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The n-order of algebraic triangulated categories

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Abstract

We quantify certain features of algebraic triangulated categories using the 'n-order', an invariant that measures how strongly n annihilates objects of the form Y/n. We show that the n-order of an algebraic triangulated category is infinite, and that the p-order of the p-local stable homotopy category is exactly $p-1$ for any prime p. In particular, the p-local stable homotopy category is not algebraic.

Introduction

Many triangulated categories arise from chain complexes in an additive or abelian category by passing to chain homotopy classes or inverting quasi-isomorphisms. Such examples are called 'algebraic' because they have underlying additive categories. Stable homotopy theory produces examples of triangulated categories by quite different means, and in this context the underlying categories are usually very 'non-additive' before passing to homotopy classes of morphisms. The purpose of this paper is to explain some systematic differences between algebraic and topological triangulated categories. There are certain properties, defined entirely in terms of the triangulated structure, which hold in all algebraic examples, but which can fail in general. The precise statements use the *n-order* of a triangulated category for n a natural number. The n-order is a non-negative integer (or infinity), and it measures, roughly speaking, 'how strongly' the relation $n \cdot Y/n = 0$ holds for the objects Y in a given triangulated category (where Y/n denotes a cone of multiplication by n on Y).

Section 1 is devoted to the concept of n-order of triangulated categories; we show that the existence of a 'mod-n reduction' for a triangulated category implies infinite n-order and apply this to the derived categories of certain structured ring spectra. Section 2 discusses algebraic triangulated categories; the main result is that every algebraic triangulated category has a mod-n reduction and thus infinite n-order. Section 3 contains our main result: for every prime p, the p-order of the p-local stable homotopy category is at most $p-1$. In particular, the p-local stable homotopy category is not algebraic for any prime p; this is folklore for $p = 2$, but seems to be a new result for odd primes p. In a companion paper, we show that, for every prime p, the p-order of every topological triangulated category is at least $p-1$ [**9**, Theorem 5.3]; so the p-order of the p-local stable homotopy category is exactly $p-1$.

The results of this paper were announced in [**8**] and the contents of this paper and its companion [**9**] were originally combined as a single paper (arXiv:1201.0899); the referee convinced the author to divide up the results into two separate papers.

1. *Order*

In this section, we now discuss our main invariant, the *n-order*, which first appeared in the expository paper [**8**]. For an object K of a triangulated category $\mathcal{T}$ and a natural number

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n we write $n \cdot K$ for the n-fold multiple of the identity morphism in the abelian group of endomorphisms in $\mathcal{T}$. We let K/n denote any cone of $n \cdot K$, that is, an object which is part of a distinguished triangle

$$K \xrightarrow{n\cdot} K \xrightarrow{\pi} K/n \longrightarrow \Sigma K.$$

In the following definition, an *extension* of a morphism $f : K \to X$ is any morphism $\bar{f} : K/n \to X$ satisfying $\bar{f}\pi = f$.

Definition 1.1. We consider an object X of a triangulated category $\mathcal{T}$ and a natural number $n \geqslant 1$. We define the *n-order* of X inductively.

(i) Every object has n-order greater than or equal to 0.

(ii) For $k \geqslant 1$, the object X has n-order greater than or equal to k if, and only if for every object K of $\mathcal{T}$ and every morphism $f : K \to X$, there exists an extension $\bar{f} : K/n \to X$ such that some (hence any) cone of $\bar{f}$ has n-order greater or equal to $k - 1$.

The *n-order* of the triangulated category $\mathcal{T}$ is the n-order of some (hence any) zero object.

Some comments about the definition are in order. Since a cone is only well-defined up to non-canonical isomorphism, we should justify that, in the inductive definition, it makes no difference whether we ask for the condition for some or for any cone of the extension $\bar{f}$. This follows from the observation (which is most easily proved by induction on k) that the property 'having n-order greater or equal to k' is invariant under isomorphism.

We write $n\text{-ord}^{\mathcal{T}}(X)$, or simply $n\text{-ord}(X)$ if the ambient triangulated category is understood, for the n-order of X, that is, the largest k (possibly infinite) such that X has n-order greater than or equal to k. We record some direct consequences of the definition.

(i) The n-order for objects is invariant under isomorphism and shift.

(ii) An object X has positive n-order if and only if every morphism $f : K \to X$ has an extension to K/n, which is equivalent to $n \cdot f = 0$. So $n\text{-ord}(X) \geqslant 1$ is equivalent to the condition $n \cdot X = 0$.

(iii) The n-order of a triangulated category is one larger than the minimum of the n-orders of all objects of the form K/n.

(iv) Let $\mathcal{S} \subseteq \mathcal{T}$ be a full triangulated subcategory and X be an object of $\mathcal{S}$. Then we have $n\text{-ord}^{\mathcal{S}}(X) \geqslant n\text{-ord}^{\mathcal{T}}(X)$. When X is a zero object this shows that the n-order of $\mathcal{S}$ is at least as large as the n-order of $\mathcal{T}$.

(v) Suppose that $\mathcal{T}$ is a $\mathbb{Z}[1/n]$-linear triangulated category, that is, multiplication by n is an isomorphism for every object of $\mathcal{T}$. Then K/n is trivial for every object K and thus $\mathcal{T}$ has infinite n-order. If, on the other hand, X is non-trivial, then $n\text{-ord}(X) = 0$.

(vi) If every object of $\mathcal{T}$ has positive n-order, then $n \cdot X = 0$ for all objects X and so $\mathcal{T}$ is a $\mathbb{Z}/n$-linear triangulated category. Suppose conversely that $\mathcal{T}$ is a $\mathbb{Z}/n$-linear triangulated category. Then induction on k shows that $n\text{-ord}(X) \geqslant k$ for all objects X, and thus every object has infinite n-order.

As an example of the typical kind of reasoning, we give the details for the fourth item. We let X be an object of the full triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ and show by induction on k that if $n\text{-ord}^{\mathcal{T}}(X) \geqslant k$, then also $n\text{-ord}^{\mathcal{S}}(X) \geqslant k$. There is nothing to show for $k = 0$, so we may assume $k \geqslant 1$. We consider any object K of $\mathcal{S}$ and morphism $f : K \to X$. We choose a distinguished triangle for $n \cdot K$ in $\mathcal{S}$, which then serves the same purpose in the larger category $\mathcal{T}$. For any extension $\bar{f} : K/n \to X$ (in $\mathcal{S}$ or, what is the same, in $\mathcal{T}$) any mapping cone $C(\bar{f})$ in $\mathcal{S}$ is also a mapping cone in $\mathcal{T}$, and so $n\text{-ord}^{\mathcal{T}}(C(\bar{f})) \geqslant k - 1$ since $n\text{-ord}^{\mathcal{T}}(X) \geqslant k$.

By the inductive hypothesis, $n\text{-ord}^{\mathcal{S}}(C(\bar{f})) \geqslant k-1$, which shows that the n-order of X in the subcategory $\mathcal{S}$ is at least k.

The last two items above show that the concept of n-order is not interesting for triangulated categories that are linear over some field. Indeed, if $\mathbb{F}$ is a field and $\mathcal{T}$ is an $\mathbb{F}$-linear triangulated category, then the number n is either invertible or zero in $\mathbb{F}$, and hence $\mathcal{T}$ is either $\mathbb{Z}[1/n]$-linear or $\mathbb{Z}/n$-linear. In either case, the n-order of $\mathcal{T}$ is infinite.

Now we prove a sufficient criterion, the existence of a 'mod-n reduction', for infinite n-order. This criterion will be used twice below, namely in Proposition 1.5 to show that the homotopy categories of certain structured ring spectra have infinite n-order, and in Theorem 2.1 to show that every algebraic triangulated category has infinite n-order.

Definition 1.2. Let $\mathcal{T}$ be a triangulated category and n be a natural number. A *mod-n reduction* of $\mathcal{T}$ consists of a triangulated category $\mathcal{T}/n$ and an exact functor $\rho^* : \mathcal{T} \to \mathcal{T}/n$ that has a right adjoint ρ^* and such that, for every object X of $\mathcal{T}$, there exists a distinguished triangle

$$X \xrightarrow{n\cdot} X \xrightarrow{\eta_X} \rho^*(\rho_* X) \longrightarrow \Sigma X, \tag{1.3}$$

where η_X is the unit of the adjunction.

Note that in the definition of a mod-n reduction, we do *not* require that the category $\mathcal{T}/n$ be $\mathbb{Z}/n$-linear. However, we always have $n \cdot (\rho^* Z) = 0$ for all objects Z of $\mathcal{T}/n$. Indeed, for $X = \rho^* Z$ the adjunction unit $\eta_{\rho^* Z} : \rho^* Z \to \rho^*(\rho_*(\rho^* Z))$ has a left inverse $\rho^*(\epsilon_Z)$, where $\epsilon_Z : \rho_*(\rho^* Z) \to Z$ is the adjunction counit. The distinguished triangle (1.3) thus splits for $X = \rho^* Z$, and so we have $n \cdot (\rho^* Z) = 0$ in $\mathcal{T}$.

Proposition 1.4. *If a triangulated category $\mathcal{T}$ has a mod-n reduction, then, for every object X of $\mathcal{T}$, the object X/n has infinite n-order. Thus, the triangulated category $\mathcal{T}$ has infinite n-order.*

Proof. We let $(\mathcal{T}/n, \rho_*)$ be a mod-n reduction and ρ^* be a right adjoint of ρ_*. Since X/n is isomorphic to $\rho^*(\rho_* X)$, it is enough to show that, for every object Z of $\mathcal{T}/n$ and all $k \geqslant 0$, the $\mathcal{T}$-object $\rho^* Z$ has n-order greater than or equal to k.

We proceed by induction on k; for $k = 0$ there is nothing to prove. Suppose that we have already shown that every $\rho^* Z$ has n-order greater than or equal to $k-1$ for some positive k. Given a morphism $f : K \to \rho^* Z$ in $\mathcal{T}$, we form its adjoint $\hat{f} : \rho_* K \to Z$ in $\mathcal{T}/n$; if we apply ρ^* we obtain an extension $\rho^*(\hat{f}) : \rho^*(\rho_* K) \to \rho^* Z$ of f. We choose a cone of $\hat{f}$, that is, a distinguished triangle

$$\rho_* K \xrightarrow{\hat{f}} Z \longrightarrow C(\hat{f}) \longrightarrow \Sigma(\rho_* K)$$

in $\mathcal{T}/n$. Since ρ^* is exact, $\rho^* C(\hat{f})$ is a cone of the extension $\rho^*(\hat{f})$ in $\mathcal{T}$. By induction, $\rho^* C(\hat{f})$ has n-order greater than or equal to $k-1$, which proves that $\rho^* Z$ has n-order greater than or equal to k. $\square$

Now we give examples of triangulated categories that have mod-n reductions, and thus infinite n-order. The examples which follow are derived categories (or homotopy categories) of structured ring spectra; for some of these examples I do not know whether they are algebraic or not.

For definiteness, we work with symmetric ring spectra [**3**], but the arguments work just as well for structured ring spectra in any one of the modern model categories of spectra with compatible smash product. A symmetric ring spectrum R has a stable model category of left R-module spectra [**3**, Corollary 5.4.2]. We denote by $\mathbf{D}(R)$ the triangulated homotopy category of R-module spectra and refer to it as the *derived category* of R. For example, for the Eilenberg–MacLane ring spectrum HA of an ordinary ring A, the derived category $\mathbf{D}(HA)$ is triangulated equivalent to the unbounded derived category of complexes of A-modules (see [**6**] or [**10**, Appendix B.1]). If $R = \mathbb{S}$ is the sphere spectrum, then $\mathbf{D}(\mathbb{S})$ is the homotopy category of symmetric spectra, hence equivalent to the stable homotopy category.

I owe the following proposition to Tyler Lawson.

PROPOSITION 1.5. *Let R be a commutative symmetric ring spectrum and $n \geqslant 1$. Suppose that there exists an R-algebra spectrum B and a distinguished triangle*

$$R \xrightarrow{n\cdot} R \xrightarrow{\eta} B \longrightarrow \Sigma R \tag{1.6}$$

in the derived category $\mathbf{D}(R)$ of R-module spectra, where $\eta : R \to B$ is the unit morphism of the R-algebra structure. Then, for every R-algebra spectrum A the derived category $\mathbf{D}(A)$ has a mod-n reduction and thus infinite n-order.

Proof. We can replace B by a stably equivalent R-algebra that is cofibrant in the stable model structure of R-algebras of [**3**, Corollary 5.4.3]; this way we can arrange that B is cofibrant as an R-module (again by Hovey, Shipley and Smith [**3**, Corollary 5.4.3]).

The smash product $A \wedge_R B$ over R is another R-algebra spectrum equipped with a homomorphism $f = A \wedge \eta : A \cong A \wedge_R R \to A \wedge_R B$ of R-algebras. Then f gives rise to a Quillen adjoint functor pair between the associated stable model categories of A-modules and $(A \wedge_R B)$-modules: the right adjoint f^* is 'restriction of scalars' and the left adjoint $f_* = B \wedge_R -$ is 'extension of scalars'. This Quillen functor pair descends to a pair of adjoint total derived functors on the level of triangulated homotopy categories

$$\mathbf{D}(A) \quad \underset{\rho^* = R(f^*)}{\overset{\rho_* = L(f_*)}{\rightleftarrows}} \quad \mathbf{D}(A \wedge_R B).$$

We claim that these data are a mod-n reduction for $\mathbf{D}(A)$. The only missing property is that, for every A-module M, the object $\rho^*(\rho_* M)$ models M/n. We can assume without loss of generality that the A-module M is cofibrant. Since R is central in A, we can view the left A-module M as an A-R-bimodule; the functor

$$M \wedge_R - \,:\, R\text{-mod} \longrightarrow A\text{-mod}$$

is a left Quillen functor, and so descends to an exact left derived functor of triangulated categories

$$M \wedge^L_R - \,:\, \mathbf{D}(R) \to \mathbf{D}(A).$$

Hence we obtain a distinguished triangle

$$M \xrightarrow{n\cdot} M \xrightarrow{M \wedge^L_R \eta} M \wedge^L_R B \longrightarrow \Sigma M \tag{1.7}$$

in $\mathbf{D}(A)$ by smashing the triangle (1.6) over R with M and using the unit isomorphism between $M \wedge^L_R R$ and M.

Since B is cofibrant as an R-module, the derived smash product $M \wedge^L_R B$ is represented by the pointset level smash product $M \wedge_R B$ which is isomorphic, as a left A-module, to $f^*(f_* M) = (A \wedge_R B) \wedge_A M$. So $\rho^*(\rho_* M)$ is isomorphic in $\mathbf{D}(A)$ to $M \wedge^L_R B$ in such a way that the adjunction unit corresponds to the morphism $M \wedge^L_R \eta : M \to M \wedge^L_R B$. So the triangle (1.7) completes the proof that $(\mathbf{D}(A \wedge_R B), \rho_*)$ is a mod-n reduction of $\mathbf{D}(A)$. $\square$

EXAMPLE 1.8. The hypothesis in Proposition 1.5 on the commutative symmetric ring spectrum R can be paraphrased by saying that the R-module spectrum R/n (or rather, some R-module spectrum of this homotopy type) can be given the structure of an R-algebra. A theorem of Angeltveit [**1**, Corollary 3.2] gives a sufficient condition for this in terms of the graded ring π_*R of homotopy groups of R: if π_*R is n-torsion-free and concentrated in even dimensions, then R/n admits an A_∞-structure compatible with the R-module structure; equivalently, there is an R-algebra spectrum whose underlying R-module has the homotopy type of R/n.

Some prominent examples of commutative ring spectra that satisfy Angeltveit's criterion are the complex cobordism spectrum MU, the complex topological K-theory spectrum KU and the Lubin–Tate spectra $E(k,\Gamma)$ for a formal group law Γ of finite height over a perfect field k. So Proposition 1.5 implies that, for every algebra spectrum A over any of these commutative ring spectra, the derived category $\mathbf{D}(A)$ has infinite n-order for every $n \geqslant 1$.

2. *Algebraic triangulated categories*

An important class of triangulated categories are the *algebraic* triangulated categories, those that admit a 'differential graded model'. In this section, we construct a mod-n reduction for every algebraic triangulated category (Theorem 2.1); by Proposition 1.4, this implies that algebraic triangulated categories have infinite n-order.

The earliest formalization of algebraic triangulated categories, via differential graded enrichments, seems to be the notion of *enhanced triangulated category* of Bondal and Kapranov [**2**, Section 3], and this is the approach that we will use. Algebraic triangulated categories can be introduced in at least two other, equivalent, ways: as the full triangulated subcategories of homotopy categories of additive categories, or as the stable categories of exact Frobenius categories. For the equivalence of these three approaches, and for more details, background and references, we refer the reader to [**2**, **4**, **5**].

We deviate from the standard conventions in two minor points. First, we grade complexes *homologically* (as opposed to cohomologically), that is, differentials decrease the degree by 1; this is more in tune with grading conventions in topology. Secondly, we use covariant representable functors (as opposed to contravariant representable functors), which amounts to the passage to opposite dg-categories.

A *differential graded category*, or simply *dg-category*, is a category $\mathcal{B}$ enriched in chain complexes of abelian groups. So a dg-category consists of a class of objects, a chain complex $\mathcal{B}(X,Y)$ of morphisms for every pair of objects and composition morphisms of chain complexes

$$\cdot\, : \mathcal{B}(Y,Z) \otimes \mathcal{B}(X,Y) \longrightarrow \mathcal{B}(X,Z)$$

for every triple of objects. The composition morphisms have to be associative and have to admit two-sided unit elements $1_X \in \mathcal{B}(X,X)_0$ for all objects X, satisfying $d(1_X) = 0$. The *homology category* $\mathbf{H}(\mathcal{B})$ (also called the 'homotopy category') has the same objects as $\mathcal{B}$, but as morphisms the 0th homology groups of the homomorphism complexes, $\mathbf{H}(\mathcal{B})(X,Y) = H_0(\mathcal{B}(X,Y))$.

A $\mathcal{B}$-*module* is a covariant dg-functor from $\mathcal{B}$ to chain complexes of abelian groups. In more detail, a $\mathcal{B}$-module M assigns to each object Z of $\mathcal{B}$ a chain complex $M(Z)$ and to each pair of objects a morphism of chain complexes

$$\cdot\, : \mathcal{B}(Y,Z) \otimes M(Y) \longrightarrow M(Z).$$

This data is required to be associative with respect to the composition in $\mathcal{B}$ and the unit cycles have to act as identities. A $\mathcal{B}$-module M is *representable* if there exists a pair (Y,u) consisting of an object Y of $\mathcal{B}$ and a *universal* 0-*cycle* $u \in M(Y)_0$ such that, for every object Z of $\mathcal{B}$, the evaluation morphism

$$\mathcal{B}(Y,Z) \longrightarrow M(Z), \quad \varphi \longmapsto \varphi \cdot u$$

is an isomorphism of chain complexes.

A dg-category $\mathcal{B}$ is *pretriangulated* if it has a zero object and the following two closure properties:

(a) (Closure under shifts) For an object X of a dg-category $\mathcal{B}$ and an integer n we define the $\mathcal{B}$-module $\mathcal{B}(X,-)[n]$ on objects by

$$(\mathcal{B}(X,Z)[n])_{n+k} = \mathcal{B}(X,Z)_k$$

with differential and action of morphisms by

$$d(f[n]) = (-1)^n \cdot (df)[n] \quad \text{respectively } \varphi \cdot (\psi[n]) = (-1)^{n|\varphi|} \cdot (\varphi\psi)[n].$$

Here we use the notation $f[n]$ when we consider an element $f \in \mathcal{B}(X,Z)_k$ as an element of $(\mathcal{B}(X,Z)[n])_{n+k}$. Then, for every object X of $\mathcal{B}$ and every integer n, the $\mathcal{B}$-module $\mathcal{B}(X,-)[n]$ is representable.

(b) (Closure under cones) Given a closed morphism $f : X \to Y$ (that is, a 0-cycle in $\mathcal{B}(X,Y)$) we consider the $\mathcal{B}$-module M defined on objects by

$$M(Z)_k = \mathcal{B}(Y,Z)_k \oplus \mathcal{B}(X,Z)_{k+1}$$

with differential and action of morphisms by

$$d(a,b) = (d(a), af - d(b)) \quad \text{respectively } \varphi \cdot (a,b) = (\varphi a, (-1)^{|\varphi|} \cdot \varphi b).$$

Then the $\mathcal{B}$-module M is representable.

If a dg-category $\mathcal{B}$ is pretriangulated, then the homology category $\mathbf{H}(\mathcal{B})$ can be canonically triangulated, as we recall now. A *shift* of an object X is any object $X[1]$ that represents the module $\mathcal{B}(X,-)[-1]$. Choices of shifts for all objects of $\mathcal{B}$ assemble canonically into an invertible shift functor $X \mapsto X[1]$ on $\mathcal{B}$. The shift functor on $\mathbf{H}(\mathcal{B})$ is induced by this shift functor on $\mathcal{B}$. The distinguished triangles arise from mapping cone sequences in $\mathcal{B}$. Given a closed morphism $f : X \to Y$, we let Cf be a *mapping cone* of f, that is, a representing object for the module in (b) above. The cone comes with a universal 0-cycle

$$(i,u) \in \mathcal{B}(Y,Cf)_0 \oplus \mathcal{B}(X,Cf)_1 = M(Cf)_0;$$

the cycle condition means that $d(i) = 0$ and $d(u) = if$. We let $p \in \mathcal{B}(Cf,X)_{-1}$ be the element characterized by

$$p \cdot (i,u) = (0, 1_X) \in \mathcal{B}(Y,X)_{-1} \oplus \mathcal{B}(X,X)_0 = M(X)_{-1}.$$

Since $(0,1_X)$ is a cycle, so is p, which is thus represented by a closed morphism $\bar{p} : Cf \to X[1]$. By definition, a triangle in $\mathbf{H}(\mathcal{B})$ is *distinguished* if it is isomorphic to the image of a triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \xrightarrow{\bar{p}} X[1]$$

for some closed morphism f in $\mathcal{B}$. A proof that this really makes $\mathbf{H}(\mathcal{B})$ into a triangulated category can be found in [**2**, Section 3, Propositon 2]. A triangulated category is *algebraic* if it is equivalent, as a triangulated category, to the homology category of some pretriangulated dg-category.

We will now show that in algebraic triangulated categories the relation $n \cdot X/n = 0$ holds 'in a very strong sense', that is, the n-order of every object of the form X/n is infinite.

THEOREM 2.1. *Let $\mathcal{T}$ be an algebraic triangulated category and $n \geqslant 1$. Then $\mathcal{T}$ has a mod-n reduction and thus has infinite n-order.*

Proof. The main idea is simple: if $\mathcal{T} = \mathbf{D}(A)$ happens to be the derived category of a differential graded ring A, then we can take $\mathcal{T}/n$ as the derived category of the differential graded ring $A \otimes \mathbb{Z}[e]$, where $\mathbb{Z}[e]$ is a flat resolution of the ring $\mathbb{Z}/n$, namely the exterior algebra on a generator e with differential $d(e) = n$. The adjoint functor pair (ρ_*, ρ^*) is derived from restriction and extension of scalars along the morphism $A \to A \otimes \mathbb{Z}[e]$. The rigorous proof is an adaptation of this idea to pretriangulated dg-categories.

We may suppose that $\mathcal{T} = \mathbf{H}(\mathcal{B})$ is the homology category of a pretriangulated dg-category $\mathcal{B}$. We consider the new dg-category $\mathbb{Z}[e] \otimes \mathcal{B}$ where $\mathbb{Z}[e]$ is the dg category with a single object whose endomorphism dg-ring is the exterior algebra, over the integers, on a one-dimensional class e with differential $d(e) = n$. In more detail, the dg-category $\mathbb{Z}[e] \otimes \mathcal{B}$ has the same objects as $\mathcal{B}$, the morphism complexes are defined by

$$(\mathbb{Z}[e] \otimes \mathcal{B})(X, Y) = \mathbb{Z}[e] \otimes \mathcal{B}(X, Y),$$

and composition is given by

$$(1 \otimes \varphi + e \otimes \psi) \cdot (1 \otimes \varphi' + e \otimes \psi') = 1 \otimes \varphi\varphi' + e \otimes (\psi\varphi' + (-1)^{|\varphi|}\varphi\psi').$$

A dg-functor $J : \mathcal{B} \to \mathbb{Z}[e] \otimes \mathcal{B}$ is given by $J(X) = X$ on objects and by $J(\varphi) = 1 \otimes \varphi$ on morphisms.

The new dg-category $\mathbb{Z}[e] \otimes \mathcal{B}$ is typically not closed under mapping cones, hence not pretriangulated. We form the pretriangulated envelope $\mathcal{B}/n = (\mathbb{Z}[e] \otimes \mathcal{B})^{\mathrm{pre}}$; see [**2**, Section 1] (where the objects are called *twisted complexes*) or [**4**, 4.5] (where this is called the *pretriangulated hull*). The envelope $\mathcal{B}/n$ contains $\mathbb{Z}[e] \otimes \mathcal{B}$ as a full dg-subcategory and the inclusion $\iota : \mathbb{Z}[e] \otimes \mathcal{B} \to \mathcal{B}/n$ is an initial example of a dg-functor from $\mathbb{Z}[e] \otimes \mathcal{B}$ to a pretriangulated dg-category. The composite morphism of dg-categories

$$i = \iota \circ J \,:\, \mathcal{B} \longrightarrow \mathbb{Z}[e] \otimes \mathcal{B} \longrightarrow (\mathbb{Z}[e] \otimes \mathcal{B})^{\mathrm{pre}} = \mathcal{B}/n$$

descends to an exact functor of triangulated homology categories

$$\rho_* = \mathbf{H}(i) \,:\, \mathbf{H}(\mathcal{B}) \longrightarrow \mathbf{H}(\mathcal{B}/n).$$

We will show that $(\mathbf{H}(\mathcal{B}/n), \rho_*)$ is a mod-n reduction.

We define a dg-functor $t : \mathbb{Z}[e] \otimes \mathcal{B} \to \mathcal{B}$ that will eventually give rise to a right adjoint to ρ_*. Given X in $\mathcal{B}$, we let $t(X)$ be a mapping cone of the closed morphism $n \cdot 1_X$. So, by definition, $t(X)$ represents the $\mathcal{B}$-module given in dimension k by

$$Z \longmapsto \mathcal{B}(X, Z)_k \oplus \mathcal{B}(X, Z)_{k+1}, \quad d(a, b) = (d(a), na - d(b)).$$

The object $t(X)$ comes with a universal 0-cycle

$$(\eta_X, g_X) \in \mathcal{B}(X, t(X))_0 \oplus \mathcal{B}(X, t(X))_1.$$

The cycle condition means that $d(\eta_X) = 0$ and $d(g_X) = n \cdot \eta_X$. On morphism complexes we let

$$t \,:\, (\mathbb{Z}[e] \otimes \mathcal{B})(X, Y) = \mathbb{Z}[e] \otimes \mathcal{B}(X, Y) \longrightarrow \mathcal{B}(t(X), t(Y))$$

be the unique chain map characterized by the relations

$$t(1 \otimes \varphi + e \otimes \psi) \cdot (\eta_X, g_X) = (\eta_Y\varphi + g_Y\psi, g_Y\varphi) \in \mathcal{B}(X, t(Y))_k \oplus \mathcal{B}(X, t(Y))_{k+1}$$

for all $\varphi \in \mathcal{B}(X, Y)_k$ and $\psi \in \mathcal{B}(X, Y)_{k-1}$. Compatibility with units and composition is straightforward, so we have really defined a dg-functor $t : \mathbb{Z}[e] \otimes \mathcal{B} \to \mathcal{B}$. The universal property of the pretriangulated envelope then provides a dg-functor $t' : \mathcal{B}/n \to \mathcal{B}$ that extends t. We let

$$\rho^* = \mathbf{H}(t') \,:\, \mathbf{H}(\mathcal{B}/n) \longrightarrow \mathbf{H}(\mathcal{B})$$

denote the induced exact functor on homology categories.

We will now make ρ^* into a right adjoint of ρ_*. The relation $t(1 \otimes \varphi)\eta_X = \eta_Y\varphi$ means that the homology classes of the closed morphisms $\eta_X : X \to t(X) = (t'(i(X))$ constitute a natural transformation $\eta : \mathrm{Id}_{\mathbf{H}(\mathcal{B})} \to \mathbf{H}(t'i) = \rho^*\rho_*$. We claim that η is the unit of an adjunction between ρ_* and ρ^*; so we need to show that the composite

$$\mathbf{H}(\mathcal{B}/n)(\rho_* X, Z) \xrightarrow{\rho^*} \mathbf{H}(\mathcal{B})(\rho^*(\rho_* X), \rho^* Z) \xrightarrow{\mathbf{H}(\mathcal{B})(\eta_X, \rho^* Z)} \mathbf{H}(\mathcal{B})(X, \rho^* Z) \tag{2.2}$$

is bijective for all X in $\mathcal{B}$ and all Z in $\mathcal{B}/n$. In the special case when $Z = \rho_* Y$ for some Y in $\mathcal{B}$ we have $\rho^*(\rho_* Y) = t(Y)$ and the composite (2.2) is the effect on H_0 of the chain map

$$u : \mathbb{Z}[e] \otimes \mathcal{B}(X, Y) = (\mathcal{B}/n)(i(X), i(Y)) \to \mathcal{B}(X, t(Y))$$

given by

$$u(1 \otimes \varphi + e \otimes \psi) = \eta_Y \varphi + g_Y \psi.$$

We can describe the inverse of u explicitly, as follows. We let $r \in \mathcal{B}(t(X), X)_0$ and $p \in \mathcal{B}(t(X), X)_{-1}$ be the elements characterized by the relations

$$r \cdot (\eta_X, g_X) = (1_X, 0) \quad \text{respectively} \quad p \cdot (\eta_X, g_X) = (0, 1_X).$$

Then $(\eta_X r + g_X p)(\eta_X, g_X) = (\eta_X, g_X)$, so we must have $\eta_X r + g_X p = 1_{t(X)}$. We define

$$v : \mathcal{B}(Y, t(X)) \to (\mathcal{B}/n)(i(Y), i(X)) \quad \text{by } v(a) = 1 \otimes ra - e \otimes pa,$$

and direct calculation shows that uv and vu are the identity maps. This shows that the map (2.2) is bijective in the special case $Z = \rho_* Y$.

We consider the class of objects Z of $\mathbf{H}(\mathcal{B}/n)$ such that the map (2.2) is bijective for all X in $\mathbf{H}(\mathcal{B})$. Since ρ_* and ρ^* are exact functors, this class forms a triangulated subcategory of $\mathbf{H}(\mathcal{B}/n)$. Moreover, the class contains all objects of the form $\rho_* Y$, by the last paragraph. The homology category $\mathbf{H}(\mathcal{B}/n)$ of the pretriangulated envelope of $\mathbb{Z}[e] \otimes \mathcal{B}$ is generated, as a triangulated category, by the objects of $\mathbf{H}(\mathbb{Z}[e] \otimes \mathcal{B})$, and these are precisely the objects of the form $\rho_* Y$. So the map (2.2) is always bijective, and this shows that ρ^* is right adjoint to ρ_* with η as the adjunction unit.

For all X in $\mathcal{B}$, the object $\rho^*(\rho_* X) = t(X)$ is a mapping cone of multiplication by n on X, by construction. Moreover, the homology class of $\eta_X : X \to t(X) = \rho^*(\rho_* X)$ is the cone inclusion. Since mapping cone sequences give rise to exact triangles in $\mathbf{H}(\mathcal{B})$, this proves the existence of a distinguished triangle (1.3). □

Remark 2.3. The construction of the mod-n reduction $\mathcal{T}/n$ in Theorem 2.1 depends on the choice of the dg-model for $\mathcal{T}$, and not just on the triangulated category $\mathcal{T}$. For a specific example to illustrate this, we can take $\mathcal{T}$ as the category of $\mathbb{F}_2$-vector spaces, with identity shift functor and the exact sequences as distinguished triangles. There are two well-known dg-models, the dg-categories of acyclic complexes of projective modules over the rings $\mathbb{F}_2[\epsilon]$ and $\mathbb{Z}/4$. We remark without proof that the construction of Theorem 2.1 applied to these two dg categories yields two mod-2 reductions that are not equivalent (even as categories).

3. *The order of Moore spectra*

A triangulated category that is well known not to be algebraic is the stable homotopy category. Indeed, the mod-2 Moore spectrum in the stable homotopy category is of the form $\mathbb{S}/2$ for $\mathbb{S}$ the sphere spectrum, and $\mathbb{S}/2$ is *not* annihilated by 2. An account of the classical argument using Steenrod operations can be found in [**8**, Proposition 4]. In particular, the 2-order of the stable homotopy category is 1, so it is not algebraic. For odd primes p, however, $p \cdot \mathbb{S}/p = 0$ and this direct argument breaks down. In this section, we show that, for every prime p the p-order of the stable homotopy category of finite spectra is at most $p - 1$. This implies that the p-local stable homotopy category is not algebraic for any prime p.

We denote by $\mathbf{SH}^c_{(p)}$ the category of *finite p-local spectra*, that is, the compact objects in the triangulated category of p-local spectra.

THEOREM 3.1. *Let p be a prime. Then the mod-p Moore spectrum $\mathbb{S}/p$ has p-order at most $p-2$ in the triangulated category $\mathbf{SH}^c_{(p)}$ of finite p-local spectra. Hence the category $\mathbf{SH}^c_{(p)}$ has p-order at most $p-1$, and is not algebraic.*

Since the stable homotopy category is a topological triangulated category, its p-order and that of any subcategory is at least $p-1$ by Schwede [9, Theorem 5.3]. By Theorem 3.1, the p-order of $\mathbf{SH}^c_{(p)}$, and hence of every triangulated category which contains it, is at most $p-1$. So we conclude that any triangulated category that sits between $\mathbf{SH}^c_{(p)}$ and the stable homotopy category has p-order exactly $p-1$.

Proof. For $p=2$ the theorem just rephrases the fact that $2\cdot\mathbb{S}/2\neq 0$. So we assume for the rest of the proof that p is an odd prime. Readers familiar with the proof of the rigidity theorem for the stable homotopy category will recognize the following arguments as a key step in [7, Theorem 3.1]. We will use mod-p cohomology operations and some knowledge about stable homotopy groups of spheres. We adopt the standard abbreviation $q=2p-2$. Then the Steenrod operation P^i has degree iq. Below we will use the Adem relation

$$P^pP^{(j-1)p} = j\cdot P^{jp} + P^{jp-1}P^1, \tag{3.2}$$

which holds for all positive j.

We say that a finite p-local spectrum X *satisfies condition* (C_i) if its mod-p cohomology is one-dimensional in dimensions jpq and $jpq+1$, connected by a Bockstein operation, for $j=0,\dots,i$, and trivial in all other dimensions, and the Steenrod operation $P^{ip}: H^0(X,\mathbb{F}_p)\to H^{ipq}(X,\mathbb{F}_p)$ is an isomorphism.

For the course of this proof we write

$$\mathbb{S}^i_{(p)} = \Sigma^i\mathbb{S}_{(p)}$$

for the i-dimensional p-local sphere spectrum.

Step 1: Let X be a finite p-local spectrum that satisfies condition (C_i) for some i between 0 and $p-1$. Then there exists a morphism

$$f\,:\,\mathbb{S}^{(i+1)pq-1}_{(p)}\longrightarrow X,$$

which is detected by the operation P^p, that is, such that in the mod-p cohomology of any mapping cone $C(f)$ the operation $P^p: H^{ipq}(C(f),\mathbb{F}_p)\to H^{(i+1)pq}(C(f),\mathbb{F}_p)$ is an isomorphism.

For the proof of this step we let $X^{(n)}$ denote a stable p-local n-skeleton of X. Since $X/X^{(ipq-1)}$ has the mod-p cohomology of the suspended Moore spectrum $\Sigma^{ipq}\mathbb{S}/p$, it is isomorphic to $\Sigma^{ipq}\mathbb{S}/p$ in the stable homotopy category. Hence there exists a morphism

$$\tilde\beta_1\,:\,\mathbb{S}^{(i+1)pq-1}_{(p)}\longrightarrow X/X^{(ipq-1)},$$

which is detected by P^p; see, for example, p. 60 of [11, Section 5] (the notation indicates that $\tilde\beta_1$ can be chosen so that the composite with the appropriate shift of the pinch map $\mathbb{S}/p\to\mathbb{S}^1_{(p)}$ is the class β_1 that generates the p-component of the stable stem of dimension $pq-2$). Now we claim that $\tilde\beta_1$ can be lifted to a morphism $f:\mathbb{S}^{(i+1)pq-1}_{(p)}\to X$.

The obstruction to lifting $\tilde\beta_1$ to X is the composite morphism

$$\mathbb{S}^{(i+1)pq-1}_{(p)}\xrightarrow{\tilde\beta_1} X/X^{(ipq-1)}\longrightarrow \Sigma X^{(ipq-1)},$$

where the second map is the connecting morphism. Since $\Sigma X^{(ipq-1)}$ has stable cells in dimensions $jpq+1$ and $jpq+2$ for $j=0,\dots,i-1$, the obstructions lie in the p-local stable stems of dimension $jpq-3$ and $jpq-2$ for $j=2,\dots,i+1$; by serious calculations it is known that indeed for $j=2,\dots,p$, the p-components of the stable stems of dimension $jpq-3$ and $jpq-2$ are trivial. We give detailed references for this calculation in the proof of [**7**, Theorem 3.1]. Since the obstruction group vanishes, there exists a morphism $f:\mathbb{S}^{(i+1)pq-1}_{(p)}\to X$ as above.

Step 2: We show that there does not exist any finite p-local spectrum X which satisfies condition (C_{p-1}). Suppose to the contrary that such an X exists. By Step 1, there is a morphism $f:\mathbb{S}^{p^2q-1}_{(p)}\to X$ which is detected by the operation P^p, and we consider its mapping cone $C(f)$. Since X satisfies (C_{p-1}), the composite Steenrod operation

$$P^pP^{(p-1)p}\;:\;H^0(C(f);\mathbb{F}_p)\longrightarrow H^{p^2q}(C(f);\mathbb{F}_p)$$

is non-trivial. On the other hand, for $i=p$ the Adem relation (3.2) becomes $P^pP^{(p-1)p}=P^{p^2-1}P^1$. Since the operation P^1 is trivial in the cohomology of $C(f)$ for dimensional reasons, we arrive at a contradiction, which means that there is no object X satisfying condition (C_{p-1}).

Step 3: We show by downward induction on i that an object X of the category $\mathbf{SH}^c_{(p)}$ which satisfies condition (C_i) has p-order at most $p-i-2$. We start the induction with $i=p-1$, where the conclusion 'X has p-order at most -1' means that there does not exist such an X, which was shown in Step 2.

For the inductive step we assume that, for some $i=0,\dots,p-2$, every object that satisfies condition (C_{i+1}) has p-order at most $p-i-3$. Let X be an object that satisfies condition (C_i). We consider a morphism $f:\mathbb{S}^{(i+1)pq-1}_{(p)}\to X$ that is detected by the operation P^p, and which exists by Step 1. We claim that any mapping cone $C(\bar f)$ of any extension $\bar f:\Sigma^{(i+1)pq-1}\mathbb{S}/p\to X$ satisfies condition (C_{i+1}), so it has p-order at most $p-i-3$. This proves that X has p-order at most $p-i-2$.

Indeed, by attaching a copy of the suspended mod-p Moore spectrum $\Sigma^{(i+1)pq-1}\mathbb{S}/p$ to X, the mod-p cohomology increases by one copy of $\mathbb{F}_p$ in dimensions $(i+1)pq$ and $(i+1)pq+1$, connected by a Bockstein operation, and it remains unchanged in all other dimensions. So the mapping cone $C(\bar f)$ has its mod-p cohomology in the right dimensions. Since f is detected by P^p, the composite Steenrod operation

$$P^pP^{ip}\;:\;H^0(C(\bar f);\mathbb{F}_p)\longrightarrow H^{(i+1)pq}(C(\bar f);\mathbb{F}_p)$$

is an isomorphism. By the Adem relation (3.2) and since P^1 acts trivially for dimensional reasons, $P^{(i+1)p}$ acts as a unit multiple of P^pP^{ip}, and thus as an isomorphism. This proves that $C(\bar f)$ satisfies condition (C_{i+1}) and completes Step 3.

Now we draw the final conclusion. The mod-p Moore spectrum $\mathbb{S}/p$ satisfies condition (C_0), and so by Step 3, for $i=0$ it has p-order at most $p-2$. □

REMARK 3.3. On a conceptual level, the fact that $\mathbb{Z}/n$ is an associative ring is responsible for the infinite n-order of algebraic triangulated categories, whereas the multiplication of the Moore spectrum $\mathbb{S}/p$ is less well-behaved. Indeed, the mod-2 Moore spectrum has no multiplication whatsoever, and the mod-3 Moore spectrum has no homotopy-associative multiplication. For primes $p\geqslant 5$, the mod-p Moore spectrum has a multiplication in the stable homotopy category which is commutative and associative. So, on the level of tensor triangulated categories, there does not seem to be any qualitative difference between the Moore spectrum $\mathbb{S}/p$ as an object of the triangulated category of finite spectra and $\mathbb{Z}/p$ as an object of the bounded derived category of finitely generated abelian groups, as long as $p\geqslant 5$. However, mod-p Moore spectra do not have models as strict ring spectra (or A_∞-ring spectra); this seems to be folklore, and a proof can be found in [**1**, Example 3.3]. With the methods of this paper, this also follows by combining Proposition 1.5 and Theorem 3.1. So the notion of p-order explains and quantifies how the higher order non-associativity eventually manifests itself in the triangulated structure of the stable homotopy category (that is, without any reference to the smash product).

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