ORBISPACES, ORTHOGONAL SPACES, AND THE UNIVERSAL COMPACT LIE GROUP

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Introduction

In this article we provide a new perspectives on unstable global homotopy theory: we interpret it as the homotopy theory of 'spaces with an action of the universal compact Lie group'. This 'universal compact Lie group' is a well known object, namely the topological monoid $\mathcal{L} = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ of linear isometric embeddings of $\mathbb{R}^\infty$ into itself. The category of $\mathcal{L}$-spaces has been much studied, for example in [1, 3, 13, 14]; in some of these source the symbol $\mathcal{L}$ refers to the linear isometries *operad*, so the monoid we denote $\mathcal{L}$ is then the monoid $\mathcal{L}(1)$ of unary operations. The underlying space of $\mathcal{L}$ is contractible, so the homotopy theory of $\mathcal{L}$-spaces with respect to 'underlying' weak equivalences is just another model for the homotopy theory of spaces. However, we shift the perspective on the homotopy theory that $\mathcal{L}$-spaces represent, and use a notion *global equivalences* of $\mathcal{L}$-spaces that is much finer than the notion of 'underlying' weak equivalence that has so far been studied. When viewed through the eyes of global equivalences, one should think of $\mathcal{L}$ as a 'universal compact Lie group' and hence of an $\mathcal{L}$-space as a 'global space' on which all compact Lie groups act simultaneously and in a compatible way. Such a statement is of course not literally correct: the topological monoid $\mathcal{L}$ is neither compact, nor a group, much less a compact Lie group.

However, we will make the case that $\mathcal{L}$ has all the moral right to be thought of as the universal compact Lie group. In fact, $\mathcal{L}$ contains a copy (in fact, many conjugate ones) of every compact Lie group in a certain way: we may choose a continuous isometric linear G-action on $\mathbb{R}^\infty$ that makes $\mathbb{R}^\infty$ into a complete G-universe. This action is a continuous injective group homomorphism $\rho : G \longrightarrow \mathcal{L}$, and we call the images $\rho(G)$ of such homomorphisms *completely universal subgroups* of $\mathcal{L}$ (compare Definition 2.10 below). Because any two complete G-universes are equivariantly isometrically isomorphic, the group $\rho(G)$ is independent, up to conjugacy by an invertible linear isometry, of the choice of ρ. So in this way every compact Lie group determines a specific conjugacy class of subgroups of $\mathcal{L}$, abstractly isomorphic to G.

One of the main results of this paper is a chain of two Quillen equivalences of model categories:

$$\mathit{orbispc} \underset{\Phi}{\overset{\Lambda}{\rightleftarrows}} \mathcal{L}\mathbf{T} \underset{\mathrm{map}^{\mathcal{L}}(Q,-)}{\overset{Q\otimes_{\mathbf{L}}-}{\leftrightarrows}} \mathit{spc}$$

The left most category is the category of *orbispaces*, i.e., the category of contravariant continuous functors from the *global orbit category* $\mathbf{O}_{\mathrm{gl}}$ to spaces. Orbispaces are equipped with a 'pointwise' model structure. Moreover, $\mathcal{L}\mathbf{T}$ is the category of $\mathcal{L}$-spaces, i.e., spaces (compactly generated and weakly Hausdorff, as usual), equipped with a continuous left $\mathcal{L}$-action. Finally, *spc* is the category of orthogonal spaces, i.e., continuous functors to spaces from the category of inner product spaces and linear isometric embeddings. Both orthogonal spaces and $\mathcal{L}$-spaces are equipped with global model structures.

The Quillen equivalence (Λ, Φ) between $\mathcal{L}$-spaces and orbispaces is then an analog of Elmendorf's theorem [8] saying that the passage from G-spaces to functors on the orbit category that collects the fixed point spaces of the various closed subgroups of G is an equivalence of homotopy theories. Indeed, the global orbit category $\mathbf{O}_{\mathrm{gl}}$ is the direct analog for the universal compact Lie group of the orbit category of a single

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compact Lie group: the objects of $\mathbf{O}_{\mathrm{gl}}$ are the completely universal subgroups of $\mathcal{L}$ and the morphism spaces in $\mathbf{O}_{\mathrm{gl}}$ are defined by

$$\mathbf{O}_{\mathrm{gl}}(K,G) \;=\; \mathrm{map}_{\mathcal{L}}(\mathcal{L}/K,\mathcal{L}/G) \;\cong\; (\mathbf{L}(\mathbb{R}^\infty_G,\mathbb{R}^\infty_K)/G)^K \ .$$

As we explain in Theorem 2.30, the Quillen equivalence between $\mathcal{L}$-spaces (with the global model structure) and orbispaces is a special case of a generalization of Elmendorf's theorem to a context of topological monoids relative to certain collections of closed submonoids, compare Proposition 1.14.

We should probably justify the terminology 'orbispaces' for functors on the global orbit category. For this we refer to the paper [10] of Gepner and Henriques, who compare the homotopy theories of 'Orb-spaces' with homotopy theories of topological stacks and of topological groupoids. The setup of [10] is relative to a specified class of 'allowed isotropy group', and Gepner and Henriques then construct a topological category whose objects are the allowed isotropy groups and such that the morphism space $\mathrm{Orb}(K,G)$ from a group K to a group G has the weak homotopy type of the homotopy orbit space of G acting by conjugation on the space of continuous homomorphisms from K to G. An *orbispace*, or *Orb-space*, is then a continuous functor from the category Orb to spaces. Our global orbit category $\mathbf{O}_{\mathrm{gl}}$ is such a category for the class of compact Lie groups, whence the terminology. So a more precise, but too lengthy name would be 'orbispaces with compact Lie group isotropy'.

1. Model structures for equivariant spaces

In this section we establish certain 'projective' model structures for spaces equipped with an action of a topological monoid. The results and methods are fairly standard, but we do not know of a reference in the generality we need, so we provide full proofs.

Before we start, let us fix some notation and conventions. By a 'space' we mean a *compactly generated space* in the sense of [16], i.e., a k-space (also called *Kelley space*) that satisfies the weak Hausdorff condition. We denote by $\mathbf{T}$ the category of compactly generated spaces. The 'classical' model structure on the category of all topological spaces was established by Quillen in [17, II.3 Thm. 1]. We use the straightforward adaptation of this model structure to the category of compactly generated spaces, which is described for example in [11, Thm. 2.4.25]. In this model structure on the category $\mathbf{T}$, the weak equivalences are the weak homotopy equivalences and fibrations are the Serre fibrations. The cofibrations are the retracts of generalized CW-complexes, i.e., cell complexes in which cells can be attached in any order and not necessarily to cells of lower dimensions.

We let M be a topological monoid, i.e., a compactly generated space equipped with an associative and unital multiplication

$$\mu \;:\; M\times M \;\longrightarrow\; M$$

that is continuous with respect to the compactly generated product topology. An M-space is then a compactly generated space X equipped with an associative and unital action

$$\alpha \;:\; M\times X \;\longrightarrow\; X$$

that is continuous with respect to the compactly generated product topology.

We let N be a submonoid of M and denote by

$$X^N \;=\; \{x\in X\mid nx=x \text{ for all } n\in N\}$$

the subspace of N-fixed points. For an individual element $n\in N$ the n-fixed subspace $\{x\in X\mid nx=x\}$ is the preimage of the diagonal under the continuous map $(\mathrm{Id},n\cdot -) : X\longrightarrow X\times X$, so it is a closed subspace of X by the weak Hausdorff condition. The N-fixed points X^N are then closed in X as an intersection of closed subsets. This means that the subspace topology on X^N is again compactly generated and so

$$X^N \xrightarrow{\ \mathrm{incl}\ } X \rightrightarrows \mathrm{map}(N,X) \tag{1.1}$$

is an equalizer diagram in the category of compactly generated spaces, where the two maps on the right are adjoint to the projection $N \times X \longrightarrow X$ respectively the composite

$$N \times X \xrightarrow{\text{incl}\times X} M \times X \xrightarrow{\alpha} X \ .$$

Definition 1.2. A submonoid N of a topological monoid M is *biclosed* if the following two conditions hold:

(i) the set N is closed in the topology of M, and
(ii) if $m \in M$ and $n \in N$ satisfy $mn \in N$, then $m \in N$.

Remark 1.3. Eventually we want to define classes of weak equivalences for M-spaces by testing on the fixed point spaces X^N for collections of submonoids. For this purpose it is no loss of generality to restrict to biclosed submonoids, as we now explain. First we observe that any intersection of biclosed submonoids of a topological monoid is again biclosed. So an arbitrary submonoid N of M has a *biclosure* $\bar{N}$, defined as the intersection of all biclosed submonoid of M that contain N, which is the smallest biclosed submonoid of M that contains N.

We will now argue that for every M-space X the N-fixed points agree with the $\bar{N}$-fixed points:

$$X^N = X^{\bar{N}} \ .$$

The *stabilizer* of a point $x \in X$ is the submonoid

$$\mathrm{stab}_M(x) \ = \ \{m \in M \mid mx = x\} \ .$$

The stabilizer is also the preimage of $\{x\}$ under the continuous map $- \cdot x : M \longrightarrow X$; since singletons in compactly generated spaces spaces are closed, the stabilizer is a closed subset of M. Moreover, if $m, n \in M$ are such that n and mn stabilize x, then

$$mx \ = \ m(nx) \ = \ (mn)x \ = \ x \ ,$$

i.e., $m \in \mathrm{stab}_M(x)$. So the point stabilizer of M-spaces are always biclosed submonoids.

More generally, for every subset $S \subseteq X$ the stabilizer

$$\mathrm{stab}_M(S) \ = \ \{m \in M \mid mx = x \text{ for all } x \in S\}$$

is the intersection of the stabilizers of all points in S, so it is another biclosed submonoid of M. If N is a submonoid of M, not necessarily biclosed, then

$$N \ \subseteq \ \mathrm{stab}_M(X^N)$$

and the stabilizer monoid on the right is biclosed. So the biclosure $\bar{N}$ of N is contained in $\mathrm{stab}_M(X^N)$, and hence $X^N = X^{\bar{N}}$.

Example 1.4. For example, every closed subgroup G of a topological monoid M is biclosed, because the assumptions $g \in G$ and $mg \in G$ imply $m = (mg)g^{-1} \in G$.

Another example relevant to global homotopy theory is the topological monoid $\mathcal{L} = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ of linear isometric self-embeddings of $\mathbb{R}^\infty$. The topology on $\mathcal{L}$ is as the inverse limit of the spaces $\mathbf{L}(\mathbb{R}^n, \mathbb{R}^\infty)$, and $\mathbf{L}(\mathbb{R}^n, \mathbb{R}^\infty)$ has the colimit topology as the union of the compact spaces $\mathbf{L}(\mathbb{R}^n, \mathbb{R}^m)$. If V is a finite dimensional inner product space, then the space $\mathbf{L}(V, \mathbb{R}^\infty)$ of linear isometric embeddings (again with the colimit topology of the sequence $\mathrm{map}(V, \mathbb{R}^m)$) is an $\mathcal{L}$-space under composition of isometries. For every linear isometric embedding $\alpha : V \longrightarrow \mathbb{R}^\infty$ the stabilizer

$$\mathrm{stab}_{\mathcal{L}}(\alpha) \ = \ \{\varphi \in \mathcal{L} \mid \varphi \circ \alpha = \alpha\}$$

is thus a biclosed submonoid of $\mathcal{L}$, compare the previous remark.

Now we show that the functor sending an M-space X to the set of N-fixed points is representable by an 'orbit space' M/N. We denote by M/N a coequalizer in the category of M-spaces

$$M \times N \underset{\mu'}{\overset{\text{proj}}{\rightrightarrows}} M \xrightarrow{\quad q \quad} M/N \ , \tag{1.5}$$

where $\mu' = \mu \circ (M \times \text{incl})$. Since the forgetful functor creates colimits, we could equivalently take a coequalizer in the underlying category of compactly generated spaces, and that inherits a unique M-action that makes the projection $q : M \longrightarrow M/N$ a homomorphism of M-spaces.

Now we let K be any compactly generated space. Since product with K is a left adjoint, the diagram

$$M \times N \times K \underset{\mu' \times K}{\overset{\text{proj} \times K}{\rightrightarrows}} M \times K \xrightarrow{q \times K} M/N \times K \ .$$

is another coequalizer of M-spaces, where M acts trivially on K. So for every M-space X, precomposition with $q \times K$ is a bijection from $M\mathbf{T}(M/N \times K, X)$ to the equalizer of the two maps

$$M\mathbf{T}(M \times K, X) \underset{M\mathbf{T}(\mu' \times K, X)}{\overset{M\mathbf{T}(\text{proj} \times K, X)}{\rightrightarrows}} M\mathbf{T}(M \times N \times K, X) \ .$$

The free-forgetful adjunction and the adjunction between $N \times -$ and $\text{map}(N, -)$ identifies this with the set of those continuous maps $f : K \longrightarrow X$ that are equalized by the two right maps in the equalizer diagram (1.1). Since the inclusion of X^N into X is an equalizer, we have shown altogether that evaluation at the class of the identity element is a bijection

$$M\mathbf{T}(M/N \times K, X) \longrightarrow \mathbf{T}(K, X^N) \tag{1.6}$$

from the set of continuous M-maps from $M/N \times K$ to X to the set of continuous maps from K to the N-fixed points of X.

One more time we do not lose any generality by restricting to biclosed submonoids. Indeed, the proof of the adjunction (1.6) did not use any property of the submonoid N. Since $X^N = X^{\bar{N}}$ for every M-space X, the M-spaces M/N and $M/\bar{N}$ represent the same functor, and so they are isomorphic. In other words, the 'orbit space' M/N only depends on the biclosure $\bar{N}$ of the submonoid N.

As we recall in [18, Prop. A.1.4 (iv)], orbit spaces of compactly generated spaces by actions of compact topological groups behave as expected, i.e., the usual quotient topology on the orbit set is compactly generated. One should beware that this need not be the case if we drop the compactness hypothesis or the existence of inverses. So even for biclosed submonoids N, the 'orbit space' M/N may not be what one expects at first sight. To construct M/N, one could start from the equivalence relation $\sim_N$ on M generated by $m \sim_N mn$ for all $m \in M$ and $n \in N$. If N is biclosed, then it is the equivalence class of the unit element 1, but the other equivalence classes may still be hard to identify.

Since M is compactly generated, the quotient topology on the set $M/\sim_N$ of equivalence classes will automatically yield a k-space, but not necessarily a weak Hausdorff space. So in a second step one has to apply the left adjoint to the inclusion of compactly generated spaces into k-spaces, but this step can change the topology and may even alter the underlying set by identifying different equivalence classes.

Example 1.7. Here are two examples of particular relevance for us where we can describe an 'orbit space' more explicitly. If G is a compact topological group and H a closed subgroup, then the set G/H of left cosets endowed with the quotient topology is compact, so it is a coequalizer in then sense of (1.5). So in this situation the orbit space notation is unambiguous.

For the monoid $\mathcal{L}$ of linear isometric self-embeddings of $\mathbb{R}^\infty$ and a finite dimensional inner product space V, the $\mathcal{L}$-space $\mathbf{L}(V, \mathbb{R}^\infty)$ ought to be a quotient of $\mathcal{L}$ by the stabilizer of any particular linear isometric embedding $\alpha : V \longrightarrow \mathbb{R}^\infty$. In fact, $\mathbf{L}(V, \mathbb{R}^\infty)$ is transitive as an $\mathcal{L}$-space in the strong sense that any two points are related by the action of an invertible element in $\mathcal{L}$. So the stabilizers of any two points in $\mathbf{L}(V,\mathbb{R}^\infty)$ are conjugate in $\mathcal{L}$.

Lemma 1.8. *For every topological monoid M, every submonoid N and every compact space K the M-space $M/N \times K$ is finite with respect to sequences of closed embeddings of M-spaces.*

Proof. We let

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots$$

be a sequence of morphisms of M-spaces that are closed embeddings of underlying spaces, and

$$f \ : \ M/N \times K \ \longrightarrow \ \mathrm{colim}_{i\geq 0}\ X_i$$

a morphism of M-spaces. The composite

$$K \ \xrightarrow{(N,-)} \ M/N \times K \ \xrightarrow{f} \ \mathrm{colim}_{i\geq 0}\ X_i$$

factors through a continuous map $g : K \longrightarrow X_i$ for some $i \geq 0$ by [11, Prop. 2.4.2] (this uses that singletons in weak Hausdorff spaces are closed). Sequential colimits of compactly generated spaces along injective continuous maps are given by the colimits of underlying sequence of sets [ref to Appendix of Lewis' thesis], so the canonical map $X_i \longrightarrow \mathrm{colim}_{i\geq 0}\ X_i$ is injective. Since the map $f(N,-) : K \longrightarrow \mathrm{colim}_{i\geq 0}\ X_i$ lands in the N-fixed points of the colimit, the factorization g lands in the N-fixed points, so it extends uniquely to a morphism of M-spaces $\tilde{g} : M/N \times K \longrightarrow X_i$ by the adjunction (1.6). Since morphisms out of $M/N \times K$ are determined by their restriction to K, the morphism $\tilde{g}$ is the desired factorization of the original morphism f. $\square$

Now we let $\mathcal{C}$ be a collection of biclosed submonoids of M that is stable under conjugacy by invertible elements of M. We call a morphism $f : X \longrightarrow Y$ of M-spaces a *$\mathcal{C}$-equivalence* (respectively *$\mathcal{C}$-fibration*) if the restriction $f^N : X^N \longrightarrow Y^N$ to N-fixed points is a weak equivalence (respectively Serre fibration) of spaces for all submonoids N of M that belong to the collection $\mathcal{C}$. A *$\mathcal{C}$-cofibration* is a morphism with the right lifting property with respect to all morphisms that are simultaneously $\mathcal{C}$-equivalences and $\mathcal{C}$-fibrations. The resulting '$\mathcal{C}$-projective model structure' is well known in the case when M is a group and $\mathcal{C}$ is a collection of closed subgroups, and the proof for monoids is not much different and fairly standard. However, I do not know a reference in the monoid case, so I provide the proof.

One aspect of the proof occurs several other times in similar contexts, namely that a certain model structure is topological. To avoid repeating the same kind of argument several times, we axiomatize it. We consider a model category $\mathcal{M}$ that is also enriched, tensored and cotensored over the category $\mathbf{T}$ of compactly generated spaces. We denote the tensor by $\times$. Given a continuous map of spaces $f : A \longrightarrow B$ and a morphism $g : X \longrightarrow Y$ in $\mathcal{M}$, we denote by $f\Box g$ the *pushout product* morphism defined as

$$f\Box g = (f \times Y) \cup (A \times g) \ : \ A \times Y \cup_{A\times X} B \times X \ \longrightarrow \ B \times Y \ .$$

We recall that the model structure is called *topological* if the following two conditions hold:

- if f is a cofibration of spaces and g is a cofibration in $\mathcal{M}$, then the pushout product morphism $f\Box g$ is also a cofibration;
- if in addition f or g is a weak equivalence, then so is the pushout product morphism $f\Box g$.

We denote by

$$D^k \ = \ \{x \in \mathbb{R}^k \ : \ \langle x,x\rangle \leq 1\} \qquad \text{and} \qquad \partial D^k \ = \ \{x \in \mathbb{R}^k \ : \ \langle x,x\rangle = 1\}$$

the unit disc in $\mathbb{R}^k$ respectively its boundary, a sphere of dimension $k-1$. In particular, $D^0 = \{0\}$ is a one-point space and $\partial D^0 = \emptyset$ is empty. We denote by

$$i_k \ : \ \partial D^k \ \longrightarrow \ D^k \qquad \text{and} \qquad j_k \ : \ D^k \times \{0\} \ \longrightarrow \ D^k \times [0,1]$$

the inclusions. Then $\{i_k\}_{k\geq 0}$ is the standard set of generating cofibrations for the Quillen model structure on the category of spaces, and $\{j_k\}_{k\geq 0}$ is the standard set of generating acyclic cofibrations, compare Theorem [11, Thm. 2.4.25]. The pushout product condition can also be stated in two different, but equivalent, adjoint forms, compare [11, Lemma 4.2.2].

Proposition 1.9. *Let $\mathcal{M}$ be a model category that is also enriched, tensored and cotensored over the category $\mathbf{T}$ of spaces. Suppose that there is a set of objects $\mathcal{G}$ of $\mathcal{M}$ with the following properties:*

(a) *The acyclic fibrations are characterized by the right lifting property with respect to the morphisms of the form $i_k \times K$ for all $k \geq 0$ and $K \in \mathcal{G}$.*
(b) *The fibrations are characterized by the right lifting property with respect to the morphisms of the form $j_k \times K$ for all $k \geq 0$ and $K \in \mathcal{G}$.*

Then the model structure is topological.

Proof. The hypothesis are saying that $\{i_k \times K\}_{k\geq 0, K\in\mathcal{G}}$ is a set of generating cofibrations for the given model structure on $\mathcal{M}$, and that $\{j_k \times K\}_{k\geq 0, K\in\mathcal{G}}$ is a set of generating acyclic cofibrations. Since the tensor bifunctor $\times$ has an adjoint in each variable, it preserves colimits in each variable. So it suffices to the check the pushout product properties when the maps f and g are from the sets of generating (acyclic) cofibrations, compare [11, Cor. 4.2.5].

The set of inclusions of spheres into discs is closed under pushout product, in the sense that $i_k \Box i_l$ is homeomorphic to i_{k+l}. So pushout product with i_k preserves the set $\{i_k \times K\}_{k\geq 0, K\in K}$ of generating cofibrations (up to isomorphism). Similarly, the pushout product of a sphere inclusion i_k with the inclusion j_l isomorphic to j_{k+l}. So pushout product with i_k preserves the set $\{j_l \times K\}_{l\geq 0, K\in K}$ of generating cofibrations; and pushout product with j_k takes the set $\{i_l \times K\}_{l\geq 0, K\in K}$ of generating cofibrations to the set of generating acyclic cofibrations. □

Proposition 1.10. *Let M be a topological monoid and $\mathcal{C}$ a collection of biclosed submonoids of M. Then the $\mathcal{C}$-equivalences, $\mathcal{C}$-cofibrations and $\mathcal{C}$-fibrations form a model structure, the* $\mathcal{C}$-projective model structure *on the category of M-spaces. This model structure is proper, cofibrantly generated and topological.*

Proof. We refer the reader to [7, 3.3] for the numbering of the model category axioms. The forgetful functor from the category of M-spaces to the category of compactly generated spaces has a left adjoint free functor $M \times -$ and a right adjoint cofree functor $\mathrm{map}(M, -)$; so the category of M-spaces is complete and cocomplete and all limits and colimits are created in the underlying category of compactly generated spaces.

Model category axioms MC2 (2-out-of-3) and MC3 (closure under retracts) are clear. One half of MC4 (lifting properties) holds by the definition of $\mathcal{C}$-cofibrations. The proof of the remaining axioms uses Quillen's small object argument, originally given in [17, II p. 3.4], and later axiomatized in various places, for example in [7, 7.12] or [11, Thm. 2.1.14]. We recall the 'standard' set of generating cofibrations and generating acyclic cofibrations. In the category of (non-equivariant) spaces, the set $\{i_k : \partial D^k \longrightarrow D^k\}_{k\geq 0}$ of inclusions of spheres into discs detects Serre fibrations that are simultaneously weak equivalences. By adjointness (i.e., the bijection (1.6)), the set

$$I_{\mathcal{C}} \;=\; \{M/N \times i_k \;:\; M/N \times \partial D^k \;\longrightarrow\; M/N \times D^k\}_{k\geq 0, N\in\mathcal{C}} \tag{1.11}$$

then detects acyclic fibrations in the $\mathcal{C}$-projective model structure on M-spaces. Similarly, the set of inclusions $\{j_k : D^k \times \{0\} \longrightarrow D^k \times [0,1]\}_{k\geq 0}$ detects Serre fibrations; so by adjointness, the set

$$J_{\mathcal{C}} \;=\; \{M/N \times j_k\}_{k\geq 0, N\in\mathcal{C}} \tag{1.12}$$

detects fibrations in the $\mathcal{C}$-projective model structure on M-spaces.

All morphisms in $I_{\mathcal{C}}$ and $J_{\mathcal{C}}$ are closed embeddings, and this property is preserved by coproducts, cobase change and sequential colimits in the category of M-spaces. Lemma 1.8 guarantees that sources and targets of all morphisms in $I_{\mathcal{C}}$ and $J_{\mathcal{C}}$ are finite (sometimes called 'finitely presented') with respect to sequences of closed embeddings of M-spaces. In particular, the sources of all these morphisms are finite with respect to sequences of morphisms in $I_{\mathcal{C}}$-cell and $J_{\mathcal{C}}$-cell.

Now we can prove the factorization axiom MC5. Every morphism in $I_{\mathcal{C}}$ and $J_{\mathcal{C}}$ is a $\mathcal{C}$-cofibration by adjointness. Hence every $I_{\mathcal{C}}$-cofibration or $J_{\mathcal{C}}$-cofibration is a $\mathcal{C}$-cofibration of M-spaces. The small object argument applied to the set $I_{\mathcal{C}}$ gives a (functorial) factorization of any morphism of M-spaces as a $\mathcal{C}$-cofibration followed by a morphism with the right lifting property with respect to $I_{\mathcal{C}}$. Since $I_{\mathcal{C}}$ detects the $\mathcal{C}$-acyclic $\mathcal{C}$-fibrations, this provides the factorizations as cofibrations followed by acyclic fibrations.

For the other half of the factorization axiom MC5 we apply the small object argument to the set $J_{\mathcal{C}}$; we obtain a (functorial) factorization of any morphism of M-spaces as a $J_{\mathcal{C}}$-cell complex followed by a morphism with the right lifting property with respect to $J_{\mathcal{C}}$. Since $J_{\mathcal{C}}$ detects the $\mathcal{C}$-fibrations, it remains to show that every $J_{\mathcal{C}}$-cell complex is a $\mathcal{C}$-equivalence. To this end we observe that the morphisms in $J_{\mathcal{C}}$ are inclusions of deformation retracts internal to the category of M-spaces. This property is inherited by coproducts and cobase changes, so every morphisms obtained by cobase changes of coproducts of morphisms in $J_{\mathcal{C}}$ is a homotopy equivalence of M-spaces, hence also a $\mathcal{C}$-equivalence. We also need to pass to sequential colimits, which is fine because $J_{\mathcal{C}}$-cell complexes are closed embeddings, and taking N-fixed points commutes with sequential colimits over closed embeddings.

It remains to prove the other half of MC4, i.e., that every $\mathcal{C}$-acyclic $\mathcal{C}$-cofibration $f : A \longrightarrow B$ has the left lifting property with respect to $\mathcal{C}$-fibrations. In other words, we need to show that the $\mathcal{C}$-acyclic $\mathcal{C}$-cofibrations are contained in the $J_{\mathcal{C}}$-cofibrations. The small object argument provides a factorization

$$A \xrightarrow{j} W \xrightarrow{q} B$$

as a $J_{\mathcal{C}}$-cell complex j followed by a $\mathcal{C}$-fibration q. In addition, q is a $\mathcal{C}$-equivalence since f is. Since f is a $\mathcal{C}$-cofibration, a lifting in

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "f"'] & W \arrow[d, "q", "\sim"'] \\ B \arrow[r, equal] \arrow[ur, dotted] & B \end{tikzcd}$$

exists. Thus f is a retract of a morphism q that has the left lifting property for $\mathcal{C}$-fibrations. So f itself has the left lifting property for $\mathcal{C}$-fibrations.

Right properness of the model structure is straightforward from right properness of the model structure on spaces, since the N-fixed point functor, for $N \in \mathcal{C}$, preserves pullbacks and takes $\mathcal{C}$-fibrations to Serre fibrations. For left properness we consider a pushout square of M-spaces

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r] & D \end{tikzcd}$$

such that i is a $\mathcal{C}$-cofibration and f is a $\mathcal{C}$-equivalence. We let N be a submonoid in the collection $\mathcal{C}$. Taking N-fixed points preserves pushouts along $\mathcal{C}$-cofibrations [justify], so the square

$$\begin{tikzcd} A^N \arrow[r, "i^N"] \arrow[d, "f^N"'] & B^N \arrow[d, "g^N"] \\ C^N \arrow[r] & D^N \end{tikzcd}$$

is a pushout. Since i is an h-cofibration of M-spaces, the fixed point map i^N is an h-cofibration of spaces. Since f^N is a weak equivalence and the model category of spaces is left proper, the cobase change g^N is a weak equivalence. Since N was any monoid from the collection $\mathcal{C}$, we conclude that g is a $\mathcal{C}$-equivalence.

The model structure is topological by Proposition 1.9. □

Now we are going to formulate a version of Elmendorf's theorem [8] for the homotopy theory of M-spaces relative to a collection $\mathcal{C}$ of biclosed submonoids. Again, this is well known for topological groups and the proof for topological monoids is essentially the same. However, I do not know a reference of the generalization of Elmendorf's theorem to this context, so I provide a proof.

Construction 1.13. Associated to the collection $\mathcal{C}$ of submonoids of M we define the topological *orbit category* $\mathbf{O}_{M,\mathcal{C}}$ as the full topological subcategory of the category of M-spaces with objects the orbits M/N for $N \in \mathcal{C}$. More precisely, we let $\mathcal{C}$ be the object set of $\mathbf{O}_{M,\mathcal{C}}$, and for $N, N' \in \mathcal{C}$ the morphism space is

$$\mathbf{O}_{M,\mathcal{C}}(N,N') \ = \ M\mathbf{T}(M/N, M/N') \ \cong \ (M/N')^N \ ,$$

where the bijection on the right hand side is by evaluation at the image of the unit element in M/N. The topology of this space is specified by the right hand side, i.e., the subspace topology of the N-fixed points of the orbit space M/N'. Composition is given by composition of M-maps.

For every M-space X the various fixed point subspaces assemble into a continuous functor $\Phi : \mathbf{O}_{M,\mathcal{C}}^{\mathrm{op}} \longrightarrow \mathbf{T}$ on the orbit category via

$$\Phi(X)(N) \ = \ X^N \ \cong \ M\mathbf{T}(M/N, X) \ ,$$

with subspace topology of X. The functoriality in N as an object of $\mathbf{O}_{M,\mathcal{C}}$ comes from bijection (1.6) and composition of M-maps between the orbit spaces.

For every small topological category J with discrete object set the category $J\mathbf{T}$ of continuous functors from J to spaces has a well-known 'projective' model structure (see for example [15, VI Thm. 5.2]) in which the weak equivalence and fibrations are those natural transformations that are weak equivalences respectively Serre fibrations at every object.

In the case of M-spaces and a collection $\mathcal{C}$ of biclosed submonoid the fixed point functor

$$\Phi \ : \ M\mathbf{T} \ \longrightarrow \ \mathbf{O}_{M,\mathcal{C}}\mathbf{T}$$

has a left adjoint Λ, with value at an $\mathbf{O}_{M,\mathcal{C}}$-space F given by a coend of the functor

$$\Lambda(F) \ = \ \int_{N \in \mathcal{C}} M/N \times F(N) \ ,$$

i.e., a coequalizer, in the category of M-spaces, of the two morphisms

$$\textstyle\coprod_{N,N' \in \mathcal{C}} M/N \times \mathbf{O}_{M,\mathcal{C}}(N,N') \times F(N') \rightrightarrows \coprod_{N \in \mathcal{C}} M/N \times F(N) \ .$$

All we will need to know about the left adjoint is that for all $N \in \mathcal{C}$ it takes the representable $\mathbf{O}_{M,\mathcal{C}}$-space $\mathbf{O}_{M,\mathcal{C}}(-,N) = \Phi(M/N)$ to M/N. Indeed, the counit $\epsilon_{M/N} : \Lambda(\Phi(M/N)) \longrightarrow M/N$ induces a bijection of morphism sets

$$\begin{aligned} M\mathbf{T}(\Lambda(\Phi(M/N),X) \ &\cong \ \mathbf{O}_{M,\mathcal{C}}\mathbf{T}(\Phi(M/N),\Phi(X)) \ = \ \mathbf{O}_{M,\mathcal{C}}\mathbf{T}(\mathbf{O}_{M,\mathcal{C}}(-,N),\Phi(X)) \\ &\cong \ \Phi(X)(M/N) \ = \ M\mathbf{T}(M/N,X) \ . \end{aligned}$$

So the counit $\epsilon_{M/N} : \Lambda(\Phi(M/N)) \longrightarrow M/N$ is an isomorphism of M-spaces.

The projective $\mathcal{C}$-model structure is defined so that the fixed point functor Φ preserves and detects weak equivalence and fibrations. So (Λ, Φ) is a Quillen adjoint functor pair.

Proposition 1.14. *Let M be a topological monoid and $\mathcal{C}$ a collection of biclosed submonoids of M.*

(i) *For every cofibrant $\mathbf{O}_{M,\mathcal{C}}$-space F the adjunction unit $F \longrightarrow \Phi(\Lambda F)$ is an isomorphism.*

(ii) *The adjoint functor pair*

$$\mathbf{O}_{M,\mathcal{C}} \underset{\Phi}{\overset{\Lambda}{\rightleftarrows}} M\mathbf{T}$$

is a Quillen equivalence with respect to the $\mathcal{C}$-projective model structure on M-spaces and the projective model structure for $\mathbf{O}_{M,\mathcal{C}}$-spaces.

Proof. (i) We let $\mathcal{G}$ denote the class of $\mathbf{O}_{M,\mathcal{C}}$-spaces for which the adjunction unit is an isomorphism. We show the following property: For every index set I, every I-indexed family N_i of monoids in $\mathcal{C}$, all numbers $n_i \geq 0$ and every pushout square $\mathbf{O}_{M,\mathcal{C}}$-spaces

$$\begin{tikzcd} \coprod_{i\in I} \mathbf{O}_{M,\mathcal{C}}(-,N_i) \times \partial D^{n_i} \arrow[r] \arrow[d] & \coprod_{i\in I} \mathbf{O}_{M,\mathcal{C}}(-,N_i) \times D^{n_i} \arrow[d] \\ F \arrow[r] & G \end{tikzcd} \tag{1.15}$$

such that F belongs to $\mathcal{G}$, the $\mathbf{O}_{M,\mathcal{C}}$-space G also belongs to $\mathcal{G}$.

As a left adjoint, Λ preserves pushout and coproducts. For every space K the functor $- \times K$ is a left adjoint, so it commutes with colimits and coends. So Λ also commutes with products with spaces. Thus Λ takes the original square to a pushout square of M-spaces:

$$\begin{tikzcd} \coprod_{i\in I} M/N_i \times \partial D^{n_i} \arrow[r] \arrow[d] & \coprod_{i\in I} M/N_i \times D^{n_i} \arrow[d] \\ \Lambda F \arrow[r] & \Lambda G \end{tikzcd}$$

The upper horizontal morphisms in this square is a closed embedding. For every biclosed submonoid P of M the P-fixed point functor $(-)^P$ commutes with disjoint unions, products with spaces and pushouts along closed embeddings. So the square

$$\begin{tikzcd} \coprod_{i\in I} (M/N_i)^P \times \partial D^{n_i} \arrow[r] \arrow[d] & \coprod_{i\in I} (M/N_i)^P \times D^{n_i} \arrow[d] \\ (\Lambda F)^P \arrow[r] & (\Lambda G)^P \end{tikzcd}$$

is a pushout in the category of compactly generated spaces. Colimits and products with spaces of $\mathbf{O}_{M,\mathcal{C}}$-spaces are formed objectwise, so letting P run through the monoids in the collection $\mathcal{C}$ shows that the square

$$\begin{tikzcd} \coprod_{i\in I} \Phi(M/N_i) \times \partial D^{n_i} \arrow[r] \arrow[d] & \coprod_{i\in I} \Phi(M/N_i) \times D^{n_i} \arrow[d] \\ \Phi(\Lambda F) \arrow[r] & \Phi(\Lambda G) \end{tikzcd}$$

is a pushout in the category of $\mathbf{O}_{M,\mathcal{C}}$-spaces. The adjunction units induce compatible maps from the original pushout square (1.15) to this last square. Since $\Phi(M/N_i) = \mathbf{O}_{M,\mathcal{C}}(-,N_i)$ and the unit $\eta_F : F \longrightarrow \Phi(\Lambda F)$ is an isomorphism, the unit $\eta_G : G \longrightarrow \Phi(\Lambda G)$ is also an isomorphism.

(ii) The adjoint functor pair (Λ, Φ) is a Quillen pair and the right adjoint Φ preserves and detects weak equivalences. Moreover, for every cofibrant $\mathbf{O}_{M,\mathcal{C}}$-space F the adjunction unit $\eta_F : F \longrightarrow \Phi(\Lambda(F))$ is an isomorphism by (i), hence a weak equivalence. So (Λ, Φ) is a Quillen equivalence, see for example [11, Cor. 1.3.16]. □

2. Global model structures for $\mathcal{L}$-spaces

In this section we define a global model structures on the category of $\mathcal{L}$-spaces and establish a Quillen equivalence to the model category of orbispaces, compare Theorem 2.30.

An *inner product space* is a finite dimensional real vector space equipped with a scalar product, i.e., a positive definite symmetric bilinear form. We denote by $\mathbf{L}$ the category with objects the inner product spaces and morphisms the linear isometric embeddings. The category $\mathbf{L}$ is a topological category in the sense that the morphism spaces come with a preferred topology: if $\varphi : V \longrightarrow W$ is one linear isometric embedding, then the action of the orthogonal group $O(W)$, by postcomposition, induces a bijection

$$O(W)/O(\varphi^\perp) \;\cong\; \mathbf{L}(V,W)\ , \quad A\cdot O(\varphi^\perp) \longmapsto A\circ\varphi\ ,$$

where $\varphi^\perp = W - \varphi(V)$ is the orthogonal complement of the image of φ. We topologize $\mathbf{L}(V,W)$ so that this bijection is a homeomorphism, and this topology is independent of φ. If $(v_1,\dots,v_k)$ is an orthonormal basis of V, then for every linear isometric embedding $\varphi : V \longrightarrow W$ the tuple $(\varphi(v_1),\dots,\varphi(v_k))$ is an orthonormal k-frame of W. This assignment is a homeomorphism from $\mathbf{L}(V,W)$ to the Stiefel manifold of k-frames in W.

An example of an inner product spaces is the vector space $\mathbb{R}^n$ with the standard scalar product

$$\langle x,y\rangle \;=\; \sum_{i=1}^n x_i y_i\ .$$

In fact, every inner product space V is isometrically isomorphic to the inner product space $\mathbb{R}^n$, for n the dimension of V. So the full topological subcategory with objects the $\mathbb{R}^n$ is a small skeleton of $\mathbf{L}$.

We start by making various topologies we use explicit and explain why certain composition and action maps are continuous. We let $\mathcal{V}$ and $\mathcal{U}$ be real inner product spaces of countable infinite dimension, for example $\mathcal{V} = \mathcal{U} = \mathbb{R}^\infty$. A map from $\mathcal{V}$ to $\mathcal{U}$ is a linear isometric embedding if and only if its restriction to every finite dimensional subvector space of $\mathcal{V}$ is a linear isometric embedding. So the set $\mathbf{L}(\mathcal{V},\mathcal{U})$ linear isometric embeddings is an inverse limit of the sets $\mathbf{L}(V,\mathcal{U})$ as V runs over the poset $s(\mathcal{V})$ of finite dimensional subvector spaces of $\mathcal{V}$, and we endow $\mathbf{L}(\mathcal{V},\mathcal{U})$ with the inverse limit topology.

Construction 2.1. We let Y be an orthogonal space. We extend the action maps

$$\mathbf{L}(V,W)\times Y(V) \;\longrightarrow Y(W)$$

which are part of the structure of an orthogonal space to the situation where V and W are allowed to be of countably infinite dimension. If $\mathcal{W}$ is an product spaces of countably infinite dimension, then as before we define

$$Y(\mathcal{W}) \;=\; \mathrm{colim}_{W\in s(\mathcal{W})}\, Y(W)$$

with colimit topology. If V is a finite dimensional inner product space, we define the action map

$$\mathbf{L}(V,\mathcal{W})\times Y(V) \;\longrightarrow\; Y(\mathcal{W})$$

by sending (φ,y) to the image of

$$Y(\bar\varphi)(y) \;\in\; Y(\varphi(V))$$

under the canonical map $Y(\varphi(V)) \longrightarrow Y(\mathcal{W})$, where $\bar\varphi : V \longrightarrow \varphi(V)$ is φ with different range. If $\mathcal{V}$ is also of countably infinite dimension, then $\mathbf{L}(\mathcal{V},\mathcal{W})\times Y(\mathcal{V})$ is the colimit of $\mathbf{L}(\mathcal{V},\mathcal{W})\times Y(V)$ for $V\in s(\mathcal{V})$; so the compatible maps

$$\mathbf{L}(\mathcal{V},\mathcal{W})\times Y(V) \;\xrightarrow{\rho^{\mathcal{V}}_V\times \mathrm{Id}}\; \mathbf{L}(V,\mathcal{W})\times Y(V) \;\xrightarrow{\mathrm{act}}\; Y(\mathcal{W})$$

assemble into an action map.

Proposition 2.2. *Let $\mathcal{U},\mathcal{V}$ and $\mathcal{W}$ be real inner product spaces of finite or countably infinite dimension.*

(i) *Then for every orthogonal space Y the action map*

$$\mathbf{L}(\mathcal{V},\mathcal{W})\times Y(\mathcal{V}) \longrightarrow Y(\mathcal{W}) , \quad (\varphi,y) \longmapsto Y(\varphi)(y)$$

is continuous.

(ii) *The composition map*

$$\circ \ : \ \mathbf{L}(\mathcal{V},\mathcal{W})\times \mathbf{L}(\mathcal{U},\mathcal{V}) \longrightarrow \mathbf{L}(\mathcal{U},\mathcal{W})$$

is continuous.

Proof. (i) There is nothing to show when both $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional. Now we suppose that $\mathcal{V}$ is finite dimensional and $\mathcal{W}$ is infinite dimensional. Since $-\times Y(\mathcal{V})$ is a left adjoint, $\mathbf{L}(\mathcal{V},\mathcal{W})\times Y(\mathcal{V})$ has the colimit topology of $\mathbf{L}(\mathcal{V},W)\times \mathbf{L}(\mathcal{V})$ for $W\in s(\mathcal{W})$. So it suffices to show that the restriction of the action map to $\mathbf{L}(\mathcal{V},W)\times Y(\mathcal{V})$ is continuous for each finite dimensional W inside $\mathcal{W}$. But this restriction factors as

$$\mathbf{L}(\mathcal{V},W)\times Y(\mathcal{V}) \xrightarrow{\text{act}} Y(W) \longrightarrow Y(\mathcal{W}) ,$$

and is thus continuous.

Finally, we assume that both $\mathcal{V}$ and $\mathcal{W}$ are infinite dimensional. Since $\mathbf{L}(\mathcal{V},\mathcal{W})\times -$ is a left adjoint, $\mathbf{L}(\mathcal{V},\mathcal{W})\times Y(\mathcal{V})$ has the colimit topology of $\mathbf{L}(\mathcal{V},\mathcal{W})\times Y(V)$ for $V\in s(\mathcal{V})$. So it suffices to show that the restriction of the action map to $\mathbf{L}(\mathcal{V},\mathcal{W})\times Y(V)$ is continuous for each finite dimensional V inside $\mathcal{V}$. But this restriction factors as

$$\mathbf{L}(\mathcal{V},\mathcal{W})\times Y(V) \xrightarrow{\rho^{\mathcal{V}}_V\times \mathrm{Id}} \mathbf{L}(V,\mathcal{W})\times Y(V) \xrightarrow{\circ} \mathbf{L}(\mathcal{U},\mathcal{W}) ,$$

where $\rho^{\mathcal{V}}_V$ is the (continuous!) restriction from $\mathcal{V}$ to V. This composite is continuous by the previous case.

(ii) If $\mathcal{U}$ is finite dimensional, then the claim is the special case of part (i) for the free orthogonal space $\mathbf{L}(\mathcal{U},-)$. It remains to treat the case when $\mathcal{U}$ is infinite dimensional. Since $\mathbf{L}(\mathcal{U},\mathcal{W})$ has the inverse limit topology, it suffices to show that the composite of the composition map with the restriction map $\mathbf{L}(\mathcal{U},\mathcal{W})\longrightarrow \mathbf{L}(U,\mathcal{W})$ is continuous for each finite dimensional U inside $\mathcal{U}$. But this composite factors as

$$\mathbf{L}(\mathcal{V},\mathcal{W})\times \mathbf{L}(\mathcal{U},\mathcal{V}) \xrightarrow{\mathrm{Id}\times \mathrm{res}} \mathbf{L}(\mathcal{V},\mathcal{W})\times \mathbf{L}(U,\mathcal{V}) \xrightarrow{\circ} \mathbf{L}(U,\mathcal{W})$$

which is continuous by the previous case. □

The special case $U=V=W=\mathbb{R}^\infty$ of the previous proposition shows that $\mathcal{L}=\mathbf{L}(\mathbb{R}^\infty,\mathbb{R}^\infty)$ is a topological monoid with respect to the inverse limit topology.

Definition 2.3. An $\mathcal{L}$-*space* is a space equipped with a continuous action of the topological monoid $\mathcal{L}$.

Example 2.4. Every orthogonal space Y gives rise to an $\mathcal{L}$-space by evaluation at $\mathbb{R}^\infty$. Indeed, for $\mathcal{V}=\mathcal{W}=\mathbb{R}^\infty$, Proposition 2.2 (i) precisely says that the action maps make $Y(\mathbb{R}^\infty)$ into an $\mathcal{L}$-space. This includes trivial $\mathcal{L}$-spaces obtained by equipping any space with the trivial $\mathcal{L}$-action.

If $\mathcal{V}$ is an inner product spaces of finite or countably infinite dimension. Then the space $\mathbf{L}(\mathcal{V},\mathbb{R}^\infty)$ becomes an $\mathcal{L}$-space by postcomposition, by Proposition 2.2 (ii). If $\mathcal{V}$ is infinite dimensional, then $\mathbf{L}(\mathcal{V},\mathbb{R}^\infty)$ does *not* arise from an orthogonal space by evaluation at $\mathbb{R}^\infty$.

We let V be a finite dimensional subspace of $\mathcal{V}$. Then the continuous restriction map

$$\rho \ : \ \mathbf{L}(\mathcal{V},\mathbb{R}^\infty) \longrightarrow \mathbf{L}(V,\mathbb{R}^\infty) , \quad \psi \longmapsto \psi|_V$$

is a morphism of $\mathcal{L}$-spaces.

Proposition 2.5. *We let $\mathcal{V}$ be a real inner product spaces of finite or countably infinite dimension and V a finite dimensional subspace of $\mathcal{V}$. Then the restriction map $\rho:\mathbf{L}(\mathcal{V},\mathbb{R}^\infty)\longrightarrow \mathbf{L}(V,\mathbb{R}^\infty)$ is a locally trivial fiber bundle with fiber homeomorphic to the space $\mathbf{L}(\mathcal{V}-V,\mathbb{R}^\infty)$.*

Proof. This is well known when $\mathcal{V}$ is finite dimensional, and the classical argument also works in our present generality. For a linear isometric embedding $\varphi : V \longrightarrow \mathbb{R}^\infty$ we denote by U_φ the open subset of $\mathbf{L}(V,\mathbb{R}^\infty)$ defined as

$$U_\varphi \;=\; \{\psi \in \mathbf{L}(V,\mathbb{R}^\infty) \;:\; \psi(V)\cap\varphi(V)^\perp = 0\}\;,$$

where $\varphi(V)^\perp$ is the orthogonal complement of the image of φ. We claim that the restriction map is trivial over the open set U_φ. To write the preimage of U_φ as a product we need the Gram-Schmidt orthonormalization. We choose an orthonormal basis $e_1,\dots,e_n$ of V and extend it to a (finite or countable) orthonormal basis $\{e_i\}_{i\geq 1}$ of $\mathcal{V}$. We view the Gram-Schmidt process as a continuous map

$$GS \;:\; \mathrm{Mono}(\mathcal{V},\mathbb{R}^\infty) \;\longrightarrow\; \mathbf{L}(\mathcal{V},\mathbb{R}^\infty)$$

from the space of $\mathbb{R}$-linear monomorphism to the space of linear isometric embeddings. Given a linear monomorphism $f : \mathcal{V} \longrightarrow \mathbb{R}^\infty$, the map $GS(f)$ is defined on the orthogonal basis by sending e_k to

$$(GS(f))(e_k) \;=\; \frac{e_k - p_{k-1}(e_k)}{|e_k - p_{k-1}(v_k)|}\;,$$

where p_{k-1} is the orthogonal projection onto the span of $f(e_1),\dots,f(e_{k-1})$. We use without proof that the Gram-Schmidt orthonormalization map is continuous [...]. We note that if f is already isometric on V, then $GS(f)$ and f agree on V. We can now define a continuous map

$$F \;:\; U_\varphi\times\mathbf{L}(\mathcal{V}-V,\varphi(V)^\perp) \;\longrightarrow\; \rho^{-1}(U_\varphi) \qquad\text{by}\qquad F(\psi,g) \;=\; GS(\psi+g)\;.$$

A continuous map in the other direction is given by

$$G \;:\; \rho^{-1}(U_\varphi) \;\longrightarrow\; U_\varphi\times\mathbf{L}(\mathcal{V}-V,\varphi(V)^\perp) \qquad\text{by}\qquad \psi \;\longmapsto\; (\rho(\psi), P\circ\psi|_{\mathcal{V}-V})\;,$$

where $P : \mathbb{R}^\infty \longrightarrow \varphi(V)^\perp$ is the orthogonal projection away from $\varphi(V)$. $\square$

We let V be a finite dimensional inner product space. The monoid $\mathcal{L}$ acts from the right on the space $\mathbf{L}(V\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)$ by

$$\mathbf{L}(V\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)\times\mathcal{L} \;\longrightarrow\; \mathbf{L}(V\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)\;,\quad (\psi,f)\;\longmapsto\;\psi\circ(V\oplus f)\;.$$

This action commutes with the left action by postcomposition, i.e., it is through morphisms of $\mathcal{L}$-spaces. Similarly, we can define a right action of $\mathcal{L}^2$ by morphisms of $\mathcal{L}$-spaces on $\mathbf{L}(V\oplus\mathbb{R}^\infty\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)$ by

$$\mathbf{L}(V\oplus\mathbb{R}^\infty\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)\times\mathcal{L}^2 \;\longrightarrow\; \mathbf{L}(V\oplus\mathbb{R}^\infty\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)\;,\quad (\psi,f,g)\;\longmapsto\;\psi\circ(V\oplus f\oplus g)\;.$$

The next proposition says, loosely speaking, that restriction to V induces isomorphisms of $\mathcal{L}$-spaces

$$\mathbf{L}(V\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)/\mathcal{L} \;\cong\; \mathbf{L}(V,\mathbb{R}^\infty) \;\cong\; \mathbf{L}(V\oplus\mathbb{R}^\infty\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)/\mathcal{L}^2\;.$$

Proposition 2.6. *Let V be a finite dimensional inner product space.*

(i) *The diagram*

$$\mathbf{L}(V\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)\times\mathcal{L} \;\underset{\mathrm{proj}}{\overset{\mathrm{act}}{\rightrightarrows}}\; \mathbf{L}(V\oplus\mathbb{R}^\infty,\mathbb{R}^\infty) \;\xrightarrow{\;\rho\;}\; \mathbf{L}(V,\mathbb{R}^\infty)$$

is a coequalizer diagram in the category of $\mathcal{L}$-spaces.

(ii) *The diagram*

$$\mathbf{L}(V\oplus\mathbb{R}^\infty\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)\times\mathcal{L}^2 \;\underset{\mathrm{proj}}{\overset{\mathrm{act}}{\rightrightarrows}}\; \mathbf{L}(V\oplus\mathbb{R}^\infty\oplus\mathbb{R}^\infty,\mathbb{R}^\infty) \;\xrightarrow{\;\rho\;}\; \mathbf{L}(V,\mathbb{R}^\infty)$$

is a coequalizer diagram in the category of $\mathcal{L}$-spaces.

Proof. (i) The restriction map ρ is a locally trivial fiber bundle by Proposition 2.5. Every fiber bundle projection is in particular a quotient map, i.e., $\mathbf{L}(V,\mathbb{R}^\infty)$ carries the quotient topology induced by the surjective continuous map ρ. On the other hand, two maps $\varphi,\psi \in \mathbf{L}(V\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)$ agree on the first summand V if and only if they are equivalent in the equivalence relation generated by the right $\mathcal{L}$-action. One direction is obvious. On the other hand, if $\varphi|_V = \psi|_V$, then we choose a linear isometry $\alpha : V\oplus\mathbb{R}^\infty \cong \mathbb{R}^\infty$ that agrees with φ and ψ on the summand V. Then $\alpha^{-1}\varphi$ is the identity on V, so $\alpha^{-1}\varphi = V\oplus\kappa$ for a unique element $\kappa\in\mathcal{L}$. Hence

$$\varphi \;=\; \alpha\circ(\alpha^{-1}\varphi) \;=\; \alpha\circ(V\oplus\kappa)\ ,$$

so φ is equivalent to κ; analogously, ψ is also equivalent to κ, and hence φ and ψ are equivalent.

We have thus shown that the quotient, in the category of all topological spaces, of the equivalence relation coming from the $\mathcal{L}$-action maps homeomorphically to $\mathbf{L}(V,\mathbb{R}^\infty)$. Since $\mathbf{L}(V,\mathbb{R}^\infty)$ is compactly generated and Hausdorff, $\mathbf{L}(V,\mathbb{R}^\infty)$ is then automatically a coequalizer of the right $\mathcal{L}$-action in the category $\mathbf{T}$ of compactly generated spaces. Colimits of $\mathcal{L}$-spaces are created in the underlying category $\mathbf{T}$, which proves the claim.

(ii) This is roughly similar to the first part, but the identification of the equivalence relation generated by the right $\mathcal{L}^2$-action is more involved. We consider the equivalence relation $\sim$ on the set $\mathbf{L}(V\oplus\mathbb{R}^\infty\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)$ generated by

$$\psi\;\sim\;\psi\circ(V\oplus f\oplus g)$$

for all $f,g\in\mathcal{L}$. We claim that two elements are equivalent if and only if they agree on the first summand V. One direction is clear from the definition because ψ and $\psi\circ(V\oplus\alpha\oplus\beta)$ agree on V.

Conversely, suppose that $\psi,\psi'\in\mathbf{L}(V\oplus\mathbb{R}^\infty\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)$ agree on V; we want to show that then $\psi\sim\psi'$. The special case $V=0$ is treated in [9, I Lemma 8.1], and we reduce the general case to this special case. Since V is finite dimensional, we can choose a linear isometry $\kappa : V\oplus\mathbb{R}^\infty\cong\mathbb{R}^\infty$ that satisfies

$$\kappa(v,0)\;=\;\psi(v,0,0)\;=\;\psi'(v,0,0)$$

for all $v\in V$. Then

$$(\kappa^{-1}\psi)(v,0,0)\;=\;(v,0)\;=\;(\kappa^{-1}\psi')(v,0,0)\ ,$$

so

$$\kappa^{-1}\psi\;=\;V\oplus\mu\qquad\text{and}\qquad\kappa^{-1}\psi'\;=\;V\oplus\mu'$$

for two linear isometric embeddings $\mu,\mu'\in\mathbf{L}(\mathbb{R}^\infty\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)$. By [9, I Lemma 8.1] all elements of $\mathbf{L}(\mathbb{R}^\infty\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)=\mathcal{L}(2)$ are equivalent under the equivalence relation generated by $\psi\sim\psi\circ(f\oplus g)$ for $f,g\in\mathcal{L}$; in particular, μ and μ' are related by a finite sequence of such elementary relations. If

$$\mu'\;=\;\mu\circ(f\oplus g)\ ,$$

then

$$\psi'\;=\;\kappa\circ(V\oplus\mu')\;=\;\kappa\circ(V\oplus\mu)\circ(V\oplus f\oplus g)\;=\;\psi\circ(V\oplus f\oplus g)\ ,$$

and so ψ' is equivalent to ψ. In general there is a finite sequence of such elementary relations connecting μ' and μ, and this gives a finite sequence of elementary relations between ψ' and ψ. This proves the claim.

The rest of the argument is as in part (i). Since the restriction map

$$\rho\;:\;\mathbf{L}(V\oplus\mathbb{R}^\infty\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)\;\longrightarrow\;\mathbf{L}(V,\mathbb{R}^\infty)$$

is a fiber bundle projection, it makes $\mathbf{L}(V,\mathbb{R}^\infty)$ a quotient space of $\mathbf{L}(V\oplus\mathbb{R}^\infty\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)$. Since the equivalence relation induced by restriction is the same as the equivalence relation from the $\mathcal{L}^2$-action, $\mathbf{L}(V,\mathbb{R}^\infty)$ is a coequalizer, in the category $\mathbf{T}$ and hence also in $\mathcal{L}\mathbf{T}$, of the $\mathcal{L}^2$-action. □

Definition 2.7. Let G be a compact Lie group. A *G-universe* is an orthogonal G-representation $\mathcal{U}$ of countably infinite dimension with the following two properties:

- the representation $\mathcal{U}$ has non-zero G-fixed points,

- if a finite dimensional G-representation V embeds into $\mathcal{U}$, then a countable infinite direct sum of copies of V also embeds into $\mathcal{U}$.

A G-universe is *complete* if every finite dimensional G-representation embeds into it.

A G-universe is characterized, up to equivariant isometry, by the set of irreducible G-representations that can be embedded into it. We let $\Lambda = \{\lambda\}$ be a complete set of pairwise non-isomorphic irreducible G-representations that embed into $\mathcal{U}$. The first condition says that Λ contains a trivial 1-dimensional representation, and the second condition is equivalent to the requirement that

$$\mathcal{U} \cong \bigoplus_{\lambda\in\Lambda}\bigoplus_{\mathbb{N}} \lambda \ .$$

Moreover, $\mathcal{U}$ is complete if and only if Λ contains (representatives of) *all* irreducible G-representations. In the following we fix, for every compact Lie group G, a complete G-universe $\mathcal{U}_G$.

For the following proposition and for the model structures on $\mathcal{L}$-spaces we need a slight generalization of the notion of a G-universe.

Definition 2.8. Let G be a compact Lie group. A *G-preuniverse* is an orthogonal G-representation $\mathcal{U}$ of countably infinite dimension with the following property: if a finite dimensional G-representation V embeds into $\mathcal{U}$, then a countable infinite direct sum of copies of V also embeds into $\mathcal{U}$.

The only difference between a preuniverse and a universe is that a preuniverse may have trivial fixed points; in contrast, a G-universe always has a non-zero G-fixed point, and hence contains a copy of $\mathbb{R}^\infty$ with trivial G-action. In the same way as universes, G-preuniverses are characterized, up to equivariant isometry, by which irreducible G-representations can be embedded into them (but now the trivial irreducible representation need not be among these).

Proposition 2.9. *Let G and K be compact Lie groups, $\mathcal{V}$ a faithful G-preuniverse and $\mathcal{U}_K$ a complete K-universe.*

(i) *For every faithful finite dimensional G-subrepresentation V of $\mathcal{V}$ the restriction morphism*

$$\rho^{\mathcal{V}}_V \ : \ \mathbf{L}(\mathcal{V},\mathcal{U}_K) \ \longrightarrow \ \mathbf{L}(V,\mathcal{U}_K)$$

is a $(K\times G)$-homotopy equivalence.

(ii) *The $(K\times G)$-space $\mathbf{L}(\mathcal{V},\mathcal{U}_K)$ is a universal space for the family $\mathcal{F}(K;G)$ of graph subgroups.*

(iii) *In the special case $K = G$, the space $\mathbf{L}(\mathcal{V},\mathcal{U}_G)$, viewed as a G-space under the conjugation action, is G-equivariantly contractible.*

Proof. (i) We choose an exhausting nested sequence

$$V \ = \ V_0 \ \subset \ V_1 \ \subset \ V_2 \ \ldots$$

of finite-dimensional G-subrepresentations of $\mathcal{V}$, starting with the given faithful representation. We claim that all the restriction maps

$$\rho_n \ : \ \mathbf{L}(V_n,\mathcal{U}_K) \ \longrightarrow \ \mathbf{L}(V_{n-1},\mathcal{U}_K)$$

are $(K\times G)$-acyclic fibrations, i.e., for every closed subgroup $\Gamma\leq K\times G$ the fixed point map

$$(\rho_n)^\Gamma \ : \ \mathbf{L}(V_n,\mathcal{U}_K)^\Gamma \ \longrightarrow \ \mathbf{L}(V_{n-1},\mathcal{U}_K)^\Gamma$$

is a weak equivalence and Serre fibration. Since G acts faithfully on V_n, the Γ-fixed points of source and target are empty whenever $\Gamma\cap(1\times G) = \{(1,1)\}$. Otherwise Γ is the graph of continuous homomorphism $\alpha : L\longrightarrow G$ with $L\leq K$. So the fixed point map $(\rho_n)^\Gamma$ is the restriction map

$$(\rho_n)^\Gamma \ : \ \mathbf{L}^L(\alpha^*(V_n),\mathcal{U}_K) \ \longrightarrow \ \mathbf{L}^L(\alpha^*(V_{n-1}),\mathcal{U}_K) \ .$$

Source and target of this map are contractible (for example by [18, Prop. I.2.4]), so the map $(\rho_n)^\Gamma$ is a weak equivalence. But $(\rho_n)^\Gamma$ is also a locally trivial fiber bundle, hence a Serre fibration.

Since ρ_n is a $(K\times G)$-acyclic fibration and $\mathbf{L}(V,\mathcal{U}_K)$ is $(K\times G)$-cofibrant, for example by [18, Prop. I.2.2 (ii)]. So we can choose a $(K\times G)$-equivariant section $s_n : \mathbf{L}(V_{n-1},\mathcal{U}_K) \longrightarrow \mathbf{L}(V_n,\mathcal{U}_K)$ to ρ_n and a homotopy

$$H_n \ : \ [0,1]\times \mathbf{L}(V_n,\mathcal{U}_K) \ \longrightarrow \ \mathbf{L}(V_n,\mathcal{U}_K)$$

from the identity to $s_n\circ p_n$ such that $p_n\circ H_n : [0,1]\times\mathbf{L}(V_n,\mathcal{U}_K)\longrightarrow \mathbf{L}(V_{n-1},\mathcal{U}_K)$ is the constant homotopy from p_n to itself. The maps

$$s_n\circ s_{n-1}\circ\cdots\circ s_1 \ : \ \mathbf{L}(V,\mathcal{U}_K) \ \longrightarrow \ \mathbf{L}(V_n,\mathcal{U}_K)$$

are then compatible, so they assemble into a continuous map

$$s_\infty \ : \ \mathbf{L}(V_n,\mathcal{U}_K) \ \longrightarrow \ \lim_n \mathbf{L}(V_n,\mathcal{U}_K) \ = \mathbf{L}(\mathcal{V},\mathcal{U}_K)$$

to the inverse limit, and s_∞ is a section to $\rho_V^{\mathcal{V}}$.

We claim that the composite $s_\infty\circ\rho_V^{\mathcal{V}}$ is homotopic to the identity. To prove the claim we construct compatible homotopies

$$K_n \ : \ [0,1]\times\mathbf{L}(\mathcal{V},\mathcal{U}_K) \ \longrightarrow \ \mathbf{L}(V_n,\mathcal{U}_K)$$

by induction on n satisfying

(i) $p_n\circ K_n = K_{n-1}$,

(ii) $K_n(t,-) = p_\infty^{(n)}$, the restriction from $\mathcal{V}$ to V_n, for all $t\in[0,\frac{1}{n+1}]$, and

(iii) $K_n(1,-) = s_n\circ s_{n-1}\circ\cdots s_1\circ\rho_V^{\mathcal{V}}$.

The induction starts by defining K_0 as the constant homotopy from $\rho_V^{\mathcal{V}} : \mathbf{L}(\mathcal{V},\mathcal{U}_K)\longrightarrow\mathbf{L}(V,\mathcal{U}_K)$ to itself. Now we assume $n\geq 1$ and suppose that the homotopies $K_0,\dots,K_{n-1}$ have already been constructed. We define K_n by

$$K_n(t,-) \ = \ \begin{cases} p_\infty^{(n)} & \text{for } t\in[0,\frac{1}{n+1}],\\ H_n(n(n+1)t-n,\,-)\circ p_\infty^{(n)} & \text{for } t\in[\frac{1}{n+1},\frac{1}{n}], \text{ and}\\ s_n\circ K_{n-1}(t,-) & \text{for } t\in[\frac{1}{n},1].\end{cases}$$

This is well-defined at the intersections of the intervals because

$$H_n\left(n(n+1)\frac{1}{n+1}-n,\,-\right)\circ p_\infty^{(n)} \ = \ H_n(0,-)\circ p_\infty^{(n)} \ = \ p_\infty^{(n)}$$

and

$$\begin{aligned} H_n\left(n(n+1)\frac{1}{n}-n,\,-\right)\circ p_\infty^{(n)} \ &= \ H_n(1,-)\circ p_\infty^{(n)} \ = \ s_n\circ p_n\circ p_\infty^{(n)}\\ &= \ s_n\circ p_\infty^{(n-1)} \ = \ s_n\circ K_{n-1}(1/n,-)\end{aligned}$$

Then condition (i) holds because

$$\begin{aligned} p_n\circ K_n(t,-) \ &= \ \begin{cases} p_n\circ p_\infty^{(n)} & \text{for } t\in[0,\frac{1}{n+1}],\\ p_n\circ H_n(n(n+1)t-n,\,-)\circ p_\infty^{(n)} & \text{for } t\in[\frac{1}{n+1},\frac{1}{n}], \text{ and}\\ p_n\circ s_n\circ K_{n-1}(t,-) & \text{for } t\in[\frac{1}{n},1],\end{cases}\\ &= \ \begin{cases} p_\infty^{(n-1)} & \text{for } t\in[0,\frac{1}{n}],\\ K_{n-1}(t,-) & \text{for } t\in[\frac{1}{n},1],\end{cases}\\ &= \ K_{n-1}(t,-)\ .\end{aligned}$$

Now we can finish the proof. By condition (i) the homotopies K_n are compatible, so they assemble into a continuous map $K_\infty : [0,1]\times\mathbf{L}(\mathcal{V},\mathcal{U}_K)\longrightarrow\mathbf{L}(\mathcal{V},\mathcal{U}_K)$. Property (ii) shows that K_∞ starts with the identity of $\mathbf{L}(\mathcal{V},\mathcal{U}_K)$ and property (iii) ensures that K_∞ ends with the morphism $s_\infty\circ\rho_V^{\mathcal{V}}$. So s_∞ is a homotopy inverse to $\rho_V^{\mathcal{V}}$.

Part (ii) follows from (i) and the fact that the target of the restriction morphism $\rho_V^{\mathcal{V}}$ is a universal $(K \times G)$-space for the family $\mathcal{F}(K;G)$, for example by [18, Prop. I.2.10 (i)].

(iii) We let V be a finite dimensional faithful G-subrepresentation of $\mathcal{V}$. By part (i) the restriction map $\rho_V^{\mathcal{V}} : \mathbf{L}(\mathcal{V}, \mathcal{U}_G) \longrightarrow \mathbf{L}(V, \mathcal{U}_G)$ is a $(G \times G)$-homotopy equivalence, hence a G-homotopy equivalence for the conjugation action on both sides. The target space $\mathbf{L}(V, \mathcal{U}_G)$ is G-equivariantly contractible (for example by [18, Prop. I.2.4]), hence so is $\mathbf{L}(\mathcal{V}, \mathcal{U}_G)$. □

Now we come to a key definition.

Definition 2.10. A compact subgroup G of the topological monoid $\mathcal{L}$ is a *universal subgroup* if it admits the structure of a Lie group (necessarily unique) such that the tautological G-action makes $\mathbb{R}^\infty$ into a G-preuniverse. A universal subgroup is a *completely universal subgroup* if the tautological G-action makes $\mathbb{R}^\infty$ into a complete G-universe.

When G is a universal subgroup of $\mathcal{L}$ we write $\mathbb{R}^\infty_G$ for the G-preuniverse given by the tautological G-action on $\mathbb{R}^\infty$. The G-action on $\mathbb{R}^\infty_G$ is automatically faithful. The next proposition shows that conjugacy classes of completely universal subgroups of $\mathcal{L}$ biject with isomorphism classes of compact Lie groups.

Proposition 2.11. *Every compact Lie group is isomorphic to a completely universal subgroup of $\mathcal{L}$. Every isomorphism between completely universal subgroups is given by conjugation by an invertible linear isometry in $\mathcal{L}$. In particular, isomorphic completely universal subgroups are conjugate in $\mathcal{L}$.*

Proof. Given a compact Lie group G we can choose a continuous isometric linear action of G on $\mathbb{R}^\infty$ that makes $\mathbb{R}^\infty$ into a complete G-universe. Such an action is a continuous monomorphism $\rho : G \longrightarrow \mathcal{L}$ and the image $\rho(G)$ is a completely universal subgroup of $\mathcal{L}$, isomorphic to G via ρ.

Now we let $G, G' \leq \mathcal{L}$ be two completely universal subgroups and $\alpha : G \longrightarrow G'$ an isomorphism. Then $\mathbb{R}^\infty_G$ and $\alpha^*(\mathbb{R}^\infty_{G'})$ are two complete G-universes, so there is a G-equivariant linear isometry $\psi : \mathbb{R}^\infty_G \longrightarrow \alpha^*(\mathbb{R}^\infty_{G'})$. This ψ is an invertible element of the monoid $\mathcal{L}$ and the G-equivariance means that $\psi \circ g = \alpha(g) \circ \psi$ for all $g \in G$. Hence α coincides with conjugation by ψ. □

The topological monoid $\mathcal{L}$ contains many other compact Lie subgroups that are not universal subgroups: any continuous, faithful linear isometric action of a compact Lie group G on $\mathbb{R}^\infty$ provides such a compact Lie subgroup. However, with respect to this action, $\mathbb{R}^\infty$ need not be a G-preuniverse, because some irreducible G-representations may occur with non-zero finite multiplicity.

Definition 2.12. A morphism $f : X \longrightarrow Y$ of $\mathcal{L}$-spaces is

- a *universal equivalence* if for every universal subgroup G of $\mathcal{L}$ the induced map
$$f^G \ : \ X^G \ \longrightarrow \ Y^G$$
is a weak homotopy equivalence;
- a *strong universal equivalence* if for every universal subgroup G of $\mathcal{L}$, the underlying G-map of f is a G-equivariant homotopy equivalence;
- a *universal fibration* if for every universal subgroup G of $\mathcal{L}$ the induced map f^G is a Serre fibration;
- a *global equivalence* if for every completely universal subgroup G of $\mathcal{L}$ the induced map f^G is a weak homotopy equivalence.
- a *strong global equivalence* if for every completely universal subgroup G of $\mathcal{L}$, the underlying G-map of f is a G-equivariant homotopy equivalence.

In other words, $f : X \longrightarrow Y$ is a strong universal equivalence (respectively strong global equivalence) if for every universal subgroup G of $\mathcal{L}$ (respectively every completely universal subgroup) there is a G-equivariant continuous map $g : Y \longrightarrow X$ such that $g \circ f : X \longrightarrow X$ and $f \circ g : Y \longrightarrow Y$ are G-equivariantly homotopic to the respective identity maps. However, there is no compatibility requirement on the homotopy inverses g and the equivariant homotopies. Clearly, the following implications hold between various kinds of equivalences of $\mathcal{L}$-spaces:

$$\begin{array}{ccccc} \mathcal{L}\text{-homotopy equivalence} & \Longrightarrow & \text{strong universal equivalence} & \Longrightarrow & \text{universal equivalence} \\ & & \Downarrow & & \Downarrow \\ & & \text{strong global equivalence} & \Longrightarrow & \text{global equivalence} \\ & & & & \Downarrow \\ & & & & \text{underlying weak equivalence} \end{array}$$

Example 2.13 (Induced $\mathcal{L}$-spaces)**.** We let G be a universal subgroup of $\mathcal{L}$ and A a left G-space. Then we can form the *induced $\mathcal{L}$-space*

$$\mathcal{L} \times_G A \ = \ (\mathcal{L} \times A)/(\varphi g, a) \sim (\varphi, ga) \ .$$

The functor $\mathcal{L} \times_G -$ is left adjoint to the restriction functor from $\mathcal{L}$-spaces to G-spaces.

When A is a one-point space, the previous construction specializes to the 'orbit $\mathcal{L}$-space' $\mathcal{L}/G \cong \mathcal{L} \times_G *$. Because

$$(\mathcal{L}/G)^K \ = \ (\mathbf{L}(\mathbb{R}^\infty_G, \mathbb{R}^\infty_K)/G)^K \ ,$$

the following proposition shows that $\mathcal{L}/G$ is the incarnation, in the world of $\mathcal{L}$-spaces, of the global classifying space of the group G. So for every completely universal subgroup K of $\mathcal{L}$, the underlying K-space of $\mathcal{L}/G$ is a classifying space for principal G-bundles over K-spaces. In particular, the underlying non-equivariant space of $\mathcal{L}/G$ is a classifying space for G.

For every finite-dimensional faithful G-representation V and every compact Lie group K the $(K \times G)$-space $\mathbf{L}(V, \mathcal{U}_K)$ is a universal space for the family $\mathcal{F}(K;G)$ of graph subgroups of $K \times G$, see for example [18, Prop. I.2.10 (i)]. We will now show that this is still true if we replace the finite-dimensional representation V by a faithful G-universe $\mathcal{V}$. The main difference between the finite-dimensional and the universe situation is that $\mathbf{L}(V, \mathcal{U}_K)$ is $(K \times G)$-cofibrant, whereas $\mathbf{L}(\mathcal{V}, \mathcal{U}_K)$ is not; still $\mathbf{L}(\mathcal{V}, \mathcal{U}_K)$ is $(K \times G)$-homotopy equivalent to a cofibrant $(K \times G)$-space.

Proposition 2.14. *Let G be a universal subgroup of the monoid $\mathcal{L}$, V a faithful G-subrepresentation of $\mathbb{R}^\infty_G$ and A a G-space. Then the restriction morphism*

$$\rho_V \times_G A \ : \ \mathcal{L} \times_G A \ = \ \mathbf{L}(\mathbb{R}^\infty_G, \mathbb{R}^\infty) \times_G A \ \longrightarrow \ \mathbf{L}(V, \mathbb{R}^\infty) \times_G A \ = \ (\mathbf{L}_{G,V} A)(\mathbb{R}^\infty)$$

is a strong universal equivalence of $\mathcal{L}$-spaces.

Proof. We let K be a completely universal subgroup of the monoid $\mathcal{L}$. Then $\rho_V : \mathbf{L}(\mathbb{R}^\infty_G, \mathbb{R}^\infty_K) \longrightarrow \mathbf{L}(V, \mathbb{R}^\infty_K)$ is a $(K \times G)$-homotopy equivalence by Proposition 2.9 (ii); so the map

$$\rho_V \times_G A \ : \ \mathbf{L}(\mathbb{R}^\infty_G, \mathbb{R}^\infty_K) \times_G A \ \longrightarrow \ \mathbf{L}(V, \mathbb{R}^\infty_K) \times_G A$$

is a K-homotopy equivalence. □

In Proposition 1.10 we established projective model structures for equivariant spaces with an action of a topological monoid, relative to a collection of biclosed submonoids. The following proposition is the special case for the topological monoid $\mathcal{L}$ and the collection $\mathcal{C}^u$ of universal subgroups.

Proposition 2.15 (Universal model structure)**.** *The universal equivalences and universal fibrations are part of a proper topological closed model category structure on the category of $\mathcal{L}$-spaces, the* universal model structure*. The cofibrations and acyclic cofibrations are generated by the morphisms by the morphisms*

$$\begin{array}{lcl} i_k \times \mathcal{L}/G \ : & \partial D^k \times \mathcal{L}/G \longrightarrow & D^k \times \mathcal{L}/G \quad \textit{respectively} \\ j_k \times \mathcal{L}/G \ : & D^k \times \{0\} \times \mathcal{L}/G \longrightarrow & D^k \times [0,1] \times \mathcal{L}/G \end{array}$$

for all universal subgroups G of $\mathcal{L}$ and all $k \geq 0$.

Let us consider two universal subgroups G, $\bar{G}$ of $\mathcal{L}$, a continuous homomorphism $\alpha : G \longrightarrow \bar{G}$ and a G-equivariant linear isometric embedding $\varphi : \alpha^*(\mathbb{R}^\infty_{\bar{G}}) \longrightarrow \mathbb{R}^\infty_G$. So explicitly, we have

$$\varphi \circ \alpha(g) \;=\; g \circ \varphi$$

for all $g \in G$, and hence φ determines the homomorphism α. Then for every $\mathcal{L}$-space Y the map

$$\varphi \cdot - \;:\; Y \;\longrightarrow\; Y$$

satisfies

$$g \cdot (\varphi \cdot x) \;=\; (g \cdot \varphi) \cdot x \;=\; (\varphi \cdot \alpha(g)) \cdot y \;=\; \varphi \cdot (\alpha(g) \cdot y)$$

for all $g \in G$ and $y \in Y$; hence it restricts to a map

$$\varphi \cdot - \;:\; Y^{\bar{G}} \;\longrightarrow\; Y^G$$

from the $\bar{G}$-fixed points to the G-fixed points. Any two G-equivariant linear isometric embeddings from $\alpha^*(\mathbb{R}^\infty_{\bar{G}})$ to $\mathbb{R}^\infty_G$ are homotopic through G-equivariant linear isometric embeddings, so up to homotopy, the comparison maps between the fixed points does not depend on the choice of φ.

Definition 2.16. An $\mathcal{L}$-space Y is *injective* if for every pair of universal subgroup $G, \bar{G}$ of $\mathcal{L}$, every isomorphism $\alpha : G \longrightarrow \bar{G}$ of Lie groups, and every G-equivariant linear isometric embedding $\varphi : \alpha^*(\mathbb{R}^\infty_{\bar{G}}) \longrightarrow \mathbb{R}^\infty_G$ the map

$$\varphi \cdot - \;:\; Y^{\bar{G}} \;\longrightarrow\; Y^G$$

is a weak equivalence.

Remark 2.17. If Y is an injective $\mathcal{L}$-space, then for every universal subgroup G of $\mathcal{L}$, the homotopy type of the fixed point space Y^G only depends on the isomorphism class of G as an abstract Lie group. Indeed, if G and $\bar{G}$ are isomorphic universal subgroups, and if G is even completely universal, then there always exists G-equivariant linear isometric embedding $\varphi : \alpha^*(\mathbb{R}^\infty_{\bar{G}}) \longrightarrow \mathbb{R}^\infty_G$ because the target is a complete universe; the G-fixed points and $\bar{G}$-fixed points can then be compared via φ. If neither G nor $\bar{G}$ are completely universal, then there exists another completely universal subgroup G' of $\mathcal{L}$ that is abstractly isomorphic to G and $\bar{G}$, and the G-fixed points can be compared to the $\bar{G}$-fixed points through the G'-fixed points.

The universal model structure on $\mathcal{L}$-space has 'too many homotopy types' for our purposes, i.e., it is not Quillen equivalent to the global model structure on orthogonal spaces. We fix this by performing a Bousfield localization on the universal model structure, with the injective $\mathcal{L}$-spaces as the local objects.

Construction 2.18. We let $j : A \longrightarrow B$ be a morphism in a topological model category. We factor j through the mapping cylinder as the composite

$$A \xrightarrow{\;c(j)\;} Z(j) = ([0,1] \times A) \cup_j B \xrightarrow{\;r(j)\;} B\ ,$$

where $c(j)$ is the 'front' mapping cylinder inclusion and $r(j)$ is the projection, which is a homotopy equivalence. In our applications we will assume that both A and B are cofibrant, and then the morphism $c(j)$ is a cofibration by the pushout product property. We then define $\mathcal{Z}(j)$ as the set of all pushout product maps

$$i_k \Box c(j) \;:\; D^k \times A \cup_{\partial D^k \times A} \partial D^k \times Z(j) \;\longrightarrow\; D^k \times Z(j)$$

for $k \geq 0$, where $i_k : \partial D^k \longrightarrow D^k$ is the inclusion.

Proposition 2.19. *Let $\mathcal{C}$ be a topological model category, $j : A \longrightarrow B$ a morphism between cofibrant objects and $f : X \longrightarrow Y$ a fibration. Then the following two conditions are equivalent:*

(i) *The square of spaces*

$$\begin{tikzcd} \mathrm{map}(B,X) \arrow[r, "\mathrm{map}(j,X)"] \arrow[d, "\mathrm{map}(B,f)"'] & \mathrm{map}(A,X) \arrow[d, "\mathrm{map}(A,f)"] \\ \mathrm{map}(B,Y) \arrow[r, "\mathrm{map}(j,Y)"'] & \mathrm{map}(A,Y) \end{tikzcd} \tag{2.20}$$

is homotopy cartesian.

(ii) *The morphism f has the right lifting property with respect to the set $\mathcal{Z}(j)$.*

Proof. The square (2.20) maps to the square

$$\begin{tikzcd} \mathrm{map}(Z(j),X) \arrow[r, "\mathrm{map}(c(j),X)"] \arrow[d, "\mathrm{map}(Z(j),f)"'] & \mathrm{map}(A,X) \arrow[d, "\mathrm{map}(A,f)"] \\ \mathrm{map}(Z(j),Y) \arrow[r, "\mathrm{map}(c(j),Y)"'] & \mathrm{map}(A,Y) \end{tikzcd} \tag{2.21}$$

via the map induced by $r(j) : Z(j) \longrightarrow B$ on the left part and the identity on the right part. Since $r(j)$ is a homotopy equivalence, the map of squares is a weak equivalence at all four corners. So the square (2.20) is homotopy cartesian if and only if the square (2.21) is homotopy cartesian.

Since A is cofibrant and f a fibration, $\mathrm{map}(A, f)$ is a Serre fibration. So the square (2.21) is homotopy cartesian if and only if the map

$$(\mathrm{map}(Z(j), f), \mathrm{map}(c(j), X)) \ : \ \mathrm{map}(Z(j), X) \ \longrightarrow \ \mathrm{map}(Z(j), Y) \times_{\mathrm{map}(A,Y)} \mathrm{map}(A, X) \tag{2.22}$$

is a weak equivalence. Since $c(j)$ is a cofibration and f is a fibration, the map (2.22) is always a Serre fibration. So (2.22) is a weak equivalence if and only if it is an acyclic fibration, which is equivalent to the right lifting property for the inclusions $i_k : \partial D^k \longrightarrow D^k$ for all $k \geq 0$. By adjointness, the map (2.22) has the right lifting property with respect to the maps i_k if and only if the morphism f has the right lifting property with respect to the set $\mathcal{Z}(j)$. □

The following proposition provides the necessary localization functor for the Bousfield localization.

Proposition 2.23. *There is an endofunctor Q of the category of $\mathcal{L}$-spaces with values in injective $\mathcal{L}$-spaces and a natural global equivalence $j_X : X \longrightarrow QX$.*

Proof. For every pair of universal subgroups $G, \bar{G}$ of $\mathcal{L}$, every homomorphism $\alpha : G \longrightarrow \bar{G}$ of Lie groups and every G-equivariant linear isometric embedding $\varphi : \alpha^*(\mathbb{R}^\infty_{\bar{G}}) \longrightarrow \mathbb{R}^\infty_G$, the morphism

$$\varphi_\sharp \ : \ \mathcal{L}/G \ \longrightarrow \ \mathcal{L}/\bar{G} \ , \quad \psi \cdot G \ \longmapsto \ (\psi \circ \varphi) \cdot \bar{G}$$

represents the natural transformation

$$\varphi \cdot - \ : \ Y^{\bar{G}} \ \longrightarrow \ Y^G \ .$$

If α is an isomorphism, then the morphism $\varphi_\sharp$ is a global equivalence by Proposition 2.14 [fix this], but it is not a cofibration in any sense.

We factor $\varphi_\sharp$ through the mapping cylinder as the composite

$$\mathcal{L}/G \ \xrightarrow{c_{G,V}} \ Z(\varphi_\sharp) = ([0,1] \times \mathcal{L}/G) \cup_{\varphi_\sharp} \mathcal{L}/\bar{G} \ \xrightarrow{r_{\varphi_\sharp}} \ \mathcal{L}/\bar{G} \ ,$$

where $c_{\varphi_\sharp}$ is the 'front' mapping cylinder inclusion and $r_{\varphi_\sharp}$ is the projection, which is a homotopy equivalence. We then define

$$K \ = \ \bigcup_{(G,\bar{G},\varphi)} \ \mathcal{Z}(\rho_{\varphi_\sharp})$$

as the set of all pushout product maps with the inclusions $\partial D^m \longrightarrow D^m$, compare Construction 2.18. Here $(G, \bar{G}, \varphi)$ runs through all triples consisting of a universal subgroups $G, \bar{G} \subset \mathcal{L}$ and G-equivariant linear isometric embeddings $\varphi : \alpha^*(\mathbb{R}^\infty_{\bar{G}}) \longrightarrow \mathbb{R}^\infty_G$. Proposition 2.19 (with Y a one-point $\mathcal{L}$-space) shows that the right lifting property with respect to the set K is equivalent to being injective.

Now we apply the countable small object argument with respect to the set K to the unique morphism from a given $\mathcal{L}$-space Y to the terminal $\mathcal{L}$-space. A countable version (i.e., with sequential colimits) of Quillen's original argument which works in our case can be found in [7, Prop. 7.17]. Dwyer and Spalinski assume that the sources of all morphisms in the set K are sequentially small, which is not the case here. However, what is really needed is only that the sources of all morphisms in the set K are sequentially small for cobase changes of coproducts of morphisms in K, compare the more general version of the small object argument in [11, Thm. 2.1.14]. In our situation, all morphisms in K are h-cofibrations, hence so are all cobase changes of coproducts, and the sources of morphisms in are small with respect to sequences of h-cofibrations.

In any case, the small object argument produces an endofunctor Q on the category of $\mathcal{L}$-spaces and a natural transformation $j_Y : Y \longrightarrow QY$ with the following properties:

(i) The object QY has the right lifting property with respect to all morphisms in K, i.e., it is injective.
(ii) The morphism j_Y is a sequential composite of cobase changes of coproducts of morphisms in K.

All morphisms in K are simultaneously h-cofibrations and global equivalences; the class of h-cofibrations that are also global equivalences is closed under coproducts, cobase change and sequential composition. So the morphism $j_Y : Y \longrightarrow QY$ is an h-cofibration and a global equivalence. □

Now we have all the ingredients to localize the universal model structure into a second 'global' model structure on the category of $\mathcal{L}$-spaces.

Theorem 2.24 (Global model structure for $\mathcal{L}$-spaces)**.** *The $\mathcal{C}^u$-cofibrations and global equivalences are part of a cofibrantly generated proper topological model structure on the category of $\mathcal{L}$-spaces, the* global model structure*. The fibrant objects in the global model structure are the injective $\mathcal{L}$-spaces.*

Proof. We construct the global model structure by applying Bousfield's localization theorem [5, Thm. 9.3] to the universal model structure. We use the localization functor Q given by Proposition 2.23. By the very definition of 'injective', a global equivalence between injective $\mathcal{L}$-spaces is already a universal equivalence. So the Q-equivalences in the sense of [5, Thm. 9.3] are precisely the global equivalences.

Now we verify the hypotheses (A1)–(A3) of Bousfield's theorem. The universal model structure is proper. If f is a universal equivalence, then Qf is a global equivalence between injective $\mathcal{L}$-spaces, hence a universal equivalence. This shows (A1).

The morphism j_{QX} is a global equivalence between injective $\mathcal{L}$-spaces, hence a universal equivalence. On the other hand, $Q(j_X) : QX \longrightarrow QQX$ is a universal equivalence since Q takes all global equivalences to universal equivalences. So j_{QX} and $Q(j_X)$ are also universal equivalences, and this proves axiom (A2).

In axiom (A3) we are given a pullback square

$$\begin{tikzcd} V \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "h"'] & Y \end{tikzcd}$$

of $\mathcal{L}$-spaces in which f is a universal fibration such that $j_X : X \longrightarrow QX$, $j_Y : Y \longrightarrow QY$ and Qh are universal equivalences. We have to show that then Qk is a universal equivalence. These hypothesis can be reformulated as follows: the $\mathcal{L}$-spaces X and Y are injective, f is a universal fibration and h is a global equivalence. We have to show that then k is a global equivalence. But this is straightforward: for every completely universal subgroup G of $\mathcal{L}$ the square

$$\begin{tikzcd} V^G \arrow[r, "k^G"] \arrow[d, "g^G"'] & X^G \arrow[d, "f^G"] \\ W^G \arrow[r, "h^G"'] & Y^G \end{tikzcd}$$

is a pullback, f^G is a Serre fibration and h^G is a weak equivalence. Since the model structure of topological spaces is right proper, the map k^G is again a weak equivalence. Hence k is a global equivalence.

This proves (A3), and thus Bousfield's theorem applied to the universal model structures provides a proper model structures with global equivalences as weak equivalences and with $\mathcal{C}^u$-cofibrations as the cofibrations. Bousfield's theorem also provides the characterization of the fibrations in this global model structure. For $Y = *$ the criterion specializes to the fact that X is fibrant if and only if it is $\mathcal{C}^u$-fibrant (an empty condition) and the morphism $j_X : X \longrightarrow QX$ is a universal equivalence. Since QX is injective, the fibrancy is equivalent to X being injective.

The cofibrations in the global model structures are the same as the cofibrations in the universal model structure, so the part of the pushout product property that involves only cofibrations (but not equivalences) holds because the universal model structure is topological [fill in the rest] □

We close this section by giving rigorous meaning to the slogan that global homotopy theory of $\mathcal{L}$-spaces is the homotopy theory of 'orbispaces with compact Lie group isotropy'. In Section 1 we establish a version of Elmendorf's theorem, saying that an equivariant homotopy type can be reassembled from fixed point data; our generalization works for topological monoids relative to a collection of biclosed submonoids. The identification of the global homotopy theory of $\mathcal{L}$-spaces with the homotopy theory of orbispaces is just a special case of this. Indeed, the global orbit category defined in the following construction is simply the orbit category, in the sense of Construction 1.13, of the topological monoid $\mathcal{L}$ relative to the collection of completely universal subgroups.

Construction 2.25 (Global orbit category)**.** We define a topological category $\mathbf{O}_{\mathrm{gl}}$, the *global orbit category.* The object of $\mathbf{O}_{\mathrm{gl}}$ are all completely universal subgroups of the monoid $\mathcal{L}$, and the space of morphisms from K to G is the space

$$\mathbf{O}_{\mathrm{gl}}(K,G) \;=\; (\mathcal{L}/G)^K \;=\; (\mathbf{L}(\mathbb{R}^\infty_G,\mathbb{R}^\infty_K)/G)^K .$$

Here the $(K\times G)$-action on $\mathcal{L}$ is by pre- and postcomposition. Then $\mathbf{O}_{\mathrm{gl}}(K,G)$ is the space of K-fixed points of the G-orbit space. Composition in $\mathbf{O}_{\mathrm{gl}}$ is induced by composition of linear isometric embeddings. Indeed, the continuous $\mathcal{L}$-action

$$\mathcal{L}\times\mathcal{L}/G \;\longrightarrow\; \mathcal{L}/G \tag{2.26}$$

is compatible with fixed points: If $\varphi\in\mathcal{L}$ is a linear isometric embedding whose orbit φG is K-fix, then the relation

$$(\psi k)\circ(\varphi G) \;=\; \psi\circ(k\varphi G) \;=\; \psi\circ\varphi G \;=\; \psi\varphi G$$

shows that the G-orbit of $\psi\varphi$ only depends on the K-orbit of ψ. So the restriction of (2.26) to $\mathcal{L}\times(\mathcal{L}/G)^K$ factors over a well-defined map

$$(\mathcal{L}/K)\times(\mathcal{L}/G)^K \;\longrightarrow\; \mathcal{L}/G \ .$$

Finally, if the K-orbit ψK is L-fix and the G-orbit φG is K-fix, then the relation

$$l(\psi\varphi G) \;=\; (l\psi)\circ(\varphi G) \;=\; (\psi k)\circ(\varphi G) \;=\; \psi\circ(k\varphi G) \;=\; \psi\circ\varphi G \;=\; \psi\varphi G$$

shows that the G-orbit of $\psi\varphi$ is L-fix. So the composition map indeed passes to a well-defined continuous composition map

$$\mathbf{O}_{\mathrm{gl}}(L,K)\times\mathbf{O}_{\mathrm{gl}}(K,G) \;=\; (\mathcal{L}/K)^L \,\times\, (\mathcal{L}/G)^K \;\longrightarrow\; (\mathcal{L}/G)^L = \mathbf{O}_{\mathrm{gl}}(L,G) \ .$$

Remark 2.27. The global orbit category refines the category Rep of compact Lie groups and conjugacy classes of continuous homomorphisms in the following sense. For all completely universal subgroups G and K, the components $\pi_0(\mathbf{O}_{\mathrm{gl}}(K,G))$ biject functorially with $\mathrm{Rep}(K,G)$. Indeed, by Proposition 2.9 the $(K\times G)$-space $\mathbf{L}(\mathbb{R}^\infty_G,\mathbb{R}^\infty_K)$ is a universal space for the family $\mathcal{F}(K;G)$ of graph subgroups. So the space $\mathbf{O}_{\mathrm{gl}}(K,G)=(\mathcal{L}/G)^K$ is a disjoint union, indexed by conjugacy classes of continuous group homomorphisms $\alpha:K\longrightarrow G$, of classifying spaces of the centralizer of the image of α, see for example [12, Prop. 5] or [18, Prop. I.6.14 (i)]. In particular, the path component category $\pi_0(\mathbf{O}_{\mathrm{gl}})$ of the global orbit category is isomorphic to the category Rep of compact Lie groups and conjugacy classes of continuous homomorphisms. The preferred bijection

$$\mathrm{Rep}(K,G)\ \longrightarrow\ \pi_0(\mathbf{O}_{\mathrm{gl}}(K,G))$$

sends the conjugacy class of $\alpha:K\longrightarrow G$ to the G-orbit of any K-equivariant linear isometric embedding of the K-universe $\alpha^*(\mathbb{R}^\infty_G)$ into the complete K-universe $\mathbb{R}^\infty_K$.

Definition 2.28. An *orbispace* is a continuous functor $Y:\mathbf{O}_{\mathrm{gl}}^{\mathrm{op}}\longrightarrow\mathbf{T}$ from from the opposite of the global orbit category to the category of spaces. We denote the category of orbispaces and natural transformations by *orbispc*.

It would be somewhat more precise (but too lengthy) to speak of 'orbispaces with compact Lie isotropy', but no confusion should arise because will not consider more general classes of allowed isotropy groups.

Construction 2.29. We introduce a *fixed point functor*

$$\Phi\ :\ \mathcal{L}\mathbf{T}\ \longrightarrow\ \mathit{orbispc}$$

from the category of $\mathcal{L}$-spaces to the category of orbispaces that will turn out to be a right Quillen equivalence with respect to the projective global model structure on the left hand side. Given an $\mathcal{L}$-space Y we define the value of the orbispace $\Phi(Y)$ at a completely universal subgroup G as the G-fixed points

$$\Phi(Y)(G)\ =\ Y^G\ \cong\ \mathrm{map}_{\mathcal{L}}(\mathcal{L}/G,Y)\ .$$

The restriction of the action map $\mathcal{L}\times Y\longrightarrow Y$ to Y^G factors over a morphism of $\mathcal{L}$-spaces

$$\mathcal{L}/G\times Y^G\ \longrightarrow\ Y$$

(with trivial $\mathcal{L}$-action on Y^G). So for a second completely universal subgroup K of $\mathcal{L}$, the restriction to K-fixed points is the action map

$$\mathbf{O}_{\mathrm{gl}}(K,G)\times\Phi(Y)(G)\ =\ (\mathcal{L}/G)^K\times Y^G\ \longrightarrow\ Y^K\ =\ \Phi(Y)(K)\ .$$

As an example of this construction we note that

$$\Phi(\mathcal{L}/G)\ =\ \mathbf{O}_{\mathrm{gl}}(-,G)\ ,$$

i.e., the fixed points of the orbit $\mathcal{L}$-space $\mathcal{L}/G$ form the orbispace represented by G.

As for continuous functors out of any topological category, the category of orbispaces supports a well-known 'projective' model structure in which the weak equivalences (respectively fibrations) are those natural transformations that are weak equivalences (respectively Serre fibrations) at every object, see for example [15, VI Thm. 5.2]. By general arguments, the fixed point functor Φ just defined has a left adjoint Λ. The following is then the special case of Proposition 1.14 for the topological monoid $\mathcal{L}$ with respect to the family $\mathcal{C}^u$ of universal subgroups. [fix this]

Theorem 2.30. *The adjoint functor pair*

$$\mathit{orbispc}\ \underset{\Phi}{\overset{\Lambda}{\rightleftarrows}}\ \mathcal{L}\mathbf{T}$$

is a Quillen equivalence between the category of $\mathcal{L}$-spaces with the global model structure and the category of orbispaces. Moreover, for every cofibrant orbispace F the adjunction unit $F\longrightarrow\Phi(\Lambda F)$ is an isomorphism.

The Quillen equivalence is also well-behaved from a monoidal perspective. We endow the category of orbispaces with the cartesian monoidal structure, i.e., the objectwise product of orbispaces. Since the morphism $\rho_{X,Y} : X \boxtimes_{\mathcal{L}} Y \longrightarrow X \times Y$ is a strong global equivalence for all orthogonal spaces X and Y, the morphisms of orbispaces

$$\Phi(\rho_{X,Y}) \ : \ \Phi(X \boxtimes_{\mathcal{L}} Y) \ \longrightarrow \Phi(X \times Y) \ \cong \ \Phi(X) \times \Phi(Y)$$

is a weak equivalence. Moreover, this morphism makes the right adjoint into a lax symmetric comonoidal functor.

Construction 2.31. The fixed point Quillen equivalence can be used to push any continuous and functorial construction for spaces to $\mathcal{L}$-spaces. In more detail, let us consider a continuous functor

$$F \ : \ \mathbf{T} \ \longrightarrow \ \mathbf{T}$$

from the category of spaces to itself. Given an $\mathcal{L}$-space Y, we take its fixed point functor ΦY and postcompose it with F. The result is the continuous composite functor

$$F \circ (\Phi Y) \ : \ \mathrm{Rep}^{\mathrm{op}} \ \longrightarrow \ \mathbf{T} \ .$$

We take a cofibrant replacement $(F \circ (\Phi Y))^c \longrightarrow F \circ (\Phi Y)$ in the model category of orbispaces (which can be done functorially by the small object argument). Then

$$\bar{F}(Y) \ = \ \Lambda((F \circ (\Phi Y))^c)$$

is an $\mathcal{L}$-space. We obtain a chain of two weak equivalences of orbispaces

$$F \circ (\Phi Y) \ \longleftarrow \ (F \circ (\Phi Y))^c \ \xrightarrow{\eta} \ \Phi(\bar{F}(Y)) \ ,$$

where η is the adjunction unit.

This shows:

Proposition 2.32. *Let $F : \mathbf{T} \longrightarrow \mathbf{T}$ be a continuous functor from the category of spaces to itself. Then there is a functor $\bar{F}$ from the category of $\mathcal{L}$-spaces to itself and a natural chain of weak equivalences of orbispaces*

$$F \circ (\Phi Y) \qquad \textit{and} \qquad \Phi(\bar{F}(Y)) \ .$$

We emphasize that the $\mathcal{L}$-space $\bar{F}(Y)$ is *not* in general obtained by applying F to the underlying space of Y with the induced $\mathcal{L}$-action, because F need not commute with fixed points of group actions. However, there is a natural map relating these two constructions. For every universal subgroup G of $\mathcal{L}$ the map $F(\mathrm{incl}) : F(Y^G) \longrightarrow F(Y)$ has image in $F(Y)^G$, and as G varies these maps define a morphism of orbispaces

$$\iota \ : \ F \circ (\Phi Y) \ \longrightarrow \ \Phi(F(Y)) \ .$$

Precomposition with the cofibrant replacement and forming of adjoint is a morphism of $\mathcal{L}$-spaces

$$\bar{F}(Y) \ = \ \Lambda((F \circ (\Phi Y))^c) \ \longrightarrow \ F(Y) \ .$$

3. $\mathcal{L}$-spaces and orthogonal spaces

The aim of this section is to compare the global homotopy theory of $\mathcal{L}$-spaces with the global homotopy theory of orthogonal spaces as developed by the author in Chapter I of [18]; we will show in Theorem 3.7. that the global model structures of orthogonal spaces and of $\mathcal{L}$-spaces are Quillen equivalent.

Definition 3.1. An *orthogonal space* is a continuous functor $Y : \mathbf{L} \longrightarrow \mathbf{T}$ to the category of spaces. A morphism of orthogonal spaces is a natural transformation. We denote by *spc* the category of orthogonal spaces.

The use of continuous functors from the category $\mathbf{L}$ to spaces has a long history in homotopy theory. The systematic use of inner product spaces (as opposed to numbers) to index objects in stable homotopy theory seems to go back to Boardman's thesis [2]. The category $\mathbf{L}$ (or its extension that also contains countably infinite dimensional inner product spaces) is denoted $\mathscr{I}$ by Boardman and Vogt [3], and this notation is also used in [14]; other sources [13] use the symbol $\mathcal{I}$. Accordingly, orthogonal spaces are sometimes referred to as $\mathscr{I}$-functors, $\mathscr{I}$-spaces or $\mathcal{I}$-spaces. Our justification for using yet another name is twofold: on the one hand, our use of orthogonal spaces comes with a shift in emphasis, away from a focus on non-equivariant homotopy types, and towards viewing an orthogonal space as representing compatible equivariant homotopy types for all compact Lie groups. Secondly, we want to stress the analogy between orthogonal spaces and orthogonal spectra, the former being an unstable global world with the latter a corresponding stable global world.

Now we define our main new concept, the notion of 'global equivalence' between orthogonal spaces. We let G be a compact Lie group. By a *G-representation* we mean a finite dimensional orthogonal representation, i.e., a real inner product space equipped with a continuous G-action by linear isometries. In other words, a G-representation consists of an inner product space V and a continuous homomorphism $\rho : G \longrightarrow O(V)$. In this context, and throughout the book, we will often use without explicit mentioning that continuous homomorphisms between Lie groups are automatically smooth. For every orthogonal space Y and every G-representation V, the value $Y(V)$ inherits a G-action from the G-action on V and the functoriality of Y. For a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ the induced map $Y(\varphi) : Y(V) \longrightarrow Y(W)$ is G-equivariant.

Definition 3.2. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is a *global equivalence* if the following condition holds: for every compact Lie group G, every G-representation V, every $k \geq 0$ and all continuous maps $\alpha : \partial D^k \longrightarrow X(V)^G$ and $\beta : D^k \longrightarrow Y(V)^G$ such that $\beta|_{\partial D^k} = f(V)^G \circ \alpha$, there is a G-representation W, a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous map $\lambda : D^k \longrightarrow X(W)^G$ such that $\lambda|_{\partial D^k} = X(\varphi)^G \circ \alpha$ and such that $f(W)^G \circ \lambda$ is homotopic, relative to ∂D^k, to $Y(\varphi)^G \circ \beta$.

In other words, for every commutative square on the left

$$
\begin{tikzcd}
\partial D^k \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & X(V)^G \arrow[d, "f(V)^G"] \\
D^k \arrow[r, "\beta"'] & Y(V)^G
\end{tikzcd}
\qquad
\begin{tikzcd}
\partial D^k \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & X(V)^G \arrow[r, "X(\varphi)^G"] & X(W)^G \arrow[d, "f(W)^G"] \\
D^k \arrow[r, "\beta"'] \arrow[urr, dashed, "\lambda"] & Y(V)^G \arrow[r, "Y(\varphi)^G"'] & Y(W)^G
\end{tikzcd}
$$

there exists the lift λ on the right hand side that makes the upper left triangle commute on the nose, and the lower right triangle up to homotopy relative to ∂D^k. In such a situation we will often refer to the pair (α, β) as a 'lifting problem' and we will say that the pair (φ, λ) *solves the lifting problem.*

Example 3.3. If $X = \underline{A}$ and $Y = \underline{B}$ are the constant orthogonal spaces with values the spaces A respectively B, and $f = \underline{g}$ the constant morphism associated to a continuous map $g : A \longrightarrow B$, then $\underline{g}$ is a global equivalence if and only if for every commutative square

$$
\begin{tikzcd}
\partial D^k \arrow[r] \arrow[d, "\mathrm{incl}"'] & A \arrow[d, "g"] \\
D^k \arrow[r] \arrow[ur, dashed, "\lambda"] & B
\end{tikzcd}
$$

there exists the lift λ that makes the upper left triangle commute, and the lower right triangle up to homotopy relative to ∂D^k. But this is one of the equivalent ways of characterizing weak equivalences of spaces. So $\underline{g}$ is a global equivalence if and only if g is a weak equivalence.

Example 3.4. Every orthogonal space Y gives rise to an $\mathcal{L}$-space by evaluation at $\mathbb{R}^\infty$. Indeed, for $\mathcal{V} = \mathcal{W} = \mathbb{R}^\infty$, Proposition 2.2 precisely says that the action maps make $Y(\mathbb{R}^\infty)$ into an $\mathcal{L}$-space. This includes trivial $\mathcal{L}$-spaces obtained by equipping any space with the trivial $\mathcal{L}$-action.

Remark 3.5. The notion of global equivalence is meant to capture the idea that for every compact Lie group G, some induced morphism

$$\mathrm{hocolim}_V\, f(V) \ : \ \mathrm{hocolim}_V\, X(V) \ \longrightarrow \ \mathrm{hocolim}_V\, Y(V)$$

is a G-weak equivalence, where '$\mathrm{hocolim}_V$' is a suitable homotopy colimit over all G-representations V along all equivariant linear isometric embeddings. This is a useful way to think about global equivalences, and it could be made precise by letting V run over the poset of finite dimensional subrepresentations of a complete G-universe and using the Bousfield-Kan construction of a homotopy colimit over this poset. However, the actual definition that we work with has the advantage that we do not have to make precise what we mean by 'all' G-representations and we do not have to define or manipulate homotopy colimits.

In many examples of interest, all the structure maps of an orthogonal space Y are closed embeddings. When this is the case, the actual colimit (over the subrepresentations of a complete universe) of the G-spaces $Y(V)$ serves the purpose of a 'homotopy colimit over all representations', and it can be used to detect global equivalences, compare [18, Prop. I.20].

Theorem I.5.10 of [18] establishes the global model structure on the category of orthogonal spaces in which the weak equivalences are the global equivalences. A morphism f is a global fibration if and only if f is a strong level fibration and for every compact Lie group G, every faithful G-representation V and equivariant linear isometric embedding $\varphi : V \longrightarrow W$ the square of G-fixed point spaces

$$\begin{tikzcd} X(V)^G \arrow[r, "X(\varphi)^G"] \arrow[d, "f(V)^G"'] & X(W)^G \arrow[d, "f(W)^G"] \\ Y(V)^G \arrow[r, "Y(\varphi)^G"'] & Y(W)^G \end{tikzcd}$$

is homotopy cartesian. This global model structure is proper, topological, compactly generated and monoidal with respect to the convolution box product of orthogonal spaces.

To compare the global model structure of $\mathcal{L}$-spaces and orthogonal spaces we use the adjoint functor pair

$$Q \otimes_{\mathbf{L}} - \ : \ spc \ \rightleftarrows \ \mathcal{L}\mathbf{T} \ : \ \mathrm{map}^{\mathcal{L}}(Q, -)$$

introduced by Lind in [13, Sec. 8]; Lind denotes the functor $Q \otimes_{\mathbf{L}} -$ by $\mathbb{Q}$. The adjoint pair arises from a continuous functor

$$Q \ : \ \mathbf{L}^{\mathrm{op}} \ \longrightarrow \ \mathcal{L}\mathbf{T} \ , \quad V \ \longmapsto \ \mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \ .$$

Here $\mathcal{L}$ acts on $\mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty)$ by postcomposition. A linear isometric embedding $\varphi : V \longrightarrow W$ induces the homomorphism of $\mathcal{L}$-spaces

$$Q(\varphi) \ = \ \mathbf{L}(\varphi \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \ : \ \mathbf{L}(W \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \ \longrightarrow \ \mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \ , \quad \psi \ \longmapsto \ \psi \circ (\varphi \otimes \mathbb{R}^\infty) \ .$$

Since orthogonal spaces are defined as the continuous functor from $\mathbf{L}$, and since the category of $\mathcal{L}$-spaces is tensored and cotensored over spaces, any continuous functor from $\mathbf{L}^{\mathrm{op}}$ induces an adjoint functor pair by an enriched end-coend construction. Indeed, the value of the left adjoint on an orthogonal space Y is given by

$$Q \otimes_{\mathbf{L}} Y \ = \ \int_{V \in \mathcal{I}} \mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \times Y(V) \ ,$$

the enriched coend of the continuous functor

$$\mathbf{L}^{\mathrm{op}} \times \mathbf{L} \ \longrightarrow \ \mathcal{L}\mathbf{T} \ , \quad (V, W) \ \longmapsto \ \mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \times Y(W) \ .$$

The functor $Q \otimes_{\mathbf{L}} -$ has a right adjoint $\mathrm{map}^{\mathcal{L}}(Q, -)$ whose value at an $\mathcal{L}$-space Z is given by

$$\mathrm{map}(Q, Z)(V) \ = \ \mathrm{map}^{\mathcal{L}}(Q(V), Z) \ ,$$

the mapping space of $\mathcal{L}$-equivariant maps from $Q(V)$ to Z. The covariant functoriality in V comes from the contravariant functoriality of Q.

The coend of contravariant functor with a representable covariant functor returns the representing object, i.e.,

$$Q \otimes_{\mathbf{L}} \mathbf{L}_V \ = \ Q(V) \ = \ \mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \ .$$

So the value on the free orthogonal space $L_{G,V}$ generated by a G-representation V comes out as

$$(3.6) \qquad Q \otimes_{\mathbf{L}} \mathbf{L}_{G,V} \ = \ Q \otimes_{\mathbf{L}} (\mathbf{L}_V/G) \ \cong \ (Q \otimes_{\mathbf{L}} \mathbf{L}_V)/G \ \cong \ Q(V)/G \ = \ \mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty)/G \ .$$

Theorem 3.7. *The adjoint functor pair*

$$\begin{tikzcd} spc \arrow[r, shift left, "Q"] & \mathcal{L}\mathbf{T} \arrow[l, shift left, "\mathrm{map}^{\mathcal{L}}(Q,-)"] \end{tikzcd}$$

is a Quillen equivalence with respect to the positive global model structures on orthogonal spaces and the global model structures on $\mathcal{L}$-spaces.

Proof. We let G be a compact Lie group and V a non-trivial faithful G-representation. Then $V \otimes \mathbb{R}^\infty$ is a faithful G-preuniverse. So there is a universal subgroup $\bar{G}$ of $\mathcal{L}$, an isomorphism of Lie groups $\alpha : G \longrightarrow \bar{G}$ and a G-equivariant linear isometry $\varphi : V \otimes \mathbb{R}^\infty \cong \alpha^*(\mathbb{R}^\infty_{\bar{G}})$. This data induces an isomorphism of $\mathcal{L}$-spaces

$$Q(V)/G \ = \ \mathbf{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty)/G \ \cong \ \mathbf{L}(\mathbb{R}^\infty_{\bar{G}}, \mathbb{R}^\infty)/\bar{G} \ .$$

Taking $\mathcal{L}$-equivariant maps out of this isomorphism gives a homeomorphism

$$\begin{aligned} \left(\mathrm{map}^{\mathcal{L}}(Q, Z)(V)\right)^G \ &= \ \left(\mathrm{map}^{\mathcal{L}}(Q(V), Z)\right)^G \ \cong \ \mathrm{map}^{\mathcal{L}}(Q(V)/G, Z) \\ &\cong \ \mathrm{map}^{\mathcal{L}}(\mathbf{L}(\mathbb{R}^\infty_{\bar{G}}, \mathbb{R}^\infty)/\bar{G}, Z) \ \cong \ Z^{\bar{G}} \ . \end{aligned}$$

This homeomorphism is natural in Z, so the functor $\mathrm{map}^{\mathcal{L}}(Q, -)$ takes universal fibrations of $\mathcal{L}$-spaces to positive strong level fibrations of orthogonal spaces. Similarly, $\mathrm{map}^{\mathcal{L}}(Q, -)$ takes acyclic fibrations in the universal model structure of $\mathcal{L}$-spaces to acyclic fibrations in the positive strong level model structure of orthogonal spaces. [finish]

We have now shown that the adjoint functor pair $(Q \otimes_{\mathbf{L}} -, \mathrm{map}^{\mathcal{L}}(Q, -))$ is a Quillen pair with respect to the two global model structures. Now we suppose that A is a flat orthogonal space and Z is an injective $\mathcal{L}$-space. Since Z is injective, the orthogonal space $\mathrm{map}^{\mathcal{L}}(Q, Z)$ is positively static. [...] This shows that the adjoint functor pair $(Q \otimes_{\mathbf{L}} -, \mathrm{map}^{\mathcal{L}}(Q, -))$ is a Quillen equivalence. □

Remark 3.8. The adjoint functor pair $((-)(\mathbb{R}^\infty), u)$ also shows up in Lind's paper, again under a different name: In Section 9, Lind defines a functor $\mathbb{O} : spc \longrightarrow \mathcal{L}\mathbf{T}$ by another enriched coend as

$$\mathbb{O}Y \ = \ \int_{V \in \mathcal{I}} Y(V) \times \mathbf{L}(V, \mathbb{R}^\infty) \ ,$$

which is naturally isomorphic, as an $\mathcal{L}$-space, to $Y(\mathbb{R}^\infty)$, see [13, Lemma 9.6]. Lind points out the symmetric monoidal structure (4.4) on this functor, but there is no mentioning that the monoidal structure is *strong* monoidal (i.e., that Proposition 4.5 below holds).

The functors $Q \otimes_{\mathbb{R}} -$ and $\mathbb{O} = (-)(\mathbb{R}^\infty)$ are closely related: A choice of unit vector $u \in \mathbb{R}^\infty$ gives rise to a linear isometric embedding $- \otimes u : W \longrightarrow W \otimes \mathbb{R}^\infty$ that is natural for linear isometric embeddings in W. So taking enriched coends over the transformation

$$Y(V) \times \mathbf{L}(- \otimes u, \mathbb{R}^\infty) \ : \ Y(V) \times \mathbf{L}(W \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \ \longrightarrow \ Y(V) \times \mathbf{L}(W, \mathbb{R}^\infty)$$

provides a natural map

$$\xi \ : \ Q \otimes_{\mathbf{L}} Y \ \longrightarrow \ \mathbb{O}Y \ = \ Y(\mathbb{R}^\infty) \ .$$

Lind shows in [13, Lemma 9.7] that for every flat orthogonal space (i.e., cofibrant $\mathcal{I}$-space in his terminology) the map $\xi : Q \otimes_{\mathbf{L}} Y \longrightarrow Y(\mathbb{R}^\infty)$ is a weak equivalence. We generalize this as follows:

Proposition 3.9. *For every flat orthogonal space Y the map $\xi : Q \otimes_{\mathbf{L}} Y \longrightarrow Y(\mathbb{R}^\infty)$ is a global equivalence of $\mathcal{L}$-spaces.*

Remark 3.10 ($\mathcal{F}$-global model structure of $\mathcal{L}$-spaces)**.** A *global family* is a class of compact Lie groups that is closed under isomorphisms, subgroups and quotients. Our Theorem 3.7 and the non-equivariant Quillen equivalence of [13, Thm. 9.9] are both special cases of a version with respect to a global family $\mathcal{F}$. We only indicate what goes into this, and leave the details to interested readers. The universal model structure of $\mathcal{L}$-space (see Proposition 2.15) has a straightforward version relative to the family $\mathcal{F}$: A morphism $f : X \longrightarrow Y$ of $\mathcal{L}$-spaces is an *$\mathcal{F}$-universal equivalence* (respectively *$\mathcal{F}$-universal fibration*) if for every universal subgroup G of $\mathcal{L}$ that belongs to $\mathcal{F}$ the induced map

$$f^G \ : \ X^G \ \longrightarrow \ Y^G$$

is a weak homotopy equivalence (respectively a Serre fibration). We can apply Proposition 1.10 to the topological monoid $\mathcal{L}$ and the collection of those universal subgroups that belong to $\mathcal{F}$; we conclude that the $\mathcal{F}$-universal equivalences and $\mathcal{F}$-universal fibrations are part of a proper topological closed model category structure on the category of $\mathcal{L}$-spaces, the *$\mathcal{F}$-universal model structure.*

We call a morphism $f : X \longrightarrow Y$ of $\mathcal{L}$-spaces is an *$\mathcal{F}$-equivalence* if for every completely universal subgroup G of $\mathcal{L}$ that belongs to $\mathcal{F}$ the induced map

$$f^G \ : \ X^G \ \longrightarrow \ Y^G$$

is a weak homotopy equivalence. In a second step we then perform the same kind of Bousfield localization as in Theorem 2.24, using the same functor Q from Proposition 2.23, to get from the the $\mathcal{F}$-universal to the $\mathcal{F}$-global model structure. The outcome is that the cofibrations of the $\mathcal{F}$-universal model structure and the $\mathcal{F}$-equivalences are part of a cofibrantly generated proper topological model structure on the category of $\mathcal{L}$-spaces, the *$\mathcal{F}$-global model structure.* Theorem 3.7 also has a relative version, with the same proof: for every global family $\mathcal{F}$, the adjoint functor pair $(Q \otimes_{\mathbf{L}} -, \mathrm{map}^{\mathcal{L}}(Q, -))$ is a Quillen equivalence with respect to the $\mathcal{F}$-global model structures on orthogonal spaces and on $\mathcal{L}$-spaces.

We claim that when $\mathcal{F} = \langle e \rangle$ is the global family of trivial groups, we recover the Quillen equivalence established by Lind in [13, Thm. 9.9]. To make the connection, we recall that orthogonal spaces are called *$\mathcal{I}$-spaces* in [13], and the category of $\mathcal{I}$-spaces is denoted $\mathcal{I}\mathscr{U}$ Moreover, our $\mathcal{L}$-spaces are called *$\mathbb{L}$-spaces* in [13], where $\mathbb{L}$ stands for the monad whose underlying functor sends A to $\mathcal{L} \times A$; the category of $\mathbb{L}$-spaces is denoted $\mathscr{U}[\mathbb{L}]$. For the trivial global family there is no difference between $\langle e \rangle$-universal equivalences and $\langle e \rangle$-equivalences, and both specialize to the morphisms of $\mathcal{L}$-spaces that are weak equivalences on underlying non-equivariant spaces. So in this case, no Bousfield localization is necessary, and the $\langle e \rangle$-universal and $\langle e \rangle$-global model structure coincide and become the non-equivariant model structure. So for the trivial global family we recover the Quillen equivalence established by Lind.

4. Monoidal properties

The Quillen equivalence between orthogonal spaces and $\mathcal{L}$-spaces of Theorem 3.7 also has nice monoidal properties. Indeed, the category of $\mathcal{L}$-spaces can be endowed with the *operadic product* $\boxtimes_{\mathcal{L}}$, defined as follows. We denote by

$$\mathcal{L}(2) \ = \ \mathbf{L}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty, \mathbb{R}^\infty)$$

the space of binary operations in the linear isometries operad. It comes with a left action of $\mathcal{L}$ and a right action of $\mathcal{L}^2$ by

$$\mathcal{L} \times \mathcal{L}(2) \times \mathcal{L}^2 \ \longrightarrow \ \mathcal{L}(2) \ , \quad (f, \psi, (g, h)) \ \longmapsto \ f \circ \psi \circ (g \oplus h) \ .$$

Given two $\mathcal{L}$-spaces X and Y we can coequalize the right $\mathcal{L}^2$-action on $\mathcal{L}(2)$ with the left $\mathcal{L}^2$-action on the product $X \times Y$ and form

$$X \boxtimes_{\mathcal{L}} Y \ = \ \mathcal{L}(2) \times_{\mathcal{L}\times\mathcal{L}} (X \times Y) \ .$$

The left $\mathcal{L}$-action on $\mathcal{L}(2)$ by postcomposition descends to an $\mathcal{L}$-action on this operadic product. Some care has to be taken when analyzing this construction: because the monoid $\mathcal{L}$ is not a group, it may be hard to figure out when two elements of $\mathcal{L}(2) \times X \times Y$ become equal in the coequalizer. The operadic product $\boxtimes_{\mathcal{L}}$ is coherently associative and commutative, but it does *not* have a unit object. The functor $Q \otimes_{\mathbf{L}} -$ from orthogonal spaces to $\mathcal{L}$-spaces is strong symmetric monoidal by Lemma 8.3 of [13].

Given two $\mathcal{L}$-spaces X and Y, we define a natural $\mathcal{L}$-linear map

$$\rho_{X,Y} \ : \ X \boxtimes_{\mathcal{L}} Y \ \longrightarrow \ X \times Y \qquad \text{by} \qquad [\varphi; x, y] \ \longmapsto \ ((\varphi i_1)\cdot x, (\varphi i_2)\cdot y) \ .$$

Here $i_1, i_2 : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty \oplus \mathbb{R}^\infty$ are the two direct summand embeddings. In the special case $Y = *$ this transformation specializes to a natural $\mathcal{L}$-linear map

$$\lambda_X \ : \ X \boxtimes_{\mathcal{L}} * \ \longrightarrow \ X \qquad \text{defined by} \qquad [\varphi; x, *] \ \longmapsto \ (\varphi i_1)\cdot x \ ;$$

the map λ_X is, however, is not always an isomorphism. The monoids (respectively commutative monoids) with respect to $\boxtimes_{\mathcal{L}}$ are essentially A_∞-monoids (respectively E_∞-monoids). We refer the reader to [1, Sec. 4] for details.

Proposition 4.1. *The universal model structure and the global model structure of $\mathcal{L}$-spaces both satisfy the pushout product property with respect to the operadic box product.*

Proof. The key observation is the following. We let G and K be compact Lie groups and $\mathcal{V}$ respectively $\mathcal{U}$ faithful preuniverses of G respectively K. Then the map

$$\mathbf{L}(\mathcal{V}, \mathbb{R}^\infty) \boxtimes_{\mathcal{L}} \mathbf{L}(\mathcal{U}, \mathbb{R}^\infty) \ \longrightarrow \ \mathbf{L}(\mathcal{V} \oplus \mathcal{U}, \mathbb{R}^\infty) \ , \quad [\varphi; \, \psi, \kappa] \ \longmapsto \ \varphi \circ (\psi \oplus \kappa)$$

is an isomorphism of $\mathcal{L}$-spaces by [9, I Lemma 5.4] (sometimes referred to as 'Hopkins' lemma'). The map is also $(G \times K)$-equivariant, and $\boxtimes_{\mathcal{L}}$ preserves colimits in both variables. So the map descends to an isomorphism of $\mathcal{L}$-spaces

$$\begin{aligned} \mathbf{L}(\mathcal{V}, \mathbb{R}^\infty)/G \boxtimes_{\mathcal{L}} \mathbf{L}(\mathcal{U}, \mathbb{R}^\infty)/K \ &\longrightarrow \ \mathbf{L}(\mathcal{V} \oplus \mathcal{U}, \mathbb{R}^\infty)/(G \times K) \\ [\varphi; \, \psi G, \kappa K] \quad &\longmapsto \ (\varphi \circ (\psi \oplus \kappa))(G \times K) \ . \end{aligned}$$

On the other hand, $\mathcal{V} \oplus \mathcal{U}$ is a faithful preuniverse for the group $G \times K$. Hence for every pair of universal subgroups G and K of $\mathcal{L}$ the product

$$\mathcal{L}/G \boxtimes_{\mathcal{L}} \mathcal{L}/K$$

is isomorphic to $\mathcal{L}/H$ where H is a universal subgroup of $\mathcal{L}$ isomorphic to $G \times K$. The explicit set of generating cofibrations for the universal model structure specified in Proposition 2.15 is thus closed under pushout product (up to isomorphism). Similarly, any pushout product of a generating cofibration with a generating acyclic cofibration is (isomorphic to) another generating acyclic cofibrations. This proves the pushout product property for the universal model structure of $\mathcal{L}$-spaces.

The cofibrations coincide in the universal and the global model structure, so we have also shown the part of the pushout product property in the global model structure that involves only cofibrations. [finish] □

The next result shows that up to global equivalence, the box product of $\mathcal{L}$-spaces coincides with the categorical product. This result has a non-equivariant precursor: Blumberg, Cohen and Schlichtkrull show in [1, Prop. 4.23] that for certain $\mathcal{L}$-spaces (those that are cofibrant in the model structure of [1, Thm. 4.15]), the morphism $\rho_{X,Y}$ is a non-equivariant weak equivalence. The following theorem shows that a much stronger conclusion holds without any hypothesis on X and Y.

Theorem 4.2. *For all $\mathcal{L}$-spaces X and Y, the morphism $\rho_{X,Y} : X \boxtimes_{\mathcal{L}} Y \longrightarrow X \times Y$ is a strong global equivalence. In particular, the morphism $\lambda_X : X \boxtimes_{\mathcal{L}} * \longrightarrow X$ is a strong global equivalence.*

Proof. We let G be a universal subgroup of $\mathcal{L}$. We choose a G-equivariant linear isometry

$$\psi \ : \ \mathbb{R}^\infty_G \oplus \mathbb{R}^\infty_G \ \cong \ \mathbb{R}^\infty_G$$

and define a continuous map

$$\psi_* \ : \ X \times Y \ \longrightarrow \ X \boxtimes_{\mathcal{L}} Y \qquad \text{by} \qquad \psi_*(x,y) \ = \ [\psi, x, y] \ .$$

The G-equivariance means explicitly that $\psi(g \oplus g) = g\psi$ for all $g \in G$, and so the map ψ_* is G-equivariant (but *not* $\mathcal{L}$-linear).

The composite $\rho_{X,Y} \circ \psi_* : X \times Y \longrightarrow X \times Y$ is given by

$$\rho_{X,Y}(\psi_*(x,y)) \ = \ ((\psi i_1) \cdot x, (\psi i_2) \cdot y) \ .$$

By Proposition 2.9 (iii) the space of G-equivariant linear isometric self-embeddings of $\mathbb{R}^\infty_G$ is contractible, so ψi_1 can be linked to the identity of $\mathbb{R}^\infty_G$ by a path of G-equivariant linear isometric self-embeddings. Such a path induces a G-equivariant homotopy from the map $(\psi i_1) \cdot - : X \longrightarrow X$ to the identity of X; similarly, $(\psi i_2) \cdot -$ is G-homotopic to the identity of Y. So altogether we conclude that $\rho_{X,Y} \circ \psi_*$ is G-homotopic to the identity.

To analyze the other composite we define a continuous map

$$\begin{aligned} H \ : \ [0,1] \times \mathbf{L}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty, \mathbb{R}^\infty) \ &\longrightarrow \ \mathbf{L}(\mathbb{R}^\infty \oplus \mathbb{R}^\infty, \mathbb{R}^\infty \oplus \mathbb{R}^\infty) \\ \text{by} \qquad H(t,\varphi)(v,w) \ &= \ \left(\varphi(v,tw),\ \varphi(0, \sqrt{1-t^2} \cdot w)\right) \ . \end{aligned}$$

Then

$$H(0,\varphi) \ = \ (\varphi i_1) \oplus (\varphi i_2) \qquad \text{and} \qquad H(1,\varphi) \ = \ i_1 \varphi \ .$$

Moreover, for every $t \in [0,1]$ the map $H(t,-)$ equivariant for the left $\mathcal{L}$-action (with diagonal action on the target) and for the right $\mathcal{L}^2$-action. So we can define a homotopy of G-equivariant maps (which are *not* $\mathcal{L}$-linear)

$$K \ : \ [0,1] \times (X \boxtimes_{\mathcal{L}} Y) \ \longrightarrow \ X \boxtimes_{\mathcal{L}} Y \qquad \text{by} \qquad K(t, [\varphi; x, y]) \ = \ [\psi H(t,\varphi), x, y] \ .$$

Then

$$\begin{aligned} K(0, [\varphi; x, y]) \ = \ [\psi H(0,\varphi), x, y] \ &= \ [\psi((\varphi i_1) \oplus (\varphi i_2)), x, y] \\ &= \ [\psi, (\varphi i_1) \cdot x, (\varphi i_2) \cdot y] \ = \ \psi_*(\rho_{X,Y}[\varphi; x, y]) \end{aligned}$$

and

$$K(1, [\varphi; x, y]) \ = \ [\psi H(1,\varphi), x, y] \ = \ [\psi i_1 \varphi, x, y] \ = \ (\psi i_1) \cdot [\varphi, x, y] \ .$$

As in the first part of this proof, ψi_1 can be linked to the identity of $\mathbb{R}^\infty_G$ by a path of G-equivariant linear isometric self-embeddings, and such a path induces another G-equivariant homotopy from the map $(\psi i_1) \cdot - : X \boxtimes_{\mathcal{L}} Y \longrightarrow X \boxtimes_{\mathcal{L}} Y$ to the identity of $X \boxtimes_{\mathcal{L}} Y$. So altogether we have exhibited a G-homotopy between $\psi_* \circ \rho_{X,Y}$ and the identity. Since the universal subgroup G was arbitrary, this shows that $\rho_{X,Y}$ is a strong global equivalence. □

The functor $(-)(\mathbb{R}^\infty)$ from orthogonal spaces to $\mathcal{L}$-spaces can be made into a strong symmetric monoidal functor. For this purpose we recall the box product of orthogonal spaces, which is a special case of Day's convolution product of enriched functors [6]. We define a *bimorphism* $b : (X,Y) \longrightarrow Z$ from a pair of orthogonal spaces (X,Y) to another orthogonal space Z as a collection of continuous maps

$$b_{V,W} \ : \ X(V) \times Y(W) \ \longrightarrow \ Z(V \oplus W),$$

for all inner product spaces V and W, such that for all linear isometric embeddings $\varphi : V \longrightarrow V'$ and $\psi : W \longrightarrow W'$ the following square commutes:

$$\begin{tikzcd} X(V) \times Y(W) \arrow[r, "b_{V,W}"] \arrow[d, "X(\varphi)\times Y(\psi)"'] & Z(V \oplus W) \arrow[d, "Z(\varphi\oplus\psi)"] \\ X(V') \times Y(W') \arrow[r, "b_{V',W'}"'] & Z(V' \oplus W') \end{tikzcd}$$

We define a box product of X and Y as a universal example of an orthogonal space with a bimorphism from X and Y. More precisely, a box product for X and Y is a pair $(X \boxtimes Y, i)$ consisting of an orthogonal space $X \boxtimes Y$ and a universal bimorphism $i : (X, Y) \longrightarrow X \boxtimes Y$, i.e., a bimorphism such that for every orthogonal space Z the map

$$spc(X \boxtimes Y, Z) \longrightarrow \operatorname{Bimor}((X, Y), Z) \ , \quad f \longmapsto fi = \{f(V \oplus W) \circ i_{V,W}\}_{V,W} \tag{4.3}$$

is bijective. Very often only the object $X \boxtimes Y$ will be referred to as the box product, but one should keep in mind that it comes equipped with a specific, universal bimorphism. We will often refer to the bijection (4.3) as the *universal property* of the box product of orthogonal spaces.

The existence of a universal bimorphism out of any pair of orthogonal spaces X and Y, and thus of a box product $X \boxtimes Y$, is a special case of the existence of Day type convolution products on certain functor categories [6]; the construction is an enriched Kan extension of the 'pointwise' cartesian product of X and Y along the direct sum functor $\oplus : \mathbf{L} \times \mathbf{L} \longrightarrow \mathbf{L}$, or more explicitly an enriched coend.

Also by the general theory of convolution products, the box product $X \boxtimes Y$ is a functor in both variables and it supports a preferred symmetric monoidal structure; so there are specific natural associativity respectively symmetry isomorphisms

$$(X \boxtimes Y) \boxtimes Z \longrightarrow X \boxtimes (Y \boxtimes Z) \qquad \text{respectively} \qquad X \boxtimes Y \longrightarrow Y \boxtimes X$$

and a strict unit, i.e., such that $\mathbf{1} \boxtimes X = X = X \boxtimes \mathbf{1}$. The upshot is that the associativity and symmetry isomorphisms make the box product of orthogonal spaces into a symmetric monoidal product with the terminal orthogonal space $\mathbf{1}$ as unit object. The box product of orthogonal spaces is *closed* symmetric monoidal in the sense that the box product is adjoint to an internal Hom orthogonal space.

Now we make the functor $(-)(\mathbb{R}^\infty)$ from orthogonal spaces to $\mathcal{L}$-spaces into a strong symmetric monoidal functor. For this we let X and Y be two orthogonal spaces. By simultaneous colimit over $V \in s(\mathbb{R}^\infty)$ and $W \in s(\mathbb{R}^\infty)$, the constituents $i_{V,W} : X(V) \times Y(W) \longrightarrow (X \boxtimes Y)(V \oplus W)$ of the universal bimorphism give rise to a continuous map

$$i_{\infty,\infty} \ : \ X(\mathbb{R}^\infty) \times Y(\mathbb{R}^\infty) \longrightarrow (X \boxtimes Y)(\mathbb{R}^\infty \oplus \mathbb{R}^\infty) \ .$$

The composite

$$\mathcal{L}(2) \times X(\mathbb{R}^\infty) \times Y(\mathbb{R}^\infty) \xrightarrow{\mathcal{L}(2)\times i_{\infty,\infty}} \mathcal{L}(2) \times (X \boxtimes Y)(\mathbb{R}^\infty \oplus \mathbb{R}^\infty) \xrightarrow{\text{act}} (X \boxtimes Y)(\mathbb{R}^\infty)$$

is $\mathcal{L}$-equivariant and coequalizes the action of $\mathcal{L} \times \mathcal{L}$, so it factors over a homomorphism of $\mathcal{L}$-spaces

$$\psi_{X,Y} \ : \ X(\mathbb{R}^\infty) \boxtimes_{\mathcal{L}} Y(\mathbb{R}^\infty) \longrightarrow (X \boxtimes Y)(\mathbb{R}^\infty) \ . \tag{4.4}$$

We omit the verification that the maps ψ are coherently commutative and associative.

Proposition 4.5. *The map $\psi_{X,Y} : X(\mathbb{R}^\infty) \boxtimes_{\mathcal{L}} Y(\mathbb{R}^\infty) \longrightarrow (X \boxtimes Y)(\mathbb{R}^\infty)$ is an isomorphism of $\mathcal{L}$-spaces for all orthogonal spaces X and Y.*

Proof. We start with the special case $X = \mathbf{L}(V,-)$ and $Y = \mathbf{L}(W,-)$ of the free orthogonal spaces represented by two inner product spaces V and W. A general property of convolution products is that the product of represented functor is again represented [ref]. In our situation, this manifests itself as follows. The orthogonal direct sum maps

$$\oplus \ : \ \mathbf{L}(V,U) \times \mathbf{L}(W,U') \ \longrightarrow \ \mathbf{L}(V\oplus W, U\oplus U')$$

form a bimorphism as U and U' vary over all inner product spaces. So the universal property provides a morphisms

$$\mathbf{L}(V,-)\boxtimes \mathbf{L}(W,-) \ \longrightarrow \ \mathbf{L}(V\oplus W,-) \ ,$$

an this canonical morphism is an isomorphism of orthogonal spaces. This isomorphism turns the map $\psi_{\mathbf{L}(V,-),\mathbf{L}(W,-}$ into the morphism of $\mathcal{L}$-spaces

$$\psi_{V,W} \ : \ \mathbf{L}(V,\mathbb{R}^\infty)\boxtimes_{\mathcal{L}} \mathbf{L}(W,\mathbb{R}^\infty) \ \longrightarrow \ \mathbf{L}(V\oplus W,\mathbb{R}^\infty) \ , \quad [\varphi;\alpha,\beta] \ \longmapsto \ \varphi\circ(\alpha\oplus\beta) \ . \tag{4.6}$$

We rewrite the left hand side: we choose linear isometries

$$\alpha \ : \ V\oplus\mathbb{R}^\infty \cong \mathbb{R}^\infty \qquad \text{and} \qquad \beta \ : \ W\oplus\mathbb{R}^\infty \cong \mathbb{R}^\infty$$

and claim that the map

$$\begin{aligned} \mathbf{L}(V\oplus\mathbb{R}^\infty\oplus W\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)/\mathcal{L}^2 \ &\longrightarrow \quad \mathbf{L}(V,\mathbb{R}^\infty)\boxtimes_{\mathcal{L}}\mathbf{L}(W,\mathbb{R}^\infty) \\ [\psi] \qquad &\longmapsto \ [\psi\circ(\alpha^{-1}\oplus\beta^{-1});\, \alpha|_V, \beta|_W] \end{aligned} \tag{4.7}$$

is an isomorphism of $\mathcal{L}$-spaces, where the right $\mathcal{L}^2$-action is through the two $\mathbb{R}^\infty$-summand. Granting this for the moment, this proves the special case because the composite

$$\mathbf{L}(V\oplus\mathbb{R}^\infty\oplus W\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)/\mathcal{L}^2 \ \longrightarrow \ \mathbf{L}(V,\mathbb{R}^\infty)\boxtimes_{\mathcal{L}}\mathbf{L}(W,\mathbb{R}^\infty) \ \xrightarrow{\psi_{V,W}} \ \mathbf{L}(V\oplus W,\mathbb{R}^\infty)$$

is restriction to $V\oplus W$, and hence an isomorphism by Proposition 2.6 (ii) (with $V\oplus W$ instead of V).

A tautological isomorphism of $\mathcal{L}$-spaces is given by

$$\begin{aligned} \mathbf{L}(V\oplus\mathbb{R}^\infty\oplus W\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)/\mathcal{L}^2 \ &\longrightarrow \ (\mathbf{L}(V\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)/\mathcal{L})\boxtimes(\mathbf{L}(W\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)/\mathcal{L}) \\ [\psi] \qquad &\longmapsto \ [\psi\circ(\alpha^{-1}\oplus\beta^{-1});\, [\alpha],[\beta]] \end{aligned}$$

and with explicit inverse given by

$$\begin{aligned} (\mathbf{L}(V\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)/\mathcal{L})\boxtimes(\mathbf{L}(W\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)/\mathcal{L}) \ &\longrightarrow \ \mathbf{L}(V\oplus\mathbb{R}^\infty\oplus W\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)/\mathcal{L}^2 \\ [\varphi;[f],[g]] \qquad &\longmapsto \qquad [\varphi\circ(f\oplus g)] \ . \end{aligned}$$

Moreover, the coequalizer diagram of Proposition 2.6 (i) provides an isomorphism

$$(\mathbf{L}(V\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)/\mathcal{L})\boxtimes(\mathbf{L}(W\oplus\mathbb{R}^\infty,\mathbb{R}^\infty)/\mathcal{L}) \ \cong \ \mathbf{L}(V,\mathbb{R}^\infty)\boxtimes\mathbf{L}(W,\mathbb{R}^\infty) \ .$$

The composite agrees with the map (4.7), so this proves the claim.

Now we prove the general case of the proposition. As functors of X and of Y, source and target of $\psi_{X,Y}$ preserve colimits and products with spaces. Every orthogonal space Y is a coend of a functor with values $Y(W)\times L(W,-)$, so the general case follows formally from the special case. □

Remark 4.8 (Global model structures for $\star$-modules)**.** Since the unit transformation $\lambda_X : X\boxtimes_{\mathcal{L}} * \longrightarrow X$ is not always an isomorphism, certain $\mathcal{L}$-spaces are distinguished. A *$\star$-module* is an $\mathcal{L}$-space X for which the morphism λ_X is an isomorphism. The category of $\star$-modules is particularly relevant because on it, the one-point $\mathcal{L}$-space is a unit object for $\boxtimes_{\mathcal{L}}$ (by definition); so when restricted to the full subcategory of $\star$-modules, the operadic product $\boxtimes_{\mathcal{L}}$ is symmetric monoidal.

For $Y = *$, a terminal orthogonal space, $Y(\mathbb{R}^\infty)$ is a one-point $\mathcal{L}$-space and $Z\boxtimes Y$ is isomorphic to Z. Under these identifications the map $\psi_{Z,*}$ specializes to the unit transformation $\lambda_{Z(\mathbb{R}^\infty)} : Z(\mathbb{R}^\infty)\boxtimes * \longrightarrow Z(\mathbb{R}^\infty)$. So Proposition 4.5 in particular shows that for every orthogonal space Z, the $\mathcal{L}$-space $Z(\mathbb{R}^\infty)$ is a $\star$-module. On the other hand, $\mathcal{L}$-spaces of the form $\mathcal{L}/G$ for a universal subgroup G of $\mathcal{L}$ are *not* $\star$-modules.

The category of $\star$-modules admits a (non-equivariant) model structure with weak equivalences defined after forgetting the $\mathcal{L}$-action, cf. [1, Thm. 4.16]; with this model structure the composite

$$\mathcal{M}_\star \xrightarrow{\text{incl}} \mathcal{L}\mathbf{T} \xrightarrow{F_\boxtimes(*,-)} \mathcal{L}\mathbf{T}$$

is a right Quillen equivalence, where $F_\boxtimes(*,-)$ is right adjoint to $-\boxtimes_\mathcal{L} *$.

In his master thesis [4], Böhme constructed a monoidal model structures on the category of $\star$-modules that has the global equivalences of ambient $\mathcal{L}$-spaces are its weak equivalences; he also showed that with these global model structures, $\mathcal{L}$-spaces and $\star$-modules are Quillen equivalent, and that the global model structure on $\star$-modules lifts to associative monoids (with respect to $\boxtimes_\mathcal{L}$). This effectively provides a global model structure on the category of A_∞-monoids, i.e., algebras over the linear isometries operad (considered as a non-symmetric operad). It remains to be seen to what extent the global model structure lifts to commutative monoids with respect to $\boxtimes_\mathcal{L}$ (i.e., to E_∞-monoids).

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