [1, Chapter I, 2.2] introduced a *calculus of fractions* for localizations, a set of axioms that guarantees that every morphism in $\mathcal{C}[\mathcal{W}^{-1}]$ is a 'fraction' $\gamma(f)/\gamma(w)$, i.e., a composite of the form
$$X \xrightarrow{\gamma(f)} B \xrightarrow{\gamma(w)^{-1}} Y$$
of one $\mathcal{C}$-morphism, and the inverse of one morphism in $\mathcal{W}$. The localizations $\mathbf{sset}[\mathrm{weq}^{-1}]$ and $\mathbf{Top}[\mathrm{weq}^{-1}]$ discussed below are examples of this situation, where one has a calculus of left fractions, and the other one a calculus of right fractions.

The universal property of a localization $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ stipulates that precomposition with γ is a bijection from the set of functors $\mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{D}$ to the set of $\mathcal{W}$-inverting functors $\mathcal{C} \longrightarrow \mathcal{D}$. As we shall now show, the universal property for functors in fact implies an analogous universal property for natural transformations; hence the universal property for *sets* of functors also holds for *categories* of functors. For this purpose we introduce new notation. Given two categories $\mathcal{C}$ and $\mathcal{D}$, we write $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for the category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$, and with natural transformations as morphisms.

Proposition 7.2. *Let $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ be a localization at a class $\mathcal{W}$ of morphisms. Then for every category $\mathcal{D}$, the restriction functor*
$$\mathrm{Fun}(\gamma, \mathcal{D}) \ : \ \mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \ \longrightarrow \ \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is an isomorphism onto the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ *spanned by the* $\mathcal{W}$*-inverting functors.*

Proof. On the level of objects, the claim is a restatement of the universal property of a localization. At the level of morphisms, we exploit the fact that natural transformations can be reinterpreted as functors, as follows. We let I denote the category with two objects 0 and 1, and with a unique non-identity morphism $a : 0 \longrightarrow 1$. The data of the natural transformation $\tau : G \longrightarrow H$ between functors $G, H : \mathcal{C} \longrightarrow \mathcal{D}$ can be recoded as a functor

$$\tau^\flat \ : \ \mathcal{C} \ \longrightarrow \ \mathrm{Fun}(I,\mathcal{D}) \ ,$$

defined as follows. On objects, $\tau^\flat(c) : I \longrightarrow \mathcal{D}$ is the functor determined given by

$$\tau^\flat(c)(0) \ = \ G(c) \ , \quad \tau^\flat(c)(1) \ = \ H(c) \quad \text{and} \quad \tau^\flat(c)(a) \ = \ \tau_c \ .$$

On morphisms, $\tau^\flat(f : c \longrightarrow d)$ is the natural transformation $\tau^\flat(c) \longrightarrow \tau^\flat(d)$ whose value at the two objects is

$$\tau^\flat(f)(0) \ = \ G(f) \qquad \text{and} \qquad \tau^\flat(f)(1) \ = \ H(f) \ .$$

This process is reversible, i.e., every functor $\mathcal{C} \longrightarrow \mathrm{Fun}(I,\mathcal{D})$ is of the form $\tau^\flat$ for a unique natural transformation of functors from $\mathcal{C}$ to $\mathcal{D}$.

Now we apply the universal property of the localization $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ to the target category $\mathrm{Fun}(I,\mathcal{D})$. We obtain that precomposition with γ is a bijection from the set of functors $\mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathrm{Fun}(I,\mathcal{D})$ to the set of $\mathcal{W}$-inverting functors $\mathcal{C} \longrightarrow \mathrm{Fun}(I,\mathcal{D})$. After translating functors to $\mathrm{Fun}(I,\mathcal{D})$ into natural transformations as described in the previous paragraph, this becomes the statement that precomposition with γ is a bijection from the set of natural transformations of functors $\mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{D}$ to the set of natural transformations between $\mathcal{W}$-inverting functors $\mathcal{C} \longrightarrow \mathcal{D}$. But this is exactly the statement that

$$\mathrm{Fun}(\gamma,\mathcal{D}) \ : \ \mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}],\mathcal{D}) \ \longrightarrow \ \mathrm{Fun}(\mathcal{C},\mathcal{D})$$

is bijective on morphism sets between all functors in the source. □

Remark 7.3. There is a slightly weaker notion of localization that takes seriously the fact that the entirety of categories does not just form a category (of categories and functors), but even a 2-category (of categories, functors and natural transformations). We could call a functor $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ a *weak localization* at a class $\mathcal{W}$ of $\mathcal{C}$-morphisms if for every category $\mathcal{D}$, the restriction functor

$$\mathrm{Fun}(\gamma,\mathcal{D}) \ : \ \mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}],\mathcal{D}) \ \longrightarrow \ \mathrm{Fun}(\mathcal{C},\mathcal{D})$$

is an equivalence onto the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ spanned by the $\mathcal{W}$-inverting functors. Since isomorphisms of categories are in particular equivalences, Proposition 7.2 shows that localizations are also weak localizations. The converse is not true: if $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ is a localization and $F : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{E}$ an equivalence of categories that is not an isomorphism (i.e., not bijective on objects), then the composite $F\gamma : \mathcal{C} \longrightarrow \mathcal{E}$ is a weak localization, but not a localization, at $\mathcal{W}$.

We will now show that the localization of the category **sset** of simplicial sets at the class of weak homotopy equivalences is equivalent to $\mathrm{Ho}(\mathbf{Top}_{\mathrm{CW}})$, the homotopy category of CW-complexes. We first construct the former localization.

Definition 7.4. A morphism $f : X \longrightarrow Y$ of simplicial sets is a *weak equivalence* if its geometric realization $|f| : |X| \longrightarrow |Y|$ is a homotopy equivalence.

Remark 7.5 (Homotopy equivalences versus weak equivalences)**.** Because geometric realization preserves the homotopy relation, every homotopy equivalence of simplicial sets is also a weak equivalence, but not conversely. We illustrate this by an example. We consider the morphism of simplicial sets

$$f\ :\partial\Delta^2\ \longrightarrow\ \Delta^1/\partial\Delta^1$$

that is determined by

$$f_2(d_2)\ =\ \mathrm{Id}_{[1]}$$

any by sending $d_0, d_1 \in (\partial\Delta^2)_1$ to the collapsed boundary of Δ^1. Under the homeomorphisms

$$|\partial\Delta^2| \cong \partial\nabla^2 \qquad \text{and} \qquad |\Delta^1/\partial\Delta^1| \cong \nabla^1/\partial\nabla^1 \cong [0,1]/\{0,1\}\ ,$$

the realization of f becomes the continuous map

$$\partial\nabla^2\ \longrightarrow\ [0,1]/\{0,1\}$$

that sends two of the three sides of $\partial\nabla^2$ to the basepoint, and maps the third side linearly onto the target. So the geometric realization of f is a homotopy equivalence, and f is a weak equivalence of simplicial sets.

However, the morphism f is *not* a homotopy equivalence of simplicial sets. Indeed, because $\Delta^1/\partial\Delta^1$ has only one vertex, every morphism $g : \Delta^1/\partial\Delta^1 \longrightarrow \partial\Delta^2$ must send the generating 1-simplex to a 1-simplex of $\partial\Delta^2$ whose two vertices are the same. But only the degenerate 1-simplices of $\partial\Delta^2$ have this property, so any such morphism g is constant at one of the three vertices of $\partial\Delta^2$. Hence the geometric realization $|g|$ is a constant map, and thus not a homotopy equivalence.

Proposition 7.6. *For every simplicial set X, the adjunction unit $\eta_X : X \longrightarrow \mathcal{S}|X|$ is a weak equivalence.*

Proof. The triangle identity of an adjunction shows that the following composite is the identity:

$$|X|\ \xrightarrow{|\eta_X|}\ |\mathcal{S}|X||\ \xrightarrow{\epsilon_{|X|}}\ |X|$$

The map $\epsilon_{|X|}$ is a weak homotopy equivalence by Theorem 6.4. Since both $|\mathcal{S}|X||$ and $|X|$ are CW-complexes by Theorem 3.8, the map $\epsilon_{|X|}$ is even a homotopy equivalence by the Whitehead Theorem. Since the composite is the identity, the map $|\eta_X|$ is a homotopy equivalence, too. So η_X is a weak equivalence. □

Construction 7.7 (Localization of simplicial sets at the weak equivalences)**.** We define the category $\mathbf{sset}[\mathrm{weq}^{-1}]$ to have all simplicial sets as its objects. Morphisms in $\mathbf{sset}[\mathrm{weq}^{-1}]$ are defined by

$$\mathrm{Hom}_{\mathbf{sset}[\mathrm{weq}^{-1}]}(X,Y)\ =\ \mathrm{Hom}_{\mathbf{Top}}(|X|,|Y|)/\text{homotopy}\ .$$

Composition in $\mathbf{sset}[\mathrm{weq}^{-1}]$ is composition of homotopy classes of continuous maps. We define a functor $\gamma : \mathbf{sset} \longrightarrow \mathbf{sset}[\mathrm{weq}^{-1}]$ on objects by $\gamma(X) = X$, and on morphisms by sending a morphism $f : X \longrightarrow Y$ to the homotopy class of the continuous map $|f| : |X| \longrightarrow |Y|$.

As we shall now explain, every morphism in $\mathbf{sset}[\mathrm{weq}^{-1}]$ is a 'fraction' of a morphism of simplicial sets and the inverse of a weak equivalence. This is a special property, not generally enjoyed by localizations of categories.

Proposition 7.8. *Let X and Y be simplicial sets, and let $\alpha : |X| \longrightarrow |Y|$ be a continuous map between the geometric realizations.*

(i) *The square of spaces and continuous maps*

(7.9)
$$\begin{array}{ccc} |X| & \xrightarrow{|\eta_X|} & |\mathcal{S}|X|| \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle |\mathcal{S}(\alpha)|} \\ |Y| & \xrightarrow[|\eta_Y|]{} & |\mathcal{S}|Y|| \end{array}$$

commutes up to homotopy.

(ii) *The relation*

(7.10)
$$[\alpha] \;=\; \gamma(\eta_Y)^{-1} \circ \gamma(\mathcal{S}(\alpha) \circ \eta_X)$$

holds as morphisms from X to Y in $\mathbf{sset}[\text{weq}^{-1}]$.

Proof. (i) We compose both ways around the square (7.9) with the adjunction counit $\epsilon_{|Y|} : |\mathcal{S}|Y|| \longrightarrow |Y|$; we get

$$\epsilon_{|Y|} \circ |\mathcal{S}(\alpha)| \circ |\eta_X| \;=\; \alpha \circ \epsilon_{|X|} \circ |\eta_X| \;=\; \alpha \;=\; \epsilon_{|Y|} \circ |\eta_Y| \circ \alpha \; .$$

We have used naturality of ϵ, and two instances of the triangle identity of an adjunction. The map $\epsilon_{|Y|}$ is a weak homotopy equivalence between CW-complexes, and hence a homotopy equivalence. We can thus 'cancel' $\epsilon_{|Y|}$ up to homotopy, and the previous equality implies that the maps

$$|\mathcal{S}(\alpha)| \circ |\eta_X| \; , \; |\eta_Y| \circ \alpha \;:\; |X| \;\longrightarrow\; |\mathcal{S}|Y||$$

are homotopic.

(ii) Because the square (7.9) commutes up to homotopy, the associated square of homotopy classes commutes. Those associated homotopy classes of all four maps are morphisms in $\mathbf{sset}[\text{weq}^{-1}]$, and they form a commutative square in $\mathbf{sset}[\text{weq}^{-1}]$:

$$\begin{array}{ccc} X & \xrightarrow{\gamma(\eta_X)} & \mathcal{S}|X| \\ {\scriptstyle [\alpha]}\downarrow & & \downarrow{\scriptstyle \gamma(\mathcal{S}(\alpha))} \\ Y & \xrightarrow[\gamma(\eta_Y)]{} & \mathcal{S}|X| \end{array}$$

Equivalently:

$$\gamma(\eta_Y) \circ [\alpha] \;=\; \gamma(\mathcal{S}(\alpha) \circ \eta_X)$$

The morphism $\eta_Y : Y \longrightarrow \mathcal{S}|Y|$ is a weak equivalence by Proposition 7.6, so $\gamma(\eta_Y)$ is an isomorphism, and the relation (7.10) follows. □

Theorem 7.11. *The functor γ :* $\mathbf{sset} \longrightarrow \mathbf{sset}[\text{weq}^{-1}]$ *is a localization at the class of weak equivalences.*

Proof. By the very definition, the realization functor takes weak equivalences of simplicial sets to homotopy equivalences between CW-complexes; the latter become isomorphisms in the homotopy category $\text{Ho}(\mathbf{Top}_{\text{CW}})$. So the functor γ takes weak equivalences in $\mathbf{sset}$ to isomorphisms in $\mathbf{sset}[\text{weq}^{-1}]$.

To establish the universal property, we let $F : \mathbf{sset} \longrightarrow \mathcal{D}$ be any functor that inverts weak equivalences. We must show that there is a unique functor $G : \mathbf{sset}[\text{weq}^{-1}] \longrightarrow \mathcal{D}$ such that $F = G\gamma : \mathbf{sset} \longrightarrow \mathcal{D}$. The uniqueness part is easy. Because γ is the identity on objects, we must have $G(X) = G(\gamma(X)) = F(X)$ on objects. For the uniqueness on morphisms we apply G to the fraction relation (7.10) to get

$$G[\alpha] \;=\; G\left(\gamma(\eta_Y)^{-1} \circ \gamma(\mathcal{S}(\alpha) \circ \eta_X)\right) \;=\; F(\eta_Y)^{-1} \circ F(\mathcal{S}(\alpha)) \circ F(\eta_X)\ . \tag{7.12}$$

So also the behavior of G on morphisms is determined by F.

The uniqueness argument also tells us how to *define* the functor $G : \mathbf{sset}[\mathrm{weq}^{-1}] \longrightarrow \mathcal{D}$ in terms of F: we must set $G(X) = F(X)$ on objects, and define G on morphism by the formula (7.12). This definition makes sense because η_Y is a weak equivalence by Proposition 7.6, so the functor F takes it to an isomorphism. We must still argue that the definition is well-defined, i.e., it does not depend on the representative α within its homotopy class. To this end we show that the functor F takes the same value on simplicially homotopic morphisms. We let $H : X \times \Delta^1 \longrightarrow Y$ be a simplicial homotopy from $f = H \circ i_0$ to $g = H \circ i_1$, where $i_0, i_1 : X \longrightarrow X \times \Delta^1$ are the two end inclusions. We write $p : X \times \Delta^1 \longrightarrow X$ for the projection to the first factor. This morphism geometrically realizes to a homotopy equivalence, so it is a weak equivalence. So by hypothesis, $F(p) : F(X \times \Delta^1) \longrightarrow F(X)$ is an isomorphism in $\mathcal{D}$. Because $p \circ i_0 = \mathrm{Id}_X = p \circ i_1$, we have $F(p) \circ F(i_0) = \mathrm{Id}_{F(X)} = F(p) \circ F(i_1)$. Since $F(p)$ is an isomorphism, we deduce $F(i_0) = F(i_1)$. Hence

$$F(f) \;=\; F(H) \circ F(i_0) \;=\; F(H) \circ F(i_1) \;=\; F(g)\ .$$

This proves the claim. If now α is homotopic to $\alpha' : |X| \longrightarrow |Y|$, then the morphisms $\mathcal{S}(\alpha), \mathcal{S}(\alpha') : \mathcal{S}|X| \longrightarrow \mathcal{S}|Y|$ are simplicially homotopic, so $F(\mathcal{S}(\alpha)) = F(\mathcal{S}(\alpha'))$ by the previous paragraph; so the assignment (7.12) is well-defined.

To show that the assignment (7.12) is functorial we consider another simplicial set Z and a continuous map $\beta : |Y| \longrightarrow |Z|$. Then

$$\begin{aligned} G[\beta] \circ G[\alpha] \;&=\; F(\eta_Z)^{-1} \circ F(\mathcal{S}(\beta)) \circ F(\eta_Y) \circ F(\eta_Y)^{-1} \circ F(\mathcal{S}(\alpha)) \circ F(\eta_X) \\ &=\; F(\eta_Z)^{-1} \circ F(\mathcal{S}(\beta\alpha)) \circ F(\eta_X) \;=\; G[\beta\alpha] \;=\; G([\beta] \circ [\alpha])\ . \end{aligned}$$

The relation $G[\mathrm{Id}_{|X|}] = \mathrm{Id}_{G(X)}$ is even easier, so G is indeed a functor.

Finally, we show that the functor G satisfies $G\gamma = F : \mathbf{sset} \longrightarrow \mathcal{D}$. This is clear on objects. To establish the relation on morphisms, we consider any morphism of simplicial sets $f : X \longrightarrow Y$. We apply the functor F to the naturality relation of the adjunction unit $\eta : \mathrm{Id}_{\mathbf{sset}} \longrightarrow \mathcal{S} \circ |-|$ to get

$$F(\mathcal{S}|f|) \circ F(\eta_X) \;=\; F(\mathcal{S}|f| \circ \eta_X) \;=\; F(\eta_Y \circ f) \;=\; F(\eta_Y) \circ F(f)\ .$$

This yields

$$G(\gamma(f)) \;=\; G[|f|] \;=_{(7.12)}\; F(\eta_Y)^{-1} \circ F(\mathcal{S}|f|) \circ F(\eta_X) \;=\; F(f)\ .$$

So the functors $G\gamma$ and F also agree on morphisms. □

Now we use the universal property to define an equivalence of categories between the localization $\mathbf{sset}[\mathrm{weq}^{-1}]$ and the homotopy category $\mathrm{Ho}(\mathbf{Top}_{\mathrm{CW}})$. The geometric realization functor $|-| : \mathbf{sset} \longrightarrow \mathbf{Top}$ lands in the full subcategory $\mathbf{Top}_{\mathrm{CW}}$, and it takes weak equivalences to homotopy equivalences, which in turn become isomorphisms in the homotopy category $\mathrm{Ho}(\mathbf{Top}_{\mathrm{CW}})$. So the composite functor

$$\mathbf{sset} \xrightarrow{|-|} \mathbf{Top}_{\mathrm{CW}} \xrightarrow{\mathrm{proj}} \mathrm{Ho}(\mathbf{Top}_{\mathrm{CW}})$$

takes weak equivalences to isomorphisms. The universal property of the localization thus provides a unique functor

$$\Phi \ : \ \mathbf{sset}[\mathrm{weq}^{-1}] \ \longrightarrow \ \mathrm{Ho}(\mathbf{Top}_{\mathrm{CW}})$$

that makes the following diagram of categories and functors commute:

$$\begin{array}{ccc} \mathbf{sset} & \xrightarrow{|-|} & \mathbf{Top}_{\mathrm{CW}} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\text{project}} \\ \mathbf{sset}[\mathrm{weq}^{-1}] & \xrightarrow[\Phi]{} & \mathrm{Ho}(\mathbf{Top}_{\mathrm{CW}}) \end{array}$$

Theorem 7.13. *The functor $\Phi : \mathbf{sset}[\mathrm{weq}^{-1}] \longrightarrow \mathrm{Ho}(\mathbf{Top}_{CW})$ is an equivalence of categories.*

Proof. If we chase through the definitions, we see that Φ is given on objects by $\Phi(S) = |X|$, and on morphisms,

$$\Phi \ : \ \mathrm{Hom}_{\mathbf{sset}[\mathrm{weq}^{-1}]}(X, Y) \ = \ \mathrm{Hom}_{\mathbf{Top}}(|X|, |Y|)/\text{homotopy} \ \longrightarrow \ \mathrm{Hom}_{\mathrm{Ho}(\mathbf{Top}_{\mathrm{CW}})}(|X|, |Y|)$$

is the identity. So Φ is fully faithful. To see that Φ is also essentially surjective, we let K be any object of $\mathrm{Ho}(\mathbf{Top}_{\mathrm{CW}})$, i.e., a space that admits a CW-structure. Then by Theorem 6.4, the adjunction counit $\epsilon_K : |\mathcal{S}(K)| \longrightarrow K$ is a weak homotopy equivalence; since source and target of ϵ_K admit CW-structures, ϵ_K is even a homotopy equivalence. So the homotopy class of ϵ_K is an isomorphism in $\mathrm{Ho}(\mathbf{Top}_{\mathrm{CW}})$

$$[\epsilon_K] \ : \ \Phi(\mathcal{S}(K)) \ = \ |\mathcal{S}(K)| \ \xrightarrow{\cong} \ K \ .$$

Hence every object of $\mathrm{Ho}(\mathbf{Top}_{\mathrm{CW}})$ is isomorphic to an object in the image of the functor Φ. □

Now we turn to topological spaces, and we show that localization of spaces at weak homotopy equivalences is equivalent to the categories $\mathbf{sset}[\mathrm{weq}^{-1}]$ and $\mathrm{Ho}(\mathbf{Top}_{\mathrm{CW}})$.

Construction 7.14 (Localization of spaces at the weak equivalences)**.** We define the category $\mathbf{Top}[\mathrm{weq}^{-1}]$ to have all topological spaces as its objects. Morphisms in $\mathbf{Top}[\mathrm{weq}^{-1}]$ are defined by

$$\mathrm{Hom}_{\mathbf{Top}[\mathrm{weq}^{-1}]}(A, B) \ = \ \mathrm{Hom}_{\mathbf{Top}}(|\mathcal{S}(A)|, |\mathcal{S}(B)|)/\text{homotopy} \ .$$

Composition in $\mathbf{Top}[\mathrm{weq}^{-1}]$ is composition of homotopy classes of continuous maps. We define a functor $\gamma : \mathbf{Top} \longrightarrow \mathbf{Top}[\mathrm{weq}^{-1}]$ on objects by $\gamma(A) = A$, and on morphisms by sending a continuous map $f : A \longrightarrow B$ to the homotopy class of the map $|\mathcal{S}(f)| : |\mathcal{S}(A)| \longrightarrow |\mathcal{S}(B)|$.

Again, every morphism in $\mathbf{Top}[\mathrm{weq}^{-1}]$ is a 'fraction' of a continuous map and the inverse of a weak equivalence; in comparison to the fraction description in $\mathbf{sset}[\mathrm{weq}^{-1}]$ in (7.10), the 'denominator' is now on the right.

Proposition 7.15. *Let A and B be topological spaces, and let $\alpha : |\mathcal{S}(A)| \longrightarrow |\mathcal{S}(B)|$ be a continuous map. Then the relation*

$$[\alpha] \ = \ \gamma(\epsilon_B \circ \alpha) \circ \gamma(\epsilon_A)^{-1} \tag{7.16}$$

holds as morphisms from A to B in $\mathbf{Top}[\mathrm{weq}^{-1}]$.

Proof. The continuous map α participates in a diagram in **Top**:

$$\begin{array}{ccc} |\mathcal{S}|\mathcal{S}(A)|| & \xrightarrow{|\mathcal{S}(\epsilon_A)|} & |\mathcal{S}(A)| \\ {\scriptstyle |\mathcal{S}(\alpha)|}\downarrow & & \downarrow{\scriptstyle \alpha} \\ |\mathcal{S}|\mathcal{S}(B)|| & \xrightarrow[|\mathcal{S}(\epsilon_B)|]{} & |\mathcal{S}(B)| \end{array} \tag{7.17}$$

However, this diagram will typically *not* commute. (It would commute by naturality of ϵ if the horizontal maps were $\epsilon_{|\mathcal{S}(A)|}$ and $\epsilon_{|\mathcal{S}(B)|}$, but those are not the maps we are looking at.) We shall now argue that the diagram (7.17) commutes *up to homotopy*. To this end we precompose both ways around the square with the continuous map $|\eta_{\mathcal{S}(A)}| : |\mathcal{S}(A)| \longrightarrow |\mathcal{S}|\mathcal{S}(A)||$. Then we get

$$\begin{aligned} |\mathcal{S}(\epsilon_B)| \circ |\mathcal{S}(\alpha)| \circ |\eta_{\mathcal{S}(A)}| \ &\sim \ |\mathcal{S}(\epsilon_B)| \circ |\eta_{\mathcal{S}(B)}| \circ \alpha \ = \ |\mathcal{S}(\epsilon_B) \circ \eta_{\mathcal{S}(B)}| \circ \alpha \\ &= \ \alpha \ = \ \alpha \circ |\mathcal{S}(\epsilon_A) \circ \eta_{\mathcal{S}(A)}| \ = \ \alpha \circ |\mathcal{S}(\epsilon_A)| \circ |\eta_{\mathcal{S}(A)}| \end{aligned}$$

The first homotopy is provided by (7.9) for $X = \mathcal{S}(A)$ and $Y = \mathcal{S}(B)$. Besides this homotopy, we have used two instances of the triangle identity of an adjunction. The morphism $\eta_{\mathcal{S}(A)}$ is a weak equivalence by Proposition 7.6, so $|\eta_{\mathcal{S}(A)}|$ is a homotopy equivalence. We can thus 'cancel' $|\eta_{\mathcal{S}(A)}|$ up to homotopy, and the previous equality implies that the maps

$$|\mathcal{S}(\epsilon_B)| \circ |\mathcal{S}(\alpha)| \ , \ \alpha \circ |\mathcal{S}(\epsilon_A)| \ : \ |\mathcal{S}|\mathcal{S}(A)|| \ \longrightarrow \ |\mathcal{S}(B)|$$

are homotopic.

Because the square (7.17) commutes up to homotopy, the associated square of homotopy classes commutes. The associated homotopy classes of all four maps are morphisms in **Top**[weq^{-1}], and they form a commutative square in **Top**[weq^{-1}]:

$$\begin{array}{ccc} |\mathcal{S}(A)| & \xrightarrow{\gamma(\epsilon_A)} & A \\ {\scriptstyle \gamma(\alpha)}\downarrow & & \downarrow{\scriptstyle [\alpha]} \\ |\mathcal{S}(B)| & \xrightarrow[\gamma(\epsilon_B)]{} & B \end{array}$$

Equivalently:

$$[\alpha] \circ \gamma(\epsilon_A) \ = \ \gamma(\epsilon_B \circ \alpha) \ .$$

The map $\epsilon_A : |\mathcal{S}(A)| \longrightarrow A$ is a weak equivalence by Theorem 6.4, so $\gamma(\epsilon_A)$ is an isomorphism. So the relation (7.16) holds. $\square$

The following proof of the localization property of $\gamma : \mathbf{Top} \longrightarrow \mathbf{Top}[\text{weq}^{-1}]$ is quite similar to the previous proof in the context of simplicial sets in Theorem 7.11.

Theorem 7.18. *The functor* $\gamma : \mathbf{Top} \longrightarrow \mathbf{Top}[\text{weq}^{-1}]$ *is a localization at the class of weak homotopy equivalences.*

Proof. The composite functor

$$| - | \circ \mathcal{S} \ : \ \mathbf{Top} \ \longrightarrow \ \mathbf{Top}$$

takes weak equivalences to homotopy equivalences between CW-complexes; the latter become isomorphisms in the homotopy category $\mathrm{Ho}(\mathbf{Top}_{\mathrm{CW}})$. So the functor γ takes weak equivalences in **Top** to isomorphisms in **Top**[weq^{-1}].

To establish the universal property, we let $F : \mathbf{Top} \longrightarrow \mathcal{D}$ be any functor that inverts weak homotopy equivalences. We must show that there is a unique functor $G : \mathbf{Top}[\mathrm{weq}^{-1}] \longrightarrow \mathcal{D}$ such that $F = G\gamma : \mathbf{Top} \longrightarrow \mathcal{D}$. The uniqueness part is easy. Because γ is the identity on objects, we must have $G(A) = G(\gamma(A)) = F(A)$ on objects. For the uniqueness on morphisms we apply G to the fraction relation (7.16) to get

$$G[\alpha] \;=\; G\left(\gamma(\epsilon_B \circ \alpha) \circ \gamma(\epsilon_A)^{-1}\right) \;=\; F(\epsilon_B) \circ F(\alpha) \circ F(\epsilon_A)^{-1} \;. \tag{7.19}$$

So also the behavior of G on morphisms is determined by F.

The uniqueness argument also tells us how to *define* the functor $G : \mathbf{Top}[\mathrm{weq}^{-1}] \longrightarrow \mathcal{D}$ in terms of F: we must set $G(A) = F(A)$ on objects, and define G on morphism by the formula (7.19). This definition makes sense because ϵ_A is a weak equivalence by Theorem 6.4, so the functor F takes it to an isomorphism. We must still argue that the definition is well-defined, i.e., it does not depend on the representative α within its homotopy class. To this end we show that the functor F takes the same value on homotopic morphisms. The argument is essentially the same as for simplicial sets in Theorem 7.11. We let $H : A \times [0,1] \longrightarrow B$ be a homotopy from $f = H(-,0)$ to $g = H(-,1)$. We write $p : A \times [0,1] \longrightarrow A$ for the projection to the first factor. This map is a homotopy equivalence, and hence a weak equivalence. So by hypothesis, $F(p) : F(A \times [0,1]) \longrightarrow F(A)$ is an isomorphism in $\mathcal{D}$. The composite with the two end inclusions $i_0, i_1 : A \longrightarrow A \times [0,1]$ satisfy $p \circ i_0 = \mathrm{Id}_A = p \circ i_1$, so we have $F(p) \circ F(i_0) = \mathrm{Id}_{F(A)} = F(p) \circ F(i_1)$. Since $F(p)$ is an isomorphism, we deduce $F(i_0) = F(i_1)$. Hence

$$F(f) \;=\; F(H) \circ F(i_0) \;=\; F(H) \circ F(i_1) \;=\; F(g) \;.$$

This proves that G is well-defined on morphisms.

To show that the assignment (7.19) is functorial we consider another space C and a continuous map $\nu : |\mathcal{S}(B)| \longrightarrow |\mathcal{S}(C)|$. Then

$$\begin{aligned} G[\nu] \circ G[\psi] \;&=\; F(\epsilon_C) \circ F(\nu) \circ F(\epsilon_B)^{-1} \circ F(\epsilon_B) \circ F(\psi) \circ F(\epsilon_A)^{-1} \\ &=\; F(\epsilon_C) \circ F(\nu\psi) \circ F(\epsilon_A)^{-1} \;=\; G[\nu\psi] \;=\; G([\nu] \circ [\psi]) \;. \end{aligned}$$

The relation $G[\mathrm{Id}_{|\mathcal{S}(A)|}] = \mathrm{Id}_{G(A)}$ is even easier, so G is indeed a functor.

Finally, we show that the functor G satisfies $G\gamma = F : \mathbf{Top} \longrightarrow \mathcal{D}$. This is clear on objects. To establish the relation on morphisms, we consider any continuous map $f : A \longrightarrow B$. We apply the functor F to the naturality relation of the adjunction counit $\epsilon : |-| \circ \mathcal{S} \longrightarrow \mathrm{Id}_{\mathbf{Top}}$ to get

$$F(\epsilon_B) \circ F(|\mathcal{S}(f)|) \;=\; F(\epsilon_B \circ |\mathcal{S}(f)|) \;=\; F(f \circ \epsilon_A) \;=\; F(f) \circ F(\epsilon_A) \;.$$

This yields

$$G(\gamma(f)) \;=\; G[|\mathcal{S}(f)|] \;=_{(7.19)}\; F(\epsilon_B) \circ F(|\mathcal{S}(f)|) \circ F(\epsilon_A)^{-1} \;=\; F(f) \;.$$

So the functors $G\gamma$ and F also agree on morphisms. □

Now we use the universal property to define an equivalence of categories between the localizations $\mathbf{sset}[\mathrm{weq}^{-1}]$ and $\mathbf{Top}[\mathrm{weq}^{-1}]$. The realization functor

$$|-| \;:\; \mathbf{sset} \;\longrightarrow\; \mathbf{Top}$$

takes weak equivalences to homotopy equivalences, which are in particular weak equivalences. So the composite functor

$$\mathbf{sset} \xrightarrow{|-|} \mathbf{Top} \xrightarrow{\gamma} \mathbf{Top}[\mathrm{weq}^{-1}]$$

takes weak equivalences to isomorphisms. Similarly, the composite functor

$$\mathbf{Top} \xrightarrow{\mathcal{S}} \mathbf{sset} \xrightarrow{\gamma} \mathbf{sset}[\mathrm{weq}^{-1}]$$

takes weak homotopy equivalences to isomorphisms. The universal properties of the localizations $\gamma : \mathbf{sset} \longrightarrow \mathbf{sset}[\mathrm{weq}^{-1}]$ and $\gamma : \mathbf{Top} \longrightarrow \mathbf{Top}[\mathrm{weq}^{-1}]$ thus provide unique functors

$$\alpha \ : \ \mathbf{sset}[\mathrm{weq}^{-1}] \ \longrightarrow \ \mathbf{Top}[\mathrm{weq}^{-1}] \qquad \text{and} \qquad \beta \ : \ \mathbf{Top}[\mathrm{weq}^{-1}] \ \longrightarrow \ \mathbf{sset}[\mathrm{weq}^{-1}]$$

that make the following diagram of categories and functors commute:

$$\begin{array}{ccccc} \mathbf{sset} & \xrightarrow{|-|} & \mathbf{Top} & \xrightarrow{\mathcal{S}} & \mathbf{sset} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\ \mathbf{sset}[\mathrm{weq}^{-1}] & \xrightarrow[\alpha]{} & \mathbf{Top}[\mathrm{weq}^{-1}] & \xrightarrow[\beta]{} & \mathbf{sset}[\mathrm{weq}^{-1}] \end{array}$$

Theorem 7.20. *The composite functors*

$$\beta \circ \alpha : \mathbf{sset}[\mathrm{weq}^{-1}] \longrightarrow \mathbf{sset}[\mathrm{weq}^{-1}] \qquad \textit{and} \qquad \alpha \circ \beta : \mathbf{Top}[\mathrm{weq}^{-1}] \longrightarrow \mathbf{Top}[\mathrm{weq}^{-1}]$$

are naturally isomorphic to the respective identity functors. In particular, α and β are equivalences of categories.

Proof. We give the argument for $\beta \circ \alpha$; the argument for the other composite is completely analogous, and we omit it. We compose the adjunction unit $\eta : \mathrm{Id}_{\mathbf{sset}} \longrightarrow \mathcal{S} \circ |-|$ with the localization functor $\gamma : \mathbf{sset} \longrightarrow \mathbf{sset}[\mathrm{weq}^{-1}]$ to obtain a natural transformation

$$\gamma \circ \eta \ : \ \gamma \ \longrightarrow \ \gamma \circ \mathcal{S} \circ |-| \ = \ \beta \circ \gamma \circ |-| \ = \ \beta \circ \alpha \circ \gamma$$

between two functors from $\mathbf{sset}$ to $\mathbf{sset}[\mathrm{weq}^{-1}]$ that both invert weak equivalences. The universal property of the localization functor for natural transformations (see Proposition 7.2) provides a unique natural transformation

$$\tau \ : \ \mathrm{Id} \ \longrightarrow \ \beta \circ \alpha$$

of endofunctors on $\mathbf{sset}[\mathrm{weq}^{-1}]$ such that

$$\tau \circ \gamma \ = \ \gamma \circ \eta \ .$$

For every simplicial set X, this in particular means that

$$\tau_X \ = \ \tau_{\gamma(X)} \ = \ \gamma(\eta_X) \ .$$

Because $\eta_X : X \longrightarrow \mathcal{S}|X|$ is a weak equivalence, $\gamma(\eta_X)$ is an isomorphism in $\mathbf{sset}[\mathrm{weq}^{-1}]$. So the natural transformation τ is in fact a natural isomorphism. □

Remark 7.21 (Kan complexes)**.** We can extend the triangle of equivalences of category from the beginning of this section to a square by adding yet another category that is equivalent to the other three:

$$\begin{array}{ccc} \mathrm{Ho}(\mathbf{Top}_{\mathrm{CW}}) & \xleftarrow{|-|} & \mathbf{sset}[\text{weak eq}^{-1}] \\ {\scriptstyle\text{inclusion}}\downarrow & & \uparrow{\scriptstyle\text{inclusion}} \\ \mathbf{Top}[\text{weak eq}^{-1}] & \xrightarrow[\mathcal{S}]{} & \mathrm{Ho}(\mathbf{sset}_{\mathrm{Kan}}) \end{array}$$

Here the lower right corner is the *homotopy category of Kan complexes*. This notion goes back to Dan Kan [3, Definition 1.1] under name of 'extension condition' for simplicial sets; simplicial sets with the extension condition are nowadays also called *Kan complexes*. For example, the singular complex $\mathcal{S}(A)$ of every space A is a Kan complex, and so is the nerve of every groupoid. In particular, for every simplicial set X, the adjunction unit $\eta_X : X \longrightarrow \mathcal{S}|X|$ is a natural weak equivalence to a Kan complex.

The key facts about Kan complexes needed for this equivalence are:

- For morphisms from an arbitrary simplicial set to a Kan complex, 'elementary homotopy' is symmetric and transitive, and hence an equivalence relation. The category $\mathrm{Ho}(\mathbf{sset}_{\mathrm{Kan}})$ has as objects the Kan complexes, and as morphisms the homotopy classes of morphisms of simplicial sets.
- If X is any simplicial set and Y a Kan complex, then geometric realization induces a bijection
$$[X, Y]_{\mathbf{sset}} \longrightarrow [|X|, |Y|]_{\mathbf{Top}} , \quad [f] \longmapsto [|f|]$$
from the set of simplicial homotopy classes of morphisms of simplicial sets to the set of homotopy classes of continuous maps. This can be viewed as a version of 'simplicial approximation'.
- For every CW-complex A and every space B, the singular complex functor induces a bijection
$$[A, B]_{\mathbf{Top}} \longrightarrow [\mathcal{S}(A), \mathcal{S}(B)]_{\mathbf{sset}} , \quad [f] \longmapsto [\mathcal{S}(f)]$$
from the set of homotopy classes of continuous maps to the set of simplicial homotopy classes of morphism of simplicial sets.
- Kan complexes yield an intrinsic characterization of weak equivalences of simplicial sets, i.e., one that does not refer to topological spaces and geometric realization. Indeed, a morphism $f : X \longrightarrow Y$ of simplicial sets is a weak equivalence in the sense of Definition 7.4 if and only if for every Kan complex K, the induced map
$$f^* : \mathrm{Hom}_{\mathbf{sset}}(Y, K)/\mathrm{homotopy} \longrightarrow \mathrm{Hom}_{\mathbf{sset}}(X, K)/\mathrm{homotopy}$$
of simplicial homotopy classes of morphisms is bijective. Showing that these two definitions of weak equivalences coincide is, however, not all that easy.

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