ORBISPACES, ORTHOGONAL SPACES, AND THE UNIVERSAL COMPACT LIE GROUP

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Introduction

The purpose of this article is to identify the homotopy theory of topological stacks with unstable global homotopy theory. At the same time, we provide a new perspective on this homotopy theory, by interpreting it as the homotopy theory of 'spaces with an action of the universal compact Lie group'. This link then provides a new way to construct and study genuine cohomology theories on stacks, orbifolds, and orbispaces.

Before describing the contents of this paper in more detail, I expand on how global homotopy theory provides orbifold cohomology theories. Stacks and orbifolds are concepts from algebraic geometry respectively geometric topology that allow us to talk about objects that locally look like the quotient of a smooth object by a group action, in a way that remembers information about the isotropy groups of the action. Such 'stacky' objects can behave like smooth objects even if the underlying spaces have singularities. As for spaces, manifolds, and schemes, cohomology theories are important invariants also for stacks and orbifolds, and examples such as ordinary cohomology or K-theory lend themselves to generalization. Special cases of orbifolds are 'global quotients', often denoted $M/\!\!/G$, for example for a smooth action of a compact Lie group G on a smooth manifold M. In such examples, the orbifold cohomology of $M/\!\!/G$ is supposed to be the G-equivariant cohomology of M. This suggests a way to *define* orbifold cohomology theories by means of equivariant stable homotopy theory, via suitable G-spectra. However, since the group G is not intrinsic and can vary, one needs equivariant cohomology theories for all groups G, with some compatibility. Global homotopy theory makes this idea precise.

As explained in [21], the well-known category of orthogonal spectra models global stable homotopy theory when studied with respect to *global equivalences*, the class of morphisms that induce isomorphisms of equivariant stable homotopy groups for all compact Lie groups. The localization is the *global stable homotopy category* $\mathcal{GH}$, a compactly generated, tensor triangulated category that houses all global stable homotopy types. The present paper explains how to pass between stacks, orbispaces and orthogonal spaces in a homotopically meaningful way; a consequence is that every global stable homotopy type (i.e., every orthogonal spectrum) gives rise to a cohomology theory on stacks and orbifolds. Indeed, by taking unreduced suspension spectra, every unstable global homotopy type is transferred into the triangulated global stable homotopy category $\mathcal{GH}$. In particular, taking morphisms in $\mathcal{GH}$ into an orthogonal spectrum E defines $\mathbb{Z}$-graded E-cohomology groups. The counterpart of a global quotient orbifold $M/\!\!/G$ in the global homotopy theory of orthogonal spaces is the semifree orthogonal space $\mathbf{L}_{G,V}M$ introduced in [21, Con. 1.1.22]. The morphisms $[\![\Sigma^\infty_+\mathbf{L}_{G,V}M, E]\!]$ in the global stable homotopy category biject with the G-equivariant E-cohomology groups of M, by an adjunction relating the global and G-equivariant stable homotopy categories. In other words,

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when evaluated on a global quotient $M/\!\!/G$, our orbifold cohomology theory defined from a global stable homotopy type precisely returns the G-equivariant cohomology of M, which is essentially the design criterion. The cohomology theories defined in this way should be thought of as 'genuine' (as opposed to 'naive'). Indeed, the global stable homotopy category forgets to the G-equivariant stable homotopy category based on a complete G-universe; the equivariant cohomology theories represented by such objects are usually called *genuine* (as opposed to *naive*). Genuine equivariant cohomology theories have much more structure than naive ones; this structure manifests itself in different forms, for example as transfer maps, stability under twisted suspension (i.e., smash product with linear representation spheres), an extension of the $\mathbb{Z}$-graded cohomology groups to an $RO(G)$-graded theory, and an equivariant refinement of additivity (the so called *Wirthmüller isomorphism*). Hence global stable homotopy types in the sense of [21] represent *genuine* (as opposed to 'naive') orbifold cohomology theories.

Now we describe the contents of this paper in more detail. There are different formal frameworks for stacks and orbifolds (algebro-geometric, smooth, topological), and these objects can be studied with respect to various notions of 'equivalence'. The approach that most easily feeds into our present context are the notions of *topological stacks* respectively *orbispaces* as developed by Gepner and Henriques in [10]. The present paper and the article [12] by Körschgen together identify the Gepner-Henriques model with the global homotopy theory of orthogonal spaces as established by the author in [21, Ch. 1]. The comparison proceeds through yet another model, the global homotopy theory of 'spaces with an action of the universal compact Lie group'. This universal compact Lie group (which is neither compact nor a Lie group) is a well known object, namely the topological monoid $\mathcal{L} = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ of linear isometric embeddings of $\mathbb{R}^\infty$ into itself. To my knowledge, the use of the monoid $\mathcal{L}$ in homotopy theory started with Boardman and Vogt's paper [2]; since then, $\mathcal{L}$-spaces have been extensively studied, for example in [1, 9, 16, 17]. The underlying space of $\mathcal{L}$ is contractible, so the homotopy theory of $\mathcal{L}$-spaces with respect to 'underlying' weak equivalences is just another model for the homotopy theory of spaces. However, we shift the perspective on the homotopy theory that $\mathcal{L}$-spaces represent, and use a notion of *global equivalences* of $\mathcal{L}$-spaces that is much finer than the notion of underlying weak equivalence that has so far been studied.

We will make the case that $\mathcal{L}$ has all the moral right to be thought of as the universal compact Lie group. Indeed, $\mathcal{L}$ contains a copy of every compact Lie group in a specific way: we may choose a continuous isometric linear G-action on $\mathbb{R}^\infty$ that makes $\mathbb{R}^\infty$ into a complete G-universe. This action is a continuous injective homomorphism $\rho : G \longrightarrow \mathcal{L}$, and we call the images $\rho(G)$ of such homomorphisms *universal subgroups* of $\mathcal{L}$, compare Definition 1.4 below. In this way every compact Lie group determines a specific conjugacy class of subgroups of $\mathcal{L}$, abstractly isomorphic to G. A morphism of $\mathcal{L}$-spaces is a *global equivalence* if it induces weak homotopy equivalences on the fixed point spaces for all universal subgroups of $\mathcal{L}$. When viewed through the eyes of global equivalences, one should think of an $\mathcal{L}$-space as a 'global space' on which all compact Lie groups act simultaneously and in a compatible way.

In Section 1 we discuss the global homotopy theory of $\mathcal{L}$-spaces, including many examples and the global model structure (see Theorem 1.20). We also discuss the operadic product of $\mathcal{L}$-spaces and show that is is fully homotopical for global equivalences (see Theorem 1.21) and satisfies the pushout product property with respect to the global model structure (see Proposition 1.22).

Section 2 compares the global homotopy theory of $\mathcal{L}$-spaces to the homotopy theory of orbispaces. To this end we introduce the global orbit category $\mathbf{O}_{\mathrm{gl}}$ (see Definition 2.1), the direct analog for the universal compact Lie group of the orbit category of a single compact Lie group: the objects of $\mathbf{O}_{\mathrm{gl}}$ are the universal subgroups of $\mathcal{L}$ and the morphism spaces in $\mathbf{O}_{\mathrm{gl}}$

are defined by

$$\mathbf{O}_{\mathrm{gl}}(K,G) \;=\; \mathrm{map}^{\mathcal{L}}(\mathcal{L}/K,\mathcal{L}/G)\ .$$

The main result is Theorem 2.5, describing a Quillen equivalence between the category of $\mathcal{L}$-spaces under the global model structure and the category of *orbispaces*, i.e., contravariant continuous functors from the global orbit category to spaces. This Quillen equivalence is an analog of Elmendorf's theorem [8] saying that taking fixed points with respect to all closed subgroups of G is an equivalence from the homotopy theory of G-spaces to functors on the orbit category.

Section 3 compares the global homotopy theory of $\mathcal{L}$-spaces to the global homotopy theory of orthogonal spaces as introduced in [21, Ch. 1]. The comparison is via an adjoint functor pair $(Q\otimes_{\mathbf{L}}-,\mathrm{map}^{\mathcal{L}}(Q,-))$ that was already used in a non-equivariant context (and with different terminology) by Lind [16, Sec. 8]. We show in Theorem 3.9 that this functor pair is a Quillen equivalence with respect to the two global model structures.

The following diagram of Quillen equivalences summarizes the results of this paper and the connection to the homotopy theory of orbispaces in the sense of Gepner and Henriques [10]:

$$\text{orthogonal spaces}\quad \mathrel{\substack{\xrightarrow{Q\otimes_{\mathbf{L}}-}\\ \xleftarrow[\mathrm{map}^{\mathcal{L}}(Q,-)]{}}}\quad \mathcal{L}\mathbf{T}\quad \mathrel{\substack{\xleftarrow{\Lambda}\\ \xrightarrow[\Phi]{}}}\quad \text{orbispaces}\quad \rightleftarrows\quad \text{Orb-spaces}$$

On the left is the category of orthogonal spaces, i.e., continuous functors to spaces from the category of finite-dimensional inner product spaces and linear isometric embeddings, equipped with the positive global model structure of [21, Prop. 1.2.23]. Next to it is the category of $\mathcal{L}$-spaces, equipped with the global model structure of Theorem 1.20. Then comes the category of orbispaces in the sense of this paper, i.e., contravariant continuous functors from the global orbit category $\mathbf{O}_{\mathrm{gl}}$ to spaces; orbispaces are equipped with a 'pointwise' (or 'projective') model structure. Finally, on the right is the category of 'Orb-spaces' in the sense of Gepner and Henriques, i.e., contravariant continuous functors from the topological category Orb defined in [10, Sec. 4.1], for the family of compact Lie groups as allowed isotropy groups. We establish the left and middle Quillen equivalence in Theorem 2.5 respectively Theorem 3.9. The right double arrows indicate a chain of two Quillen equivalences established by Körschgen in [12]; this chain arises from a zig-zag of Dwyer-Kan equivalences between our indexing category $\mathbf{O}_{\mathrm{gl}}$ and the category Orb used by Gepner and Henriques, see [12, 1.1 Cor.]. In their paper [10], Gepner and Henriques furthermore compare their homotopy theory of Orb-spaces to the homotopy theories of topological stacks and of topological groupoids.

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1. Global model structures for $\mathcal{L}$-spaces

In this section we define global equivalences of $\mathcal{L}$-spaces and establish the global model structure, see Theorem 1.20. In the following sections, we will show that this global model structure is Quillen equivalent to the model category of orbispaces, compare Theorem 2.5, and to the category of orthogonal spaces, with respect to the positive global model structure, compare Theorem 3.9.

We denote by $\mathcal{L}=\mathbf{L}(\mathbb{R}^\infty,\mathbb{R}^\infty)$ the monoid, under composition, of linear isometric self-embeddings of $\mathbb{R}^\infty$, i.e., those $\mathbb{R}$-linear self-maps that preserve the scalar product. The space $\mathcal{L}$ carries the compactly-generated function space topology. The space $\mathcal{L}$ is contractible by [2, Sec. 1, Lemma] or Remark A.12, and composition

$$\circ \ : \ \mathcal{L} \times \mathcal{L} \ \longrightarrow \ \mathcal{L}$$

is continuous, see for example Proposition A.2. Appendix A reviews the definition of the function space topology in more detail, and collects other point-set topological properties of spaces of linear isometric embeddings.

Definition 1.1. An *$\mathcal{L}$-space* is an $\mathcal{L}$-object in the category $\mathbf{T}$ of compactly generated spaces.

More explicitly, an $\mathcal{L}$-space is a compactly generated space X in the sense of [18], equipped with an action of the monoid $\mathcal{L}$, and such that the action map

$$\mathcal{L} \times X \ \longrightarrow \ X \ , \quad (\varphi, x) \ \longmapsto \ \varphi \cdot x$$

is continuous for the Kelleyfied product topology on the source. We write $\mathcal{L}\mathbf{T}$ for the category of $\mathcal{L}$-spaces and $\mathcal{L}$-equivariant continuous maps.

Example 1.2. Examples of interesting $\mathcal{L}$-spaces are the global classifying spaces $\mathcal{L}/G$ for a compact Lie subgroup G of $\mathcal{L}$, see Example 1.9. Many more examples arise from orthogonal spaces via 'evaluation at $\mathbb{R}^\infty$' as explained in Construction 3.2. The point of the Quillen equivalence between $\mathcal{L}$-spaces and orthogonal spaces (Theorem 3.9 in combination with Proposition 3.7) is precisely that up to global equivalence, *all* $\mathcal{L}$-spaces arise from orthogonal spaces by evaluation at $\mathbb{R}^\infty$.

In particular, the orthogonal spaces discussed in Chapters 1 and 2 of [21] provide a host of examples of $\mathcal{L}$-spaces, and our Quillen equivalence translates all homotopical statements about their global homotopy types into corresponding properties of the associated $\mathcal{L}$-spaces. Explicit examples include global projective spaces and Grassmannians [21, Ex.. 1.1.28, 2.3.12–2.3.16], cofree global homotopy types [21, Thm. 1.2.23], free ultra-commutative monoids [21, Ex. 2.1.5], global equivariant refinements $\mathbf{O}$, $\mathbf{SO}$, $\mathbf{U}$, $\mathbf{SU}$, $\mathbf{Sp}$, $\mathbf{Spin}$, $\mathbf{Spin}^c$ of the infinite families of classical Lie group [21, Ex. 2.3.6–2.3.10], unordered frames [21, Ex. 2.3.23], different global refinements $\mathbf{bO}$, $\mathbf{BO}$ and $\mathbf{B}^\circ\mathbf{O}$ of the classifying space of the infinite orthogonal group [21, Sec. 2.4], global versions $\mathbf{BOP}$, $\mathbf{BUP}$ and $\mathbf{BSpP}$ of the infinite loop spaces of the real, complex and symplectic equivariant K-theory spectra [21, Ex. 2.4.1, 2.4.34], and the underlying 'global infinite loop space' $\Omega^\bullet X$ of a stable global homotopy type X [21, Con. 4.1.6].

Definition 1.3. Let G be a compact Lie group. A *complete G-universe* is an orthogonal G-representation $\mathcal{U}$ of countably infinite dimension such that every finite-dimensional G-representation admits a G-equivariant linear isometric embedding into $\mathcal{U}$.

Proposition A.7 (ii) shows that an orthogonal representation of a compact Lie group G on a countably infinite dimensional inner product space is necessarily the orthogonal direct sum of finite-dimensional G-invariant subspaces. By further decomposing the summands into irreducible pieces, the orthogonal G-representation can be written as the orthogonal direct sum of finite-dimensional irreducible orthogonal G-representations. If the given representation is a complete G-universe, then every irreducible G-representation must occur infinitely often. So we conclude that a complete G-universe is equivariantly isometrically isomorphic to

$$\bigoplus_{\lambda \in \Lambda} \bigoplus_{\mathbb{N}} \lambda \ ,$$

where $\Lambda = \{\lambda\}$ is a complete set of pairwise non-isomorphic irreducible G-representations. The set Λ is finite if G is finite, and countably infinite if G has positive dimension.

Now we come to a key definition.

Definition 1.4. A *compact Lie subgroup* of the topological monoid $\mathcal{L}$ is a subgroup that is compact in the subspace topology and admits the structure of a Lie group (necessarily unique). A compact Lie subgroup is a *universal subgroup* if the tautological G-action makes $\mathbb{R}^\infty$ into a complete G-universe.

Since $\mathcal{L}$ is a Hausdorff space, every compact subgroup is necessarily closed. As we show in Proposition A.8, a compact subgroup G of $\mathcal{L}$ is a Lie group if and only if there exists a finite-dimensional G-invariant subspace of $\mathbb{R}^\infty$ on which G acts faithfully. There are compact subgroups of $\mathcal{L}$ that are not Lie groups, compare Example A.9. When G is a compact Lie subgroup of $\mathcal{L}$ we write $\mathbb{R}^\infty_G$ for the tautological G-representation on $\mathbb{R}^\infty$, which is automatically faithful.

The next proposition shows that conjugacy classes of universal subgroups of $\mathcal{L}$ biject with isomorphism classes of compact Lie groups.

Proposition 1.5. *Every compact Lie group is isomorphic to a universal subgroup of $\mathcal{L}$. Every isomorphism between universal subgroups is given by conjugation by an invertible linear isometry in $\mathcal{L}$. In particular, isomorphic universal subgroups are conjugate in $\mathcal{L}$.*

Proof. Given a compact Lie group G we can choose a continuous isometric linear action of G on $\mathbb{R}^\infty$ that makes it a complete G-universe. Such an action is adjoint to a continuous injective monoid homomorphism $\rho : G \longrightarrow \mathcal{L}$. Since G is compact and $\mathcal{L}$ is a Hausdorff space, the map ρ is a closed embedding, and the image $\rho(G)$ is a universal subgroup of $\mathcal{L}$, isomorphic to G via ρ.

Now we let $\alpha : G \longrightarrow \bar{G}$ be an isomorphism between two universal subgroups of $\mathcal{L}$. Then $\mathbb{R}^\infty_G$ and $\alpha^*(\mathbb{R}^\infty_{\bar{G}})$ are two complete G-universes, so there is a G-equivariant linear isometry $\varphi : \alpha^*(\mathbb{R}^\infty_{\bar{G}}) \longrightarrow \mathbb{R}^\infty_G$. This φ is an invertible element of the monoid $\mathcal{L}$ and the G-equivariance means that $\varphi \circ \alpha(g) = g \circ \varphi$ for all $g \in G$. Hence α coincides with conjugation by φ. □

The topological monoid $\mathcal{L}$ contains many other compact Lie subgroups that are not universal subgroups: any continuous, faithful linear isometric action of a compact Lie group G on $\mathbb{R}^\infty$ provides such a compact Lie subgroup. However, with respect to this action, $\mathbb{R}^\infty$ need not be a complete G-universe, because some irreducible G-representations may occur only with finite multiplicity.

Definition 1.6. A morphism $f : X \longrightarrow Y$ of $\mathcal{L}$-spaces is a *global equivalence* if for every universal subgroup G of $\mathcal{L}$ the induced map $f^G : X^G \longrightarrow Y^G$ on G-fixed points is a weak homotopy equivalence.

The class of global equivalences of $\mathcal{L}$-spaces is closed under various constructions; we collect some of these properties in the next proposition. We call a morphism $f : A \longrightarrow B$ of $\mathcal{L}$-spaces an *h-cofibration* if it has the homotopy extension property, i.e., given a morphism of $\mathcal{L}$-spaces $\varphi : B \longrightarrow X$ and a homotopy of $\mathcal{L}$-equivariant maps $H : A \times [0,1] \longrightarrow X$ such that $H(-,0) = \varphi f$, there is a homotopy $\bar{H} : B \times [0,1] \longrightarrow X$ such that $\bar{H} \circ (f \times [0,1]) = H$ and $\bar{H}(-,0) = \varphi$. The class of h-cofibrations is closed under retracts, cobase change, coproducts and sequential compositions because it can be characterized by a right lifting property.

Proposition 1.7. (i) *A coproduct of global equivalences is a global equivalence.*
(ii) *A product of global equivalences is a global equivalence.*
(iii) *Let $e_n : X_n \longrightarrow X_{n+1}$ and $f_n : Y_n \longrightarrow Y_{n+1}$ be morphisms of $\mathcal{L}$-spaces that are closed embeddings of underlying spaces, for $n \geq 0$. Let $\psi_n : X_n \longrightarrow Y_n$ be global equivalences of $\mathcal{L}$-spaces that satisfy $\psi_{n+1} \circ e_n = f_n \circ \psi_n$ for all $n \geq 0$. Then the induced morphism $\psi_\infty : X_\infty \longrightarrow Y_\infty$ between the colimits of the sequences is a global equivalence.*

(iv) *Let* $f_n : Y_n \longrightarrow Y_{n+1}$ *be a global equivalence of* $\mathcal{L}$*-spaces that is objectwise a closed embedding, for* $n \geq 0$*. Then the canonical morphism* $f_\infty : Y_0 \longrightarrow Y_\infty$ *to the colimit of the sequence* $\{f_n\}_{n\geq 0}$ *is a global equivalence.*

(v) *Let*

$$\begin{tikzcd} C \arrow[d, "\gamma"'] & A \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ \bar{C} & \bar{A} \arrow[l, "\bar{g}"] \arrow[r, "\bar{f}"'] & \bar{B} \end{tikzcd}$$

be a commutative diagram of $\mathcal{L}$*-spaces such that* g *and* $\bar{g}$ *are h-cofibrations. If the morphisms* α, β *and* γ *are global equivalences, then so is the induced morphism of pushouts*

$$\gamma \cup \beta \ : \ C \cup_A B \ \longrightarrow \ \bar{C} \cup_{\bar{A}} \bar{B} \ .$$

(vi) *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

be a pushout square of $\mathcal{L}$*-spaces such that* f *is a global equivalence. If in addition* f *or* g *is an h-cofibration, then the morphism* k *is a global equivalence.*

Proof. Part (i) holds because fixed points commute with disjoint unions and a disjoint union of weak equivalences is a weak equivalence. Part (ii) holds because fixed points commute with products and a product of weak equivalences is a weak equivalence.

Fixed points commute with sequential colimits along closed embeddings, compare [21, Prop. B.1 (ii)]. Moreover, a colimit of weak equivalences along two sequences of closed embeddings is another weak equivalence (compare [21, Prop. A.7 (i)]), so together this implies part (iii). Similarly, a sequential composite of weak equivalences that are simultaneously closed embeddings is another weak equivalence (compare [21, Prop. A.7 (ii)]), so together this implies part (iv).

(v) Let G be a universal subgroup of $\mathcal{L}$. Then the three vertical maps in the following commutative diagram are weak equivalences:

$$\begin{tikzcd} C^G \arrow[d, "\gamma^G"'] & A^G \arrow[l, "g^G"'] \arrow[r, "f^G"] \arrow[d, "\alpha^G"] & B^G \arrow[d, "\beta^G"] \\ \bar{C}^G & \bar{A}^G \arrow[l, "\bar{g}^G"] \arrow[r, "\bar{f}^G"'] & \bar{B}^G \end{tikzcd}$$

Since g and $\bar{g}$ are h-cofibrations of $\mathcal{L}$-spaces, the maps g^G and $\bar{g}^G$ are h-cofibrations of non-equivariant spaces. The induced map of the horizontal pushouts is thus a weak equivalence by the gluing lemma, see for example [3, Appendix, Prop. 4.8 (b)]. Since g^G and $\bar{g}^G$ are h-cofibrations, they are in particular closed embeddings, compare [13, Prop. 8.2] or [21, Prop. A.34]. So taking G-fixed points commutes with the horizontal pushout (compare [21, Prop. B.1 (i)]), and we conclude that also the map

$$(\gamma \cup \beta)^G \ : \ (C \cup_A B)^G \ \longrightarrow \ (\bar{C} \cup_{\bar{A}} \bar{B})^G$$

is a weak equivalence. This proves part (v).

(vi) In the commutative diagram

$$\begin{tikzcd} C \arrow[d, equal] & A \arrow[l, "g"'] \arrow[r, equal] \arrow[d, equal] & A \arrow[d, "f"] \\ C & A \arrow[l, "g"] \arrow[r, "f"'] & B \end{tikzcd}$$

all vertical morphisms are global equivalences. If g is an h-cofibration, we can apply part (v) to this square to get the desired conclusion. If f is an h-cofibration, we apply part (v) after interchanging the roles of left and right horizontal morphisms. □

Several interesting morphisms of $\mathcal{L}$-spaces that come up in this paper satisfy the following stronger form of 'global equivalence':

Definition 1.8. A morphism $f : X \longrightarrow Y$ of $\mathcal{L}$-spaces is a *strong global equivalence* if for every universal subgroup G of $\mathcal{L}$, the underlying G-map of f is a G-equivariant homotopy equivalence.

In other words, a morphism of $\mathcal{L}$-spaces $f : X \longrightarrow Y$ is a strong global equivalence if for every universal subgroup G there is a G-equivariant continuous map $g : Y \longrightarrow X$ such that $g \circ f : X \longrightarrow X$ and $f \circ g : Y \longrightarrow Y$ are G-equivariantly homotopic to the respective identity maps. However, there is no compatibility requirement on the homotopy inverses and the equivariant homotopies, and they need not be equivariant for the full monoid $\mathcal{L}$. Clearly, every strong global equivalence is in particular a global equivalence.

Example 1.9 (Induced $\mathcal{L}$-spaces and global classifying spaces)**.** We let G be a compact Lie subgroup of $\mathcal{L}$ and A a left G-space. Then we can form the *induced $\mathcal{L}$-space*

$$\mathcal{L} \times_G A \;=\; (\mathcal{L} \times A)/(\varphi g, a) \sim (\varphi, ga) \; .$$

The functor $\mathcal{L} \times_G -$ is left adjoint to the restriction functor from $\mathcal{L}$-spaces to G-spaces.

The G-representation $\mathbb{R}^\infty_G$ has a finite-dimensional faithful G-subrepresentation V, by Proposition A.8. The following Proposition 1.10 shows in particular that for every universal subgroup K of $\mathcal{L}$ the restriction map

$$\mathcal{L}/G \;=\; \mathbf{L}(\mathbb{R}^\infty_G, \mathbb{R}^\infty_K)/G \;\;\longrightarrow\;\; \mathbf{L}(V, \mathbb{R}^\infty_K)/G$$

is a K-homotopy equivalence. The K-space $\mathbf{L}(V, \mathcal{U}_K)/G$ is a classifying space for principal G-bundles over K-spaces, see for example [21, Prop. 1.1.30]. So under the tautological K-action, $\mathcal{L}/G$ is a classifying space for principal G-bundles over K-spaces. We conclude that $\mathcal{L}/G$ is an incarnation, in the world of $\mathcal{L}$-spaces, of the global classifying space of the group G, or of the stack of principal G-bundles. In particular, the underlying non-equivariant space of $\mathcal{L}/G$ is a classifying space for G.

Proposition 1.10. *Let G be a compact Lie subgroup of the monoid $\mathcal{L}$, V a faithful finite-dimensional G-subrepresentation of $\mathbb{R}^\infty_G$ and A a G-space. Then the restriction morphism*

$$\rho_V \times_G A \;:\; \mathcal{L} \times_G A \;=\; \mathbf{L}(\mathbb{R}^\infty_G, \mathbb{R}^\infty) \times_G A \;\longrightarrow\; \mathbf{L}(V, \mathbb{R}^\infty) \times_G A$$

is a strong global equivalence of $\mathcal{L}$-spaces.

Proof. We let K be a universal subgroup of the monoid $\mathcal{L}$. Then $\rho_V : \mathbf{L}(\mathbb{R}^\infty_G, \mathbb{R}^\infty_K) \longrightarrow \mathbf{L}(V, \mathbb{R}^\infty_K)$ is a $(K \times G)$-homotopy equivalence by Proposition A.10; so the map

$$\rho_V \times_G A \;:\; \mathbf{L}(\mathbb{R}^\infty_G, \mathbb{R}^\infty_K) \times_G A \;\longrightarrow\; \mathbf{L}(V, \mathbb{R}^\infty_K) \times_G A$$

is a K-homotopy equivalence. □

Now we turn to the construction of model structures on the category of $\mathcal{L}$-spaces, with the global model structure of Theorem 1.20 as the ultimate aim. The 'classical' model structure on the category of all topological spaces was established by Quillen in [20, II.3 Thm. 1]. We use the straightforward adaptation of this model structure to the category $\mathbf{T}$ of compactly generated spaces, which is described for example in [11, Thm. 2.4.25]. In this model structure, the weak equivalences are the weak homotopy equivalences and fibrations are the Serre fibrations. The cofibrations are the retracts of generalized CW-complexes, i.e., cell complexes in which cells can be attached in any order and not necessarily to cells of lower dimensions.

Now we let $\mathcal{C}$ be a collection of closed subgroups of the linear isometries monoid $\mathcal{L}$. We call a morphism $f : X \longrightarrow Y$ of $\mathcal{L}$-spaces a *$\mathcal{C}$-equivalence* (respectively *$\mathcal{C}$-fibration*) if the restriction $f^G : X^G \longrightarrow Y^G$ to G-fixed points is a weak equivalence (respectively Serre fibration) of spaces for all subgroups G that belong to the collection $\mathcal{C}$. A *$\mathcal{C}$-cofibration* is a morphism with the left lifting property with respect to all morphisms that are simultaneously $\mathcal{C}$-equivalences and $\mathcal{C}$-fibrations. The resulting '$\mathcal{C}$-projective model structure' would be well known if $\mathcal{L}$ were a topological group. Despite the fact that $\mathcal{L}$ is only a topological monoid, the standard proof for topological groups (see for example [21, Prop. B.7]) goes through essentially unchanged, and hence we omit it.

Proposition 1.11. *Let $\mathcal{C}$ be a collection of closed subgroups of $\mathcal{L}$. Then the $\mathcal{C}$-equivalences, $\mathcal{C}$-cofibrations and $\mathcal{C}$-fibrations form a model structure, the* $\mathcal{C}$-projective model structure *on the category of $\mathcal{L}$-spaces. This model structure is proper, cofibrantly generated and topological.*

For easier reference we recall the 'standard' sets of generating cofibrations and acyclic cofibrations of the $\mathcal{C}$-projective model structure. We let $I_{\mathcal{C}}$ be the set of morphisms of $\mathcal{L}$-spaces

$$\mathcal{L}/G \times \text{incl} \ : \ \mathcal{L}/G \times \partial D^k \ \longrightarrow \ \mathcal{L}/G \times D^k \tag{1.12}$$

for all G in $\mathcal{C}$ all $k \geq 0$. We let $J_{\mathcal{C}}$ denote the set of morphisms

$$\mathcal{L}/G \times \text{incl} \ : \ \mathcal{L}/G \times D^k \times \{0\} \ \longrightarrow \ \mathcal{L}/G \times D^k \times [0,1] \tag{1.13}$$

for all G in $\mathcal{C}$ and all $k \geq 0$. Then the right lifting property with respect to the set $I_{\mathcal{C}}$ (respectively $J_{\mathcal{C}}$) is equivalent to being a $\mathcal{C}$-acyclic fibration (respectively $\mathcal{C}$-fibration).

If we want a model structure on the category of $\mathcal{L}$-spaces with the global equivalences as weak equivalences, then one possibility is the *universal projective* model structure, i.e., the $\mathcal{C}^u$-projective model structure of the previous Proposition 1.11, for the collection $\mathcal{C}^u$ of universal subgroups. We explain in Remark 2.7 below why this model structure is not the most convenient one for our purposes. We instead favor the following *global model structure*, which arises from the projective model structure for the collection $\mathcal{C}^L$ of all compact Lie subgroups (as opposed to only universal subgroups) by left Bousfield localization at the class of global equivalences.

Let G and $\bar{G}$ be two compact Lie subgroups of $\mathcal{L}$ and $\varphi \in \mathcal{L}$ a linear isometric embedding. We denote by

$$\text{stab}(\varphi) \ = \ \{(g,\gamma) \in G \times \bar{G} \ : \ g \circ \varphi = \varphi \circ \gamma\}$$

the stabilizer group of φ with respect to the action of $G \times \bar{G}$ on $\mathcal{L}$ by post- respectively precomposition. Since $\mathcal{L}$ is a Hausdorff space and composition of $\mathcal{L}$ is continuous, the stabilizer $\text{stab}(\varphi)$ is a closed subgroup of $G \times \bar{G}$, hence a compact Lie group in its own right. The two projections from $\text{stab}(\varphi)$ to G and $\bar{G}$ are continuous homomorphisms.

Definition 1.14. Let G and $\bar{G}$ be two compact Lie subgroups of $\mathcal{L}$. A *correspondence* from G to $\bar{G}$ is a linear isometric embedding $\varphi \in \mathcal{L}$ such that the two projections

$$G \longleftarrow \operatorname{stab}(\varphi) \longrightarrow \bar{G}$$

are isomorphisms.

So a linear isometric embedding $\varphi \in \mathcal{L}$ is a correspondence from G to $\bar{G}$ if and only if for every $g \in G$ there is a unique $\gamma \in \bar{G}$ such that $g \circ \varphi = \varphi \circ \gamma$, and conversely for every $\gamma \in \bar{G}$ there is a unique $g \in G$ such that $g \circ \varphi = \varphi \circ \gamma$. Since a linear isometric embedding is injective, the condition $\varphi \circ \gamma = g \circ \varphi$ shows that γ is determined by φ and g.

We write $\varphi : G \rightsquigarrow \bar{G}$ for a correspondence between compact Lie subgroups of $\mathcal{L}$. Such a correspondence effectively embeds the tautological $\bar{G}$-representation into the tautological G-representation. More precisely, if $\alpha : G \longrightarrow \bar{G}$ denotes the isomorphism provided by φ, i.e., $\varphi \circ \alpha(g) = g \circ \varphi$ for all $g \in G$, then $\varphi : \alpha^*(\mathbb{R}^\infty_{\bar{G}}) \longrightarrow \mathbb{R}^\infty_G$ is a G-equivariant linear isometric embedding. If φ happens to be bijective, then $\alpha(g) = \varphi^{-1} g \varphi$; so informally speaking, one can think of a correspondence as an element of $\mathcal{L}$ that 'conjugates G isomorphically onto $\bar{G}$'. The caveat is that 'conjugation by φ' does not have a literal meaning unless φ is bijective.

Let $\varphi : G \rightsquigarrow \bar{G}$ be a correspondence between two compact Lie subgroups of $\mathcal{L}$. Then for every $g \in G$ there is a $\gamma \in \bar{G}$ such that $g \circ \varphi = \varphi \circ \gamma$. So for every $\mathcal{L}$-space X and every $\bar{G}$-fixed element $x \in X^{\bar{G}}$, we have

$$g \cdot (\varphi \cdot x) \;=\; (g \circ \varphi) \cdot x \;=\; (\varphi \circ \gamma) \cdot x \;=\; \varphi \cdot x\ .$$

Hence the continuous map $\varphi \cdot - : X \longrightarrow X$ restricts to a map

$$\varphi \cdot - \;:\; X^{\bar{G}} \longrightarrow X^G$$

from the $\bar{G}$-fixed points to the G-fixed points. We denote by $\mathcal{C}^L$ the collection of compact Lie subgroups of $\mathcal{L}$.

Definition 1.15. A morphism $f : X \longrightarrow Y$ of $\mathcal{L}$-spaces is a *global fibration* if it is a $\mathcal{C}^L$-fibration and for every correspondence $\varphi : G \rightsquigarrow \bar{G}$ between compact Lie subgroups of $\mathcal{L}$ the map

$$(f^{\bar{G}}, \varphi \cdot -) \;:\; X^{\bar{G}} \longrightarrow Y^{\bar{G}} \times_{Y^G} X^G$$

is a weak equivalence. An $\mathcal{L}$-space X is *injective* if for every correspondence $\varphi : G \rightsquigarrow \bar{G}$ between compact Lie subgroups of $\mathcal{L}$ the map

$$\varphi \cdot - \;:\; X^{\bar{G}} \longrightarrow X^G$$

is a weak equivalence.

Equivalently, a morphism f is a global fibration if and only if f is a $\mathcal{C}^L$-fibration and for every correspondence $\varphi : G \rightsquigarrow \bar{G}$ the square of fixed point spaces

$$(1.16)\qquad \begin{tikzcd} X^{\bar{G}} \arrow[r, "\varphi\cdot -"] \arrow[d, "f^{\bar{G}}"'] & X^G \arrow[d, "f^G"] \\ Y^{\bar{G}} \arrow[r, "\varphi\cdot -"'] & Y^G \end{tikzcd}$$

is homotopy cartesian. Moreover, an $\mathcal{L}$-space is injective if and only if the unique morphism to a terminal $\mathcal{L}$-space is a global fibration.

Proposition 1.17. (i) *Every compact Lie subgroup of $\mathcal{L}$ admits a correspondence from a universal subgroup of $\mathcal{L}$.*

(ii) *Every global equivalence that is also a global fibration is a $\mathcal{C}^L$-equivalence.*
(iii) *Every global equivalence between injective $\mathcal{L}$-spaces is a $\mathcal{C}^L$-equivalence.*

Proof. (i) We let $\bar{G}$ be a compact Lie subgroup of $\mathcal{L}$. Proposition 1.5 provides a universal subgroup G of $\mathcal{L}$ and an isomorphism $\alpha : G \longrightarrow \bar{G}$. Since $\mathbb{R}^\infty_G$ is a complete G-universe, there is a G-equivariant linear isometric embedding $\varphi : \alpha^*(\mathbb{R}^\infty_{\bar{G}}) \longrightarrow \mathbb{R}^\infty_G$. The fact that φ is G-equivariant precisely means that $\varphi \circ \alpha(g) = g \circ \varphi$ for all $g \in G$. So φ is a correspondence from G to $\bar{G}$.

(ii) We let f be a global equivalence of $\mathcal{L}$-spaces that is also a global fibration. We let $\bar{G}$ be any compact Lie subgroups and $\varphi : G \rightsquigarrow \bar{G}$ a correspondence from a universal subgroup as provided by part (i). Then both vertical maps in the commutative square (1.16) are Serre fibrations because f is a $\mathcal{C}^L$-fibration. The right vertical map is also a weak equivalence because f is a global equivalence. Since f is a global fibration, the square is also homotopy cartesian, so the left vertical map is a weak equivalence. Since $\bar{G}$ was any compact Lie subgroup, this shows that f is a $\mathcal{C}^L$-equivalence.

(iii) We let $f : X \longrightarrow Y$ be a global equivalence between injective $\mathcal{L}$-spaces. We let $\bar{G}$ be any compact Lie subgroup and $\varphi : G \rightsquigarrow \bar{G}$ a correspondence from a universal subgroup as provided by part (i). Then both horizontal maps in the commutative square (1.16) are weak equivalences. The right vertical map is also a weak equivalence because f is a global equivalence. Hence the left vertical map is a weak equivalence. Since $\bar{G}$ was any compact Lie subgroup, this shows that f is a $\mathcal{C}^L$-equivalence. □

The sets $I = I_{\mathcal{C}^L}$ and $J = J_{\mathcal{C}^L}$ of generating cofibrations and acyclic cofibrations for the $\mathcal{C}^L$-projective model structure were defined in (1.12) respectively (1.13). We add another set of morphisms K that detects when the squares (1.16) are homotopy cartesian. Given a correspondence $\varphi : G \rightsquigarrow \bar{G}$ between compact Lie subgroups of $\mathcal{L}$, the map

$$\varphi_\sharp \ : \ \mathcal{L}/G \ \longrightarrow \ \mathcal{L}/\bar{G} \ , \quad \psi \cdot G \ \longmapsto \ (\psi \circ \varphi) \cdot \bar{G}$$

is a morphism of $\mathcal{L}$-spaces that represents the natural transformation

$$\varphi \cdot - \ : \ Y^{\bar{G}} \ \longrightarrow \ Y^G \ .$$

Proposition 1.18. *Let $\varphi : G \rightsquigarrow \bar{G}$ be a correspondence between compact Lie subgroups of $\mathcal{L}$. Then the morphism $\varphi_\sharp : \mathcal{L}/G \longrightarrow \mathcal{L}/\bar{G}$ is a strong global equivalence of $\mathcal{L}$-spaces.*

Proof. Proposition A.8 provides a finite-dimensional faithful $\bar{G}$-subrepresentation V of $\mathbb{R}^\infty_{\bar{G}}$. The space $\varphi(V)$ is then a finite-dimensional faithful G-subrepresentation of $\mathbb{R}^\infty_G$. We obtain a commutative square of $\mathcal{L}$-spaces

$$\begin{tikzcd}
\mathbf{L}(\mathbb{R}^\infty_G, \mathbb{R}^\infty)/G \arrow[r, "\varphi_\sharp"] \arrow[d, "\rho_{\varphi(V)}"'] & \mathbf{L}(\mathbb{R}^\infty_{\bar{G}}, \mathbb{R}^\infty)/\bar{G} \arrow[d, "\rho_V"] \\
\mathbf{L}(\varphi(V), \mathbb{R}^\infty)/G \arrow[r, "\cong", "\psi G \mapsto \psi(\varphi|_V)\bar{G}"'] & \mathbf{L}(V, \mathbb{R}^\infty)/\bar{G}
\end{tikzcd}$$

in which the vertical maps are restrictions. Since φ restricts to an isomorphism from V to $\varphi(V)$, the lower horizontal map is an isomorphism. Proposition 1.10 shows that the two vertical maps are strong global equivalences of $\mathcal{L}$-spaces. So the upper horizontal morphism is a strong global equivalence as well. □

We factor the global equivalence $\varphi_\sharp$ associated to a correspondence $\varphi : G \rightsquigarrow \bar{G}$ through the mapping cylinder as the composite

$$\mathcal{L}/G \ \xrightarrow{c(\varphi)} \ Z(j) = (\mathcal{L}/G \times [0,1]) \cup_{\varphi_\sharp} \mathcal{L}/\bar{G} \ \xrightarrow{r(\varphi)} \ \mathcal{L}/\bar{G} \ ,$$

where $c(\varphi)$ is the 'front' mapping cylinder inclusion and $r(\varphi)$ is the projection, which is a homotopy equivalence of $\mathcal{L}$-spaces. We then define $\mathcal{Z}(\varphi)$ as the set of all pushout product maps

$$c(\varphi)\Box i_k \ : \ \mathcal{L}/G \times D^k \cup_{\mathcal{L}/G\times\partial D^k} Z(\varphi)\times \partial D^k \ \longrightarrow \ Z(\varphi)\times D^k$$

for $k \geq 0$, where $i_k : \partial D^k \longrightarrow D^k$ is the inclusion. We then define

$$K \ = \ \bigcup_{G,\bar{G},\varphi} \mathcal{Z}(\varphi) \ ,$$

indexed by all triples consisting of two compact Lie subgroups of $\mathcal{L}$ and a correspondence between them. By [21, Prop. 1.2.16], the right lifting property with respect to the union $J \cup K$ then characterizes the global fibrations, i.e., we have shown:

Proposition 1.19. *A morphism of $\mathcal{L}$-spaces is a global fibration if and only if it has the right lifting property with respect to the set $J \cup K$.*

Now we have all the ingredients to establish the global model structure of $\mathcal{L}$-spaces. As the proof will show, the set $I = I_{\mathcal{C}^L}$ defined in (1.12) is a set of generating cofibrations. By Proposition 1.19, the set $J \cup K = J_{\mathcal{C}^L} \cup K$ is a set of generating acyclic cofibrations.

Theorem 1.20 (Global model structure for $\mathcal{L}$-spaces)**.** *The $\mathcal{C}^L$-cofibrations, global fibrations and global equivalences form a cofibrantly generated proper topological model structure on the category of $\mathcal{L}$-spaces, the* global model structure*. The fibrant objects in the global model structure are the injective $\mathcal{L}$-spaces. Every $\mathcal{C}^L$-cofibration is an h-cofibration of $\mathcal{L}$-spaces and a closed embedding of underlying spaces.*

Proof. We start with the last statement and let $f : A \longrightarrow B$ be a $\mathcal{C}^L$-cofibration of $\mathcal{L}$-spaces. For every $\mathcal{L}$-space X the evaluation map $\mathrm{ev} : X^{[0,1]} \longrightarrow X$ sending a path ω to $\omega(0)$ is an acyclic fibration in the $\mathcal{C}^L$-projective model structure of Proposition 1.11. Given a morphism $\varphi : B \longrightarrow X$ and a homotopy $H : A \times [0,1] \longrightarrow X$ starting with φf, we let $\hat{H} : A \longrightarrow X^{[0,1]}$ be the adjoint and choose a lift in the commutative square:

$$\begin{tikzcd} A \arrow[r, "\hat{H}"] \arrow[d, "f"'] & X^{[0,1]} \arrow[d, "\mathrm{ev}"] \\ B \arrow[r, "\varphi"'] \arrow[ur, dashed, "\lambda"] & X \end{tikzcd}$$

The adjoint of the lift λ is then the desired homotopy extending φ and H. So the morphism f is an h-cofibration. Every h-cofibration of $\mathcal{L}$-spaces is in particular an h-cofibration of underlying non-equivariant spaces, and hence a closed embedding by [13, Prop. 8.2] or [21, Prop. A.34].

Now we turn to the model category axioms, where we use the numbering as in [7, 3.3]. The category of $\mathcal{L}$-spaces is complete and cocomplete, so axiom MC1 holds. Global equivalences satisfy 2-out-of-3 axiom MC2. Global equivalences, $\mathcal{C}^L$-cofibrations and global fibrations are closed under retracts, so axiom MC3 holds.

The $\mathcal{C}^L$-model structure of Proposition 1.11 shows that every morphism of $\mathcal{L}$-spaces can be factored as $f \circ i$ for a $\mathcal{C}^L$-cofibration i followed by a $\mathcal{C}^L$-equivalence f that is also a $\mathcal{C}^L$-fibration. For every correspondence $\varphi : G \rightsquigarrow \bar{G}$ between compact Lie subgroups, both vertical maps in the commutative square of fixed point spaces (1.16) are then weak equivalences, so the square is homotopy cartesian. The morphism f is thus a global fibration and a global equivalence, so this provides one of the factorizations as required by MC5. For the other half of the factorization axiom MC5 we apply the small object argument (see for example [7, 7.12] or [11, Thm. 2.1.14]) to the set $J \cup K$. All morphisms in J are $\mathcal{C}^L$-cofibrations and $\mathcal{C}^L$-equivalences. Since $\mathcal{L}/G$ is $\mathcal{C}^L$-cofibrant for every compact Lie subgroup G of $\mathcal{L}$, the morphisms in K are also $\mathcal{C}^L$-cofibrations, and we argued above that they are global equivalences. The small object argument provides a functorial factorization of any given morphism of $\mathcal{L}$-spaces as a composite

$$X \xrightarrow{i} W \xrightarrow{q} Y$$

where i is a sequential composition of cobase changes of coproducts of morphisms in $J \cup K$, and q has the right lifting property with respect to $J \cup K$. Since all morphisms in $J \cup K$ are $\mathcal{C}^L$-cofibrations and global equivalences, the morphism i is a $\mathcal{C}^L$-cofibration and a global equivalence, using the various closure properties of the class of global equivalences listed in Proposition 1.7. Moreover, q is a global fibration by Proposition 1.19.

Now we show the lifting properties MC4. By Proposition 1.17 (ii) a morphism that is both a global equivalence and a global fibration is a $\mathcal{C}^L$-equivalence, and hence an acyclic fibration in the $\mathcal{C}^L$-projective model structure. So every morphism that is simultaneously a global equivalence and a global fibration has the right lifting property with respect to $\mathcal{C}^L$-cofibrations. Now we let $j : A \longrightarrow B$ be a $\mathcal{C}^L$-cofibration that is also a global equivalence and we show that it has the left lifting property with respect to all global fibrations. We factor $j = q \circ i$, via the small object argument for $J \cup K$, where $i : A \longrightarrow W$ is a $(J \cup K)$-cell complex and $q : W \longrightarrow B$ a global fibration. Then q is a global equivalence since j and i are, and hence an acyclic fibration in the $\mathcal{C}^L$-projective model structure, again by Proposition 1.17 (ii). Since j is a $\mathcal{C}^L$-cofibration, a lifting in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & W \arrow[d, "q", "\sim"'] \\ B \arrow[r, equal] \arrow[ur, dotted] & B \end{tikzcd}$$

exists. Thus j is a retract of the morphism i that has the left lifting property with respect to global fibrations. But then j itself has this lifting property. This finishes the verification of the model category axioms. Alongside we have also specified sets of generating cofibrations I and generating acyclic cofibrations $J \cup K$. Fixed points commute with sequential colimits along closed embeddings (compare [21, Prop. B.1 (ii)]) and $\mathcal{C}^L$-cofibrations are closed embeddings. So the sources and targets of all morphisms in I and $J \cup K$ are small with respect to sequential colimits of $\mathcal{C}^L$-cofibrations. So the global model structure is cofibrantly generated.

Every $\mathcal{C}^L$-cofibration is in particular an h-cofibration of $\mathcal{L}$-spaces, and hence an h-cofibration of underlying G-spaces for every universal subgroup G. So left properness follows from the gluing lemma for G-weak equivalences, compare [21, Prop. B.6]. Since global equivalences are detected by fixed points with respect to universal subgroups, right properness follows from right properness of the Quillen model structure of spaces and the fact that fixed points preserve pullbacks.

The global model structure is topological by [21, Prop. B.5], where we take $\mathcal{G}$ as the set of $\mathcal{L}$-spaces $\mathcal{L}/G$ for all compact Lie subgroups G of $\mathcal{L}$, and we take $\mathcal{Z}$ as the set of acyclic cofibrations $c(\varphi) : \mathcal{L}/G \longrightarrow (\mathcal{L}/G \times [0,1]) \cup_{\varphi_\sharp} \mathcal{L}/\bar{G}$ for $(G, \bar{G}, \varphi)$ as in the definition of the set K. □

We end this section with a brief discussion on the interaction of the global model structure with the operadic product of $\mathcal{L}$-spaces. We denote by

$$\mathcal{L}(2) \;=\; \mathbf{L}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty,\, \mathbb{R}^\infty)$$

the space of binary operations in the linear isometries operad. It comes with a left action of $\mathcal{L}$ and a right action of $\mathcal{L}^2$ by

$$\mathcal{L} \times \mathcal{L}(2) \times \mathcal{L}^2 \longrightarrow \mathcal{L}(2) , \quad (f, \psi, (g, h)) \longmapsto f \circ \psi \circ (g \oplus h) .$$

Given two $\mathcal{L}$-spaces X and Y we can coequalize the right $\mathcal{L}^2$-action on $\mathcal{L}(2)$ with the left $\mathcal{L}^2$-action on the product $X \times Y$ and form

$$X \boxtimes_{\mathcal{L}} Y = \mathcal{L}(2) \times_{\mathcal{L} \times \mathcal{L}} (X \times Y) .$$

The left $\mathcal{L}$-action on $\mathcal{L}(2)$ by postcomposition descends to an $\mathcal{L}$-action on this operadic product. Some care has to be taken when analyzing this construction: because the monoid $\mathcal{L}$ is not a group, it may be hard to figure out when two elements of $\mathcal{L}(2) \times X \times Y$ become equal in the coequalizer. The operadic product $\boxtimes_{\mathcal{L}}$ is coherently associative and commutative, but it does *not* have a unit object. The monoids (respectively commutative monoids) with respect to $\boxtimes_{\mathcal{L}}$ are essentially A_∞-monoids (respectively E_∞-monoids). We refer the reader to [1, Sec. 4] for more details.

The next result shows that up to global equivalence the operadic product of $\mathcal{L}$-spaces coincides with the categorical product. Given two $\mathcal{L}$-spaces X and Y, we define a natural $\mathcal{L}$-linear map

$$\rho_{X,Y} : X \boxtimes_{\mathcal{L}} Y \longrightarrow X \times Y \qquad \text{by} \qquad [\varphi; x, y] \longmapsto ((\varphi i_1) \cdot x, (\varphi i_2) \cdot y) .$$

Here $i_1, i_2 : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty \oplus \mathbb{R}^\infty$ are the two direct summand embeddings. The following theorem has a non-equivariant precursor: Blumberg, Cohen and Schlichtkrull show in [1, Prop. 4.23] that for certain $\mathcal{L}$-spaces (those that are cofibrant in the model structure of [1, Thm. 4.15]), the morphism $\rho_{X,Y}$ is a non-equivariant weak equivalence. We show that a much stronger conclusion holds without any hypothesis on X and Y.

Theorem 1.21. *For all $\mathcal{L}$-spaces X and Y, the morphism $\rho_{X,Y} : X \boxtimes_{\mathcal{L}} Y \longrightarrow X \times Y$ is a strong global equivalence. In particular, the functor $X \boxtimes_{\mathcal{L}} -$ preserves global equivalences.*

Proof. We let G be a universal subgroup of $\mathcal{L}$. We choose a G-equivariant linear isometry

$$\psi : \mathbb{R}^\infty_G \oplus \mathbb{R}^\infty_G \cong \mathbb{R}^\infty_G$$

and define a continuous map

$$\psi_* : X \times Y \longrightarrow X \boxtimes_{\mathcal{L}} Y \qquad \text{by} \qquad \psi_*(x, y) = [\psi, x, y] .$$

The G-equivariance means explicitly that $\psi(g \oplus g) = g\psi$ for all $g \in G$, and so the map ψ_* is G-equivariant (but *not* $\mathcal{L}$-linear).

The composite $\rho_{X,Y} \circ \psi_* : X \times Y \longrightarrow X \times Y$ is given by

$$\rho_{X,Y}(\psi_*(x, y)) = ((\psi i_1) \cdot x, (\psi i_2) \cdot y) .$$

Since $\mathbb{R}^\infty_G$ is a complete G-universe, the space of G-equivariant linear isometric self-embeddings of $\mathbb{R}^\infty_G$ is contractible, see for example [15, II Lemma 1.5]; so there is a path of G-equivariant linear isometric self-embeddings linking ψi_1 to the identity of $\mathbb{R}^\infty_G$. Such a path induces a G-equivariant homotopy from the map $(\psi i_1) \cdot - : X \longrightarrow X$ to the identity of X; similarly, $(\psi i_2) \cdot -$ is G-homotopic to the identity of Y. So altogether we conclude that $\rho_{X,Y} \circ \psi_*$ is G-homotopic to the identity.

To analyze the other composite we define a continuous map

$$H : \mathbf{L}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty, \mathbb{R}^\infty) \times [0, 1] \longrightarrow \mathbf{L}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty, \mathbb{R}^\infty \oplus \mathbb{R}^\infty)$$
$$\text{by} \qquad H(\varphi, t)(v, w) = \Big(\varphi(v, tw),\ \varphi(0, \sqrt{1 - t^2} \cdot w)\Big) .$$

Then

$$H(\varphi,0) \;=\; (\varphi i_1)\oplus(\varphi i_2) \qquad\text{and}\qquad H(\varphi,1) \;=\; i_1\varphi\ .$$

Moreover, for every $t\in[0,1]$ the map $H(-,t)$ is equivariant for the left $\mathcal{L}$-action (with diagonal action on the target) and for the right $\mathcal{L}^2$-action. So we can define a homotopy of G-equivariant maps (which are *not* $\mathcal{L}$-linear)

$$K\ :\ (X\boxtimes_{\mathcal{L}} Y)\times[0,1]\ \longrightarrow\ X\boxtimes_{\mathcal{L}} Y \qquad\text{by}\qquad K([\varphi;x,y],t)\;=\;[\psi H(\varphi,t);x,y]\ .$$

Then

$$\begin{aligned}K([\varphi;x,y],0)\;=\;[\psi H(\varphi,0);x,y]\;&=\;[\psi((\varphi i_1)\oplus(\varphi i_2));x,y]\\ &=\;[\psi;(\varphi i_1)\cdot x,(\varphi i_2)\cdot y]\;=\;\psi_*(\rho_{X,Y}[\varphi;x,y])\end{aligned}$$

and

$$K([\varphi;x,y],1)\;=\;[\psi H(\varphi,1);x,y]\;=\;[\psi i_1\varphi;x,y]\;=\;(\psi i_1)\cdot[\varphi;x,y]\ .$$

As in the first part of this proof, ψi_1 can be linked to the identity of $\mathbb{R}^\infty_G$ by a path of G-equivariant linear isometric self-embeddings, and such a path induces another G-equivariant homotopy from the map $(\psi i_1)\cdot - : X\boxtimes_{\mathcal{L}} Y\longrightarrow X\boxtimes_{\mathcal{L}} Y$ to the identity of $X\boxtimes_{\mathcal{L}} Y$. So altogether we have exhibited a G-homotopy between $\psi_*\circ\rho_{X,Y}$ and the identity. Since the universal subgroup G was arbitrary, this shows that $\rho_{X,Y}$ is a strong global equivalence. □

The following 'pushout product property' is the concise way to formulate the compatibility of the global model structure and operadic product.

Proposition 1.22. *The global model structure on the category of $\mathcal{L}$-spaces satisfies the pushout product property with respect to the operadic box product: for all $\mathcal{C}^L$-cofibrations of $\mathcal{L}$-spaces $i:A\longrightarrow B$ and $j:X\longrightarrow Y$, the pushout product morphism*

$$i\Box j\;=\;(i\boxtimes_{\mathcal{L}} Y)\cup(B\boxtimes_{\mathcal{L}} j)\ :\ (A\boxtimes_{\mathcal{L}} Y)\cup_{A\boxtimes_{\mathcal{L}} X}(B\boxtimes_{\mathcal{L}} X)\ \longrightarrow\ B\boxtimes_{\mathcal{L}} Y$$

is a $\mathcal{C}^L$-cofibration. If moreover i or j is a global equivalence, then so is $i\Box j$.

Proof. The key observation is the following. We let G and K be compact Lie groups and $\mathcal{V}$ respectively $\mathcal{U}$ faithful orthogonal representations of G respectively K of countably infinite dimension. Then the map

$$\mathbf{L}(\mathcal{V},\mathbb{R}^\infty)\boxtimes_{\mathcal{L}}\mathbf{L}(\mathcal{U},\mathbb{R}^\infty)\ \longrightarrow\ \mathbf{L}(\mathcal{V}\oplus\mathcal{U},\mathbb{R}^\infty)\ ,\quad [\varphi;\,\psi,\kappa]\ \longmapsto\ \varphi\circ(\psi\oplus\kappa)$$

is an isomorphism of $\mathcal{L}$-spaces by [9, I Lemma 5.4] (sometimes referred to as 'Hopkins' lemma'). The map is also $(G\times K)$-equivariant, and $\boxtimes_{\mathcal{L}}$ preserves colimits in both variables. So the map descends to an isomorphism of $\mathcal{L}$-spaces

$$\begin{aligned}\mathbf{L}(\mathcal{V},\mathbb{R}^\infty)/G\boxtimes_{\mathcal{L}}\mathbf{L}(\mathcal{U},\mathbb{R}^\infty)/K\ &\longrightarrow\ \mathbf{L}(\mathcal{V}\oplus\mathcal{U},\mathbb{R}^\infty)/(G\times K)\\ [\varphi;\,\psi G,\kappa K]\ &\longmapsto\ (\varphi\circ(\psi\oplus\kappa))(G\times K)\ .\end{aligned}$$

On the other hand, $\mathcal{V}\oplus\mathcal{U}$ is a faithful orthogonal representation for the group $G\times K$. We choose a linear isometry $\mathcal{V}\oplus\mathcal{U}\cong\mathbb{R}^\infty$. Conjugation with this isometry turns the action of $(G\times K)$ on $\mathcal{V}\oplus\mathcal{U}$ into a continuous group monomorphism $G\times K\longrightarrow\mathcal{L}$; the image is thus a compact Lie subgroup $H\subset\mathcal{L}$ isomorphic to $G\times K$, and the operadic product $\mathcal{L}/G\boxtimes_{\mathcal{L}}\mathcal{L}/K$ is isomorphic to $\mathcal{L}/H$.

Since the operadic product preserves colimits in both variables, it suffices to show the pushout product property for $\mathcal{C}^L$-cofibrations in the generating set $I=I_{\mathcal{C}^L}$ for the global model structure, compare [11, Cor. 4.2.5]. This set consists of the morphisms

$$\mathcal{L}/G\times i_k\ :\ \mathcal{L}/G\times\partial D^k\ \longrightarrow\ \mathcal{L}/G\times D^k$$

for all $k \geq 0$, where G runs through all compact Lie subgroups of $\mathcal{L}$. Since $i_k \Box i_m$ is isomorphic to i_{k+m}, the pushout product

$$(\mathcal{L}/G \times i_k) \boxtimes_{\mathcal{L}} (\mathcal{L}/K \times i_m)$$

of two generating cofibrations is isomorphic to $\mathcal{L}/H \times i_{k+m}$ for a compact Lie subgroup H, and hence also a cofibration.

It remains to show that for every pair of $\mathcal{C}^L$-cofibrations $i : A \longrightarrow B$ and $j : X \longrightarrow Y$ such that j is also a global equivalence, the pushout product morphism is again a global equivalence. The morphism $A \boxtimes_{\mathcal{L}} j : A \boxtimes_{\mathcal{L}} X \longrightarrow A \boxtimes_{\mathcal{L}} Y$ is a global equivalence by Theorem 1.21. Since j is a $\mathcal{C}^L$-cofibration, it is also an h-cofibration of $\mathcal{L}$-spaces (by Theorem 1.20), and hence $A \boxtimes_{\mathcal{L}} j$ is again an h-cofibration. The cobase change

$$(B \boxtimes_{\mathcal{L}} X) \longrightarrow (A \boxtimes_{\mathcal{L}} Y) \cup_{A \boxtimes_{\mathcal{L}} X} (B \boxtimes_{\mathcal{L}} X)$$

of $A \boxtimes_{\mathcal{L}} j$ is then a global equivalence by Proposition 1.7 (vi). The composite of this cobase change with the pushout product morphism $i \Box j$ is the morphism $B \boxtimes_{\mathcal{L}} j$, and hence a global equivalence by Theorem 1.21. Hence $i \Box j$ is global equivalence. □

Remark 1.23 (Global model structures for $\star$-modules)**.** Since the unit transformation of the operadic product of $\mathcal{L}$-spaces is not always an isomorphism, certain $\mathcal{L}$-spaces are distinguished. A $\star$*-module* is an $\mathcal{L}$-space X for which the unit morphism

$$X \boxtimes_{\mathcal{L}} * \longrightarrow X , \qquad [\varphi; x, *] \longmapsto (\varphi i) \cdot x$$

is an isomorphism, where $i : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty \oplus \mathbb{R}^\infty$ is the embedding as the first direct summand. The category of $\star$-modules is particularly relevant because on it, the one-point $\mathcal{L}$-space is a unit object for $\boxtimes_{\mathcal{L}}$ (by definition); so when restricted to the full subcategory of $\star$-modules, the operadic product $\boxtimes_{\mathcal{L}}$ is symmetric monoidal. One can show that for every orthogonal space A, the $\mathcal{L}$-space $A(\mathbb{R}^\infty)$ (defined in Construction 3.2 below) is a $\star$-module, so these come in rich supply. On the other hand, $\mathcal{L}$-spaces of the form $\mathcal{L}/G$ for a compact Lie subgroup G of $\mathcal{L}$ are *not* $\star$-modules.

The category of $\star$-modules admits a (non-equivariant) model structure with weak equivalences defined after forgetting the $\mathcal{L}$-action, cf. [1, Thm. 4.16]. In [4], Böhme constructs a monoidal model structure on the category of $\star$-modules that has the global equivalences of ambient $\mathcal{L}$-spaces as its weak equivalences; he also shows that with these global model structures, $\mathcal{L}$-spaces and $\star$-modules are Quillen equivalent, and that the global model structure on $\star$-modules lifts to associative monoids (with respect to $\boxtimes_{\mathcal{L}}$). This effectively provides a global model structure on the category of A_∞-monoids, i.e., algebras over the linear isometries operad (considered as a non-symmetric operad). It remains to be seen to what extent the global model structure lifts to commutative monoids with respect to $\boxtimes_{\mathcal{L}}$ (i.e., to E_∞-monoids).

2. $\mathcal{L}$-SPACES AND ORBISPACES

In this section we give rigorous meaning to the slogan that global homotopy theory of $\mathcal{L}$-spaces is the homotopy theory of 'orbispaces with compact Lie group isotropy'. To this end we formulate a version of Elmendorf's theorem [8] for the homotopy theory of $\mathcal{L}$-spaces, saying that an $\mathcal{L}$-equivariant global homotopy type can be reassembled from fixed point data.

Definition 2.1 (Global orbit category)**.** The *global orbit category* $\mathbf{O}_{\mathrm{gl}}$ is the topological category whose objects are all universal subgroups of the monoid $\mathcal{L}$, and the space of morphisms from K to G is

$$\mathbf{O}_{\mathrm{gl}}(K, G) = \mathrm{map}^{\mathcal{L}}(\mathcal{L}/K, \mathcal{L}/G) ,$$

the space of $\mathcal{L}$-equivariant maps from $\mathcal{L}/K$ to $\mathcal{L}/G$. Composition in $\mathbf{O}_{\mathrm{gl}}$ is composition of morphisms of $\mathcal{L}$-spaces.

Since $\mathcal{L}/K$ represents the functor of taking K-fixed points, the morphism space $\mathbf{O}_{\mathrm{gl}}(K,G)$ is homeomorphic to

$$(\mathcal{L}/G)^K \ = \ (\mathbf{L}(\mathbb{R}^\infty_G, \mathbb{R}^\infty_K)/G)^K \ .$$

Remark 2.2. The global orbit category refines the category Rep of compact Lie groups and conjugacy classes of continuous homomorphisms in the following sense. For all universal subgroups G and K, the components $\pi_0(\mathbf{O}_{\mathrm{gl}}(K,G))$ biject functorially with $\mathrm{Rep}(K,G)$. Indeed, by Proposition A.10 the space $\mathbf{L}(\mathbb{R}^\infty_G, \mathbb{R}^\infty_K)$ is $(K\times G)$-equivariantly homotopy equivalent to $\mathbf{L}(V, \mathbb{R}^\infty_K)$ for a finite-dimensional faithful G-representation V. That latter space is a universal space for the family of graph subgroups, compare [21, Prop. 1.1.26]. So the space $\mathbf{O}_{\mathrm{gl}}(K,G) = (\mathcal{L}/G)^K$ is a disjoint union, indexed by conjugacy classes of continuous group homomorphisms $\alpha : K \longrightarrow G$, of classifying spaces of the centralizer of the image of α, see for example [14, Prop. 5] or [21, Prop. 1.5.12 (i)]. In particular, the path component category $\pi_0(\mathbf{O}_{\mathrm{gl}})$ of the global orbit category is equivalent to the category Rep of compact Lie groups and conjugacy classes of continuous homomorphisms. The preferred bijection

$$\mathrm{Rep}(K,G) \ \longrightarrow \ \pi_0(\mathbf{O}_{\mathrm{gl}}(K,G))$$

sends the conjugacy class of $\alpha : K \longrightarrow G$ to the G-orbit of any K-equivariant linear isometric embedding of the K-universe $\alpha^*(\mathbb{R}^\infty_G)$ into the complete K-universe $\mathbb{R}^\infty_K$.

Definition 2.3. An *orbispace* is a continuous functor $Y : \mathbf{O}_{\mathrm{gl}}^{\mathrm{op}} \longrightarrow \mathbf{T}$ from the opposite of the global orbit category to the category of spaces. We denote the category of orbispaces and natural transformations by *orbispc*.

For every small topological category J with discrete object set the category $J\mathbf{T}$ of continuous functors from J to spaces has a 'projective' model structure [19, Thm. 5.4] in which the weak equivalence and fibrations are those natural transformations that are weak equivalences respectively Serre fibrations at every object. In the special case $J = \mathbf{O}_{\mathrm{gl}}^{\mathrm{op}}$, this provides a projective (or objectwise) model structure on the category of orbispaces.

Construction 2.4. We introduce a *fixed point functor*

$$\Phi \ : \ \mathcal{L}\mathbf{T} \ \longrightarrow \ \mathit{orbispc}$$

from the category of $\mathcal{L}$-spaces to the category of orbispaces that will turn out to be a right Quillen equivalence with respect to the global model structure on the left hand side. Given an $\mathcal{L}$-space Y we define the value of the orbispace $\Phi(Y)$ at a universal subgroup K as

$$\Phi(Y)(K) \ = \ \mathrm{map}^{\mathcal{L}}(\mathcal{L}/K, Y) \ ,$$

the space of $\mathcal{L}$-equivariant maps. The $\mathbf{O}_{\mathrm{gl}}$-functoriality is by composition of morphisms of $\mathcal{L}$-spaces. In other words, $\Phi(Y)$ is the contravariant hom-functor represented by Y, restricted to the global orbit category. In particular,

$$\Phi(\mathcal{L}/G) \ = \ \mathbf{O}_{\mathrm{gl}}(-,G) \ ,$$

i.e., the fixed points of the orbit $\mathcal{L}$-space $\mathcal{L}/G$ form the orbispace represented by G. Since $\mathcal{L}/K$ represents the functor of taking K-fixed points, the map

$$\Phi(Y)(K) \ \longrightarrow \ Y^K \ , \quad f \ \longmapsto \ f(K)$$

is a homeomorphism.

The fixed point functor Φ has a left adjoint

$$\Lambda \ : \ orbispc \ \longrightarrow \ \mathcal{L}\mathbf{T} \ ,$$

with value at an orbispace X given by the coend

$$\Lambda(X) \ = \ \int^{G\in\mathbf{O}_{\mathrm{gl}}} \mathcal{L}/G \times X(G) \ ,$$

i.e., a coequalizer, in the category of $\mathcal{L}$-spaces, of the two morphisms

$$\textstyle\coprod_{K,G} \mathcal{L}/K \times \mathbf{O}_{\mathrm{gl}}(K,G) \times X(G) \ \rightrightarrows \ \coprod_G \mathcal{L}/G \times X(G) \ .$$

All we will need to know about the left adjoint is that for every universal subgroup G of $\mathcal{L}$, it takes the representable orbispace $\mathbf{O}_{\mathrm{gl}}(-,G) = \Phi(\mathcal{L}/G)$ to $\mathcal{L}/G$. Indeed, the counit $\epsilon_{\mathcal{L}/G}$: $\Lambda(\Phi(\mathcal{L}/G)) \longrightarrow \mathcal{L}/G$ induces a bijection of morphism sets

$$\begin{aligned} \mathcal{L}\mathbf{T}(\Lambda(\Phi(\mathcal{L}/G)),X) \ &\cong \ orbispc(\Phi(\mathcal{L}/G),\Phi(X)) \\ &= \ orbispc(\mathbf{O}_{\mathrm{gl}}(-,G),\Phi(X)) \ \cong \ \Phi(X)(G) \ = \ \mathcal{L}\mathbf{T}(\mathcal{L}/G,X) \ . \end{aligned}$$

So the counit $\epsilon_{\mathcal{L}/G} : \Lambda(\Phi(\mathcal{L}/G)) \longrightarrow \mathcal{L}/G$ is an isomorphism of $\mathcal{L}$-spaces.

Theorem 2.5. *The adjoint functor pair*

$$\Lambda \ : \ orbispc \ \rightleftarrows \ \mathcal{L}\mathbf{T} \ : \ \Phi$$

is a Quillen equivalence between the category of $\mathcal{L}$-spaces with the global model structure and the category of orbispaces with the projective model structure. Moreover, for every cofibrant orbispace X the adjunction unit $X \longrightarrow \Phi(\Lambda X)$ is an isomorphism.

Proof. Every fibration in the global model structure of $\mathcal{L}$-spaces in particular restricts to a Serre fibration on fixed points of every universal subgroup, so the right adjoint sends fibrations in the global model structure of $\mathcal{L}$-spaces to fibrations in the projective model structure of orbispaces. The right adjoint also takes global equivalences of $\mathcal{L}$-spaces to objectwise weak equivalences of orbispaces, by the very definition of 'global equivalences'. So Φ is a right Quillen functor.

We now show that for every cofibrant orbispace X the adjunction unit $X \longrightarrow \Phi(\Lambda X)$ is an isomorphism. We let $\mathcal{G}$ denote the class of orbispaces for which the adjunction unit is an isomorphism. We show the following property: For every index set I, every I-indexed family H_i of universal subgroups of $\mathcal{L}$, all numbers $n_i \geq 0$ and every pushout square of orbispaces

$$\begin{tikzcd} \coprod_{i\in I} \mathbf{O}_{\mathrm{gl}}(-,H_i) \times \partial D^{n_i} \arrow[r] \arrow[d] & \coprod_{i\in I} \mathbf{O}_{\mathrm{gl}}(-,H_i) \times D^{n_i} \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{2.6}$$

such that X belongs to $\mathcal{G}$, the orbispace Y also belongs to $\mathcal{G}$.

As a left adjoint, Λ preserves pushout and coproducts. For every space A the functor $- \times A$ is a left adjoint, so it commutes with colimits and coends. So Λ also commutes with products with spaces. Thus Λ takes the original square to a pushout square of $\mathcal{L}$-spaces:

$$\begin{tikzcd} \coprod_{i\in I} \mathcal{L}/H_i \times \partial D^{n_i} \arrow[r] \arrow[d] & \coprod_{i\in I} \mathcal{L}/H_i \times D^{n_i} \arrow[d] \\ \Lambda X \arrow[r] & \Lambda Y \end{tikzcd}$$

The upper horizontal morphism in this square is a closed embedding. For every universal subgroup G of $\mathcal{L}$ the G-fixed point functor commutes with disjoint unions, products with spaces and pushouts along closed embeddings. So the square

$$\begin{tikzcd} \coprod_{i\in I}(\mathcal{L}/H_i)^G \times \partial D^{n_i} \arrow[r] \arrow[d] & \coprod_{i\in I}(\mathcal{L}/H_i)^G \times D^{n_i} \arrow[d] \\ (\Lambda X)^G \arrow[r] & (\Lambda Y)^G \end{tikzcd}$$

is a pushout in the category of compactly generated spaces. Colimits and products of orbispaces with spaces are formed objectwise, so letting G run through all universal subgroups shows that the square

$$\begin{tikzcd} \coprod_{i\in I}\Phi(\mathcal{L}/H_i) \times \partial D^{n_i} \arrow[r] \arrow[d] & \coprod_{i\in I}\Phi(\mathcal{L}/H_i) \times D^{n_i} \arrow[d] \\ \Phi(\Lambda X) \arrow[r] & \Phi(\Lambda Y) \end{tikzcd}$$

is a pushout in the category of orbispaces. The adjunction units induce compatible maps from the original pushout square (2.6) to this last square. Since $\Phi(\mathcal{L}/H_i) = \mathbf{O}_{\mathrm{gl}}(-, H_i)$ and the unit $\eta_X : X \longrightarrow \Phi(\Lambda X)$ is an isomorphism, the unit $\eta_Y : Y \longrightarrow \Phi(\Lambda Y)$ is also an isomorphism.

The right adjoint Φ preserves and detects all weak equivalences, and for every cofibrant orbispace X, the adjunction unit $X \longrightarrow \Phi(\Lambda X)$ is an isomorphism, hence in particular a weak equivalence of orbispaces. So the pair (Λ, Φ) is a Quillen equivalence, for example by [11, Cor. 1.3.6]. $\square$

Remark 2.7 (Universal projective versus global model structure)**.** The Quillen equivalence of Theorem 2.5 can be factored as a composite of two composable Quillen equivalences as follows:

$$\mathit{orbispc} \underset{\Phi}{\overset{\Lambda}{\rightleftarrows}} (\mathcal{L}\mathbf{T})_{\text{u-proj}} \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} (\mathcal{L}\mathbf{T})_{\mathrm{gl}}$$

The middle model category is the category of $\mathcal{L}$-spaces, equipped with the *universal projective* model structure, i.e., the projective model structure for the collection of universal subgroups. This intermediate model structure has the same weak equivalences as the global model structure (namely the global equivalences), but it has fewer cofibrations and more fibrations. The identity functor is thus a left Quillen equivalence from the universal projective model structure to the global model structure on the category of $\mathcal{L}$-spaces.

Our reasons for emphasizing the global model structure (as opposed to the universal projective model structure) for $\mathcal{L}$-spaces are twofold. On the one hand, the global model structure can be more easily compared to the global homotopy theory of orthogonal spaces, as a certain adjoint functor already used by Lind in [16] is a Quillen pair for the global model structure on $\mathcal{L}$-spaces (but *not* for the universal projective model structure), compare Theorem 3.9 below. Another reason is that the global model structure of $\mathcal{L}$-spaces is monoidal with respect to the operadic $\boxtimes_{\mathcal{L}}$-product, in the sense of Proposition 1.22 above. The universal projective model structure, in contrast, does *not* satisfy the pushout product property. Indeed, for every universal subgroup G of $\mathcal{L}$, the $\mathcal{L}$-space $\mathcal{L}/G$ is cofibrant in the universal projective model structure. If K is another universal subgroup of $\mathcal{L}$, then we showed in the proof of Proposition 1.22 that the operadic product

$$\mathcal{L}/G \boxtimes_{\mathcal{L}} \mathcal{L}/K$$

is isomorphic to $\mathcal{L}/H$ where H is a compact Lie subgroup of $\mathcal{L}$ isomorphic to $G\times K$. However, under the isomorphism $H \cong G \times K$, the tautological action of H on $\mathbb{R}^\infty$ becomes the direct sum $\mathcal{U}_G \oplus \mathcal{U}_K$ of a complete G-universe and a complete K-universe. While $\mathcal{U}_G \oplus \mathcal{U}_K$ is a universe for $G \times K$, it is typically *not* complete. So the operadic product $\mathcal{L}/G \boxtimes_{\mathcal{L}} \mathcal{L}/K$ is cofibrant in the global model structure, but typically *not* in the universal projective model structure.

3. $\mathcal{L}$-spaces and orthogonal spaces

The aim of this section is to compare the global homotopy theory of $\mathcal{L}$-spaces to the global homotopy theory of orthogonal spaces as developed by the author in [21]: we will show in Theorem 3.9 that the global model structure on the category of $\mathcal{L}$-spaces is Quillen equivalent to the positive global model structure on the category of orthogonal spaces, established in [21, Prop. 1.2.23].

We denote by $\mathbf{L}$ the category with objects the finite-dimensional inner product spaces and morphisms the linear isometric embeddings. If V and W are two finite-dimensional inner product spaces, then the function space topology on $\mathbf{L}(V,W)$ agrees with the topology as the Stiefel manifold of $\dim(V)$-frames in W, by Proposition A.5 (i). Moreover, composition of linear isometric embeddings is continuous, so $\mathbf{L}$ is then a topological category. Every inner product space V is isometrically isomorphic to $\mathbb{R}^n$ with the standard scalar product, for n the dimension of V, so the category $\mathbf{L}$ has a small skeleton.

Definition 3.1. An *orthogonal space* is a continuous functor $Y : \mathbf{L} \longrightarrow \mathbf{T}$ to the category of spaces. A morphism of orthogonal spaces is a natural transformation. We denote by *spc* the category of orthogonal spaces.

The category $\mathbf{L}$ (or its extension that also contains countably infinite dimensional inner product spaces) is denoted $\mathscr{I}$ by Boardman and Vogt [2], and this notation is also used in [17]; other sources [16] use the symbol $\mathcal{I}$. Accordingly, orthogonal spaces are sometimes referred to as $\mathscr{I}$-functors, $\mathscr{I}$-spaces or $\mathcal{I}$-spaces.

Construction 3.2 (Evaluation at $\mathbb{R}^\infty$)**.** We let Y be an orthogonal space. We extend the action maps

$$\mathbf{L}(V,W) \times Y(V) \longrightarrow Y(W) \tag{3.3}$$

which are part of the structure of an orthogonal space to the situation where V and W are allowed to be of countably infinite dimension. If $\mathcal{W}$ is an inner product spaces of countably infinite dimension, then we let $s(\mathcal{W})$ denote the poset of finite-dimensional subspaces of $\mathcal{W}$, ordered under inclusion. Then we define

$$Y(\mathcal{W}) \ = \ \operatorname{colim}_{W\in s(\mathcal{W})} Y(W) \ ,$$

the colimit in the category $\mathbf{T}$ of compactly generated spaces. If V is a finite-dimensional inner product space, we define the action map

$$\mathbf{L}(V,\mathcal{W}) \times Y(V) \ \longrightarrow \ Y(\mathcal{W})$$

from the action maps (3.3) of the functor Y by passing to colimits over the poset $s(\mathcal{W})$; this is legitimate because $- \times Y(V)$ preserves colimits and $\mathbf{L}(V,\mathcal{W})$ is the colimit of the spaces $\mathbf{L}(V,W)$ over the poset $s(\mathcal{W})$, by Proposition A.5 (ii).

If $\mathcal{V}$ is also of countably infinite dimension, then $\mathbf{L}(\mathcal{V},\mathcal{W})\times Y(\mathcal{V})$ is the colimit of $\mathbf{L}(\mathcal{V},\mathcal{W})\times Y(V)$ for $V \in s(\mathcal{V})$ because $\mathbf{L}(\mathcal{V},\mathcal{W}) \times -$ preserves colimits. So the compatible maps

$$\mathbf{L}(\mathcal{V},\mathcal{W}) \times Y(V) \ \xrightarrow{\rho^{\mathcal{V}}_V \times \mathrm{Id}} \ \mathbf{L}(V,\mathcal{W}) \times Y(V) \ \xrightarrow{\mathrm{act}} \ Y(\mathcal{W})$$

assemble into a continuous action map.

Every orthogonal space Y gives rise to an $\mathcal{L}$-space by evaluation at $\mathbb{R}^\infty$. Indeed, for $\mathcal{V} = \mathcal{W} = \mathbb{R}^\infty$, the above construction precisely says that the action maps make $Y(\mathbb{R}^\infty)$ into an $\mathcal{L}$-space. If V is a finite-dimensional inner product space, then Proposition A.5 (ii) shows that $\mathbf{L}(V,\mathbb{R}^\infty)$ carries the weak topology with respect to the closed subspaces $\mathbf{L}(V,W)$ for $W \in s(\mathbb{R}^\infty)$. So

$$\mathbf{L}(V,-)(\mathbb{R}^\infty) \ = \ \mathbf{L}(V,\mathbb{R}^\infty) \ .$$

If $\mathcal{V}$ is an inner product space of countably infinite dimension, then the space $\mathbf{L}(\mathcal{V},\mathbb{R}^\infty)$ becomes an $\mathcal{L}$-space by postcomposition, but it does *not* arise from an orthogonal space by evaluation at $\mathbb{R}^\infty$.

We recall from [21, Def. 1.1.2] the notion of *global equivalence* of orthogonal spaces. We let G be a compact Lie group. For every orthogonal space Y and every finite-dimensional orthogonal G-representation V, the value $Y(V)$ inherits a G-action from the G-action on V and the functoriality of Y. For a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ the induced map $Y(\varphi) : Y(V) \longrightarrow Y(W)$ is G-equivariant.

Definition 3.4. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is a *global equivalence* if the following condition holds: for every compact Lie group G, every G-representation V, every $k \geq 0$ and all continuous maps $\alpha : \partial D^k \longrightarrow X(V)^G$ and $\beta : D^k \longrightarrow Y(V)^G$ such that $\beta|_{\partial D^k} = f(V)^G \circ \alpha$, there is a G-representation W, a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous map $\lambda : D^k \longrightarrow X(W)^G$ such that $\lambda|_{\partial D^k} = X(\varphi)^G \circ \alpha$ and such that $f(W)^G \circ \lambda$ is homotopic, relative to ∂D^k, to $Y(\varphi)^G \circ \beta$.

In other words, for every commutative square on the left

$$\begin{tikzcd} \partial D^k \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & X(V)^G \arrow[d, "f(V)^G"] \\ D^k \arrow[r, "\beta"'] & Y(V)^G \end{tikzcd} \qquad \begin{tikzcd} \partial D^k \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & X(V)^G \arrow[r, "X(\varphi)^G"] & X(W)^G \arrow[d, "f(W)^G"] \\ D^k \arrow[r, "\beta"'] \arrow[urr, dashed, "\lambda"] & Y(V)^G \arrow[r, "Y(\varphi)^G"'] & Y(W)^G \end{tikzcd}$$

there exists the lift λ on the right hand side that makes the upper left triangle commute on the nose, and the lower right triangle up to homotopy relative to ∂D^k.

An orthogonal space is *closed* if all its structure maps are closed embeddings. The following is a reformulation of [21, Prop. 1.1.7].

Proposition 3.5. *A morphism $f : X \longrightarrow Y$ between closed orthogonal spaces is a global equivalence if and only if $f(\mathbb{R}^\infty) : X(\mathbb{R}^\infty) \longrightarrow Y(\mathbb{R}^\infty)$ is a global equivalence of $\mathcal{L}$-spaces.*

Proposition 1.2.23 of [21] establishes the positive global model structure on the category of orthogonal spaces in which the weak equivalences are the global equivalences. A morphism f is a 'positive global fibration' (i.e., a fibration in the positive global model structure) if and only if for every compact Lie group G, every faithful G-representation V with $V \neq 0$ and every equivariant linear isometric embedding $\varphi : V \longrightarrow W$ the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a Serre fibration and the square of G-fixed point spaces

$$\begin{tikzcd} X(V)^G \arrow[r, "X(\varphi)^G"] \arrow[d, "f(V)^G"'] & X(W)^G \arrow[d, "f(W)^G"] \\ Y(V)^G \arrow[r, "Y(\varphi)^G"'] & Y(W)^G \end{tikzcd}$$

is homotopy cartesian. This positive global model structure is proper, topological, compactly generated and monoidal with respect to the convolution box product of orthogonal spaces.

To compare the global model structures of $\mathcal{L}$-spaces and orthogonal spaces we use the adjoint functor pair

$$Q \otimes_{\mathbf{L}} - \ : \ spc \ \rightleftarrows \ \mathcal{L}\mathbf{T} \ : \ \mathrm{map}^{\mathcal{L}}(Q, -)$$

introduced by Lind in [16, Sec. 8]; Lind denotes the functor $Q \otimes_{\mathbf{L}} -$ by $\mathbb{Q}$. The adjoint pair arises from a continuous functor

$$Q \ : \ \mathbf{L}^{\mathrm{op}} \ \longrightarrow \ \mathcal{L}\mathbf{T} \ , \quad V \ \longmapsto \ \mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \ .$$

Here $\mathcal{L}$ acts on $\mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty)$ by postcomposition. A linear isometric embedding $\varphi : V \longrightarrow W$ induces the morphism of $\mathcal{L}$-spaces

$$Q(\varphi) \ = \ \mathbf{L}(\varphi \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \ : \ \mathbf{L}(W \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \ \longrightarrow \ \mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \ , \quad \psi \ \longmapsto \ \psi \circ (\varphi \otimes \mathbb{R}^\infty) \ .$$

Since orthogonal spaces are defined as the continuous functor from $\mathbf{L}$, and since the category of $\mathcal{L}$-spaces is tensored and cotensored over spaces, any continuous functor from $\mathbf{L}^{\mathrm{op}}$ induces an adjoint functor pair by an enriched end-coend construction. Indeed, the value of the left adjoint on an orthogonal space Y is given by

$$Q \otimes_{\mathbf{L}} Y \ = \ \int^{V \in \mathbf{L}} Y(V) \times \mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \ ,$$

the enriched coend of the continuous functor

$$\mathbf{L} \times \mathbf{L}^{\mathrm{op}} \ \longrightarrow \ \mathcal{L}\mathbf{T} \ , \quad (V, W) \ \longmapsto \ Y(V) \times \mathbf{L}(W \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \ .$$

The functor $Q \otimes_{\mathbf{L}} -$ has a right adjoint $\mathrm{map}^{\mathcal{L}}(Q, -)$ whose value at an $\mathcal{L}$-space Z is given by

$$\mathrm{map}(Q, Z)(V) \ = \ \mathrm{map}^{\mathcal{L}}(Q(V), Z) \ ,$$

the mapping space of $\mathcal{L}$-equivariant maps from $Q(V)$ to Z. The covariant functoriality in V comes from the contravariant functoriality of Q. The coend of a contravariant functor with a representable covariant functor returns the value at the representing object, i.e.,

$$Q \otimes_{\mathbf{L}} \mathbf{L}(V, -) \ = \ Q(V) \ = \ \mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \ .$$

So the value on the semifree orthogonal space $\mathbf{L}_{G,V} = \mathbf{L}(V, -)/G$ generated by a G-representation V comes out as

$$Q \otimes_{\mathbf{L}} \mathbf{L}_{G,V} \ \cong \ (Q \otimes_{\mathbf{L}} \mathbf{L}(V, -))/G \ \cong \ Q(V)/G \ = \ \mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty)/G \ .$$

By [16, Lemma 8.3], the functor $Q \otimes_{\mathbf{L}} -$ from orthogonal spaces to $\mathcal{L}$-spaces is strong symmetric monoidal for the box product of orthogonal spaces (compare [21, Sec. 1.3]) and the operadic product $\boxtimes_{\mathcal{L}}$ of $\mathcal{L}$-spaces.

Proposition 3.6. *Let Y be an injective $\mathcal{L}$-space, G a compact Lie subgroup of $\mathcal{L}$ and V a non-zero finite-dimensional faithful G-representation. Then for every G-equivariant linear isometric embedding $\kappa : V \otimes \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty_G$ the G-equivariant evaluation map*

$$\mathrm{map}^{\mathcal{L}}(Q, Y)(V) \ = \ \mathrm{map}^{\mathcal{L}}(\mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty), Y) \ \longrightarrow \ Y \ , \quad h \mapsto h(\kappa)$$

is a G-weak equivalence.

Proof. We show that the evaluation map restricts to a weak equivalence on G-fixed points. Since the hypothesis are stable under passage to closed subgroups of G, applying this to closed subgroups provides the claim.

We choose a linear isometry $\psi : V \otimes \mathbb{R}^\infty \cong \mathbb{R}^\infty$ and transport the G-action on $V \otimes \mathbb{R}^\infty$ to a G-action on $\mathbb{R}^\infty$ by conjugating with ψ. Since G is compact and V is faithful, the conjugation homomorphism

$$c_\psi \ : \ G \longrightarrow \mathcal{L}$$

is then a closed embedding, and hence an isomorphism onto its image $\bar{G} = c_\psi(G)$. In particular, $\bar{G}$ is a compact Lie subgroup of $\mathcal{L}$. Moreover,

$$(\kappa\psi^{-1}) \circ c_\psi(g) \ = \ (\kappa\psi^{-1}) \circ (\psi g \psi^{-1}) \ = \ \kappa g \psi^{-1} \ = \ g(\kappa\psi^{-1})$$

for all $g \in G$, so $\kappa\psi^{-1} : G \rightsquigarrow \bar{G}$ is a correspondence. Evaluation at κ factors as the composite

$$\left(\mathrm{map}^{\mathcal{L}}(Q, Y)(V)\right)^G \ \underset{\cong}{\xrightarrow{h \mapsto h(\psi)}} \ Y^{\bar{G}} \ \xrightarrow{(\kappa\psi^{-1})\cdot -} \ Y^G \ ;$$

the first map is a homeomorphism, and the second map is a weak equivalence because Y is injective. So evaluation at κ is a weak equivalence. □

The functor $Q \otimes_{\mathbf{L}} -$ is closely related to evaluation of an orthogonal space at $\mathbb{R}^\infty$. Indeed, a choice of unit vector $u \in \mathbb{R}^\infty$ gives rise to a natural linear isometric embedding $- \otimes u : V \longrightarrow V \otimes \mathbb{R}^\infty$. As V ranges over all finite-dimensional inner product spaces, the composite maps

$$Y(V) \times \mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \ \xrightarrow{Y(V) \times \mathbf{L}(-\otimes u, \mathbb{R}^\infty)} \ Y(V) \times \mathbf{L}(V, \mathbb{R}^\infty) \ \xrightarrow{\text{act}} \ Y(\mathbb{R}^\infty)$$

are compatible with the coend relations, and assembly into a natural map

$$\xi_Y \ : \ Q \otimes_{\mathbf{L}} Y \ \longrightarrow \ Y(\mathbb{R}^\infty) \ .$$

Lind shows in [16, Lemma 9.7] that for every flat orthogonal space (i.e., cofibrant $\mathcal{I}$-space in his terminology) the map $\xi_Y : Q \otimes_{\mathbf{L}} Y \longrightarrow Y(\mathbb{R}^\infty)$ is a weak equivalence. We generalize this as follows:

Proposition 3.7. *For every flat orthogonal space Y the map $\xi_Y : Q \otimes_{\mathbf{L}} Y \longrightarrow Y(\mathbb{R}^\infty)$ is a global equivalence of $\mathcal{L}$-spaces.*

Proof. We start with the special case where Y is one of the generating objects for the flat cofibrations. In other words, we let $Y = \mathbf{L}_{G,V} = \mathbf{L}(V, -)/G$ be the semifree orthogonal space for a compact Lie group G and a faithful G-representation V. In this case, the $\mathcal{L}$-space $Q \otimes_{\mathbf{L}} \mathbf{L}_{G,V}$ is isomorphic to $\mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty)/G$ and $\mathbf{L}_{G,V}(\mathbb{R}^\infty)$ is isomorphic to $\mathbf{L}(V, \mathbb{R}^\infty)/G$. Under these identifications the morphism $\xi_{\mathbf{L}_{G,V}}$ becomes the restriction morphism

$$\mathbf{L}(- \otimes u, \mathbb{R}^\infty)/G \ : \ \mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty)/G \ \longrightarrow \ \mathbf{L}(V, \mathbb{R}^\infty)/G$$

induced by the G-equivariant linear isometric embedding $- \otimes u : V \longrightarrow V \otimes \mathbb{R}^\infty$. If $V = 0$, then this morphism is an isomorphism of $\mathcal{L}$-spaces. If $V \neq 0$, then for every universal subgroup K of $\mathcal{L}$, the restriction map

$$\mathbf{L}(- \otimes u, \mathbb{R}^\infty_K) \ : \ \mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty_K) \ \longrightarrow \ \mathbf{L}(V, \mathbb{R}^\infty_K)$$

is a $(K \times G)$-homotopy equivalence by Proposition A.10. So the map descends to a K-homotopy equivalence on G-orbit spaces, and $\mathbf{L}(- \otimes u, \mathbb{R}^\infty)/G$ is a global equivalence of $\mathcal{L}$-spaces. This proves the special case $Y = \mathbf{L}_{G,V}$.

Now we suppose that the orthogonal space Y arises as the colimit of a sequence

$$\emptyset = Y_0 \ \longrightarrow \ Y_1 \ \longrightarrow \dots \ \longrightarrow \ Y_k \ \longrightarrow \ \dots \tag{3.8}$$

in which each Y_k is obtained from Y_{k-1} as a pushout of orthogonal spaces

$$\begin{tikzcd} \coprod_I \mathbf{L}_{G_i,V_i} \times \partial D^{k_i} \arrow[r, "\mathrm{incl}"] \arrow[d] & \coprod_I \mathbf{L}_{G_i,V_i} \times D^{k_i} \arrow[d] \\ Y_{n-1} \arrow[r] & Y_n \end{tikzcd}$$

for some indexing set I, compact Lie groups G_i, faithful G_i-representations V_i, and numbers $k_i \geq 0$. We show by induction that the morphisms $\xi_{Y_n} : Q \otimes_{\mathbf{L}} Y_n \longrightarrow Y_n(\mathbb{R}^\infty)$ are global equivalences. The induction starts with $n = 0$, where there is nothing to show.

For the inductive step we assume that the morphism $\xi_{Y_{n-1}}$ is a global equivalence, and we show that then ξ_{Y_n} is a global equivalence as well. The functors $Q \otimes_{\mathbf{L}} -$ and evaluation at $\mathbb{R}^\infty$ preserve colimits. The morphism ξ_{Y_n} is thus induced by the commutative diagram

$$\begin{tikzcd} Q \otimes_{\mathbf{L}} Y_{n-1} \arrow[d, "\xi_{Y_{n-1}}"'] & \coprod_I Q \otimes_{\mathbf{L}} \mathbf{L}_{G_i,V_i} \times \partial D^{k_i} \arrow[l] \arrow[r, "\mathrm{incl}"] \arrow[d, "\coprod \xi_{\mathbf{L}_{G_i,V_i}} \times \partial D^{k_i}"] & \coprod_I Q \otimes_{\mathbf{L}} \mathbf{L}_{G_i,V_i} \times D^{k_i} \arrow[d, "\coprod \xi_{\mathbf{L}_{G_i,V_i}} \times D^{k_i}"] \\ Y_{n-1}(\mathbb{R}^\infty) & \coprod_I \mathbf{L}_{G_i,V_i}(\mathbb{R}^\infty) \times \partial D^{k_i} \arrow[l] \arrow[r, "\mathrm{incl}"'] & \coprod_I \mathbf{L}_{G_i,V_i}(\mathbb{R}^\infty) \times D^{k_i} \end{tikzcd}$$

by passage to pushouts in the horizontal direction. The left vertical map is a global equivalence by hypothesis. The middle and right vertical maps are global equivalences by the special case above and parts (i) and (ii) of Proposition 1.7. The inclusion of $\coprod_I \mathbf{L}_{G_i,V_i} \times \partial D^{k_i}$ into $\coprod_I \mathbf{L}_{G_i,V_i} \times D^{k_i}$ is an h-cofibration of orthogonal spaces. Since both $Q \otimes_{\mathbf{L}} -$ and evaluation at $\mathbb{R}^\infty$ preserve h-cofibrations, the two right horizontal morphisms in the diagram are h-cofibrations of $\mathcal{L}$-spaces. So the induced map on pushouts ξ_{Y_n} is a global equivalence of $\mathcal{L}$-spaces by Proposition 1.7 (v).

Since Y is a colimit of the sequence (3.8), $Q \otimes_{\mathbf{L}} Y$ is a colimit of the sequence $Q \otimes_{\mathbf{L}} Y_n$, and $Y(\mathbb{R}^\infty)$ is a colimit of the sequence $Y_n(\mathbb{R}^\infty)$. Moreover, since each morphism $Y_{n-1} \longrightarrow Y_n$ is an h-cofibration of orthogonal spaces, the morphisms $Q \otimes_{\mathbf{L}} Y_{n-1} \longrightarrow Q \otimes_{\mathbf{L}} Y_n$ and $Y_{n-1}(\mathbb{R}^\infty) \longrightarrow Y_n(\mathbb{R}^\infty)$ are h-cofibrations of $\mathcal{L}$-spaces, and hence closed embedding. Since global equivalences are homotopical for sequential colimits along closed embeddings (by Proposition 1.7 (iii)), we conclude that the morphism ξ_Y is a global equivalence.

Every flat orthogonal space is a retract of a flat orthogonal space of the form considered in the previous paragraph. Since global equivalences are closed under retracts, the morphism ξ_Y is a global equivalence for every flat orthogonal space. $\square$

Theorem 3.9. *The adjoint functor pair*

$$Q \otimes_{\mathbf{L}} - \ : \ spc \ \rightleftarrows \ \mathcal{L}\mathbf{T} \ : \ \mathrm{map}^{\mathcal{L}}(Q, -)$$

is a Quillen equivalence with respect to the positive global model structure on orthogonal spaces and the global model structure on $\mathcal{L}$-spaces.

Proof. This theorem is a global sharpening of Lind's non-equivariant Quillen-equivalence [16, Thm. 9.9], and some of our arguments are 'globalizations' of Lind's. We let G be a compact Lie group and V a non-trivial faithful G-representation. Then $V \otimes \mathbb{R}^\infty$ is a faithful G-representation of countably infinite dimension. We choose a linear isometry $\psi : V \otimes \mathbb{R}^\infty \cong \mathbb{R}^\infty$ and transport the G-action on $V \otimes \mathbb{R}^\infty$ to a G-action on $\mathbb{R}^\infty$ by conjugating with ψ. Since G is compact, the conjugation homomorphism

$$c_\psi \ : \ G \ \longrightarrow \ \mathcal{L}$$

is then a closed embedding, and hence an isomorphism onto its image $\bar{G} = c_\psi(G)$, which is then a compact Lie subgroup of $\mathcal{L}$. Moreover, $\psi : V \otimes \mathbb{R}^\infty \cong c_\psi^*(\mathbb{R}^\infty_{\bar{G}})$ is G-equivariant, by construction. Evaluation at ψ is then a homeomorphism, natural in Y,

$$\left(\mathrm{map}^{\mathcal{L}}(Q,Y)(V)\right)^G \ = \ (\mathrm{map}^{\mathcal{L}}(\mathbf{L}(V\otimes\mathbb{R}^\infty,\mathbb{R}^\infty),Y))^G \ \longrightarrow \ Y^{\bar{G}} \ , \quad h \ \longmapsto \ h(\psi) \ .$$

Now we let $f : X \longrightarrow Y$ be a morphism of $\mathcal{L}$-spaces that is a fibration (respectively acyclic fibration) in the $\mathcal{C}^L$-projective model structure of Proposition 1.11. Then by the above, the morphism of orthogonal spaces $(\mathrm{map}^{\mathcal{L}}(Q,f)(V))^G$ is isomorphic to the map

$$f^{\bar{G}} \ : \ X^{\bar{G}} \ \longrightarrow \ Y^{\bar{G}} \ ,$$

which is a a Serre fibration (respectively a Serre fibration and weak equivalence). So $\mathrm{map}^{\mathcal{L}}(Q,f)$ is a fibration (respectively acyclic fibration) in the positive strong level model structure of orthogonal spaces, compare the proof of [21, Prop. 1.2.23]. Since the acyclic fibrations agree in the global and $\mathcal{C}^L$-projective model structure of $\mathcal{L}$-spaces, and they agree in the positive global and strong level model structures of $\mathcal{L}$-spaces, this shows in particular that the right adjoint preserves acyclic fibrations.

Now we suppose that $f : X \longrightarrow Y$ is a global fibration between injective $\mathcal{L}$-spaces. Then f is in particular a $\mathcal{C}^L$-fibration, and hence $\mathrm{map}^{\mathcal{L}}(Q,f)$ is a positive strong level fibration by the previous paragraph. We let G be a universal subgroup of $\mathcal{L}$ and $\varphi : V \longrightarrow W$ a G-equivariant linear isometric embedding of G-representations, where V is non-zero and the action on V (and hence also on W) is faithful. We choose a G-equivariant linear isometric embedding $\psi : W \otimes \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty_G$; the composite $\kappa = \psi \circ (\varphi \otimes \mathbb{R}^\infty) : V \otimes \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty_G$ is then also a G-equivariant linear isometric embedding. In the commutative diagram

$$\begin{tikzcd}
(\mathrm{map}^{\mathcal{L}}(Q,X)(V))^G \arrow[r, "(\mathrm{map}^{\mathcal{L}}(Q,X)(\varphi))^G"'] \arrow[d, "(\mathrm{map}^{\mathcal{L}}(Q,f)(V))^G"'] \arrow[rr, bend left, "h\mapsto h(\kappa)"] & (\mathrm{map}^{\mathcal{L}}(Q,X)(W))^G \arrow[r, "h\mapsto h(\psi)"'] \arrow[d, "(\mathrm{map}^{\mathcal{L}}(Q,f)(W))^G"] & X^G \arrow[d, "f^G"] \\
(\mathrm{map}^{\mathcal{L}}(Q,Y)(V))^G \arrow[r, "(\mathrm{map}^{\mathcal{L}}(Q,Y)(\varphi))^G"] \arrow[rr, bend right, "h\mapsto h(\kappa)"'] & (\mathrm{map}^{\mathcal{L}}(Q,Y)(W))^G \arrow[r, "h\mapsto h(\psi)"] & Y^G
\end{tikzcd}$$

the right and composite horizontal maps are weak equivalences by Proposition 3.6. So the left horizontal maps $(\mathrm{map}^{\mathcal{L}}(Q,X)(\varphi))^G$ and $(\mathrm{map}^{\mathcal{L}}(Q,Y)(\varphi))^G$ are weak equivalences, and we conclude that the left square is homotopy cartesian. Since every compact Lie group is isomorphic to a universal subgroup of $\mathcal{L}$, this proves that the morphism $\mathrm{map}^{\mathcal{L}}(Q,f)$ is a fibration in the positive global model structure of orthogonal spaces. Now we know that the right adjoint $\mathrm{map}^{\mathcal{L}}(Q,-)$ preserves acyclic fibrations, and it takes fibrations between fibrant objects in the global model structure of $\mathcal{L}$-spaces to fibrations in the positive global model structure of orthogonal spaces. By a criterion of Dugger [6, Cor. A.2], this proves that the adjoint functor pair $(Q \otimes_{\mathbf{L}} -, \mathrm{map}^{\mathcal{L}}(Q,-))$ is a Quillen pair with respect to the two global model structures.

It remains to show that the Quillen pair is a Quillen equivalence. This is a consequence of the following two facts:

(a) The right adjoint $\mathrm{map}^{\mathcal{L}}(Q,-)$ reflects global equivalences between injective $\mathcal{L}$-spaces, and
(b) for every positive flat orthogonal space A, there is a global equivalence of $\mathcal{L}$-spaces $q : Q \otimes_{\mathbf{L}} A \longrightarrow X$ to an injective $\mathcal{L}$-space such that the composite

$$A \ \xrightarrow{\eta_A} \ \mathrm{map}^{\mathcal{L}}(Q, Q\otimes_{\mathbf{L}} A) \ \xrightarrow{\mathrm{map}^{\mathcal{L}}(Q,q)} \ \mathrm{map}^{\mathcal{L}}(Q,X) \tag{3.10}$$

is a global equivalence of orthogonal spaces.

Indeed, the first property guarantees that the right derived functor of the right adjoint reflects isomorphisms in the homotopy category, and the second condition ensures that the unit of the derived adjunction is a natural isomorphism. So the derived functors are equivalences of categories, and the Quillen pair is a Quillen equivalence.

(a) We let $f : X \longrightarrow Y$ be a morphism between injective $\mathcal{L}$-spaces such that $\mathrm{map}^{\mathcal{L}}(Q, f)$ is a global equivalence of orthogonal spaces. Since $\mathrm{map}^{\mathcal{L}}(Q, -)$ is a right Quillen functor, the orthogonal spaces $\mathrm{map}^{\mathcal{L}}(Q, X)$ and $\mathrm{map}^{\mathcal{L}}(Q, Y)$ are fibrant in the positive global model structure, i.e., they are positively static. So the global equivalence $\mathrm{map}^{\mathcal{L}}(Q, f)$ is in fact a positive strong level equivalence by the positive analog of [21, Prop. 1.2.14 (i)], i.e., for every compact Lie group G and every non-zero faithful G-representation V the map

$$(\mathrm{map}^{\mathcal{L}}(Q, f)(V))^G \ : \ (\mathrm{map}^{\mathcal{L}}(Q, X)(V))^G \ \longrightarrow \ (\mathrm{map}^{\mathcal{L}}(Q, Y)(V))^G$$

is a weak equivalence. We specialize to the case where G is a universal subgroup of $\mathcal{L}$ and we choose a faithful, non-zero, finite-dimensional G-representation V and a G-equivariant linear isometric embedding $\kappa : V \otimes \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty_G$. In the commutative square

$$\begin{tikzcd}
(\mathrm{map}^{\mathcal{L}}(Q, X)(V))^G \arrow[r, "(\mathrm{map}^{\mathcal{L}}(Q,f)(V))^G"] \arrow[d, "h\mapsto h(\kappa)"'] & (\mathrm{map}^{\mathcal{L}}(Q, Y)(V))^G \arrow[d, "h\mapsto h(\kappa)"] \\
X^G \arrow[r, "f^G"'] & Y^G
\end{tikzcd}$$

the two vertical maps are weak equivalences by Proposition 3.6 and the upper horizontal map is a weak equivalence by the above. So f^G is a weak equivalence, i.e., f is a global equivalence of $\mathcal{L}$-spaces.

(b) Given a positive flat orthogonal space A, we choose a fibrant replacement $i : A \longrightarrow B$ in the positive global model structure of orthogonal spaces; in other words, i is a positive flat cofibration and global equivalence, and B is a positively static orthogonal space. Then we choose a global equivalence of $\mathcal{L}$-spaces $p : B(\mathbb{R}^\infty) \longrightarrow X$ whose target is injective. The composite

$$Q \otimes_{\mathbf{L}} A \ \xrightarrow{Q\otimes_{\mathbf{L}} i} \ Q \otimes_{\mathbf{L}} B \ \xrightarrow{\xi_B} \ B(\mathbb{R}^\infty) \ \xrightarrow{p} \ X$$

is then the desired global equivalence $q : Q \otimes_{\mathbf{L}} A \longrightarrow X$ with injective target. This exploits that as a left Quillen functor, $Q \otimes_{\mathbf{L}} -$ takes acyclic cofibrations to global equivalences, and that ξ_B is a global equivalence by Proposition 3.7. It remains to show that the composite (3.10) is a global equivalence.

We let G be a universal subgroup of $\mathcal{L}$, and we let V be a finite-dimensional faithful G-subrepresentation of $\mathbb{R}^\infty_G$. We choose a G-equivariant linear isometric embedding $\kappa : V \otimes \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty_G$ such that the composite

$$V \ \xrightarrow{-\otimes u} \ V \otimes \mathbb{R}^\infty \ \xrightarrow{\kappa} \ \mathbb{R}^\infty_G$$

is the inclusion. We let $q^\sharp : B \longrightarrow \mathrm{map}^{\mathcal{L}}(Q, X)$ denote the adjoint of the composite $p \circ \xi_B : Q \otimes_{\mathbf{L}} B \longrightarrow X$. Then the following square of G-equivariant maps commutes:

$$\begin{tikzcd}
B(V) \arrow[r, "q^\sharp(V)"] \arrow[d, "B(\mathrm{incl})"'] & (\mathrm{map}^{\mathcal{L}}(Q, X)(V)) \arrow[r, equal] & \mathrm{map}^{\mathcal{L}}(\mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty), X) \arrow[d, "h\mapsto h(\kappa)"] \\
B(\mathbb{R}^\infty_G) \arrow[rr, "p"'] & & X
\end{tikzcd}$$

The left vertical map induces a weak equivalence of G-fixed points because B is flat, hence closed, and positively static. The lower horizontal map induces a weak equivalence of G-fixed points because p is a global equivalence and G is a universal subgroup. The right vertical map induces a weak equivalence on G-fixed points by Proposition 3.6. So we conclude that the map

$$(q^\sharp(V))^G \ : \ B(V)^G \ \longrightarrow \ \big(\mathrm{map}^{\mathcal{L}}(Q,X)(V)\big)^G$$

is a weak equivalence. Every pair consisting of a compact Lie group and a finite-dimensional faithful representation is isomorphic to a pair (G,V) as above. So the morphism $q^\sharp$ is a positive strong level equivalence of orthogonal spaces, and hence a global equivalence. The composite (3.10) agrees with $q^\sharp \circ i : A \longrightarrow \mathrm{map}^{\mathcal{L}}(Q,X)$, so it is a global equivalence because i and $q^\sharp$ are.

Now we have verified conditions (a) and (b), and this completes the proof that the adjoint functor pair $(Q\otimes_{\mathbf{L}} -, \mathrm{map}^{\mathcal{L}}(Q,-))$ is a Quillen equivalence. □

Remark 3.11 ($\mathcal{F}$-global model structure of $\mathcal{L}$-spaces)**.** Our results all have versions with respect to a *global family* $\mathcal{F}$, i.e., is a class of compact Lie groups that is closed under isomorphisms, subgroups and quotients. We only indicate what goes into this, and leave the details to interested readers. The global model structure of $\mathcal{L}$-spaces (see Theorem 1.20) has a straightforward version relative to the family $\mathcal{F}$. We denote by $\mathcal{L}\cap\mathcal{F}$ the collection of compact Lie subgroups of $\mathcal{L}$ that also belong to the global family $\mathcal{F}$. Proposition 1.11 then provides the $(\mathcal{L}\cap\mathcal{F})$-projective model structure on the category of $\mathcal{L}$-spaces.

We call a morphism $f : X \longrightarrow Y$ of $\mathcal{L}$-spaces an $\mathcal{F}$*-equivalence* if for every universal subgroup G of $\mathcal{L}$ that belongs to $\mathcal{F}$ the induced map

$$f^G \ : \ X^G \ \longrightarrow \ Y^G$$

is a weak homotopy equivalence. We call f an $\mathcal{F}$*-global fibration* if it is an $(\mathcal{L}\cap\mathcal{F})$-fibration and for every correspondence $\varphi : G \rightsquigarrow \bar{G}$ between groups in $\mathcal{L}\cap\mathcal{F}$ the map

$$(f^{\bar{G}}, \varphi\cdot -) \ : \ X^{\bar{G}} \ \longrightarrow \ Y^{\bar{G}} \times_{Y^G} X^G$$

is a weak equivalence. Essentially the same proof as for Theorem 1.20, but with all relevant groups restricted to the global family $\mathcal{F}$, then shows that the $\mathcal{F}$-equivalences, $\mathcal{F}$-global fibrations and $(\mathcal{L}\cap\mathcal{F})$-cofibrations form a cofibrantly generated proper topological model structure on the category of $\mathcal{L}$-spaces, the $\mathcal{F}$*-global model structure*.

The Quillen equivalence of Theorem 2.5 has a direct analog for a global family $\mathcal{F}$, with virtually the same proof. We let $\mathbf{O}_{\mathrm{gl}}^{\mathcal{F}}$ be the full topological subcategory of the global orbit category with objects the universal subgroups of $\mathcal{L}$ that belong to the family $\mathcal{F}$. Then taking fixed points is a right Quillen equivalence

$$\Phi^{\mathcal{F}} \ : \ \mathcal{L}\mathbf{T} \ \longrightarrow \ \mathcal{F}\text{-}orbispc$$

from the category of $\mathcal{L}$-spaces with the $\mathcal{F}$-global model structure to the category of $\mathcal{F}$-orbispaces, i.e., contravariant continuous functors from $\mathbf{O}_{\mathrm{gl}}^{\mathcal{F}}$ to spaces. Theorem 3.9 also has a relative version, with the same proof: for every global family $\mathcal{F}$, the adjoint functor pair $(Q\otimes_{\mathbf{L}} -, \mathrm{map}^{\mathcal{L}}(Q,-))$ is a Quillen equivalence with respect to the positive $\mathcal{F}$-global model structure on orthogonal spaces (the $\mathcal{F}$-based analog of [21, Prop. 1.2.23]) and the $\mathcal{F}$-global model structure on $\mathcal{L}$-spaces.

When $\mathcal{F} = \langle e\rangle$ is the global family of trivial groups, we recover the Quillen equivalence established by Lind in [16, Thm. 9.9]. To make the connection, we recall that orthogonal spaces are called $\mathcal{I}$*-spaces* in [16], and the category of $\mathcal{I}$-spaces is denoted $\mathcal{I}\mathscr{U}$. Moreover, our $\mathcal{L}$-spaces are called $\mathbb{L}$*-spaces* in [16], where $\mathbb{L}$ stands for the monad whose underlying functor sends A to $\mathcal{L} \times A$; the category of $\mathbb{L}$-spaces is denoted $\mathscr{U}[\mathbb{L}]$. For the trivial global family there is no difference between $(\mathcal{L} \cap \langle e \rangle)$-equivalences and $\langle e \rangle$-equivalences, and both specialize to the morphisms of $\mathcal{L}$-spaces that are weak equivalences on underlying non-equivariant spaces.

Appendix A. Topology of linear isometries

In this appendix we collect some facts about the topology on spaces of linear isometries; while these facts may be well-known to experts, not all are particularly well documented in the literature. I would like to thank Andrew Blumberg and Mike Mandell for key hints on the proof of Proposition A.5.

By a 'space' we mean a *compactly generated space* in the sense of [18], i.e., a k-space (also called *Kelley space*) that satisfies the weak Hausdorff condition. We write $\mathbf{T}$ for the category of compactly generated spaces and continuous maps. We emphasize that in contrast to some other paper on the subject, the weak Hausdorff condition is subsumed in 'compactly generated'. Besides [18], some other general references on compactly generated spaces are the Appendix A of Lewis' thesis [13], or Appendix A of the author's book [21].

The category $\mathbf{T}$ is complete and cocomplete. Given two compactly generated spaces X and Y, we write $X \times Y$ for the product in $\mathbf{T}$, i.e., the Kelleyfication of the usual product topology. If X or Y is locally compact, then the usual product topology is already a k-space, and hence compactly generated, so in this case no Kelleyfication is needed. We denote by $\mathrm{map}(X, Y)$ the set of continuous maps from X to Y, endowed with the Kelleyfication of the compact-open topology. Then $\mathrm{map}(-,-)$ is the internal function object in $\mathbf{T}$, i.e., for all compactly generated spaces X, Y and Z, the map

$$\mathrm{map}(X \times Y, Z) \longrightarrow \mathrm{map}(X, \mathrm{map}(Y, Z)) \ , \quad F \longmapsto \{x \longmapsto F(x,-)\}$$

is a homeomorphism, compare [13, App. A, Thm. 5,5] or [21, Thm. A.23].

An *inner product space* is an $\mathbb{R}$-vector space equipped with a scalar product, i.e., a positive definite symmetric bilinear form. We will only be concerned with inner product spaces of finite or countably infinite dimension. A finite-dimensional inner product space is given the metric topology induced from the scalar product. An infinite dimensional inner product space is given the weak topology from the system of its finite-dimensional subspaces. In infinite dimensions, the weak topology is strictly finer than the metric topology.

If V and W are inner product spaces, we denote by $\mathbf{L}(V, W)$ the set of linear isometric embeddings, i.e., $\mathbb{R}$-linear maps that preserve the inner product. A key property of the above topologies is that all $\mathbb{R}$-linear maps are automatically continuous. So $\mathbf{L}(V, W)$ is a subset of the space $\mathrm{map}(V, W)$ of continuous maps, and we endow $\mathbf{L}(V, W)$ with the subspace topology. The following proposition shows in particular that $\mathbf{L}(V, W)$ becomes a compactly generated space in this topology.

Proposition A.1. *For all inner product spaces V and W, the set $\mathbf{L}(V, W)$ of linear isometric embeddings is closed in the space $\mathrm{map}(V, W)$ of continuous maps. So $\mathbf{L}(V, W)$ is a Hausdorff k-space in the subspace topology from $\mathrm{map}(V, W)$.*

Proof. Since the evaluation maps, the vector space structure maps and the norm are continuous, so are the two maps

$$\alpha \ : \ \mathrm{map}(V, W) \times \mathbb{R} \times V \times V \longrightarrow W \ , \quad (f, \lambda, v, v') \longmapsto f(\lambda v + v') - \lambda f(v) - f(v')$$

and

$$\beta \ : \ \mathrm{map}(V, W) \times V \longrightarrow \mathbb{R} \ , \quad (f, v) \longmapsto |f(v)| - |v| \ .$$

Their adjoints

$$\tilde{\alpha} \ : \ \mathrm{map}(V, W) \longrightarrow \mathrm{map}(\mathbb{R} \times V \times V, W)$$

respectively

$$\tilde{\beta} \ : \ \mathrm{map}(V,W) \ \longrightarrow \ \mathrm{map}(V,\mathbb{R})$$

are thus continuous as well. A map $f : V \longrightarrow W$ is a linear isometric embedding if and only if $\tilde{\alpha}(f)$ and $\tilde{\beta}(f)$ are the zero maps. So

$$\mathbf{L}(V,W) \ = \ \tilde{\alpha}^{-1}(0) \cap \tilde{\beta}^{-1}(0)$$

is a closed subset of $\mathrm{map}(V,W)$. □

Since $\mathbf{L}(V,W)$ is defined as a subspace of the internal function space, a map $f : A \longrightarrow \mathbf{L}(V,W)$ from a compactly generated space is continuous if and only if its adjoint

$$f^\sharp \ : \ A \times V \ \longrightarrow \ W \ , \quad f^\sharp(a,v) \ = \ f(a)(v)$$

is continuous.

Proposition A.2. *For all inner product spaces U,V and W, the composition map*

$$\circ \ : \ \mathbf{L}(V,W) \times \mathbf{L}(U,V) \ \longrightarrow \ \mathbf{L}(U,W)$$

is continuous.

Proof. For all compactly generated spaces X and Y, the evaluation map $\mathrm{ev}_{X,Y} : \mathrm{map}(X,Y) \times X \longrightarrow Y$ is continuous. So the composite

$$\text{(A.3)} \qquad \mathrm{map}(V,W) \times \mathrm{map}(U,V) \times U \ \xrightarrow{\mathrm{map}(V,W)\times \mathrm{ev}_{U,V}} \ \mathrm{map}(V,W) \times V \ \xrightarrow{\mathrm{ev}_{V,W}} \ W$$

is continuous. The composition map

$$\circ \ : \ \mathrm{map}(V,W) \times \mathrm{map}(U,V) \ \longrightarrow \ \mathrm{map}(U,W)$$

is adjoint to (A.3), so it is is continuous as well. The claim now follows by restricting to subspaces of linear isometric embeddings. □

We write $\mathbb{R}^\infty$ for the $\mathbb{R}$-vector space of those sequences $(x_1, x_2, x_3, \dots)$ of real numbers such that almost all coordinates are zero. A scalar product on $\mathbb{R}^\infty$ is given by the familiar formula

$$\langle x, y \rangle \ = \ \sum\nolimits_{n \geq 1} x_n \cdot y_n \ .$$

So $\mathbb{R}^\infty$ is of countably infinite dimension, and the standard basis (i.e., the with one coordinate 1 and all other coordinates 0) is an orthonormal basis. We identify $\mathbb{R}^n$ with the subspace of $\mathbb{R}^\infty$ consisting of those tuples $(x_1, x_2, x_3, \dots)$ such that $x_i = 0$ for all $i > n$; then $\mathbb{R}^\infty$ carries the weak topology with respect to the nested subspaces $\mathbb{R}^n$.

We will also want to know that whenever V is finite-dimensional, then the function space topology on $\mathbf{L}(V,W)$ is in fact the familiar 'Stiefel manifold topology'. The following proposition is needed for this identification; it is a special case of Proposition 9.5 in [13, App. A].

Proposition A.4. *For every compact topological space K the canonical map*

$$\kappa \ : \ \mathrm{colim}_{n \geq 0}\, \mathrm{map}(K, \mathbb{R}^n) \ \longrightarrow \ \mathrm{map}(K, \mathbb{R}^\infty)$$

is a homeomorphism.

Proof. Since $\mathbb{R}^\infty$ is a colimit of the sequence of closed embeddings $\mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ between compactly generated spaces, every continuous map $f : K \longrightarrow \mathbb{R}^\infty$ factors through $\mathbb{R}^n$ for some $n \geq 0$, compare [13, App. A, Lemma 9.4] or [21, Prop. A.15 (i)]. So the canonical map κ is a continuous bijection.

It remains to show that the inverse of the canonical map is continuous. Since source and target of κ are k-spaces, continuity of κ^{-1} can be probed by continuous maps from compact spaces, i.e., it suffices to show that for every continuous map $f : L \longrightarrow \mathrm{map}(K, \mathbb{R}^\infty)$ from a compact space the composite $\kappa^{-1} \circ f$ is continuous. For this it is enough to show that f factors through $\mathrm{map}(K, \mathbb{R}^n)$ for some $n \geq 0$.

This, however, is now easy: we let $f^\sharp : L \times K \longrightarrow \mathbb{R}^\infty$ be the continuous adjoint of f, defined by $f^\sharp(l,k) = f(l)(k)$. Since L and K are compact, so is $L \times K$, and so $f^\sharp$ factors through $\mathbb{R}^n$ for some $n \geq 0$. Hence f factors through $\mathrm{map}(K, \mathbb{R}^n)$, and this completes the proof. □

Proposition A.5. *Let V and W be inner product spaces.*

(i) *If V and W are finite-dimensional, then $\mathbf{L}(V, W)$ is homeomorphic to the Stiefel manifold of $\dim(V)$-frames in W, and hence compact.*

(ii) *If V is finite-dimensional, then $\mathbf{L}(V, \mathbb{R}^\infty)$ carries the weak topology with respect to the filtration by the compact closed subspaces $\mathbf{L}(V, \mathbb{R}^n)$.*

(iii) *Let*

$$V_1 \subset V_2 \subset \ldots \subset V_n \subset \ldots$$

be a nested exhausting sequence of finite-dimensional subspaces of $\mathbb{R}^\infty$. Then $\mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ is an inverse limit, in the category of compactly generated spaces, of the tower of restriction maps

$$\ldots \longrightarrow \mathbf{L}(V_n, \mathbb{R}^\infty) \longrightarrow \ldots \longrightarrow \mathbf{L}(V_2, \mathbb{R}^\infty) \longrightarrow \mathbf{L}(V_1, \mathbb{R}^\infty) .$$

Proof. (i) We denote by $\mathbf{L}^{St}(V, W)$ the set of linear isometric embeddings endowed with the Stiefel manifold topology, making it a closed manifold. The evaluation map

$$\mathbf{L}^{St}(V, W) \times V \longrightarrow W$$

is continuous, hence so is its adjoint, the inclusion

$$\mathbf{L}^{St}(V, W) \longrightarrow \mathrm{map}(V, W) .$$

Since $\mathbf{L}^{St}(V, W)$ is compact and $\mathrm{map}(V, W)$ is a Hausdorff space, the inclusion is a closed embedding. So the Stiefel manifold topology coincides with the subspace topology of the function topology.

(ii) We let A be a subset of $\mathbf{L}(V, \mathbb{R}^\infty)$ such that $A \cap \mathbf{L}(V, \mathbb{R}^n)$ is closed in $\mathbf{L}(V, \mathbb{R}^n)$ for all $n \geq 0$; we wish to show that then A is itself closed. For this purpose we let $\rho : \mathbf{L}(V, \mathbb{R}^\infty) \longrightarrow \mathrm{map}(S(V), \mathbb{R}^\infty)$ denote the composite

$$\mathbf{L}(V, \mathbb{R}^\infty) \xrightarrow{\text{incl}} \mathrm{map}(V, \mathbb{R}^\infty) \xrightarrow{\text{restr}} \mathrm{map}(S(V), \mathbb{R}^\infty) ,$$

where the last map is restriction to the unit sphere $S(V)$. We observe that

$$\rho(A) \cap \mathrm{map}(S(V), \mathbb{R}^n) = \rho_n(A \cap \mathbf{L}(V, \mathbb{R}^n)) , \tag{A.6}$$

where

$$\rho_n : \mathbf{L}(V, \mathbb{R}^n) \longrightarrow \mathrm{map}(S(V), \mathbb{R}^n)$$

is restriction to the unit sphere. The space $\mathbf{L}(V, \mathbb{R}^n)$ is compact by part (i); so the closed subset $A \cap \mathbf{L}(V, \mathbb{R}^n)$ is itself compact. Then $\rho_n(A \cap \mathbf{L}(V, \mathbb{R}^n))$ is a quasi-compact subset of the Hausdorff space $\mathrm{map}(S(V), \mathbb{R}^n)$, and hence closed. So $\rho(A) \cap \mathrm{map}(S(V), \mathbb{R}^n)$ is closed for all $n \geq 0$, by (A.6). Since the unit sphere $S(V)$ is compact, Proposition A.4 shows that the space $\mathrm{map}(S(V), \mathbb{R}^\infty)$ has the weak topology with respect to the filtration by the closed subspaces $\mathrm{map}(S(V), \mathbb{R}^n)$. So the set $\rho(A)$ is closed in $\mathrm{map}(S(V), \mathbb{R}^\infty)$. Since ρ is continuous and injective, the set

$$A = \rho^{-1}(\rho(A))$$

is closed in $\mathbf{L}(V, \mathbb{R}^\infty)$.

(iii) The category of compactly generated spaces is cartesian closed, with $\mathrm{map}(-,-)$ as the internal function object. So the natural bijections

$$\mathbf{T}^{\mathrm{op}}(\mathrm{map}(Y,Z),X) \;=\; \mathbf{T}(X,\mathrm{map}(Y,Z)) \;\cong\; \mathbf{T}(X\times Y,Z) \;\cong\; \mathbf{T}(Y,\mathrm{map}(X,Z))$$

show that for every compactly generated space Z, the functor

$$\mathrm{map}(-,Z) \;:\; \mathbf{T} \;\longrightarrow\; \mathbf{T}^{\mathrm{op}}$$

is a left adjoint. So $\mathrm{map}(-,Z)$ takes colimits in $\mathbf{T}$ to colimits in $\mathbf{T}^{\mathrm{op}}$, which are limits in $\mathbf{T}$. Since $\mathbb{R}^\infty$ is a colimit of the sequence $\{V_n\}_{n\geq 0}$, the space $\mathrm{map}(\mathbb{R}^\infty,\mathbb{R}^\infty)$ is an inverse limit of the tower of spaces $\{\mathrm{map}(V_n,\mathbb{R}^\infty)\}_{n\geq 0}$.

Now we can prove the claim. We let $f_n : T \longrightarrow \mathbf{L}(V_n,\mathbb{R}^\infty)$ be a compatible family of continuous maps from a compactly generated space T. Then the composite maps

$$T \xrightarrow{\;f_n\;} \mathbf{L}(V_n,\mathbb{R}^\infty) \xrightarrow{\;\mathrm{incl}\;} \mathrm{map}(V_n,\mathbb{R}^\infty)$$

are compatible. Since $\mathrm{map}(\mathbb{R}^\infty,\mathbb{R}^\infty)$ is an inverse limit of the tower, there is a unique continuous map $f : T \longrightarrow \mathrm{map}(\mathbb{R}^\infty,\mathbb{R}^\infty)$ such that the square

$$\begin{tikzcd}
T \arrow[r, "f"] \arrow[d, "f_n"'] & \mathrm{map}(\mathbb{R}^\infty,\mathbb{R}^\infty) \arrow[d, "\mathrm{restr}"] \\
\mathbf{L}(V_n,\mathbb{R}^\infty) \arrow[r, "\mathrm{incl}"'] & \mathrm{map}(V_n,\mathbb{R}^\infty)
\end{tikzcd}$$

commutes for all $n \geq 0$. For every $t \in T$ the map $f(t) : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ restricts to a linear isometric embedding on every finite-dimensional subspace of $\mathbb{R}^\infty$; so $f(t)$ is itself a linear isometric embedding. Hence the map f lands in the subspace $\mathbf{L}(\mathbb{R}^\infty,\mathbb{R}^\infty)$ of $\mathrm{map}(\mathbb{R}^\infty,\mathbb{R}^\infty)$, and can be viewed as a continuous map $f : T \longrightarrow \mathbf{L}(\mathbb{R}^\infty,\mathbb{R}^\infty)$ that restricts to the original map f_n for all $n \geq 0$. This proves that $\mathbf{L}(\mathbb{R}^\infty,\mathbb{R}^\infty)$ has the universal property of an inverse limit of the tower of restriction maps $\mathbf{L}(V_n,\mathbb{R}^\infty) \longrightarrow \mathbf{L}(V_{n-1},\mathbb{R}^\infty)$. □

In this paper we are very much interested in subgroups of the linear isometries monoid that happen to be compact Lie groups. The following proposition collects some useful facts about continuous actions of compact groups by linear isometries on $\mathbb{R}^\infty$.

Proposition A.7. *Let G be a compact topological group acting continuously on $\mathbb{R}^\infty$ by linear isometries.*

- (i) *Every finite-dimensional subspace of $\mathbb{R}^\infty$ is contained in a finite-dimensional G-invariant subspace.*
- (ii) *The space $\mathbb{R}^\infty$ is the orthogonal direct sum of finite-dimensional G-invariant subspaces.*

Proof. (i) We let V be a finite-dimensional subspace of $\mathbb{R}^\infty$ and denote by

$$\rho \;:\; G \;\longrightarrow\; \mathbf{L}(V,\mathbb{R}^\infty)$$

the adjoint to the action map

$$G\times V \;\longrightarrow\; \mathbb{R}^\infty \;, \quad (g,v) \;\longmapsto\; gv \;.$$

Since G acts continuously on $\mathbb{R}^\infty$, the adjoint ρ is continuous with respect to the function space topology. By Proposition A.5 (ii), the space $\mathbf{L}(V,\mathbb{R}^\infty)$ carries the weak topology with respect to the nested sequence of closed subspaces $\mathbf{L}(V,\mathbb{R}^n)$. So the compact subset $\rho(G)$ is contained in $\mathbf{L}(V,\mathbb{R}^n)$ for some $n \geq 0$. In more concrete terms, this means that $G\cdot V \subseteq \mathbb{R}^n$. The $\mathbb{R}$-linear span of the set $G\cdot V$ is then the desired finite-dimensional G-invariant subspace.

(ii) We construct pairwise orthogonal finite-dimensional G-invariant subspaces W_n such that $\mathbb{R}^n$ is contained in the direct sum of $W_1, \dots, W_n$ for all $n \geq 1$. The construction is inductive, starting with $W_0 = 0$. The inductive step uses part (i) to obtain a finite-dimensional G-invariant subspace V of $\mathbb{R}^\infty$ that contains $W_1, \dots, W_{n-1}$ and $\mathbb{R}^n$. Then we let W_n be the orthogonal complement of $W_1 \oplus \dots \oplus W_{n-1}$ in V. Since $W_1, \dots, W_{n-1}$ are G-invariant and G acts by linear isometries, W_n is again G-invariant. Moreover, since V is finite-dimensional, it is the orthogonal direct sum of $W_1 \oplus \dots \oplus W_{n-1}$ and W_n. Since the finite sums $W_1 \oplus \dots \oplus W_n$ exhaust $\mathbb{R}^\infty$, the latter is the orthogonal direct sum of all the subspaces W_n for $n \geq 1$. □

Proposition A.8. *Let G be a compact subgroup of the linear isometries monoid $\mathcal{L}$. Then G admits the structure of a Lie group (necessarily unique) if and only if there is a finite-dimensional G-invariant subspace of $\mathbb{R}^\infty$ on which G acts faithfully.*

Proof. We start by assuming that there is a G-invariant finite-dimensional subspace V of $\mathbb{R}^\infty$ on which G acts faithfully. Then the continuous composite

$$G \xrightarrow{\text{incl}} \mathcal{L} = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty) \xrightarrow{\text{restr}} \mathbf{L}(V, \mathbb{R}^\infty)$$

factors through an injective continuous group homomorphism

$$\rho : G \longrightarrow \mathbf{L}(V, V)$$

that encodes the G-action on V. By Proposition A.5 (i), the space $\mathbf{L}(V, V)$ carries the Stiefel manifold topology, i.e., $\mathbf{L}(V, V) = O(V)$ is the orthogonal group of V with the Lie group topology. Since G is compact, the homomorphism $\rho : G \longrightarrow O(V)$ is a closed map, hence a homeomorphism onto its image, which is a closed subgroup of $O(V)$. Every closed subgroup of a Lie group carries the structure of a Lie group, compare [5, Ch. I, Thm. 3.11]; so G admits the structure of a Lie group. A topological group admits at most one structure of Lie group, see for example [5, Ch. I, Prop. 3.12]; so the Lie group structure is necessarily unique.

Now we suppose that conversely G admits the structure of a Lie group. Proposition A.7 (ii) provides finite-dimensional pairwise orthogonal G-invariant subspaces W_n that together span $\mathbb{R}^\infty$. We let

$$H_n = \{g \in G : gv = v \text{ for all } v \in W_1 \oplus \dots \oplus W_n\}$$

denote the kernel of the G-action on $W_1 \oplus \dots \oplus W_n$; this is a closed normal subgroup of G with $H_{n+1} \subseteq H_n$. In a compact Lie group, every infinite descending chain of closed subgroups is eventually constant. So there is an $m \geq 0$ such that $H_n = H_m$ for all $n \geq m$. Since the subspaces W_n span $\mathbb{R}^\infty$, the subgroup H_m is the trivial subgroup. Hence $W_1 \oplus \dots \oplus W_m$ is the desired finite-dimensional faithful G-invariant subspace of $\mathbb{R}^\infty$. □

Example A.9. The linear isometries monoid $\mathcal{L}$ has compact subgroups that are not Lie groups, and for which $\mathbb{R}^\infty$ has no faithful finite-dimensional subrepresentation. As an example we consider the unitary representation $\mathbb{C}(n)$ of the additive group $\mathbb{Z}_p^\wedge$ of p-adic integers on the complex numbers through the finite quotient $\mathbb{Z}/p^n\mathbb{Z}$, with a generator of $\mathbb{Z}/p^n\mathbb{Z}$ acting by multiplication by $e^{2\pi i/p^n}$. Then the direct sum action of $\mathbb{Z}_p^\wedge$ on $\bigoplus_{n\geq 1} \mathbb{C}(n)$ is faithful and through $\mathbb{C}$-linear isometries. An identification of the underlying $\mathbb{R}$-vector space of $\bigoplus_{n\geq 1} \mathbb{C}(n)$ with $\mathbb{R}^\infty$ turns this into a faithful, continuous isometric action of $\mathbb{Z}_p^\wedge$ on $\mathbb{R}^\infty$. The image of this action is then a compact subgroup of $\mathcal{L}$ isomorphic to $\mathbb{Z}_p^\wedge$.

Proposition A.10. *Let G and K be compact Lie groups, $\mathcal{V}$ an orthogonal G-representation of countably infinite dimension and $\mathcal{U}_K$ a complete K-universe. For every faithful finite-dimensional G-subrepresentation V of $\mathcal{V}$ the restriction morphism*

$$\rho_V^{\mathcal{V}} : \mathbf{L}(\mathcal{V}, \mathcal{U}_K) \longrightarrow \mathbf{L}(V, \mathcal{U}_K)$$

is a $(K\times G)$-homotopy equivalence.

Proof. We choose an exhausting nested sequence

$$V \;=\; V_0 \;\subset\; V_1 \;\subset\; V_2 \;\dots$$

of finite-dimensional G-subrepresentations of $\mathcal{V}$, starting with the given faithful representation. We claim that all the restriction maps

$$p_n \;:\; \mathbf{L}(V_n,\mathcal{U}_K) \;\longrightarrow\; \mathbf{L}(V_{n-1},\mathcal{U}_K)$$

are $(K\times G)$-acyclic fibrations, i.e., for every closed subgroup $\Gamma\leq K\times G$ the fixed point map

$$(p_n)^\Gamma \;:\; \mathbf{L}(V_n,\mathcal{U}_K)^\Gamma \;\longrightarrow\; \mathbf{L}(V_{n-1},\mathcal{U}_K)^\Gamma$$

is a weak equivalence and Serre fibration. Since G acts faithfully on V_n, the Γ-fixed points of source and target are empty whenever $\Gamma\cap(1\times G)\neq\{(1,1)\}$. Otherwise Γ is the graph of a continuous homomorphism $\alpha : L\longrightarrow G$ defined on a closed subgroup L of K. So the fixed point map is the restriction map

$$(p_n)^\Gamma \;:\; \mathbf{L}^L(\alpha^*(V_n),\mathcal{U}_K) \;\longrightarrow\; \mathbf{L}^L(\alpha^*(V_{n-1}),\mathcal{U}_K)\;.$$

Source and target of this map are contractible (for example by [15, II Lemma 1.5]), so the map $(p_n)^\Gamma$ is a weak equivalence. But $(p_n)^\Gamma$ is also a locally trivial fiber bundle, hence a Serre fibration.

Now we know that p_n is a $(K\times G)$-acyclic fibration, and moreover $\mathbf{L}(V_n,\mathcal{U}_K)$ is $(K\times G)$-cofibrant for every $n\geq 0$, for example by [21, Prop. 1.1.19 (ii)]. Moreover, the space $\mathbf{L}(\mathcal{V},\mathcal{U}_K)$ is the inverse limit of the tower of restriction maps $p_n : \mathbf{L}(V_n,\mathcal{U}_K)\longrightarrow\mathbf{L}(V_{n-1},\mathcal{U}_K)$, by Proposition A.5. So the following Proposition A.11, applied to the projective model structure on $(K\times G)$-spaces (compare [21, Prop. B.7]) shows that the restriction map

$$\rho^{\mathcal{V}}_V \;:\; \mathbf{L}(\mathcal{V},\mathcal{U}_K) \;\longrightarrow\; \mathbf{L}(V,\mathcal{U}_K)$$

is a $(K\times G)$-homotopy equivalence. □

Proposition A.11. *Let $\mathcal{C}$ be a topological model category and*

$$\dots\;\to\; X_n \;\xrightarrow{p_n}\; \dots\;\to\; X_2 \;\xrightarrow{p_2}\; X_1 \;\xrightarrow{p_1}\; X_0$$

an inverse system of acyclic fibrations between cofibrant objects. Then the canonical map

$$p_\infty \;:\; X_\infty \;=\; \lim_n X_n \;\longrightarrow\; X_0$$

is a homotopy equivalence.

Proof. Since p_n is an acyclic fibration and X_{n-1} is cofibrant, we can find a section $s_n : X_{n-1}\longrightarrow X_n$ to p_n by choosing a lift in the left square below:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X_n \arrow[d, "p_n"] \\
X_{n-1} \arrow[r, equal] \arrow[ur, dashed, "s_n"] & X_{n-1}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X_n\times\{0,1\} \arrow[r, "\mathrm{Id}+s_n\circ p_n"] \arrow[d] & X_n \arrow[d, "p_n"] \\
X_n\times[0,1] \arrow[r, "p_n\circ\mathrm{proj}"'] \arrow[ur, dashed, "H_n"] & X_{n-1}
\end{tikzcd}$$

Since X_n is cofibrant, the inclusion $X_n\times\{0,1\}\longrightarrow X_n\times[0,1]$ is a cofibration, and a choice of lift in the right diagram provides a homotopy

$$H_n \;:\; X_n\times[0,1] \;\longrightarrow\; X_n$$

from the identity to $s_n \circ p_n$ such that $p_n \circ H_n : X_n \times [0,1] \longrightarrow X_{n-1}$ is the constant homotopy from p_n to itself. The morphisms

$$s_n \circ s_{n-1} \circ \cdots \circ s_1 \ : \ X_0 \ \longrightarrow \ X_n$$

are then compatible with the inverse system defining X_∞, so they assemble into a morphism

$$s_\infty \ : \ X_0 \ \longrightarrow \ \lim_n X_n$$

to the inverse limit, and s_∞ is a section to p_∞.

We claim that the composite $s_\infty \circ p_\infty$ is homotopic to the identity. To prove the claim we construct compatible homotopies

$$K_n \ : \ X_\infty \times [0,1] \ \longrightarrow \ X_n$$

by induction on n satisfying

(i) $p_n \circ K_n = K_{n-1}$,
(ii) $K_n(-,t) = p_\infty^{(n)}$, the canonical morphism $X_\infty \longrightarrow X_n$, for all $t \in [0, \frac{1}{n+1}]$, and
(iii) $K_n(-,1) = s_n \circ s_{n-1} \circ \cdots s_1 \circ p_\infty$.

The induction starts by defining K_0 as the constant homotopy from $p_\infty : X_\infty \longrightarrow X_0$ to itself. Now we assume $n \geq 1$ and suppose that the homotopies $K_0, \ldots, K_{n-1}$ have already been constructed. We exploit that the functor $X_\infty \times -$ is a left adjoint, so $X_\infty \times [0,1]$ is the pushout of the objects $X_\infty \times [0, \frac{1}{n+1}]$, $X_\infty \times [\frac{1}{n+1}, \frac{1}{n}]$ and $X_\infty \times [\frac{1}{n}, 1]$ along two copies of X_∞, embedded via the points $\frac{1}{n+1}$ and $\frac{1}{n}$ of $[0,1]$. So we can define K_n by

$$K_n(-,t) \ = \ \begin{cases} p_\infty^{(n)} & \text{for } t \in [0, \frac{1}{n+1}], \\ H_n(-,\, n(n+1)t - n) \circ p_\infty^{(n)} & \text{for } t \in [\frac{1}{n+1}, \frac{1}{n}], \text{ and} \\ s_n \circ K_{n-1}(-,t) & \text{for } t \in [\frac{1}{n}, 1]. \end{cases}$$

This is well-defined at the intersections of the intervals because

$$H_n\left(-,\, n(n+1)\frac{1}{n+1} - n\right) \circ p_\infty^{(n)} \ = \ H_n(-,0) \circ p_\infty^{(n)} \ = \ p_\infty^{(n)}$$

and

$$\begin{aligned} H_n\left(-,\, n(n+1)\frac{1}{n} - n\right) \circ p_\infty^{(n)} \ &= \ H_n(-,1) \circ p_\infty^{(n)} \ = \ s_n \circ p_n \circ p_\infty^{(n)} \\ &= \ s_n \circ p_\infty^{(n-1)} \ = \ s_n \circ K_{n-1}(-,1/n) \end{aligned}$$

Then condition (i) holds because

$$\begin{aligned} p_n \circ K_n(-,t) \ &= \ \begin{cases} p_n \circ p_\infty^{(n)} & \text{for } t \in [0, \frac{1}{n+1}], \\ p_n \circ H_n(-,\, n(n+1)t - n) \circ p_\infty^{(n)} & \text{for } t \in [\frac{1}{n+1}, \frac{1}{n}], \text{ and} \\ p_n \circ s_n \circ K_{n-1}(-,t) & \text{for } t \in [\frac{1}{n}, 1], \end{cases} \\ &= \ \begin{cases} p_\infty^{(n-1)} & \text{for } t \in [0, \frac{1}{n}], \\ K_{n-1}(-,t) & \text{for } t \in [\frac{1}{n}, 1], \end{cases} \\ &= \ K_{n-1}(-,t) \ . \end{aligned}$$

Now we can finish the proof. By condition (i) the homotopies K_n are compatible, so they assemble into a morphism $K_\infty : X_\infty \times [0,1] \longrightarrow X_\infty$. Property (ii) shows that K_∞ starts with the identity of X_∞ and property (iii) ensures that K_∞ ends with the morphism $s_\infty \circ p_\infty$. So s_∞ is a homotopy inverse to p_∞. □

Remark A.12. If we specialize Proposition A.10 to $\mathcal{V} = \mathcal{U}_K = \mathbb{R}^\infty$ and ignore all group actions, it shows in particular that the restriction map $\mathcal{L} = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty) \longrightarrow \mathbf{L}(0, \mathbb{R}^\infty)$ is a homotopy equivalence to a one-point space. In other words, the underlying space of $\mathcal{L}$ is contractible.

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