SIMPLICIAL STRUCTURES ON MODEL CATEGORIES AND FUNCTORS

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Abstract. We produce a highly structured way of associating a simplicial category to a model category which improves on work of Dwyer and Kan and answers a question of Hovey. We show that model categories satisfying a certain axiom are Quillen equivalent to simplicial model categories. A simplicial model category provides higher order structure such as composable mapping spaces and homotopy colimits. We also show that certain homotopy invariant functors can be replaced by weakly equivalent simplicial, or "continuous," functors. This is used to show that if a simplicial model category structure exists on a model category then it is unique up to simplicial Quillen equivalence.

1. Introduction. In [DK] Dwyer and Kan showed that a simplicial category, called the hammock localization, can be associated to any Quillen model category [Qui]. This simplicial category captures higher order information, for example fibration and cofibration sequences and mapping spaces, see [Qui, I 3], which is not captured by the ordinary homotopy category. Hovey carried this further by showing that the homotopy category of simplicial sets acts on the homotopy category of any model category [Hov, 5.5.3]. Hovey then wondered if in fact every model category is Quillen equivalent to a simplicial model category [Hov, 8.9]. Quillen equivalence is the appropriate notion of equivalence for model categories, so this would be the most highly structured way of associating a simplicial category to any model category. The following existence result is proved in Theorem 3.6.

Theorem 1.1. *If $\mathcal{C}$ is a left proper, cofibrantly generated model category that satisfies Realization Axiom 3.4, then $\mathcal{C}$ is Quillen equivalent to a simplicial model category.*

Throughout this paper we use a slightly stronger notion of cofibrantly generated model category than is standard; see Definition 8.1. We also have the following uniqueness result, which is proved as Corollary 6.2. Assume that $\mathcal{C}$ and $\mathcal{D}$ are model categories which either satisfy the hypotheses of Theorem 1.1 or satisfy the hypotheses of one of the general localization machines in [Hir] or [Smi], see also [Dug].

Manuscript received February 7, 2000; revised August 3, 2000.

Research of the first author supported in part by an AMS Centennial Fellowship; research of the third author supported in part by NSF grants.

American Journal of Mathematics 123 (2001), 551–575.

THEOREM 1.2. *Under these hypotheses, if $\mathcal{C}$ and $\mathcal{D}$ are Quillen equivalent simplicial model categories, then $\mathcal{C}$ and $\mathcal{D}$ are simplicially Quillen equivalent.*

By considering the identity functor, this shows that a simplicial model category structure on a model category is unique up to simplicial Quillen equivalence, see Corollary 5.3. This strengthens Dwyer and Kan's analogous result on the homotopy categories in [DK].

To prove Theorem 1.2, in Section 6 we consider replacing functors between simplicial model categories by simplicial, or "continuous," functors. We show that a homotopy invariant functor F can be replaced by a naturally weakly equivalent simplicial functor, see Corollary 6.5. We also show that Quillen adjunctions between simplicial model categories, the appropriate notion of functors between model categories, can be replaced by simplicial Quillen adjunctions, see Proposition 6.1. This answers another part of Hovey's problem [Hov, 8.9].

Another reason to construct replacement simplicial model categories is to have a simple definition of a homotopy colimit. The original definition in [BK, XII] generalizes to define a homotopy colimit in any simplicial model category, see [Hir, 20]. So the simplicial replacements considered here provide new situations where a simple homotopy colimit can be defined. The Bousfield-Kan type homotopy colimit on the replacement simplicial model category can be transported to the original model category via the Quillen equivalence.

Showing that stable model categories have simplicial replacements was the original motivation for this work, see Section 4.

PROPOSITION 1.3. *Any proper, cofibrantly generated, stable model category is Quillen equivalent to a simplicial model category.*

The category of unbounded differential graded modules over a differential graded algebra is one particular example of a stable model category that was not previously known to have a Quillen equivalent simplicial replacement. This example is treated explicitly in Corollary 4.6 and answers another question of Hovey [Hov, 8.9].

For a model category $\mathcal{C}$, our candidate for a Quillen equivalent simplicial model category is based on the category of simplicial objects in $\mathcal{C}$, $s\mathcal{C}$. Reedy [Ree] establishes the Reedy model category on $s\mathcal{C}$, but it is neither simplicial nor Quillen equivalent to $\mathcal{C}$, see [DKS, 2.6] or Corollary 7.4. So we localize the Reedy model category to create the realization model category. Instead of using general machinery to produce the localization model category, we explicitly define the cofibrations, weak equivalences, and fibrations and then check that they form a model category. This avoids unnecessary hypotheses. In Theorem 3.6 we show that if $\mathcal{C}$ is a left proper, cofibrantly generated model category that satisfies Realization Axiom 3.4, then the realization structure on $s\mathcal{C}$ is a simplicial model category that is Quillen equivalent to the original model category $\mathcal{C}$.

More generally, we show that there is at most one model category on $s\mathcal{C}$ that satisfies certain properties, see Theorem 3.1. When this model category exists on

$s\mathcal{C}$ it is Quillen equivalent to the original model category $\mathcal{C}$, and we refer to it as the canonical model category structure on $s\mathcal{C}$. If $\mathcal{C}$ satisfies the hypotheses of Theorem 3.6 as listed above, then the canonical model category structure on $s\mathcal{C}$ exists and is simplicial since it agrees with the realization model category. The applications in Sections 5 and 6 rely only on the existence of the canonical model category on $s\mathcal{C}$ and the fact that it is simplicial.

In [Dug], Dugger has also developed a way to produce replacement simplicial model categories. His approach is similar to ours, but he uses the two general localization machines that exist for left proper, cellular model categories, see [Hir] and for left proper, cofibrantly generated, combinatorial model categories, see [Smi]. Hence, these hypotheses also ensure the existence of the simplicial, canonical model category on $s\mathcal{C}$. So the applications of Sections 5 and 6 also apply under the conditions investigated in [Dug].

One drawback to these general machines is that the fibrations cannot always be identified in concrete terms. Our approach here is to explicitly define the fibrations and then verify the model category axioms. This approach requires a slightly stronger notion of "cofibrantly generated," see Definition 8.1. Then for left proper, cofibrantly generated model categories, Realization Axiom 3.4 is equivalent to having the explicit definition of the fibrations, see Proposition 3.7.

Organization. In Section 2 we recall the simplicial structure on $s\mathcal{C}$ and the Reedy model category structure on $s\mathcal{C}$. In Section 3, we define the canonical model category structure on $s\mathcal{C}$, the realization model category structure on $s\mathcal{C}$, and state the main theorems. In Section 4 we consider examples including simplicial model categories, stable model categories, and unbounded differential graded modules over a differential graded algebra. In Sections 5 and 6 we consider the applications mentioned above: the uniqueness of simplicial model category structures and replacing functors by simplicial functors. In Section 7, we show that the Reedy model category structure only partially satisfies the compatibility axiom SM7. This also gives several statements that are needed in later proofs. In Section 8 we verify the main theorem, Theorem 3.6, which states that the realization structure on $s\mathcal{C}$ is a simplicial model category that is Quillen equivalent to the original model category, $\mathcal{C}$.

2. The Reedy model category for simplicial objects in $\mathcal{C}$.

Here we define the canonical simplicial structure on the category of simplicial objects of $\mathcal{C}$, $s\mathcal{C}$. This is our candidate category for replacing $\mathcal{C}$ by a simplicial model category. We also recall the definition of a simplicial model category and the Reedy model category structure on $s\mathcal{C}$.

Let $s\mathcal{C}$ denote the simplicial objects in $\mathcal{C}$, i.e. the functors $\Delta^{op} \to \mathcal{C}$. Let $\mathcal{S}$ denote the category of simplicial sets. For any category $\mathcal{C}$ with small limits and colimits, $s\mathcal{C}$ is tensored and cotensored over $\mathcal{S}$, compare [Qui, II 1]. For a set S and $X \in \mathcal{C}$, let $X \cdot S = \coprod_{s\in S} X$. For X in $s\mathcal{C}$ and K in $\mathcal{S}$ define $X \otimes K$ in $s\mathcal{C}$ as the simplicial object with nth simplicial degree $(X \otimes K)_n = X_n \cdot K_n$. For A in $\mathcal{C}$ denote $cA \otimes K$ as $A \otimes K$ in $s\mathcal{C}$ where $c\colon \mathcal{C} \to s\mathcal{C}$ is the constant object functor. Note $cA = A \otimes \Delta[0]$. The cotensor X^K in $s\mathcal{C}$ is also defined in [Qui, II 1]. In this paper we mainly use the degree zero part in $\mathcal{C}$ of this cotensor, and denote it X^K. From this simplicial tensor one can define simplicial mapping spaces, $\mathrm{map}(X, Y)$ in $\mathcal{S}$ for $X, Y \in s\mathcal{C}$ with nth simplicial degree $\mathrm{map}(X, Y)_n = s\mathcal{C}(X \otimes \Delta[n], Y)$. So $s\mathcal{C}$ is also enriched over $\mathcal{S}$.

We now recall the definition of a simplicial model category, which asks that the simplicial structure be compatible with the model category structure.

Definition 2.1. A *simplicial model category* is a model category $\mathcal{C}$ that is enriched, cotensored and tensored over $\mathcal{S}$ and satisfies the following axiom:

AXIOM 2.2. [Qui, II.2 SM7] *If $f\colon A \to B$ is a cofibration in $\mathcal{C}$ and $i\colon K \to L$ is a cofibration in $\mathcal{S}$ then*

$$q\colon A \otimes L \coprod_{A \otimes K} B \otimes K \to B \otimes L$$

(1) *is a cofibration;*
(2) *if f is a weak equivalence, then so is q;*
(3) *if i is a weak equivalence, then so is q.*

The first model category we consider on $s\mathcal{C}$ is the Reedy model category structure, see [Ree, Theorem A] or [DKS, 2.4]. Before defining the Reedy model category structure we need to define latching and matching objects. Let $\mathcal{L}_n$ be the category with objects the maps $[j] \to [n] \in \Delta^{op}$ with $j < n$ and with morphisms the commuting triangles. Let $l\colon \mathcal{L}_n \to \Delta^{op}$ be the forgetful functor. Given $X\colon \Delta^{op} \to \mathcal{C}$, an object in $s\mathcal{C}$, define $L_n X = \mathrm{colim}_{\mathcal{L}_n} l^* X$. $L_n X$ is the nth *latching object* of X. Similarly, let $\mathcal{M}_n$ be the category with objects the maps $[n] \to [j] \in \Delta^{op}$ with $j < n$ and with morphisms the commuting triangles. Let $m\colon \mathcal{M}_n \to \Delta^{op}$ be the forgetful functor. Given $X\colon \Delta^{op} \to \mathcal{C}$, an object in $s\mathcal{C}$, define $M_n X = \lim_{\mathcal{M}_n} m^* X$. $M_n X$ is the nth *matching object* of X.

Definition 2.3. A map $f\colon X \to Y$ in $s\mathcal{C}$ is a *level weak equivalence* if $X_n \to Y_n$ is a weak equivalence in $\mathcal{C}$ for each n. It is a *Reedy cofibration* if the induced map $X_n \coprod_{L_n X} L_n Y \to Y_n$ is a cofibration in $\mathcal{C}$ for each n. Similarly, f is a *Reedy fibration* if the induced map $X_n \to Y_n \prod_{M_n Y} M_n X$ is a fibration in $\mathcal{C}$.

Note that a map $X \to Y$ in $s\mathcal{C}$ is a Reedy trivial cofibration (resp. Reedy trivial fibration) if and only if all the maps $X_n \coprod_{L_n X} L_n Y \to Y_n$ are acyclic cofibrations in $\mathcal{C}$ (resp. all the maps $X_n \to Y_n \prod_{M_n Y} M_n X$ are acyclic fibrations in $\mathcal{C}$). The following theorem is due to Reedy [Ree, Theorem A]. See also [DKS, 2.4] or [Hov, 5.2.5].

THEOREM 2.4. *The category $s\mathcal{C}$ equipped with the level weak equivalences, Reedy cofibrations, and Reedy fibrations is a model category, referred to as the* Reedy model category.

This Reedy model category structure on $s\mathcal{C}$ with the canonical simplicial structure described above satisfies properties (1) and (2) of Axiom 2.2 (SM7) but does not satisfy property (3). This is stated in Corollary 7.4. So this model category is not a simplicial model category, but is a stepping stone for defining the model category structure on $s\mathcal{C}$ that is simplicial.

3. Statement of results. Here we define the realization model category structure on $s\mathcal{C}$. This is the model category structure on $s\mathcal{C}$ which is simplicial and also Quillen equivalent to the original model category on $\mathcal{C}$, see Theorem 3.6. We first show that there is at most one model category on $s\mathcal{C}$ with certain properties, which we call the canonical model category, see Theorem 3.1. We then show that the canonical model category coincides with the realization model category when it exists.

Denote the set of morphisms in the homotopy category of the Reedy model category on $s\mathcal{C}$ by $[X,Y]^{\mathrm{Ho(Reedy)}}$. Call a map in $s\mathcal{C}$ a *realization weak equivalence* if for all Z in $\mathcal{C}$ it induces an isomorphism on $[-,cZ]^{\mathrm{Ho(Reedy)}}$, where c is the constant functor. An object in $s\mathcal{C}$ is *homotopically constant* if each of the simplicial operators d_i, s_i is a weak equivalence.

THEOREM 3.1. *Let $\mathcal{C}$ be a model category. Then there is at most one model category structure on $s\mathcal{C}$ such that:*

- *every level equivalence is a weak equivalence,*
- *the cofibrations are the Reedy cofibrations, and*
- *the fibrant objects are the homotopically constant, Reedy fibrant objects.*

When this model category exists, we refer to it as the canonical model category on $s\mathcal{C}$. *Moreover, when it exists the weak equivalences coincide with the realization weak equivalences.*

Proof. First assume this canonical model category exists. Then since Reedy cofibrations are cofibrations and level equivalences are weak equivalences, a Reedy cylinder object ([Qui, I.1 Def. 4], [Hov, 1.2.4]) for a Reedy cofibrant object is also a cylinder object in the canonical model category. This shows using [Qui, I.1 Cor. 1] that for A Reedy cofibrant and X homotopically constant and Reedy fibrant the homotopy classes of maps coincide in the homotopy category of the Reedy model category and the homotopy category of the canonical model category, $[A,X]^{\mathrm{Ho(Reedy)}} \cong [A,X]^{\mathrm{Ho(can.)}}$. Since level equivalences are weak equivalences in both cases this means that for arbitrary A and homotopically constant X, $[A,X]^{\mathrm{Ho(Reedy)}} \cong [A,X]^{\mathrm{Ho(can.)}}$.

A map $f\colon A \to B$ is a weak equivalence in the canonical model category if and only if for each homotopically constant X, $[f,X]^{\mathrm{Ho(can.)}}$ is a bijection. Or, equivalently, $[f, X]^{\mathrm{Ho(Reedy)}}$ is a bijection. Since X is level equivalent to $c(X_0)$, this is equivalent to $[f, cZ]^{\mathrm{Ho(Reedy)}}$ being a bijection for each Z in $\mathcal{C}$. So the weak equivalences are the realization weak equivalences.

Since the cofibrations and weak equivalences are determined, the fibrations are determined by the right lifting property. Hence there is at most one model category on $s\mathcal{C}$ with the above properties. □

This specifies the model category of interest on $s\mathcal{C}$ because when the canonical model category exists on $s\mathcal{C}$ it is Quillen equivalent to the original model category $\mathcal{C}$, see Proposition 3.9.

Remark 3.2. In [CS, 11.3] and [Hir, 21], for any model category $\mathcal{C}$ a homotopy colimit functor is constructed which is the total left derived functor of colimit. Using this definition we could have defined the realization weak equivalences as those maps whose homotopy colimit is an isomorphism. We use "realization" instead of "hocolim" to avoid conflict with the terminology of [Dug]. Specifically, let hocolim: Ho(Reedy) $\to$ Ho$(\mathcal{C})$ be the total left derived functor of colimit. Then $[A, cZ]^{\mathrm{Ho(Reedy)}}$ is isomorphic to $[\operatorname{hocolim} A, Z]^{\mathrm{Ho}(\mathcal{C})}$. So $f\colon A \to B$ is a realization weak equivalence if and only if hocolim$f$ is an isomorphism. In the rest of this paper though we only assume the existence of the homotopy colimit for simplicial model categories, which follows from [BK, XII], see also [Hir, 20].

Now we demonstrate conditions which ensure the existence of the canonical model category structure on $s\mathcal{C}$.

Definition 3.3. A Reedy fibration $f\colon X \to Y$ in $s\mathcal{C}$ is an *equifibered Reedy fibration* if the map $X_{m+1} \xrightarrow{(d_i, f_{m+1})} X_m \times_{Y_m} Y_{m+1}$ is a weak equivalence for each m and for each simplicial face operator d_i with $0 \le i \le m+1$.

AXIOM 3.4. (Realization Axiom) *If $f\colon X \to Y$ in $s\mathcal{C}$ is an equifibered Reedy fibration and a realization weak equivalence then f is a level weak equivalence.*

See Section 4 for examples where Axiom 3.4 is verified. See Definition 8.1 for a definition of a cofibrantly generated model category. A model category is *left proper* if the pushout of a weak equivalence along a cofibration is a weak equivalence. Let Ev: $s\mathcal{C} \to \mathcal{C}$ be the evaluation functor given by Ev$\,X = X_0$. Note Ev is right adjoint to c, the constant functor.

Definition 3.5. A pair L, R of adjoint functors between two model categories is a *Quillen adjoint pair* if L, the left adjoint, preserves cofibrations and trivial cofibrations. Equivalently, R preserves fibrations and trivial fibrations. Such an adjoint pair induces adjoint total derived functors on the homotopy categories, see [Qui, I.4 Thm. 3]. A Quillen adjoint pair is a *Quillen equivalence* if the total derived functors induce an equivalence on the homotopy categories.

THEOREM 3.6. *If $\mathcal{C}$ is a left proper, cofibrantly generated model category that satisfies the Realization Axiom, then the following hold.*

(1) *The canonical model category on $s\mathcal{C}$ exists. Moreover, it is cofibrantly generated and the fibrations are the equifibered Reedy fibrations. It is also referred to as the* realization model category.

(2) *The realization model category structure on $s\mathcal{C}$ satisfies Axiom 2.2 (SM7). Hence it is a simplicial model category.*

(3) *The adjoint functor pair $c\colon \mathcal{C} \leftrightarrows s\mathcal{C}\colon \mathrm{Ev}$ induces a Quillen equivalence of the model category on $\mathcal{C}$ and the realization model category on $s\mathcal{C}$.*

Moreover, the realization model category structure agrees with the canonical model category on $s\mathcal{C}$.

This theorem is proved in Section 8. Recall that our definition of cofibrantly generated is slightly stronger than standard; see Definition 8.1. Since the weak equivalences and cofibrations of the realization model category agree with those of the canonical model category, these two model categories agree when they exist. Thus, under the hypotheses of this theorem, the canonical model category is a simplicial model category. In fact, one can show that if the canonical model category exists and is cofibrantly generated in the sense of Definition 8.1 then it is a simplicial model category.

The next proposition shows that Realization Axiom 3.4 must hold if the fibrations in the canonical model category on $s\mathcal{C}$ are to be the equifibered Reedy fibrations.

PROPOSITION 3.7. *Assume $\mathcal{C}$ is a left proper, cofibrantly generated model category and the canonical model category on $s\mathcal{C}$ exists. Then the fibrations in the canonical model structure coincide with the equifibered Reedy fibrations if and only if $\mathcal{C}$ satisfies Realization Axiom 3.4.*

Proof. If the Realization Axiom holds, then part 1 of Theorem 3.6 gives the characterization of the fibrations as equifibered Reedy fibrations. For the other implication, an equifibered Reedy fibration that is also a realization weak equivalence is a trivial fibration in the canonical model structure by assumption. But a trivial fibration has the right lifting property with respect to the Reedy cofibrations, and hence is a level equivalence. Thus the Realization Axiom holds. □

Remark 3.8. As mentioned in the introduction, Dugger [Dug] also has conditions on a model category $\mathcal{C}$ which ensure that $s\mathcal{C}$ has a model category structure, called the hocolim model category, which agrees with the canonical model category and is simplicial. In particular, Proposition 3.7 can be used to explicitly describe the fibrations for some of Dugger's examples.

We end this section by stating a few of the properties that follow just from the existence of the canonical model category structure. Note that Theorem 3.6 (3) follows from Theorem 3.6 (1) and the first statement below since the realization model category and the canonical model category agree when they exist.

PROPOSITION 3.9. *If the canonical model category on* $s\mathcal{C}$ *exists then*

(1) *The model category on* $\mathcal{C}$ *is Quillen equivalent to the canonical model category on* $s\mathcal{C}$ *via the adjoint functor pair* (c, Ev).

(2) *A map between fibrant objects is a weak equivalence if and only if it is a level equivalence.*

(3) *The fibrations between fibrant objects are the Reedy fibrations.*

Proof. For the second statement, note that c preserves cofibrations and trivial cofibrations. By adjointness Ev preserves fibrations and trivial fibrations, and hence also weak equivalences between fibrant objects. But, if $\mathrm{Ev} f$ is a weak equivalence then f is a level equivalence since fibrant objects are homotopically constant.

To show that the adjoint functor pair (c, Ev) induces a Quillen equivalence, we use the criterion in [HSS, 4.1.7] since Ev preserves and detects weak equivalences between fibrant objects. So we must show for any cofibrant object X in $\mathcal{C}$ that $X \to \mathrm{Ev}(cX)^f$ is a weak equivalence where $(cX)^f$ is a fibrant replacement of cX in $s\mathcal{C}$. Take $(cX)^f$ to be the Reedy fibrant replacement of cX; it is homotopically constant and hence also a fibrant replacement in the canonical model category. Then $(cX)^f$ and cX are level equivalent so $X \to \mathrm{Ev}(cX)^f$ is indeed a weak equivalence in $\mathcal{C}$.

Since fibrations have the right lifting property with respect to level trivial Reedy cofibrations, a fibration is a Reedy fibration. So we assume $f\colon X \to Y$ is a Reedy fibration between two fibrant objects and show that it is a fibration. Factor $f = pi$ with i a trivial cofibration and p a fibration. Then i is a weak equivalence between fibrant objects, hence a level equivalence by part two. Thus i is a trivial Reedy cofibration so it has the left lifting property with respect to f. This implies that f is a retract of p, and hence a fibration in $s\mathcal{C}$. □

4. Examples. In this section we give a criterion for simplicial model categories to satisfy the Realization Axiom and verify the Realization Axiom for stable model categories. So for the left proper, cofibrantly generated model categories among these examples, Theorem 3.6 shows that $\mathcal{C}$ is Quillen equivalent to the simplicial, canonical model category on $s\mathcal{C}$. We mention one particular example, the category $\mathcal{D}$ of unbounded differential graded modules over a differential graded algebra.

Simplicial model categories. One source of model categories satisfying Realization Axiom 3.4 is given by simplicial model categories where the realization factors through simplicial sets, see below. These examples are of interest for Sections 5 and 6, where we discuss replacing functors between simplicial model categories by simplicial functors and discuss the uniqueness of simplicial model category structures.

For a simplicial model category $\mathcal{C}$, define a functor Sing: $\mathcal{C} \to s\mathcal{C}$ by $(\operatorname{Sing} X)_n = X^{\Delta[n]}$. Then $|-|: s\mathcal{C} \to \mathcal{C}$ is the left adjoint to Sing. These functors are investigated further in Section 5.

Definition 4.1. For a simplicial model category $\mathcal{C}$, say that *the realization factors through simplicial sets* if the following hold.

(1) There is a functor $U: \mathcal{C} \to \mathcal{S}$ such that f is a weak equivalence in $\mathcal{C}$ if and only if Uf is a weak equivalence in $\mathcal{S}$.

(2) U preserves fibrations.

(3) For any object $X \in s\mathcal{C}$, $U|X|$ is naturally weakly equivalent to $|\bar{U}X|$ where $\bar{U}$ is the prolongation of U defined by applying U to each level in $s\mathcal{C}$.

Examples of such model categories include topological spaces with $U = \operatorname{Sing}$ and the standard model category on simplicial objects in a category $\mathcal{C}$ with an underlying set functor, such as simplicial groups [Qui, II.4].

A model category is *right proper* if the pullback of a weak equivalence along a fibration is a weak equivalence. A *proper* model category is one that is both right and left proper.

PROPOSITION 4.2. *If $\mathcal{C}$ is a proper, cofibrantly generated simplicial model category where the realization factors through simplicial sets, as above, then $\mathcal{C}$ satisfies Realization Axiom 3.4. Hence the canonical model category on $s\mathcal{C}$ exists, is simplicial, and is Quillen equivalent to $\mathcal{C}$ by Theorem 3.6.*

Hence, under these hypotheses on $\mathcal{C}$, the applications in Sections 5 and 6 apply. These statements basically follow because the Realization Axiom holds for simplicial sets.

LEMMA 4.3. *The model category of simplicial sets, $\mathcal{S}$, satisfies Realization Axiom 3.4.*

Below we verify that Lemma 4.3 is a special case of the following proposition, essentially due to Puppe [Pup].

PROPOSITION 4.4. *Let I be a small category and $X \to Y$ be a map of I-diagrams of simplicial sets such that for each $i_1 \to i_2 \in I$ the square*

$$\begin{array}{ccc} X(i_1) & \longrightarrow & Y(i_1) \\ \downarrow & & \downarrow \\ X(i_2) & \longrightarrow & Y(i_2) \end{array}$$

is homotopy cartesian. Then for each object $i \in I$, the square

$$\begin{array}{ccc} X(i) & \longrightarrow & Y(i) \\ \downarrow & & \downarrow \\ \operatorname{hocolim}_I X & \longrightarrow & \operatorname{hocolim}_I Y \end{array}$$

is homotopy cartesian.

Proof of Lemma 4.3. In the proposition take $I = \Delta$, the simplicial indexing category. An equifibered Reedy fibration $f\colon X \to Y$, viewed as a map of Δ-diagrams, satisfies the hypotheses of Proposition 4.4, and f is a realization weak equivalence precisely when $\operatorname{hocolim}_\Delta X \to \operatorname{hocolim}_\Delta Y$ is a weak equivalence by Remark 3.2. Therefore, for such f and for every $i \in \Delta$ the map $X(i) \to Y(i)$ is a weak equivalence, i.e., f is a level weak equivalence. □

A proof of Proposition 4.4 in this generality appears in [Rez] where it is generalized to simplicial sheaves. Alternatively, one can adapt the argument of [Far, App. HL], where the Proposition is stated under the additional hypothesis that the nerve of the indexing category I and all $Y(i)$ are path-connected. This implies that the homotopy colimit of Y is also connected, and so the conclusion as given in [Far, App. HL] in terms of homotopy fibres is equivalent to the conclusion of Proposition 4.4. Proposition 4.4 avoids explicit reference to homotopy fibres, and in this form the connectivity hypotheses are irrelevant. It can be proved, as in [Far, App. HL], by first checking the special cases of a homotopy pushout, a (possibly infinite) disjoint union and a sequential homotopy colimit; an arbitrary homotopy colimit is built from these three ingredients, so the result follows.

Puppe's original result is about simplicial objects in the category of topological spaces; we could have derived the Realization Axiom for simplicial sets directly from his result, although some care would be needed, since he effectively works in a different model category (in which the "weak equivalences" of spaces are plain homotopy equivalences) and he uses the version of geometric realization of simplicial spaces in which degeneracies are not collapsed.

Proof of Proposition 4.2. Let $f\colon X \to Y$ be an equifibered Reedy fibration and a realization weak equivalence in $s\mathcal{C}$. Since $\mathcal{C}$ is a right proper model category, the condition for an equifibered Reedy fibration is invariant under level equivalences. By definition level equivalences are realization equivalences. Hence, we can assume that X and Y are Reedy cofibrant. For simplicial model categories, the realization $|-|$ is weakly equivalent to the homotopy colimit on Reedy cofibrant objects. This follows from the generalization of [BK, XII] to general simplicial model categories, see [Hir, 20.6.1]. So $|f|$ is a weak equivalence in $\mathcal{C}$ by Remark 3.2, since f is a realization weak equivalence. By properties (1) and (2) of Definition 4.1, this means that $U|f|$ and $|\bar{U}f|$ are weak equivalences. Thus, $\bar{U}f$ is a realization weak equivalence of bisimplicial sets, by Remark 3.2 and the fact that all bisimplicial sets are Reedy cofibrant. Since U preserves fibrations and weak equivalences, it preserves homotopy pullback squares, and hence $\bar{U}$ preserves equifibered Reedy fibrations. So, by Lemma 4.3, $\bar{U}f$ is a level equivalence. Thus f is a level equivalence. □

Stable model categories. Recall from [Qui, I.2] that the homotopy category of a pointed model category supports a suspension functor Σ with a right adjoint loop functor Ω. A pointed model category $\mathcal{C}$ is *stable* if Σ and Ω are inverse equivalences on the homotopy category.

PROPOSITION 4.5. *Any proper, cofibrantly generated, stable model category $\mathcal{C}$ satisfies Realization Axiom 3.4. Hence the canonical model category on $s\mathcal{C}$ exists, is simplicial, and is Quillen equivalent to $\mathcal{C}$ by Theorem 3.6.*

Proof. First note that since $\mathcal{C}$ is stable the Reedy model category on $s\mathcal{C}$ is also stable. This follows since Reedy cofibrations and fibrations are level cofibrations and fibrations and colimits and limits are taken levelwise. So the suspension and loop functors in the Reedy model category are level equivalent to the levelwise suspension and loop in $\mathcal{C}$.

Now given a realization weak equivalence $f\colon X \to Y$ in $s\mathcal{C}$ that is an equifibered Reedy fibration, we must show that f is a level equivalence. Since $\mathcal{C}$ is right proper, the level homotopy fiber of f is weakly equivalent to F, the fiber of f. In a stable model category fiber sequences induce long exact sequences after applying $[-, cZ]^{\mathrm{Ho(Reedy)}}$. So $[F, cZ]^{\mathrm{Ho(Reedy)}}$ is trivial for any Z in $\mathcal{C}$. Since f is equifibered, F is homotopically constant and hence level equivalent to $c(F_0)$. Thus id_F is trivial in Ho(Reedy). This implies that F is level trivial, and hence that f is a level equivalence since $\mathcal{C}$ is stable. □

Differential graded modules. A cofibrantly generated model category, $\mathcal{D}$, of differential graded modules over a differential graded algebra, A, is constructed in [SSa, 5], see also [Hov, 2.3.11]. The weak equivalences and fibrations are the quasi-isomorphisms and surjections of the underlying $\mathbb{Z}$-graded chain complexes. Since $\mathcal{D}$ is stable and proper, the realization axiom follows by Proposition 4.5. Thus, the following corollary follows from Theorem 3.6.

COROLLARY 4.6. *The proper, cofibrantly generated model category $\mathcal{D}$ of differential graded modules over a differential graded algebra A is Quillen equivalent to the simplicial model category $s\mathcal{D}$ with the realization model category structure.*

This answers a problem stated by Hovey [Hov, 8.9], which asks for a simple simplicial model category that is Quillen equivalent to unbounded chain complexes of R-modules, Ch(R). Here A is the differential graded algebra that is R concentrated in degree zero.

To make this example even more explicit, one can show that the *total complex* functor T is weakly equivalent to the homotopy colimit. Let $X \in s\mathcal{D}$ be a simplicial object of differential graded A-modules. We denote by $X_{s,t}$ the group in simplicial level s and chain degree t. The total complex of X is the chain complex with levels $TX_n = \oplus_{s+t=n} X_{s,t}$ and with total differential $d_{tot} = (-1)^s d + d'$. Here d is the internal chain differential in each simplicial level and $d' = \Sigma(-1)^i d_i$. TX is again a differential graded A-module. Then a map f is a realization weak equivalence in $s\mathcal{D}$ if and only if Tf is a quasi-isomorphism.

5. Uniqueness of simplicial model category structures. In this section we consider categories $\mathcal{C}$ that already have a given simplicial model category structure. We then show that $\mathcal{C}$ is Quillen equivalent to $s\mathcal{C}$ via simplicial functors, see Theorem 5.2. As stated in Corollary 5.3, this implies that simplicial model category structures on a fixed model category are unique up to simplicial Quillen equivalence. See also Corollary 6.2 for a generalization of this result. For these two statements we only need to assume that the canonical model category on $s\mathcal{C}$ exists and is a simplicial model category. We refer to this as assuming the existence of the simplicial, canonical model category. So the hypotheses considered in [Dug] work equally as well as the hypotheses considered in Theorem 3.6. Also, Proposition 4.2 provides many examples of simplicial model categories where the simplicial, canonical model category on $s\mathcal{C}$ exists.

First we recall the definition of a simplicial functor.

Definition 5.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories enriched over simplicial sets. Then a simplicial functor $F: \mathcal{C} \to \mathcal{D}$ consists of a map $F: \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$ of objects together with maps of simplicial sets $F: \mathrm{map}_{\mathcal{C}}(X, Y) \to \mathrm{map}_{\mathcal{D}}(FX, FY)$ that are associative and unital, see [Qui, II 1].

Since the vertices of the simplicial set $\mathrm{map}_{\mathcal{C}}(X, Y)$ are the morphisms in the category $\mathcal{C}$, the restriction of a simplicial functor F to vertices is an ordinary functor. If the categories $\mathcal{C}$ and $\mathcal{D}$ are also tensored over simplicial sets, then endowing an ordinary functor with a simplicial structure is equivalent to giving a transformation $K \otimes FX \to F(K \otimes X)$ that is natural in the simplicial set K and in $X \in \mathcal{C}$ and that satisfies certain associativity and unity conditions, see [Hir, 11.6].

For $\mathcal{C}$ a simplicial model category we now recall the adjoint functors Sing: $\mathcal{C} \to s\mathcal{C}$ and $|-|: s\mathcal{C} \to \mathcal{C}$. For X an object in $\mathcal{C}$, $\mathrm{Sing}(X)$ is the simplicial object with $\mathrm{Sing}(X)_n = X^{\Delta[n]}$. For Y an object in $s\mathcal{C}$, $|Y|$ is a coend [ML, IX.6] or the coequalizer of the following diagram induced by the simplicial operators.

$$\amalg_{m,n} X_m \otimes \Delta[n] \rightrightarrows \amalg_n X_n \otimes \Delta[n]$$

Throughout this section X^K, for X in $\mathcal{C}$ and K a simplicial set, refers to the adjoint of the simplicial action on $\mathcal{C}$. The simplicial structure on $s\mathcal{C}$ is still as in Section 2 and [Qui, II 1].

THEOREM 5.2. *Let $\mathcal{C}$ be a simplicial model category such that the simplicial, canonical model category on $s\mathcal{C}$ exists. Then the adjoint functors* Sing *and* $|-|$ *are simplicial and induce a Quillen equivalence between $\mathcal{C}$ and the simplicial, canonical model category structure on $s\mathcal{C}$.*

Since the structures on $s\mathcal{C}$ are independent of any simplicial structure on $\mathcal{C}$, this gives the following uniqueness statement for simplicial model category structures.

COROLLARY 5.3. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two simplicial model categories with the same underlying model category $\mathcal{C}$ such that the simplicial, canonical model category on $s\mathcal{C}$ exists. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are simplicially Quillen equivalent.*

Proof. Apply Theorem 5.2 to both $\mathcal{C}_1$ and $\mathcal{C}_2$. Then they are both simplicially Quillen equivalent to $s\mathcal{C}$. □

To prove Theorem 5.2 we first prove that Sing and $|-|$ are simplicial.

PROPOSITION 5.4. *For $\mathcal{C}$ a simplicial model category,* Sing: $\mathcal{C} \to s\mathcal{C}$ *and* $|-|$: $s\mathcal{C} \to \mathcal{C}$ *are simplicial functors.*

Proof. To show that $|-|$ is a simplicial functor we show that $K \otimes_{\mathcal{C}} |X|$ is isomorphic to $|K \otimes_{s\mathcal{C}} X|$. Here $\otimes_{\mathcal{C}}$ and $\otimes_{s\mathcal{C}}$ are the simplicial actions in the respective categories. These are not to be confused with the coends, see [ML], $\otimes_{\Delta}$ and $\otimes_{\Delta\times\Delta}$ which follow. Since the left adjoint $|-|$ is a strong simplicial functor, that is, the natural transformation is an isomorphism, it follows that the right adjoint Sing is also a simplicial functor.

Let $\bar{\Delta}$: $\Delta \to \mathcal{S}$ be the functor such that $\bar{\Delta}(n) = \Delta[n]$, the simplicial n-simplex. Then $|X|$ is isomorphic to the coend $X \otimes_{\Delta} \bar{\Delta}$ and for any simplicial set K, $K \cong (K \otimes_{\Delta} \bar{\Delta})$. Because $\otimes_{\mathcal{C}}$ commutes with colimits, $K \otimes_{\mathcal{C}} |X| \cong (K \otimes_{\Delta} \bar{\Delta}) \otimes_{\mathcal{C}} (X \otimes_{\Delta} \bar{\Delta}) \cong (K \cdot X) \otimes_{\Delta\times\Delta} \bar{\Delta} \times \bar{\Delta}$. Here $(K \cdot X)(m, n) = K_m \cdot X_n$. The functor $\bar{\Delta} \times \bar{\Delta}$ is the left Kan extension of $\bar{\Delta}$ across the diagonal functor δ: $\Delta \to \Delta \times \Delta$. So $(K \cdot X) \otimes_{\Delta\times\Delta} \bar{\Delta} \times \bar{\Delta} \cong \delta^*(K \cdot X) \otimes_{\Delta} \bar{\Delta}$. But $\delta^*(K \cdot X)$ is the functor describing $K \otimes_{s\mathcal{C}} X$, so this gives an isomorphism of the last step with $|K \otimes_{s\mathcal{C}} X|$. This produces the required isomorphism. □

Proof of Theorem 5.2. First note that $M_n(\operatorname{Sing} X) = X^{\dot{\Delta}[n]}$ where $\dot{\Delta}[n]$ denotes the boundary of the simplicial n-simplex. So if f: $X \to Y$ is a Reedy (trivial) fibration then $\operatorname{Sing} X \to \operatorname{Sing} Y$ is a Reedy (trivial) fibration because the induced map $X_n \to M_n X \times_{M_n Y} Y_n$ is equivalent to the map $X^{\Delta[n]} \to X^{\dot{\Delta}[n]} \times_{Y^{\dot{\Delta}[n]}} Y^{\Delta[n]}$ which is a (trivial) fibration by the adjoint form of SM7, see SM7(a) [Qui, II 2]. The trivial fibrations in $s\mathcal{C}$ are the Reedy trivial fibrations. Since the fibrations in $s\mathcal{C}$ between fibrant objects are Reedy fibrations by Proposition 3.9, this shows that Sing preserves trivial fibrations and fibrations between fibrant objects. Hence, by [Dug, A.2], Sing also preserves fibrations. By adjointness, $|-|$ preserves cofibrations and trivial cofibrations.

Since $|-|$ preserves trivial cofibrations it preserves weak equivalences between cofibrant objects. It also detects weak equivalences between cofibrant objects by Remark 3.2 since $|-|$ is weakly equivalent to the homotopy colimit on Reedy cofibrant objects, by [BK, XII] and [Hir, 20.6.1]. Hence by the dual of the criterion for Quillen equivalences in [HSS, 4.1.7], we only need to check that for fibrant objects X in $\mathcal{C}$, $|(\operatorname{Sing} X)^c| \to X$ is a weak equivalence where $(\operatorname{Sing} X)^c \to \operatorname{Sing} X$ is a trivial fibration from a cofibrant object in $s\mathcal{C}$. By the simplicial model category structure on $s\mathcal{C}$, $\operatorname{Sing} X$ is homotopically constant. Since $(\operatorname{Sing} X)^c$ is level equivalent to $\operatorname{Sing} X$, it is also homotopically constant.

Consider the following commuting square

$$\begin{array}{ccccc} |c(\operatorname{Sing} X)^c_0| & \longrightarrow & |(\operatorname{Sing} X)^c| & \longrightarrow & |(\operatorname{Sing} X)^c| \\ \downarrow & & \downarrow & & \downarrow \\ |cX| & \longrightarrow & |\operatorname{Sing} X| & \longrightarrow & X. \end{array}$$

The left vertical map is a weak equivalence since $|cY| \cong Y$. The top map is a weak equivalence since $(\operatorname{Sing} X)^c$ is homotopically constant. Finally, the bottom composite is the identity map. Hence the right-hand map is a weak equivalence as required. □

6. Simplicial functors. In this section we again consider categories $\mathcal{C}$ that already have a given simplicial model category structure. Since we have simplicial replacements for model categories, we now consider simplicial replacements of functors. We show that a functor that preserves weak equivalences between fibrant objects can be replaced by a simplicial functor that is weakly equivalent to the given functor on fibrant objects. We also show that a Quillen adjoint pair between simplicial model categories can be replaced by a simplicial Quillen adjoint pair. Combined with Theorem 5.2 this shows that if two simplicial model categories have Quillen equivalent underlying model categories then they are in fact simplicially Quillen equivalent, see Corollary 6.2.

For a functor $F\colon \mathcal{C} \to \mathcal{D}$, let $\bar{F}\colon s\mathcal{C} \to s\mathcal{D}$ be the prolongation of F defined by applying F at each level.

PROPOSITION 6.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories for which the simplicial, canonical model structures on $s\mathcal{C}$ and $s\mathcal{D}$ exist. Let $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ be a Quillen adjoint pair of functors. Then $\bar{L}$ and $\bar{R}$ are a simplicial Quillen adjoint pair between the simplicial model categories $s\mathcal{C}$ and $s\mathcal{D}$. Moreover, if L, R form a Quillen equivalence, so do $\bar{L}, \bar{R}$.*

This answers Hovey's question in [Hov, 8.9] about replacing Quillen adjunctions by Quillen equivalent simplicial Quillen adjunctions. Indeed, if $\mathcal{C}$ and $\mathcal{D}$ are *simplicial* model categories, then Theorem 5.2 shows that $\mathcal{C}$ and $\mathcal{D}$ are simplicially Quillen equivalent to $s\mathcal{C}$ and $s\mathcal{D}$. So using Proposition 6.1 one can replace a Quillen adjunction by a zig-zag of simplicial Quillen adjunctions through $s\mathcal{C}$ and $s\mathcal{D}$ where the "backwards" adjunction is a Quillen equivalence.

Proof. First $\bar{L}$ is a simplicial functor. The necessary natural transformation, $\bar{L}(X)\otimes K \to \bar{L}(X\otimes K)$ is given on each level by the canonical maps $\coprod_{\sigma\in K_n} L(X_n) \to L(\coprod_{\sigma\in K_n} X_n)$.

Since R preserves fibrations, trivial fibrations, and limits, $\bar{R}$ preserves Reedy fibrations and Reedy trivial fibrations. So $\bar{R}$ preserves trivial fibrations and fibrations between fibrant objects. By [Dug, A.2] this implies $\bar{R}$ also preserves fibrations. Hence $\bar{L}, \bar{R}$ are a Quillen adjoint pair. The last statement follows from Theorem 5.2 and the two out of three property for equivalences of categories, since Quillen equivalences are Quillen adjoint functors that induce equivalences of homotopy categories [Hov, 1.3.13]. □

Corollary 6.2. *Suppose that $\mathcal{C}$ and $\mathcal{D}$ are simplicial model categories for which the simplicial, canonical model structures on $s\mathcal{C}$ and $s\mathcal{D}$ exist. If there is a Quillen equivalence between the underlying model categories $\mathcal{C}$ and $\mathcal{D}$, then $\mathcal{C}$ and $\mathcal{D}$ are simplicially Quillen equivalent.*

Proof. By Theorem 5.2, $\mathcal{C}$ and $\mathcal{D}$ are simplicially Quillen equivalent respectively to $s\mathcal{C}$ and $s\mathcal{D}$. By Proposition 6.1, the Quillen equivalence between $\mathcal{C}$ and $\mathcal{D}$ can be lifted to a simplicial Quillen equivalence between $s\mathcal{C}$ and $s\mathcal{D}$. □

Next we turn to constructing simplicial functor replacements. Constructing simplicial cofibrant and fibrant replacement functors is independent of the rest of this paper, see also [Far, I.C.11] or [Hir]. This construction is delayed to the end of the section. These simplicial replacement functors are then building blocks for replacing general functors by simplicial ones. In this section one can use the usual definition of cofibrantly generated (see e.g. [Hov, 2.1.17]), which is weaker than Definition 8.1.

Proposition 6.3. *For $\mathcal{C}$ any simplicial, cofibrantly generated model category there is a simplicial functorial factorization of any map $f\colon X \to Y$ as a cofibration followed by a trivial fibration and as a trivial cofibration followed by a fibration. In particular, this produces simplicial cofibrant and fibrant replacement functors.*

Proposition 6.4. *Assume $\mathcal{C}$, $\mathcal{D}$ are cofibrantly generated, simplicial model categories such that the simplicial, canonical model categories on $s\mathcal{C}$ and $s\mathcal{D}$ exist and are cofibrantly generated. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor that preserves weak equivalences between fibrant objects. Then $G = |Q\bar{F}\,\mathrm{Sing}\,(-)|$ is a simplicial functor, where Q is a simplicial cofibrant replacement functor in the simplicial, canonical model category on $s\mathcal{D}$. Moreover, there is a zig-zag of natural transformations between F and G that induce weak equivalences on fibrant objects in $\mathcal{C}$.*

Corollary 6.5. *Assume $\mathcal{C}$, $\mathcal{D}$ are as above. If F preserves all weak equivalences then $H = |Q\bar{F}\,\mathrm{Sing}\,R(-)|$ is a simplicial functor where Q and R are simplicial* *cofibrant and fibrant replacement functors in $s\mathcal{D}$ and $\mathcal{C}$ respectively. Moreover, for any X, FX and HX are naturally weakly equivalent.*

Proof of Proposition 6.4. G is a simplicial functor because each of its composites is simplicial by Propositions 5.4, 6.1, and 6.3.

The first step in the zig-zag between F and G uses the natural transformation $c \to \mathrm{Sing}$. This induces $|Q\bar{F}c(-)| \to |Q\bar{F}\,\mathrm{Sing}\,(-)| = G(-)$. Note that for X fibrant $cX \to \mathrm{Sing}\,X$ is a level equivalence between level fibrant objects by the simplicial model category structure on $\mathcal{C}$. Since $|-|$ preserves trivial cofibrations by Theorem 5.2, $|-|$ preserves weak equivalences between cofibrant objects. So, since F preserves weak equivalences between fibrant objects, $|Q\bar{F}(cX)| \to |Q\bar{F}\,\mathrm{Sing}\,X| = GX$ is an equivalence for X fibrant.

To relate this to FX, note that $\bar{F}(cX) = cFX$. Since $QY \xrightarrow{p} Y$ is a level equivalence, $QcFX$ is homotopically constant. Thus, $c\mathrm{Ev}\,QcFX \to QcF$ is a level equivalence between cofibrant objects. Hence $|c\mathrm{Ev}\,QcFX| \to |Q\bar{F}cX|$ is also a weak equivalence for any X. $|c\mathrm{Ev}\,QcFX| \to \mathrm{Ev}\,QcFX$ is an isomorphism. Since p is a level equivalence, $\mathrm{Ev}\,QcFX \to FX$ is also an equivalence. Combining this with the first step finishes the proof. □

Proof of Proposition 6.3. Given $f\colon X \to Y$ in $\mathcal{C}$ we construct a simplicial functorial factorization, $X \to Ff \to Y$, as a cofibration followed by a trivial fibration. The other factorization is similar. Let I be a set of generating cofibrations in $\mathcal{C}$. Define the first stage, $F^1 f$, as the pushout in the following square.

$$\begin{array}{ccc} \amalg_{A_i \to B_i \in I} A_i \otimes (\,\mathrm{map}_{\mathcal{C}}(A_i, X) \times_{\mathrm{map}_{\mathcal{C}}(A_i,Y)} \mathrm{map}_{\mathcal{C}}(B_i, Y)) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \amalg_{A_i \to B_i \in I} B_i \otimes (\,\mathrm{map}_{\mathcal{C}}(A_i, X) \times_{\mathrm{map}_{\mathcal{C}}(A_i,Y)} \mathrm{map}_{\mathcal{C}}(B_i, Y)) & \longrightarrow & F^1 f \end{array}$$

By [Hir, 12.4.23], any object that is small with respect to the regular I-cofibrations is small with respect to all cofibrations. So each A_i is small relative to the cofibrations. Let κ be the regular cardinal such that each A_i is κ-small with respect to the cofibrations. Let $F^{\alpha+1} f = F^1(F^\alpha f \to Y)$ and for any limit ordinal $\beta < \kappa$ let $F^\beta = \mathrm{colim}_\alpha F^\alpha$. Then we claim that $F = F^\kappa$ is a cofibrant replacement functor which is also a simplicial functor.

We need to show that $X \to Ff$ is a cofibration and that $Ff \to Y$ is a trivial fibration. Since $\mathcal{C}$ is a simplicial model category the left map in the square above is a cofibration. Since pushouts and colimits preserve cofibrations this shows that $X \to Ff$ is a cofibration. To show that $Ff \to Y$ is a trivial fibration we need to show that it has the right lifting property with respect to any map $A_i \to B_i \in I$. Because A_i is κ-small with respect to cofibrations, the map $A_i \to Ff$ factors through some stage, $F^\alpha f$. Then, by construction, there is a lift $B_i \to F^{\alpha+1} f \to Ff$.

We now show that F is simplicial. The colimit of a diagram of simplicial functors is again a simplicial functor. Since the composition of simplicial functors is again simplicial, we only need to show that F^1 is a simplicial functor. But F^1 itself is a colimit of functors which are simplicial, so we are done. □

7. Reedy model category. In this section we show that the Reedy model category satisfies conditions (1) and (2) but not (3) of Axiom 2.2, (SM7). These properties are also used in the proofs in Section 8.

The simplicial structure defined at the beginning of Section 3, as with any simplicial structure, can be extended to morphisms. Using this structure on morphisms simplifies some of the notation and adjointness properties that come up in verifying Axiom 2.2, (SM7), for both the Reedy and realization model categories. See [HSS, 5.3] for more about this structure on morphisms.

Definition 7.1. Given $f\colon X \to Y \in s\mathcal{C}$ and $i\colon K \to L \in \mathcal{S}$ define the *pushout product* $f \,\square\, i\colon X \otimes L \coprod_{X\otimes K} Y \otimes K \to Y \otimes L$ as the natural map from the pushout. For f in $\mathcal{C}$ define $f \,\square\, i$ as $cf \,\square\, i$ where $c\colon \mathcal{C} \to s\mathcal{C}$ is the constant functor. Define $f^{\square i}\colon X^L \to Y^L \prod_{Y^K} X^K$ as the natural map to the pullback in $s\mathcal{C}$ or its 0th level in $\mathcal{C}$ where the context will determine which category is meant.

Note that using this definition the map that appears in Axiom 2.2, (SM7), can be rewritten as the pushout product, $q = f \,\square\, i$. Also, note that $- \,\square\, i$ is adjoint to $(-)^i$. Next we rewrite the matching maps using this new notation. Since $X^{\Delta[n]} = X_n$ and $X^{\dot\Delta[n]} = M_nX$, we have:

LEMMA 7.2. *Let $f\colon X \to Y$ be a map in $s\mathcal{C}$. The matching map $M_nf\colon X_n \to Y_n \times_{M_nY} M_nX$ is the map $f^{\square i_n}$ with $i_n\colon \dot\Delta[n] \to \Delta[n]$ the boundary inclusion.*

PROPOSITION 7.3. *If g is a Reedy (trivial) fibration and i is a cofibration in $\mathcal{S}$ then $g^{\square i}$ in $s\mathcal{C}$ is a Reedy (trivial) fibration and hence its 0th level $g^{\square i}$ in $\mathcal{C}$ is a (trivial) fibration.*

Proof. We need to consider the matching maps of $g^{\square i}$, that is $(g^{\square i})^{\square i_n}$ in $\mathcal{C}$ by Lemma 7.2. Since $i \,\square\, i_n$ is a cofibration in $\mathcal{S}$, it is enough to show that $g^{\square i}$ is a (trivial) fibration in $\mathcal{C}$. In fact it is enough to show this for each i_n since they generate the cofibrations in $\mathcal{S}$ by [Hov, 3.2.2]. But $g^{\square i_n}$ is a (trivial) fibration by Lemma 7.2 since g is a Reedy (trivial) fibration. □

A corollary of this proposition is that although the Reedy model category is not simplicial it does satisfy the first two properties of Axiom 2.2, (SM7).

COROLLARY 7.4. *Given $f\colon X \to Y$ a Reedy cofibration in $s\mathcal{C}$ and $i\colon K \to L$ a cofibration in $\mathcal{S}$ then $f \,\square\, i\colon X \otimes L \coprod_{X\otimes K} Y \otimes K \to Y \otimes L$ is a Reedy cofibration. Moreover, if f is also a level weak equivalence, then so is $f \,\square\, i$. But if i is a weak equivalence and f is not, then $f \,\square\, i$ is not necessarily a weak equivalence.*

Proof. The first two statements follow by adjointness from Proposition 7.3. For all three statements, see also [DKS, 2.6] and compare with [Hov, 5.4.1]. □

8. Realization model category. In this section we prove Theorem 3.6, which states that the realization model structure on $s\mathcal{C}$ is a model category that is simplicial and Quillen equivalent to the original model category $\mathcal{C}$.

To verify the axioms for the realization model category on $s\mathcal{C}$ we assume that $\mathcal{C}$ is a cofibrantly generated model category. We now recall a version of the definition of cofibrantly generated model category from [DHK], or see [Hov, 2.1.17], [SSa, 2.2], or [Hir]. For a cocomplete category $\mathcal{C}$ and a class I of maps, the *I-injectives* are the maps with the right lifting property with respect to the maps in I. The *I-cofibrations* are the maps with the left lifting property with respect to the I-injectives. Finally, the *regular I-cofibrations* (called *relative I-cell complexes* in [Hov, 2.1]) are the (possibly transfinite) compositions of pushouts of maps in I. In particular all isomorphisms are regular I-cofibrations, see the remark following [Hov, 2.1.9].

Definition 8.1. A model category $\mathcal{C}$ is *cofibrantly generated* if it is complete and cocomplete and there exists a set of cofibrations I and a set of trivial cofibrations J such that:

(1) the fibrations are precisely the J-injectives,

(2) the acyclic fibrations are precisely the I-injectives,

(3) the domain and range of each map in I and each map in J is *small* relative to the regular I-cofibrations, and

(4) the domain and range of each map in I is cofibrant.

Moreover, here the (trivial) cofibrations are the I (J)-cofibrations.

For the definition of *small* see the above mentioned references. The crucial reason for requiring a cofibrantly generated model category is the small object argument, Proposition 8.2, as in [Qui], see also [DHK] or [Hov, 2.1.14]. The smallness requirements here are stronger than what is necessary for the small object argument to apply to I and J; we added the requirement that the ranges of I and J are also small. We use this to show that the domains of the new generators defined in 8.3 for $s\mathcal{C}$ have small domains so the small object argument will apply in $s\mathcal{C}$. Since $\mathcal{C}$ is also assumed to be left proper, we could replace J by a set J' of regular I-cofibrations and the smallness condition for J' would follow by [Hir, 12.3.8]. The maps in I are required to be between cofibrant objects so that Proposition 8.12 holds.

PROPOSITION 8.2. (Small object argument) *Let $\mathcal{C}$ be a cocomplete category and I a set of maps in $\mathcal{C}$ whose domains are small relative to the regular I-cofibrations. Then:*

(1) *there is a functorial factorization of any map f in $\mathcal{C}$ as $f = pi$ with p an I-injective and i a regular I-cofibration. And thus,*

(2) *every I-cofibration is a retract of a regular I-cofibration.*

We now begin to verify the model category axioms for the realization model structure on $s\mathcal{C}$. We assume that $\mathcal{C}$ is a left proper, cofibrantly generated model category that satisfies the Realization Axiom 3.4. For the factorizations we use Proposition 8.2. We characterize the (trivial) fibrations as the maps with the right lifting property with respect to a set of maps, J (I). Let I_C be a set of generating cofibrations for $\mathcal{C}$ and J_C be a set of generating trivial cofibrations for $\mathcal{C}$. In the category of simplicial sets, let I_∂ be the set of inclusions of boundaries into simplices, $i_n\colon \dot{\Delta}[n] \to \Delta[n]$ for each n. Let I_F be the set of inclusions of faces into simplices, $\delta_i\colon \Delta[m] \to \Delta[m+1]$ for each m and $0 \le i \le m+1$.

Definition 8.3. Let $I = I_C \,\square\, I_\partial$ denote the set of maps

$$f \,\square\, i_n\colon A \otimes \Delta[n] \coprod_{A\otimes\dot{\Delta}[n]} B \otimes \dot{\Delta}[n] \to B \otimes \Delta[n]$$

for each n and $f\colon A \to B$ any map in I_C. Let $J' = J_C \,\square\, I_\partial$ denote the set of maps

$$f \,\square\, i_n\colon A \otimes \Delta[n] \coprod_{A\otimes\dot{\Delta}[n]} B \otimes \dot{\Delta}[n] \to B \otimes \Delta[n]$$

for each n and $f\colon A \to B$ any map in J_C. Let $J'' = I_C \,\square\, I_F$ denote the set of maps

$$f \,\square\, \delta_i\colon A \otimes \Delta[m+1] \coprod_{A\otimes\Delta[m]} B \otimes \Delta[m] \to B \otimes \Delta[m+1]$$

for each m and i with $0 \le i \le m+1$ and $f\colon A \to B$ any map in I_C. Let J be the union of the two sets J' and J''.

LEMMA 8.4. *The domains of I and J are small relative to the regular I-cofibrations.*

Proof. We prove the statement for J, the statement for I follows similarly. A finite colimit of small objects is small since finite limits commute with small filtered colimits, [ML, IX 2]. The domains of J can be built by finite colimits from objects $X \otimes \Delta[n]$ for X a domain or range of a map in $I_\mathcal{C}$ or $J_\mathcal{C}$. Since $s\mathcal{C}(X \otimes \Delta[n], Y) \cong \mathcal{C}(X, Y^{\Delta[n]}) \cong \mathcal{C}(X, Y_n)$ and X is small relative to regular $I_\mathcal{C}$-cofibrations by Definition 8.1, $X \otimes \Delta[n]$ is small relative to maps in $s\mathcal{C}$ that are regular $I_\mathcal{C}$-cofibrations on each level. But each level of a regular I-cofibration is a regular $I_\mathcal{C}$-cofibration. This is because each level of a map in I is just a direct sum of copies of maps in $I_\mathcal{C}$ or identity maps. Identity maps and coproducts of regular cofibrations are regular cofibrations. So each level of each map in I is a regular $I_\mathcal{C}$-cofibration. Hence this is also true of the regular I-cofibrations. □

Since the domains are small we can use the small object argument, Proposition 8.2, to factor any map into an I (J)-cofibration followed by an I (J)-injective. This applies directly to I by Lemma 8.4. For J, since the domains of J are small relative to the regular I-cofibrations, they are small with respect to all cofibrations including the regular J-cofibrations by [Hir, 13.3.3]. Hence Proposition 8.2

applies. To see that this gives us the needed factorization we show in the next propositions that an I (J)-cofibration is a realization (trivial) cofibration and that a J (I)-injective is a realization (trivial) fibration.

PROPOSITION 8.5. *The J-injective maps are the equifibered Reedy fibrations. In other words, the equifibered Reedy fibrations are the maps with the right lifting property with respect to J. The Reedy fibrations are the maps with the right lifting property with respect to J'. Moreover, the J-injective objects are the homotopically constant Reedy fibrant objects.*

Proof. A Reedy fibration is a map f whose matching maps are fibrations. These matching maps are $f^{\square i_n}$ with $i_n \in I_\partial$ by Lemma 7.2. That is, $f^{\square i_n}$ has the right lifting property with respect to each map in J_C. By adjointness, this is equivalent to f having the right lifting property with respect to the maps in $J_C \square I_\partial = J'$.

Given a Reedy fibration $f\colon X \to Y$, then $f^{\square \delta_i}\colon X_{m+1} \to X_m \times_{Y_m} Y_{m+1}$ is a fibration by Proposition 7.3. So a Reedy fibration f is equifibered if and only if $f^{\square \delta_i}$ is a trivial fibration. By adjunction $f^{\square \delta_i}$ is a trivial fibration if and only if f has the right lifting property with respect to $J'' = I_C \square I_F$. So f is an equifibered Reedy fibration if and only if f has the right lifting property with respect to J. The last statement of the proposition follows since $f\colon Z \to *$ is an equifibered Reedy fibration if and only if Z is Reedy fibrant and for each n and i the map $d_i\colon Z_{n+1} \to Z_n$ is a trivial fibration. □

Next we turn to the I-cofibrations and I-injectives.

PROPOSITION 8.6. *The I-injective maps are the Reedy trivial fibrations. Also, the Reedy trivial fibrations are the equifibered Reedy fibrations that are also realization weak equivalences. Hence, the I-cofibrations are the Reedy cofibrations.*

Proof. Much as in the previous proof, a map f is a Reedy trivial fibration if the matching maps $f^{\square i_n}$ are trivial fibrations. That is $f^{\square i_n}$ has the right lifting property with respect to each map in I_C. By adjointness, this is equivalent to f having the right lifting property with respect to the maps in $I_C \square I_\partial = I$.

By the Realization Axiom 3.4, an equifibered Reedy fibration that is also a realization weak equivalence is a level equivalence, and hence a Reedy trivial fibration. Conversely, for f a Reedy trivial fibration, the maps $f_n\colon X_n \to Y_n$ are trivial fibrations. Since f_{n+1} factors as $X_{n+1} \to X_n \times_{Y_n} Y_{n+1} \to Y_{n+1}$ and the second map here is the pullback of a trivial fibration, the map $X_{n+1} \to X_n \times_{Y_n} Y_{n+1}$ is a weak equivalence. So a Reedy trivial fibration is equifibered. Then, since level equivalences are realization weak equivalences, this shows that a Reedy trivial fibration is a realization trivial fibration, i.e., an equifibered Reedy fibration that is also a realization weak equivalence. □

Now we are left with verifying that the J-cofibrations are Reedy cofibrations and realization weak equivalences.

PROPOSITION 8.7. *A J-cofibration is a Reedy cofibration and a realization weak equivalence.*

Proof. A J-cofibration has the left lifting property with respect to the J-injective maps, the equifibered Reedy fibrations. Since any Reedy fibration that is also a level equivalence is equifibered, a J-cofibration has the left lifting property with respect to the Reedy trivial fibrations. Hence a J-cofibration is a Reedy cofibration.

Each J-cofibration is a retract of a directed colimit of pushouts of maps in J by Proposition 8.2. The maps in J' are level equivalences, hence the maps built from J' are Reedy trivial cofibrations. These level equivalences are realization weak equivalences. So we only need to consider J''-cofibrations. Since the maps in I_F are trivial cofibrations of simplicial sets, they are I_Λ-cofibrations where $I_\Lambda = \{\lambda_n\colon \Lambda^k[n] \to \Delta[n]\}$ is the set of inclusions of the horns into simplices. Hence J''-cofibrations are $(I_\mathcal{C} \,\square\, I_\Lambda)$-cofibrations. Below, in Proposition 8.12, we show that any $(I_\mathcal{C} \,\square\, I_\Lambda)$-cofibration is a realization weak equivalence. □

To finish our verification of the realization model category structure we need to use a different characterization of the realization weak equivalences.

Definition 8.8. A map $f'\colon A' \to B'$ is a *cofibrant replacement* of a map $f\colon A \to B$ if A' and B' are cofibrant objects, f' is a cofibration, and there exist level equivalences $i_A\colon A' \to A$ and $i_B\colon B' \to B$ such that $f i_A = i_B f'$.

PROPOSITION 8.9. *A map $f\colon A \to B$ in $s\mathcal{C}$ is a realization weak equivalence if and only if for some cofibrant replacement $f'\colon A' \to B'$, and for each homotopically constant, Reedy fibrant object Z in $s\mathcal{C}$,* $\operatorname{map}(B', Z) \to \operatorname{map}(A', Z)$ *is a weak equivalence.*

The following lemmas are used to prove this proposition.

LEMMA 8.10. *The map $Z^{\lambda_n}\colon Z^{\Delta[n]} \to Z^{\Lambda^k[n]}$ in $\mathcal{C}$ is a trivial fibration for Z any homotopically constant Reedy fibrant object in $s\mathcal{C}$.*

Proof. Z^{λ_n} is a fibration, by Corollary 7.4. Since $\Lambda^k[1] = \Delta[0]$, Z^{λ_1} is the map $d_k\colon Z_1 \to Z_0$, which is a trivial fibration since Z is a homotopically constant Reedy fibrant object. This proves the lemma for $n = 1$. We proceed by induction.

$Z^{\Lambda^k[n]}$ is the pullback of a punctured n-cube where each arrow is of the form $Z^{\delta_i}\colon Z^{\Delta[m]} \to Z^{\Delta[m-1]}$, that is, $Z_m \to Z_{m-1}$ for $m < n$. These maps are fibrations by Corollary 7.4 and they are weak equivalences because Z is homotopically constant. By induction the map from the object at the puncture of each contained punctured k-cube, for $k < n$, to the pullback is a trivial fibration. For any such punctured n-cube, the added maps from the pullback are trivial fibrations. That is, the maps from the pullback, $Z^{\Lambda^k[n]}$, to each $Z^{\Delta[n-1]} = Z_{n-1}$ are trivial fibrations. Since each δ_i factors as $\Delta[n-1] \to \Lambda^k[n] \to \Delta[n]$, this proves the lemma holds for n by the two out of three property for weak equivalences. □

LEMMA 8.11. *For K any simplicial set and Z any homotopically constant Reedy fibrant object, Z^K is homotopically constant and Reedy fibrant.*

Proof. First note that by an adjoint of SM7 (i), which is verified for the Reedy model category in Corollary 7.4, Z^K is Reedy fibrant. Hence by Proposition 8.5, Z^K is J'-injective and we only need to show that Z^K is J''-injective to finish the proof.

Here we say "(f,g) has the lifting property," as shorthand for f has the left lifting property with respect to g. This also extends to sets of maps. By Lemma 8.10, (i, Z^{λ_n}) has the lifting property for i in $I_{\mathcal{C}}$, λ_n in I_Λ, and Z any homotopically constant, Reedy fibrant object. Let H be the class of maps $Z \to *$ for such Z. Then, by adjointness $(I_{\mathcal{C}} \,\square\, I_\Lambda, H)$ has the lifting property. But then pushouts, colimits and retracts of maps in $I_{\mathcal{C}} \square I_\Lambda$ also have the left lifting property with respect to H. That is, $((I_{\mathcal{C}} \square I_\Lambda)$-cofibrations, $H)$ has the lifting property. For i a cofibration and j a trivial cofibration of simplicial sets , the pushout product $j \square i$ is an I_Λ-cofibration. So $f \square j \square i$ is an $(I_{\mathcal{C}} \square I_\Lambda)$-cofibration for f in $I_{\mathcal{C}}$. Hence $(I_{\mathcal{C}} \,\square\, I_\Lambda \,\square\, I_\partial, Z)$ has the lifting property. Consider the cofibration $i\colon \emptyset \to K$. By adjointness this shows that $(I_{\mathcal{C}} \,\square\, I_\Lambda, Z^K)$ has the lifting property. Hence Z^K is J''-injective. □

Proof of Proposition 8.9. Our first claim is that $\pi_0 \operatorname{map}(A,X)$ is naturally isomorphic to $[A,X]^{\mathrm{Ho(Reedy)}}$ for A Reedy cofibrant and X homotopically constant and Reedy fibrant. Indeed the maps $X \cong X^{\Delta[0]} \xrightarrow{f} X^{\Delta[1]} \xrightarrow{p} X^{\Delta[0] \amalg \Delta[0]} \cong X \times X$ produce $X^{\Delta[1]}$ as a path object for X. Here f is a level equivalence by Lemma 8.11 since it is a map between homotopically constant objects whose 0th level is given by the equivalence $s_1\colon X_0 \to X_1$ and Proposition 7.3 shows that p is a Reedy fibration. This implies the claim.

Since f is a realization weak equivalence if and only if its cofibrant replacement is, we can restrict to the case when f is its own cofibrant replacement. Then requiring that $\operatorname{map}(f,Z)$ be a weak equivalence for all homotopically constant Reedy fibrant objects Z is equivalent to requiring that for all simplicial sets K, $\pi_0 \operatorname{map}(K, \operatorname{map}(f,Z)) \cong \pi_0 \operatorname{map}(f, Z^K)$ be a bijection for all such Z. By Lemma 8.11 and the above, this is equivalent to $[B,Z^K]^{\mathrm{Ho(Reedy)}} \to [A,Z^K]^{\mathrm{Ho(Reedy)}}$ being a bijection for all such K and Z.

As Z runs through all homotopically constant Reedy fibrant objects and K runs through all simplicial sets, $(Z^K)_0$ runs through all fibrant objects in $\mathcal{C}$. Since $c(Z^K)_0 \to Z^K$ is a level equivalence, this is equivalent to $[B,cX]^{\mathrm{Ho(Reedy)}} \to [A,cX]^{\mathrm{Ho(Reedy)}}$ being a bijection for all X in $\mathcal{C}$. □

The following proposition finishes the identification of the J-cofibrations as realization weak equivalences. It is also useful in checking that $s\mathcal{C}$ is a simplicial model category.

PROPOSITION 8.12. *Any $(I_{\mathcal{C}} \square I_\Lambda)$-cofibration is a realization weak equivalence.*

Proof. By the proof above of Lemma 8.11, $(I_{\mathcal{C}} \square I_{\Lambda} \square I_{\partial}, Z)$ has the lifting property for Z homotopically constant and Reedy fibrant. Then by adjointness, $(I_{\partial}, \mathrm{map}\,(I_{\mathcal{C}} \square I_{\Lambda}, Z))$ also has the lifting property for any such Z. That is, any map in $\mathrm{map}\,(I_{\mathcal{C}} \square I_{\Lambda}, Z)$ is a trivial fibration. Since the maps in $I_{\mathcal{C}}$ are assumed to be between cofibrant objects, the maps in $I_{\mathcal{C}} \square I_{\Lambda}$ are Reedy cofibrations between Reedy cofibrant objects. So they are their own cofibrant replacements. Hence the maps in $I_{\mathcal{C}} \square I_{\Lambda}$ are realization weak equivalences by Proposition 8.9. Since the maps in $I_{\mathcal{C}} \square I_{\Lambda}$ are Reedy cofibrations, to finish this proof it is enough to show that Reedy cofibrations that are realization weak equivalences are preserved under pushouts, directed colimits, and retracts.

Since $\mathcal{C}$ is left proper, if g is a pushout of a Reedy cofibration f then one can choose a cofibrant replacement g' for g as a pushout of the cofibrant replacement f' of f. Hence $\mathrm{map}\,(g', Z)$ is a pullback of $\mathrm{map}\,(f', Z)$. We show in the next paragraph that if f' is a Reedy cofibration then $\mathrm{map}\,(f', Z)$ is a fibration. So if f is a Reedy cofibration and realization weak equivalence then $\mathrm{map}\,(f', Z)$ and hence also $\mathrm{map}\,(g', Z)$ is a trivial fibration. Thus, g is a realization weak equivalence. Since retracts and directed limits of trivial fibrations are also trivial fibrations, it follows that retracts and directed colimits also preserve Reedy cofibrations that are realization weak equivalences.

Since $(I_{\mathcal{C}} \square I_{\partial} \square I_{\Lambda}, Z)$ has the lifting property, so does $((I_{\mathcal{C}} \square I_{\partial})$-cofibrations, $Z^{I_{\Lambda}})$ for Z any homotopically constant Reedy fibrant object. By adjointness this shows that for any Reedy cofibration i, $\mathrm{map}\,(i, Z)$ is a fibration since it has the right lifting property with respect to I_{Λ}. □

Proof of Theorem 3.6 (1). As always, we assume that $\mathcal{C}$ is a left proper, cofibrantly generated model category that satisfies Realization Axiom 3.4. The category $s\mathcal{C}$ has all limits and colimits since $\mathcal{C}$ does. The two out of three axiom for weak equivalences and the retract axiom for the cofibrations and weak equivalences are easily checked. The retract axiom for fibrations follows from Proposition 8.5. The two factorizations follow from Propositions 8.5, 8.6 and 8.7 by Proposition 8.2. One lifting property follows from Proposition 8.6 since the realization trivial fibrations are the Reedy trivial fibrations. So only the lifting of a realization trivial cofibration with respect to an equifibered Reedy fibration is left. Assume $f \colon X \to Y$ is a Reedy cofibration and a realization weak equivalence. Factor $f = pi$ where i is a J-cofibration and p is J-injective. Since f and i are realization weak equivalences, p is also a realization weak equivalence. Since f is a Reedy cofibration, Propositions 8.5 and 8.6 show that it has the left lifting property with respect to p. Thus, f is a retract of i. Hence f is a J-cofibration and so it has the left lifting property with respect to any equifibered Reedy fibration. This finishes the proof that the realization model structure on $s\mathcal{C}$ is a model category. □

COROLLARY 8.13. *Let I and J be as defined in Definition 8.3. The realization model category on $s\mathcal{C}$ is cofibrantly generated, with I a set of generating cofibrations and J a set of generating trivial cofibrations.*

We now prove Theorem 3.6 (2), which states that the realization model category structure on $s\mathcal{C}$ satisfies Axiom 2.2, (SM7). Hence, it is a simplicial model category.

Proof of Theorem 3.6 (2). Given $f\colon A \to B$ a Reedy cofibration in $s\mathcal{C}$ and $i\colon K \to L$ a cofibration in $\mathcal{S}$, $f \square i$ is a Reedy cofibration by Corollary 7.4. So we are left with showing that if f or i is also a weak equivalence then so is $f \square i$.

First consider the case where i is a trivial cofibration. Since the pushout product of a trivial cofibration and a cofibration of simplicial sets is a trivial cofibration, $((I_{\mathcal{C}} \square I_{\partial})$-cofibrations) $\square(I_{\Lambda}$-cofibrations) is contained in $(I_{\mathcal{C}} \square I_{\Lambda})$-cofibrations. So by Proposition 8.12, $f \square i$ is a realization weak equivalence for f any Reedy cofibration.

Next consider the case where f is a realization weak equivalence. Since trivial cofibrations are preserved under pushouts, retracts and colimits, it is enough to show that for f in J, $f \square i$ is a realization weak equivalence. For f in J' this follows from Corollary 7.4. For f in $J'' = I_{\mathcal{C}} \square I_F$ this follows from the previous paragraph by associativity, since the maps in I_F are trivial cofibrations. □

Recall that the Quillen equivalence of $\mathcal{C}$ and $s\mathcal{C}$, Theorem 3.6 part (3), follows from Proposition 3.9 since the realization model category agrees with the canonical model category on $s\mathcal{C}$.

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