

MORITA THEORY IN ABELIAN, DERIVED AND STABLE MODEL CATEGORIES

STEFAN SCHWEDE

These notes are based on lectures given at the Workshop on *Structured ring spectra and their applications*. This workshop took place January 21-25, 2002, at the University of Glasgow and was organized by Andy Baker and Birgit Richter.

CONTENTS

1. Introduction	1
2. Morita theory in abelian categories	2
3. Morita theory in derived categories	6
3.1. The derived category	6
3.2. Derived equivalences after Rickard and Keller	13
3.3. Examples	19
4. Morita theory in stable model categories	21
4.1. Stable model categories	22
4.2. Symmetric ring and module spectra	25
4.3. Characterizing module categories over ring spectra	32
4.4. Morita context for ring spectra	34
4.5. Examples	38
References	42

1. INTRODUCTION

The paper [Mo58] by Kiiti Morita seems to be the first systematic study of equivalences between module categories. Morita treats both contravariant equivalences (which he calls *dualities* of module categories) and covariant equivalences (which he calls *isomorphisms* of module categories) and shows that they always arise from suitable bimodules, either via contravariant hom functors (for ‘dualities’) or via covariant hom functors and tensor products (for ‘isomorphisms’). The term ‘Morita theory’ is now used for results concerning equivalences of various kinds of module categories. The authors of the obituary article [AGH] consider Morita’s theorem “probably one of the most frequently used single results in modern algebra”.

In this survey article, we focus on the covariant form of Morita theory, so our basic question is:

When do two ‘rings’ have ‘equivalent’ module categories ?

We discuss this question in different contexts:

- (Classical) When are the module categories of two rings equivalent as categories ?

Date: November 14, 2003.

- (Derived) When are the derived categories of two rings equivalent as triangulated categories ?
- (Homotopical) When are the module categories of two ring spectra Quillen equivalent as model categories ?

There is always a related question, which is in a sense more general:

What characterizes the category of modules over a ‘ring’ ?

The answer is, *mutatis mutandis*, always the same: modules over a ‘ring’ are characterized by the existence of a ‘small generator’, which plays the role of the free module of rank one. The precise meaning of ‘small generator’ depends on the context, be it an abelian category, a derived category or a stable model category. We restrict our attention to categories which have a single small generator; this keeps things simple, while showing the main ideas. Almost everything can be generalized to categories (abelian, derived or stable model categories) with a *set of small generators*. One would have to talk about *ringoids* (also called *rings with many objects*) and their differential graded and spectral analogues.

Background: for a historical perspective on Morita’s work we suggest a look at the obituary article [AGH] by Arhangel’skii, Goodearl, and Huisgen-Zimmermann. The history of Morita theory for derived categories and ‘tilting theory’ is summarized in Section 3.1 of the book by König and Zimmermann [KZ]. Both sources contain lots of further references.

For general background material on derived and triangulated categories, see [SGA 4 $\frac{1}{2}$] (Appendix by Verdier), [GM], [Ver96], or [Wei94]. We freely use the language of model categories, alongside with the concepts of Quillen adjoint pair and Quillen equivalence. For general background on model categories see Quillen’s original article [Qui67], a modern introduction [DwSp95], or [Hov99] for a more complete overview.

Acknowledgments: The Morita theory in stable model categories which I describe in Section 4 is based on joint work with Brooke Shipley spread over many years and several papers; I would like to take this opportunity to thank her for the pleasant and fruitful collaboration. I would also like to thank Andy Baker and Birgit Richter for organizing the wonderful workshop *Structured ring spectra and their applications* in Glasgow.

2. MORITA THEORY IN ABELIAN CATEGORIES

To start, we review the covariant Morita theory for modules; this is essentially the content of Section 3 of [Mo58]. This and related material is treated in more detail in [Ba, II §3], [AF92, §22] or [Lam, §18].

Definition 2.1. Let \mathcal{A} be an abelian category with infinite sums. An object M of \mathcal{A} is *small* if the hom functor $\mathcal{A}(M, -)$ preserves sums; M is a *generator* if every object of \mathcal{A} is an epimorphic image of a sum of (possibly infinitely many) copies of M .

The emphasis in the smallness definition is on *infinite* sums; finite sums are isomorphic to finite products, so they are automatically preserved by the hom functor. For modules over a ring, smallness is closely related to finite generation: every finitely generated module is small and for projective modules, ‘small’ and ‘finitely generated’ are equivalent concepts. Rentschler [Ren69] gives an example of a small module which is not finitely generated.

A generator can equivalently be defined by the property that the functor $\mathcal{A}(M, -)$ is faithful, compare [Ba, II Prop. 1.1]. A small projective generator is called a *progenerator*. The main example is when $\mathcal{A} = \mathbf{Mod}\text{-}R$ is the category of right modules over a ring R . Then the free module of rank one is a small projective generator. In this case, a general R -module

M is a generator for $\mathbf{Mod}\text{-}R$ if and only if the free R -module of rank one is an epimorphic image of a sum of copies of M .

Here is one formulation of the classical Morita theorem for rings:

Theorem 2.2. *For two rings R and S , the following conditions are equivalent.*

- (1) *The categories of right R -modules and right S -modules are equivalent.*
- (2) *The category of right S -modules has a small projective generator whose endomorphism ring is isomorphic to R .*
- (3) *There exists an R - S -bimodule M such that the functor*

$$- \otimes_R M : \mathbf{Mod}\text{-}R \longrightarrow \mathbf{Mod}\text{-}S$$

is an equivalence of categories.

If these conditions hold, then R and S are said to be Morita equivalent.

Here are some elementary remarks on Morita equivalence. Condition (1) above is symmetric in R and S . So if an R - S -bimodule M realizes an equivalence of module categories, then the inverse equivalence is also realized by an S - R -bimodule N . Since the equivalences are inverse to each other, $M \otimes_S N$ is then isomorphic to R as an R -bimodule and $N \otimes_R M$ is isomorphic to S as an S -bimodule. Moreover, M and N are then projective as right modules, and N is isomorphic to $\text{Hom}_S(M, S)$ as a bimodule.

If R is Morita equivalent to S , then the opposite ring R^{op} is Morita equivalent to S^{op} . Indeed, suppose the R - S -bimodule M and the S - R -bimodule N satisfy

$$M \otimes_S N \cong R \quad \text{and} \quad N \otimes_R M \cong S$$

as bimodules. Since the category of right R^{op} -modules is isomorphic to the category of left R -modules, we can view M as an S^{op} - R^{op} -bimodule and N as an R^{op} - S^{op} -bimodule, and then they provide the equivalence of categories between $\mathbf{Mod}\text{-}R^{op}$ and $\mathbf{Mod}\text{-}S^{op}$. Similarly, if R is Morita equivalent to S and R' is Morita equivalent to S' , then $R \otimes S$ is Morita equivalent to $R' \otimes S'$. Here, and in the rest of the paper, undecorated tensor products are taken over \mathbb{Z} .

Invariants which are preserved under Morita equivalence include all concepts which can be defined from the category of modules without reference to the ring. Examples are the number of isomorphism classes of projective modules, of simple modules or of indecomposable modules, or the algebraic K -theory of the ring. The center $Z(R) = \{r \in R \mid rs = sr \text{ for all } s \in R\}$ is also Morita invariant, since the center of R is isomorphic to the endomorphism ring of the identity functor of $\mathbf{Mod}\text{-}R$. A ring isomorphism

$$Z(R) \longrightarrow \text{End}(\text{Id}_{\mathbf{Mod}\text{-}R})$$

is obtained as follows: if $r \in Z(R)$ is a central element, then for every R -module M , multiplication by r is R -linear. So the collection of R -homomorphisms $\{\times r : M \longrightarrow M\}_{M \in \mathbf{Mod}\text{-}R}$ is a natural transformation from the identity functor to itself. For more details, see [Ba, II Prop 2.1] or [Lam, Remark 18.43]. In particular, if two commutative rings are Morita equivalent, then they are already isomorphic.

There is a variation of the Morita theorem 2.2 relative to a commutative ring k , with essentially the same proof. In this version R and S are k -algebras, condition (1) refers to a k -linear equivalence of module categories, condition (2) requires an isomorphism of k -algebras and in part (3), M has to be a k -symmetric bimodule, i.e., the scalars from the ground ring k act in the same way from the left (through R) and from the right (through S).

We sketch the **proof of Theorem 2.2** because it serves as the blueprint for analogous results in the contexts of differential graded rings and ring spectra.

Suppose (1) holds and let

$$F : \mathbf{Mod}\text{-}R \longrightarrow \mathbf{Mod}\text{-}S$$

be an equivalence of categories. The free R -module of rank one is a small projective generator of the category of R -modules. Being projective, small or a generator are categorical conditions, so they are preserved by an equivalence of categories. So the S -module FR is a small projective generator of the category of S -modules. Since F is an equivalence of categories, it is in particular an additive fully faithful functor. So F restricts to an isomorphism of rings

$$F : R \cong \mathrm{End}_R(R) \xrightarrow{\cong} \mathrm{End}_S(FR) .$$

Now assume condition (2) and let P be a small projective S -module which generates the category $\mathbf{Mod}\text{-}S$. After choosing an isomorphism $f : R \cong \mathrm{End}_S(P)$, we can view P as an R - S -bimodule by setting $r \cdot x = f(r)(x)$ for $r \in R$ and $x \in P$. We show that P satisfies the conditions of (3) by showing that the adjoint functors $- \otimes_R P$ and $\mathrm{Hom}_S(P, -)$ are actually inverse equivalences.

The adjunction unit is the R -linear map

$$X \longrightarrow \mathrm{Hom}_S(P, X \otimes_R P) , \quad x \longmapsto (y \longmapsto x \otimes y) .$$

For $X = R$, the map adjunction unit coincides with the isomorphism f , so it is bijective. Since P is small, source and target commute with sums, finite or infinite, so the unit is bijective for every free R -module. Since P is projective over S , both sides of the adjunction unit are right exact as functors of X . Every R -module is the cokernel of a morphism between free R -modules, so the adjunction unit is bijective in general.

The adjunction counit is the S -linear evaluation map

$$\mathrm{Hom}_S(P, Y) \otimes_R P \longrightarrow Y , \quad \phi \otimes x \longmapsto \phi(x) .$$

For $Y = P$, the counit is an isomorphism since the right action of R on $\mathrm{Hom}_S(P, P)$ arises from the isomorphism $R \cong \mathrm{Hom}_S(P, P)$. Now the argument proceeds as for the adjunction unit: both sides are right exact and preserves sums, finite or infinite, in the variable Y . Since P is a generator, every S -module is the cokernel of a morphism between direct sums of copies of P , so the counit is bijective in general.

Condition (1) is a special case of (3), so this finishes the proof of the Morita theorem.

Example 2.3. The easiest example of a Morita equivalence involves matrix algebras. Any free R -module of finite rank $n \geq 1$ is a small projective generator for the category of right R -modules. The endomorphism ring

$$\mathrm{End}_R(R^n) \cong M_n(R)$$

is the ring of $n \times n$ matrices with entries in R . So R and the matrix ring $M_n(R)$ are Morita equivalent. The bimodules which induce the equivalences of module categories can both be taken to be R^n , but viewed as ‘row vectors’ (or $1 \times n$ matrices) and ‘column vectors’ (or $n \times 1$ matrices) respectively.

Matrix rings do not provide the most general kind of Morita equivalences, as the example below shows. However, every ring Morita equivalent to R is isomorphic to a ring of the form $eM_n(R)e$ where $e \in M_n(R)$ is a *full idempotent* in the $n \times n$ matrix ring, i.e., we have $e^2 = e$ and $M_n(R)eM_n(R) = M_n(R)$. Indeed, if P is a small projective generator for a ring R , then

P is a summand of a free module of finite rank n , say. Thus P is isomorphic to the image of an idempotent $n \times n$ matrix e , and then $\text{End}_R(P) \cong eM_n(R)e$ as rings.

Example 2.4. The following example of a Morita equivalence which is not of matrix algebra type was pointed out to me by M. Künzler and N. Strickland. Consider a *commutative* ring R and an *invertible* R -module Q . In other words, there exists another R -module Q' and an isomorphism of R -modules $Q \otimes_R Q' \cong R$. Then tensor product with Q over R is a self-equivalence of the category of right R -modules (with quasi-inverse the tensor product with Q'). This self-equivalence is not isomorphic to the identity functor unless Q is free of rank one.

Because tensor product with an invertible module Q is an equivalence of categories, it follows that Q is a progenerator, with endomorphism ring isomorphic to R . Moreover, the ‘inverse’ module Q' is isomorphic to the R -linear dual $Q^* = \text{Hom}_R(Q, R)$. Now we consider the direct sum $P = R \oplus Q$, which is another small projective generator for $\text{Mod-}R$. Then R is Morita equivalent to the endomorphism ring of P ,

$$\text{End}_R(P) = \text{Hom}_R(R \oplus Q, R \oplus Q).$$

As an R -module, $\text{End}_R(P)$ is thus isomorphic to $R \oplus Q \oplus Q^* \oplus R$. So if Q is not free, then $\text{End}_R(P)$ is not free over its center, hence not a matrix algebra.

For a specific example we consider the ring

$$R = \mathbb{Z}[u]/(u^2 - 5u).$$

We set $Q = (2, u) \triangleleft R$, the ideal generated by 2 and u . Then Q is not free as an R -module, but it is invertible because the evaluation map

$$\text{Hom}_R(Q, R) \otimes_R Q \longrightarrow R, \quad \phi \otimes x \longmapsto \phi(x)$$

is an isomorphism. Note that the inclusion $Q \longrightarrow R$ becomes an isomorphism after inverting 2; so after inverting 2 the module $P = R \oplus Q$ is free of rank 2 and hence the ring $\text{End}_R(P)[\frac{1}{2}]$ is isomorphic to the ring of 2×2 matrices over $R[\frac{1}{2}]$.

The implication (2) \implies (1) in the Morita theorem 2.2 can be stated in a more general form, and then it gives a characterization of module categories as the cocomplete abelian categories with a small projective generator.

Theorem 2.5. *Let \mathcal{A} be an abelian category with infinite sums and a small projective generator P . Then the functor*

$$\mathcal{A}(P, -) : \mathcal{A} \longrightarrow \text{Mod-End}_{\mathcal{A}}(P)$$

is an equivalence of categories.

Proof. We give the same proof as in Bass’ book [Ba, II Thm. 1.3]. Let us say that an object X of \mathcal{A} is *good* if the map

$$\mathcal{A}(P, -) : \mathcal{A}(X, Y) \longrightarrow \text{Hom}_{\text{End}_{\mathcal{A}}(P)}(\mathcal{A}(P, X), \mathcal{A}(P, Y)) \tag{2.6}$$

is bijective for every object Y of \mathcal{A} . We note that:

- The generator P is good since $\mathcal{A}(P, P)$ is the free $\text{End}_{\mathcal{A}}(P)$ -module of rank one.
- The class of good objects is closed under sums, finite or infinite: since P is small, $\mathcal{A}(P, -)$ preserves sums and both sides of the map (2.6) take direct sums in X to direct products.

- If $f : X \rightarrow X'$ is a morphism between good objects in \mathcal{A} , then the cokernel of f is also good. This uses that P is projective, so $\mathcal{A}(P, -)$ is an exact functor and both sides of the map (2.6) are right exact in X .

Since P is a generator, every object can be written as the cokernel of a morphism between sums of copies of P . So every object of \mathcal{A} is good, which precisely means that the hom functor $\mathcal{A}(P, -)$ is full and faithful.

It remains to check that every $\text{End}_{\mathcal{A}}(P)$ -module is isomorphic to a module in the image of the functor $\mathcal{A}(P, -)$. The free $\text{End}_{\mathcal{A}}(P)$ -module of rank one is the image of P . Since $\mathcal{A}(P, -)$ commutes with sums, every free module is in the image, up to isomorphism. Finally, every $\text{End}_{\mathcal{A}}(P)$ -module X has a presentation, so it occurs in an exact sequence of $\text{End}_{\mathcal{A}}(P)$ -modules

$$\bigoplus_I \text{End}_{\mathcal{A}}(P) \xrightarrow{g} \bigoplus_J \text{End}_{\mathcal{A}}(P) \rightarrow X \rightarrow 0.$$

Since $\mathcal{A}(P, -)$ is full, the homomorphism g is isomorphic to $\mathcal{A}(P, f)$ for some morphism $f : \bigoplus_I P \rightarrow \bigoplus_J P$ in \mathcal{A} . Since the functor $\mathcal{A}(P, -)$ is exact, X is the image of the cokernel of f . Thus $\mathcal{A}(P, -)$ is an equivalence of categories. \square

Theorem 2.5 can be applied to the abelian category of right modules over a ring S ; then we conclude that for every small projective generator P of $\text{Mod-}S$ the functor

$$\text{Hom}_S(P, -) : \text{Mod-}S \rightarrow \text{Mod-End}_S(P)$$

is an equivalence of categories. This shows again that condition (2) in the Morita theorem 2.2 implies condition (1).

3. MORITA THEORY IN DERIVED CATEGORIES

Morita theory for derived categories is about the question:

When are the derived categories $\mathcal{D}(R)$ and $\mathcal{D}(S)$ of two rings R and S equivalent?

Here the derived category $\mathcal{D}(R)$ is defined from (\mathbb{Z} -graded and unbounded) chain complexes of right R -modules by formally inverting the quasi-isomorphisms, i.e., the chain maps which induce isomorphisms of homology groups. Of course, if R and S are Morita equivalent, then they are also derived equivalent. But it turns out that derived equivalences happen under more general circumstances.

Rickard [Ric89a, Ric91] developed a Morita theory for derived categories based on the notion of a tilting complex. Rickard's theorem did not come out of the blue, and he had built on previous work of several other people on *tilting modules*. Section 3.1 of the book by König and Zimmermann [KZ] gives a summary of the history in this area; this book also contains many more details, examples and references on the use of derived categories in representation theory.

We follow Keller's approach from [Kel94a], based on the (differential graded) endomorphism ring of a tilting complex. A similar approach to and more applications of Morita theory in derived categories can be found in the paper [DG02] by Dwyer and Greenlees.

3.1. The derived category. In this section, R is any ring. All chain complexes are \mathbb{Z} -graded and homological, i.e., the differential decreases the degree by 1.

Definition 3.1. A chain complex C of R -modules is *cofibrant* if there exists an exhaustive increasing filtration by subcomplexes

$$0 = C^0 \subseteq C^1 \subseteq \dots \subseteq C^m \subseteq \dots$$

such that each subquotient C^n/C^{n-1} consists of projective modules and has trivial differential. The (*unbounded*) *derived category* $\mathcal{D}(R)$ of the ring R has as objects the cofibrant complexes of R -modules and as morphisms the chain homotopy classes of chain maps.

Our definition of the derived category is different from the usual one. The more traditional way is to start with the homotopy category of all complexes, not necessarily cofibrant; then one uses a calculus of fractions to formally invert the class of quasi-isomorphisms. These two ways of constructing $\mathcal{D}(R)$ lead to equivalent categories.

The *shift functor* in $\mathcal{D}(R)$ is given by shifting a complex, i.e.,

$$(A[1])_n = A_{n-1}$$

with differential $d : (A[1])_n = A_{n-1} \longrightarrow A_{n-2} = (A[1])_{n-1}$ the *negative* of the differential of the original complex A . The *mapping cone* $C\varphi$ of a chain map $\varphi : A \longrightarrow B$ is defined by

$$(C\varphi)_n = B_n \oplus A_{n-1}, \quad d(x, y) = (dx + \varphi(y), -dy). \quad (3.2)$$

The mapping cone comes with an inclusion $i : B \longrightarrow C\varphi$ and a projection $p : C\varphi \longrightarrow A[1]$ which induce an isomorphism $(C\varphi)/B \cong A[1]$; if A and B are cofibrant, then so are the shift $A[1]$ and the mapping cone.

Remark 3.3. The following remarks are meant to give a better feeling for the notion of ‘cofibrant complex’ and the unbounded derived category.

- (i) The concept of a ‘cofibrant complex’ is closely related to, but stronger than, a complex of projective modules. Indeed, if C is a cofibrant complex, then in every dimension $k \in \mathbb{Z}$, each subquotient C_k^n/C_k^{n-1} is projective. So C_k^n splits as the sum of the subquotients,

$$C_k^n \cong \bigoplus_{i=1}^n C_k^i/C_k^{i-1}.$$

Since C_k is the union of the submodules C_k^n , the module C_k also splits as the sum of the countably many subquotients C_k^i/C_k^{i-1} . In particular, C_k is a sum of projective modules. Hence every cofibrant complex is dimensionwise projective. If C is a complex of projective modules which is *bounded below*, then it is also cofibrant. For example, if C is trivial in negative dimensions, then as the filtration we can simply take the (stupid) truncations of C , i.e.,

$$C_n^i = \begin{cases} C_n & \text{for } n < i \\ 0 & \text{for } n \geq i. \end{cases}$$

So for bounded below complexes, ‘cofibrant’ is equivalent to ‘dimensionwise projective’.

On the other hand, not every complex which is dimensionwise projective is also cofibrant. The standard example is the complex C in which C_k is the free $\mathbb{Z}/4$ -module of rank one for all $k \in \mathbb{Z}$, and where every differential $d : C_k \longrightarrow C_{k-1}$ is multiplication by 2.

- (ii) Every quasi-isomorphism between cofibrant complexes is a chain homotopy equivalence. For every complex of R -modules X , there is a cofibrant complex X^c and a quasi-isomorphism $X^c \longrightarrow X$; together these two facts essentially prove that the derived category $\mathcal{D}(R)$ enjoys the universal property of the localization of the category of chain complexes of R -modules with the class of quasi-isomorphisms inverted. These properties are very analogous to the properties that CW-complexes

have among all topological spaces: every weak equivalences between CW-complexes is a homotopy equivalence and every space admits a CW-approximation. This analogy is made precise in [KM, Part III].

- (iii) The concept of a cofibrant chain complex is closely related to that of a *K-projective* complex as defined by Spaltenstein [Spa88, Sec. 1.1] (who attributes this notion to J. Bernstein; Keller [KZ, 8.1.1] calls this *homotopically projective*). A chain complex is K-projective if every chain map into an acyclic complex (i.e., a complex with trivial homology) is chain null-homotopic.

Every cofibrant complex is K-projective. Conversely, every K-projective complex X is chain homotopy equivalent to a cofibrant complex. Indeed, we can choose a cofibrant replacement, i.e., a cofibrant complex X^c and a quasi-isomorphism $q : X^c \rightarrow X$; the mapping cone Cq is then acyclic. We have a short exact sequence of chain homotopy classes of chain maps in Cq ,

$$[X^c[1], Cq] \rightarrow [Cq, Cq] \rightarrow [X, Cq];$$

both X and X^c are K-projective, so the left and right groups are trivial. Thus the identity map of Cq is null-homotopic and so the mapping cone of q is chain contractible. Thus the map $q : X^c \rightarrow X$ is a chain homotopy equivalence.

- (iv) We use the term ‘cofibrant’ complex because they are the cofibrant objects in the *projective* model category structure on chain complexes of R -modules [Hov99, 2.3.11]. In this model structure, the weak equivalences are the quasi-isomorphisms, the fibrations are the surjections and the cofibrations are the injections whose cokernel is cofibrant in the sense of Definition 3.1. In particular, every chain complex is fibrant in the projective model structure.

Since every object is fibrant and because the fibrations (= surjections) and weak equivalences (= quasi-isomorphisms) already have well-established names, it is unnecessary, and overly complicated, to use the language of model categories in order to work with the derived category of a ring.

- (v) There is a ‘dual’ approach to the derived category $\mathcal{D}(R)$ as the homotopy category of ‘fibrant’ complexes (attention: these are *not* the fibrant objects in the projective model structure – there every complex is fibrant). This uses the notion of a ‘K-injective’ [Spa88, Sec. 1.1] or ‘homotopically injective’ [KZ, 8.1.1] complex, or the *injective model structure* on the category of complexes of R -modules [Hov99, 2.3.13]. It is often useful to have both descriptions available. Given arbitrary chain complexes C and D , we choose a cofibrant/K-projective resolution $C^c \xrightarrow{\sim} C$ and a fibrant/K-injective resolution $D \xrightarrow{\sim} D^f$. Then the maps induce isomorphisms of chain homotopy classes of chain maps

$$[C^c, D] \xrightarrow{\cong} [C^c, D^f] \xleftarrow{\cong} [C, D^f].$$

There is an additive functor

$$[0] : \mathbf{Mod}\text{-}R \rightarrow \mathcal{D}(R)$$

which is a fully faithful embedding onto the full subcategory of the derived category consisting of the complexes whose homology is concentrated in dimension zero. So we can think of the R -modules as sitting inside the derived category $\mathcal{D}(R)$. Had we defined the derived category from the category of all complexes of R -modules by formally inverting the quasi-isomorphisms, then we could define the complex $M[0]$ by putting the R -module M in

dimension 0, and taking trivial chain modules everywhere else. With our present definition of $\mathcal{D}(R)$ we let $M[0]$ be a choice of resolution P_\bullet of M by projective R -modules. Such a resolution is unique up to chain homotopy equivalence and it is cofibrant when viewed as a chain complex (by Remark 3.3 (i) above). Moreover, every R -linear map $M \rightarrow N$ is covered by a unique chain homotopy class between the chosen projective resolutions. In other words, we really get a functor from R -modules to the derived category $\mathcal{D}(R)$, together with a natural isomorphism $H_0(M[0]) \cong M$.

The usual definition of Ext-groups involves a choice of projective resolution P_\bullet of the source module M , and then $\text{Ext}_R^n(M, N)$ can be defined as the chain homotopy classes of chain maps from the resolution P_\bullet to N , shifted up into dimension n . We get the same result if N is also replaced by a projective resolution; this says that Ext-groups can be obtained from the derived category via

$$\text{Ext}_R^n(M, N) \cong \mathcal{D}(R)(M[0], N[n]) . \quad (3.4)$$

The derived category of a ring has more structure. The category $\mathcal{D}(R)$ is additive since the homotopy relation for chain maps is additive. But $\mathcal{D}(R)$ is no longer an abelian category such as the category of chain complexes. Indeed, notions such as ‘monomorphism’, ‘epimorphisms’, ‘kernels’ for chain maps do not interact well with the passage to chain homotopy classes. The *distinguished triangles* in $\mathcal{D}(R)$ are what is left of the abelian structure on the category of chain complexes, and $\mathcal{D}(R)$ is an example of a *triangulated category*.

The distinguished triangles are the diagrams which are isomorphic in $\mathcal{D}(R)$ to a mapping cone triangle. More precisely, a diagram in $\mathcal{D}(R)$ of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \quad (3.5)$$

is called a *distinguished triangle* if and only if there exists a chain map $\varphi : A \rightarrow B$ between cofibrant complexes and isomorphisms $\iota_1 : A \cong X$, $\iota_2 : B \cong Y$ and $\iota_3 : C\varphi \cong Z$ in $\mathcal{D}(R)$ such that the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\varphi} & B & \xrightarrow{i} & C\varphi & \xrightarrow{p} & A[1] \\ \iota_1 \downarrow & & \downarrow \iota_2 & & \downarrow \iota_3 & & \downarrow \iota_1[1] \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \end{array}$$

commutes in $\mathcal{D}(R)$.

We do not want to reproduce the complete definition of a triangulated category here; the data of a triangulated category consists of

- (i) an additive category \mathcal{T} ,
- (ii) a self-equivalence $[1] : \mathcal{T} \rightarrow \mathcal{T}$ called the *shift* functor and
- (iii) a class of *distinguished triangles*, i.e., a collection of diagrams in \mathcal{T} of the form (3.5).

This data is subject to several axioms which can be found for example in [Ver96], [Wei94, Sec. 10.2] [Nee01] or [KZ, 2.3].

Distinguished triangles are the source of many long exact sequences that come up in nature. Indeed, the axioms which we have suppressed imply in particular that for every distinguished triangle of the form (3.5) and every object W of \mathcal{T} the sequence of abelian morphism groups

$$\mathcal{T}(W, X) \xrightarrow{f_*} \mathcal{T}(W, Y) \xrightarrow{g_*} \mathcal{T}(W, Z) \xrightarrow{h_*} \mathcal{T}(W, X[1])$$

is exact. One of the axioms also says that one can ‘rotate’ triangles, i.e., a sequence (3.5) is a distinguished triangle if and only if the sequence

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is a distinguished triangle. So if we keep rotating a distinguished triangle in both directions and take morphisms from a fixed object W , we end up with a long exact sequence of abelian groups

$$\begin{aligned} \dots \mathcal{T}(W, X) \xrightarrow{f_*} \mathcal{T}(W, Y) \xrightarrow{g_*} \mathcal{T}(W, Z) \xrightarrow{h_*} \mathcal{T}(W, X[1]) \\ \xrightarrow{-f[1]*} \mathcal{T}(W, Y[1]) \xrightarrow{-g[1]*} \mathcal{T}(W, Z[1]) \xrightarrow{-h[1]*} \mathcal{T}(W, X[2]) \dots \end{aligned} \quad (3.6)$$

The axioms of a triangulated category also guarantee a similar long exact sequence when taking morphism *from* a triangle (and its rotations) *into* a fixed object W .

In the derived category $\mathcal{D}(R)$, the long exact sequence (3.6) becomes something more familiar when we take $W = R[0]$, the free R -module of rank one, viewed as a complex concentrated in dimension zero. For every chain complex C , cofibrant or not, the chain homotopy classes of morphisms from R to C are naturally isomorphic to the homology module H_0C ; so the long exact sequence (3.6) specializes to the long exact sequence of homology modules

$$\dots \longrightarrow H_0(A) \xrightarrow{H_0(\varphi)} H_0(B) \longrightarrow H_0(C\varphi) \xrightarrow{\delta} H_{-1}(A) \longrightarrow \dots$$

Small generators. We will often require infinite direct sums in a triangulated category. The unbounded derived category $\mathcal{D}(R)$ has direct sums, finite and infinite. Indeed, the direct sum of any number of cofibrant complexes is again cofibrant (take the direct sum of the filtrations which are required in Definition 3.1), and this also represents the direct sum in $\mathcal{D}(R)$. This is one point where it is important to allow unbounded complexes. There are variants of the derived category which start with complexes which are bounded or bounded below. One also gets triangulated categories in much the same way as for $\mathcal{D}(R)$, but for example the countable family $\{R[-n]\}_{n \geq 0}$ has no direct sum in the bounded or bounded below derived categories.

In the Morita equivalence questions, a suitably defined notion of ‘small generator’ pops up regularly. The following concepts for triangulated categories are analogous to the ones for abelian categories in Definition 2.1.

Definition 3.7. Let \mathcal{T} be a triangulated category with infinite coproducts. An object M of \mathcal{T} is *small* if the hom functor $\mathcal{T}(M, -)$ preserves sums; M is a *generator* if there is no proper full triangulated subcategory of \mathcal{T} (with shift and triangles induced from \mathcal{T}) which contains M and is closed under infinite sums.

As in abelian categories, the hom functor $\mathcal{T}(M, -)$ automatically preserves finite sums. What we call ‘small’ is sometimes called *compact* or *finite* in the literature on triangulated categories. A triangulated category with infinite coproducts and a set of small generators is often called *compactly generated*.

The class of small objects in any triangulated category is closed under shifting in either direction, taking finite sums and taking direct summands. Moreover, if two of the three objects in a distinguished triangle are small, then so is the third one (one has to exploit that the morphisms from a distinguished triangle into a fixed object give rise to a long exact sequence).

There is a convenient criterion for when a *small* object M generates a triangulated category \mathcal{T} with infinite coproducts: M generates \mathcal{T} in the sense of Definition 3.7 if and only if it ‘detects objects’, i.e., an object X of \mathcal{T} is trivial if and only if there are no graded maps from M to X , i.e. $\mathcal{T}(M[n], X) = 0$ for all $n \in \mathbb{Z}$. For the equivalence of the two conditions, see for example [SS03, Lemma 2.2.1].

The complex $R[0]$ consisting of the free module of rank one concentrated in dimension 0 is a small generator for the derived category $\mathcal{D}(R)$. Indeed, morphisms in $\mathcal{D}(R)$ out of the complex $R[0]$ represent homology, i.e., there is a natural isomorphism

$$\mathcal{D}(R)(R[n], C) \cong H_n(C)$$

for every cofibrant complex C . Since homology commutes with infinite sums, the complex $R[0]$ is small in $\mathcal{D}(R)$. Moreover, if all the morphism groups $\mathcal{D}(R)(R[n], C)$ are trivial as n ranges over the integers, then the complex C is acyclic, hence contractible, and so it is trivial in the derived category $\mathcal{D}(R)$. In other words, mapping out of shifted copies of the complex $R[0]$ detects whether an object in $\mathcal{D}(R)$ is trivial or not, so $R[0]$ is also a generator, by the previous criterion.

There is a nice characterization of the small objects in the derived category of a ring. Every bounded complex of finitely generated projective modules is built from summands of the small object $R[0]$ by shifts and extensions in triangles. Since the class of small objects is closed under these operations, a bounded complex of finitely generated projective modules is small. Conversely, these are the only small objects, up to isomorphism in $\mathcal{D}(R)$:

Theorem 3.8. *Let R be a ring. A complex of R -modules is small in the derived category $\mathcal{D}(R)$ if and only if it is quasi-isomorphic to a bounded complex of finitely generated projective R -modules.*

The proof that every small object in $\mathcal{D}(R)$ is quasi-isomorphic to a bounded complex of finitely generated projective modules is more involved. It is a special case of a result about triangulated categories \mathcal{T} with a set of small generators. Neeman [Nee92] showed that every small object in \mathcal{T} is a direct summand of an iterated extension of finitely many shifted generators. The proof can also be found in [Kel94a, 5.3].

There are non-trivial triangulated categories in which only the zero objects are small, see for example [Kel94b] or [HS99, Cor. B.13]. If a triangulated category has a set of generators, then the coproduct of all of them is a single generator. However, an infinite coproduct of non-trivial small objects is not small. So the property of having a single small generator is something special. In fact we see in Theorem 4.16 below that this condition characterizes the module categories over ring spectra among the stable model categories. A triangulated category need not have a *set* of generators whatsoever (one could consider all objects, but in general these form a proper class). For example $K(\mathbb{Z})$, the homotopy category of chain complexes of abelian groups, is not generated by a set [Nee01, E.3].

Equivalences of triangulated categories. A functor between triangulated categories is called exact if it commutes with shift and preserves distinguished triangles. More precisely, $F: \mathcal{S} \rightarrow \mathcal{T}$ is *exact* if it is equipped with a natural isomorphism $\iota_X: F(X[1]) \cong F(X)[1]$ such that for every distinguished triangle (3.5) the sequence

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\iota_X \circ F(h)} F(X)[1]$$

is again a distinguished triangle. An exact functor is automatically additive. An *equivalence of triangulated categories* is an equivalence of categories which is exact and whose inverse functor is also exact.

Exact equivalences between derived categories preserve all concepts which can be defined from $\mathcal{D}(R)$ using only the triangulated structure. One such invariant is the Grothendieck group $K_0(R)$, defined as the free abelian group generated by the isomorphism classes of finitely generated projective R -modules, modulo the relation

$$[P] + [Q] = [P \oplus Q] .$$

For any compactly generated triangulated category \mathcal{T} , the Grothendieck group $K_0(\mathcal{T})$ is defined as the free abelian group generated by the isomorphism classes of small objects in \mathcal{T} , modulo the relation

$$[X] + [Z] = [Y]$$

for every distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

involving small objects X , Y and Z (the morphisms in the triangle do not affect the relation). The split triangle

$$X \xrightarrow{(1,0)} X \oplus Y \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} Y \xrightarrow{0} X[1]$$

is always distinguished, so the relation $[X \oplus Y] = [X] + [Y]$ holds in $K_0(\mathcal{T})$. So for every ring R , the assignment

$$K_0(R) \longrightarrow K_0(\mathcal{D}(R)) , \quad [P] \longmapsto [P[0]] \tag{3.9}$$

defines a group homomorphism. This is in fact an isomorphism, see [Gr77, Sec. 7]. The inverse takes the class in $K_0(\mathcal{D}(R))$ of a bounded complex C of finitely generated projective modules to its ‘Euler characteristic’,

$$\sum_{n \in \mathbb{Z}} (-1)^n [C_n] \in K_0(R) .$$

It is much less obvious that constructions such as the center of a ring, Hochschild and cyclic homology and the higher Quillen K -groups are also invariants of the derived category. In contrast to the Grothendieck group K_0 , there is no construction which produces these groups from the triangulated structure of $\mathcal{D}(R)$ only. The proof that two derived equivalent rings share these invariants uses the ‘tilting theory’ which we outline in the next section. More precisely, if R and S are derived equivalent flat algebras over some commutative ground ring, then there exists a *two-sided tilting complex*, i.e., a chain complex C of R - S -bimodules such that the functor $- \otimes_R C$ induces a (possibly different) derived equivalence [Ric91]. Tensor product with the bimodule complex C then induces an equivalence of K -theory spaces by the work of Thomason-Trobaugh [TT, Thm. 1.9.8]. Without the flatness assumption, the Waldhausen categories of small, cofibrant chain complexes can be related through an intermediate category of differential graded modules, to still obtain an equivalence of K -theory spaces; for more details we refer to [DuSh]. Similarly, Hochschild homology and cohomology (see [Ric91, Prop. 2.5], or, including the Gerstenhaber bracket, see [Kel03]) and cyclic homology (see [Kel96, Kel98]) are isomorphic for derived equivalent rings which are flat algebras over some commutative ground ring. The invariance of the center under derived equivalence is established in [Ric89a, Prop. 9.2] or [KZ, Prop. 6.3.2].

There is a general argument which we will use several times to verify that certain triangulated functors are equivalences, so we state it as a separate proposition. This Proposition 3.10 is a version of ‘Beilinson’s Lemma’ [Bei78] and is typically applied when F is the total derived functor of a suitable left adjoint. In the following proposition, it is crucial that the

functor F be defined and exact on the entire triangulated category \mathcal{S} . It is easy to find non-equivalent triangulated categories \mathcal{S} and \mathcal{T} with infinite sums and small generators P and Q respectively such that

$$\mathcal{S}(P, P)_* \cong \mathcal{T}(Q, Q)_*$$

as graded rings. For example one can take a differential graded ring A with a non-trivial triple Massey product and consider derived categories $\mathcal{S} = \mathcal{D}(A)$ and $\mathcal{T} = \mathcal{D}(H^*A)$ (where the cohomology ring of A is given the trivial differential).

Proposition 3.10. *Let $F: \mathcal{S} \rightarrow \mathcal{T}$ be an exact functor between triangulated categories with infinite sums. Suppose that F preserves infinite sums and \mathcal{S} has a small generator P such that*

- (i) FP is a small generator of \mathcal{T} and
- (ii) for all integers n , the map

$$F : \mathcal{S}(P[n], P) \rightarrow \mathcal{T}(FP[n], FP)$$

is bijective

Then F is an equivalence of categories.

Proof. We consider the full subcategory of \mathcal{S} consisting of those Y for which the map

$$F : \mathcal{S}(P[n], Y) \rightarrow \mathcal{T}(FP[n], FY) \tag{3.11}$$

is bijective for all $n \in \mathbb{Z}$. By assumption this subcategory contains P . Since F is exact, the subcategory is closed under extensions. Since P and FP are small and F preserves coproducts, this subcategory is also closed under coproducts. Since P generates \mathcal{S} , the map (3.11) is thus bijective for arbitrary Y .

Similarly for arbitrary but fixed Y the full subcategory of \mathcal{S} consisting of those X for which the map $F : \mathcal{S}(X, Y) \rightarrow \mathcal{T}(FX, FY)$ is bijective is closed under extensions and coproducts. By the first part, it also contains P , so this subcategory is all of \mathcal{S} . In other words, F is full and faithful.

Now we consider the full subcategory of \mathcal{T} of objects which are isomorphic to an object in the image of F . This subcategory contains the generator FP and it is closed under shifts and coproducts since these are preserved by F . We claim that this subcategory is also closed under extensions. Since FP generates \mathcal{T} , this shows that F is essentially surjective and hence an equivalence.

To prove the last claim we consider a distinguished triangle

$$X \xrightarrow{f} Y \rightarrow Z \rightarrow X[1].$$

Since the subcategory under consideration is closed under isomorphism and shift in either direction we can assume that $X = F(X')$ and $Y = F(Y')$ are objects in the image of F . Since F is full there exists a map $f' : X' \rightarrow Y'$ satisfying $F(f') = f$. We can then choose a mapping cone for the map f' and a compatible map from Z to $F(\text{Cone}(f'))$ which is necessarily an isomorphism. \square

3.2. Derived equivalences after Rickard and Keller. In this section we state and prove Rickard's "Morita theory for derived categories". Rickard shows in [Ric89a, Thm. 6.4] that the existence of a *tilting complex* is necessary and sufficient for an equivalence between the unbounded derived categories of two rings. A tilting complex is a special small generator of the derived category, see Definition (3.12) below. The idea to use differential graded algebras in the proof is due to Keller [Kel94a], and we closely follow his approach.

The notion of a tilting complex comes up naturally when we examine the properties of the preferred generator $R[0]$ of the derived category $\mathcal{D}(R)$. First of all, the free R -module of rank one, considered as a complex concentrated in dimension zero, is a small generator of the derived category $\mathcal{D}(R)$. Since R is a free module, it has no self-extensions. Because Ext groups can be identified with morphisms in the derived category (see (3.4)), this means that the graded self-maps of the complex $R[0]$ are concentrated in dimension zero:

$$\mathcal{D}(R)(R[n], R) = 0 \text{ for } n \neq 0.$$

A tilting complex is any complex which also has these properties. Hence the definition is made so that the image of $R[0]$ under an equivalence of triangulated categories is a tilting complex.

Definition 3.12. A *tilting complex* for a ring R is a bounded complex T of finitely generated projective R -modules which generates the derived category $\mathcal{D}(R)$ and whose graded ring of self maps $\mathcal{D}(R)(T, T)_*$ is concentrated in dimension zero.

Special kinds of tilting complexes are the *tilting modules*; we give examples of tilting modules and tilting complexes in Section 3.3. The following theorem is due to Rickard [Ric89a, Thm. 6.4].

Theorem 3.13. *For two rings R and S the following conditions are equivalent.*

- (1) *The unbounded derived categories of R and S are equivalent as triangulated categories.*
- (2) *There is a tilting complex T in $\mathcal{D}(S)$ whose endomorphism ring $\mathcal{D}(S)(T, T)$ is isomorphic to R .*

Moreover, conditions (1) and (2) are implied by the condition

- (3) *There exists a chain complex of R - S -bimodules M such that the derived tensor product functor*

$$- \otimes_R^L M : \mathcal{D}(R) \longrightarrow \mathcal{D}(S)$$

is an equivalence of categories.

If R or S is flat as an abelian group, then all three conditions are equivalent.

Instead of using the unbounded derived category, one can replace condition (1) by an equivalence between the full subcategories of homologically bounded below or small objects inside the derived categories, see for example [Ric89a, Thm. 6.4]. There is a version relative to a commutative ring k . Then R and S are k -algebras, conditions (1) and (3) then refer to k -linear equivalences of derived categories, condition (2) requires an isomorphism of k -algebras and in the addendum, one of R or S has to be flat as a k -module.

Remark 3.14. A derived equivalence F from $\mathcal{D}(R)$ to $\mathcal{D}(S)$ which is not already a Morita equivalence maps the R -modules inside $\mathcal{D}(R)$ (i.e., complexes with homology concentrated in dimension 0) “transversely” to the S -modules inside $\mathcal{D}(S)$; more precisely, for an R -module M , the complex $F(M[0])$ can have non-trivial homology in several, or even in infinitely many dimensions; we give an example in 3.26 below. However, the homology of $F(M[0])$ is always bounded below.

A related point is that we can *not* recover the module category $\text{Mod-}R$ from $\mathcal{D}(R)$, viewed as an abstract triangulated category. This is because we cannot make sense of “complexes with homology concentrated in dimension 0” unless we specify a homology functor like H_0 , or we single out the class of complexes with homology in non-negative dimensions. This sort of extra structure is called a *t-structures* [BBD, 1.3] on a triangulated category. Every

t-structure has a *heart*, an abelian category which plays the role of complexes in $\mathcal{D}(R)$ with homology concentrated in dimension zero.

The most involved part of the tilting theorem is the implication (2) \implies (1), i.e., showing that a tilting complex gives rise to a derived equivalence. The proof we give is due to Keller; in the original paper [Kel94a], his setup is more general (he works in differential graded categories in order to allow ‘many generator’ versions). In the special case of interest for us, the exposition simplifies somewhat [KZ, Ch. 8]. Given a tilting complex T in $\mathcal{D}(S)$, the comparison between the derived categories of R and S passes through the derived category of a certain *differential graded ring* (generalizing the derived category of an ordinary ring), namely the *endomorphism DG ring* $\text{End}_S(T)$ of the tilting complex T (generalizing the endomorphism ring). So we start by introducing these new characters.

Definition 3.15. A *differential graded ring* is a \mathbb{Z} -graded ring A together with a differential d of degree -1 which satisfies the Leibniz rule

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b) \quad (3.16)$$

for all homogeneous elements $a, b \in A$. A *differential graded right module* (or *DG module* for short) over a differential graded ring A consists of a graded right A -module together with a differential d of degree -1 which satisfies the Leibniz rule (3.16), but where now a is a homogeneous element of the module and b is a homogeneous element of A . A *homomorphism* of DG modules is a homomorphism of graded A -modules which is also a chain map. A *chain homotopy* between homomorphisms of DG modules is a homomorphism of graded A -modules of degree 1 which is also a chain homotopy.

A differential graded A -module M is *cofibrant* if there exists an exhaustive increasing filtration by sub DG modules

$$0 = M^0 \subseteq M^1 \subseteq \dots \subseteq M^n \subseteq \dots$$

such that each subquotient M^n/M^{n-1} is a direct summand of a direct sum of shifted copies of A . The *derived category* $\mathcal{D}(A)$ of the differential graded ring A has as objects the cofibrant DG modules over A and as morphisms the chain homotopy classes of DG module homomorphisms.

Up to chain homotopy equivalence, the cofibrant DG modules are the ones which have Keller’s ‘property (P)’ in [Kel94a, 3.1]. A cofibrant differential graded module is sometimes called ‘semi-free’ or a ‘cell module’ [KM, Part III] (up to direct summands).

Remark 3.17. We need some facts about differential graded rings and modules which are not very difficult to prove, but which we do not want to discuss in detail.

- (i) Several of the remarks from 3.3 carry over from rings to DG rings. A cofibrant DG A -module is projective as a graded A -module, ignoring the differential. Every quasi-isomorphism between cofibrant DG modules is a chain homotopy equivalence, and every DG module can be approximated up to quasi-isomorphism by a cofibrant one. A DG module is called *homotopically projective* if every homomorphism into an acyclic DG module is null-homotopic. Then a DG module is homotopically projective if and only if it is chain homotopy equivalent, as a DG module, to a cofibrant DG module.
- (ii) The derived category $\mathcal{D}(A)$ of a differential graded ring A is naturally a triangulated category. The shift functor is again given by reindexing a DG module, and distinguished triangles arise from mapping cones as for the derived category of a ring. The only thing to note is that for a homomorphism $f : M \rightarrow N$ of DG modules over

A , the mapping cone becomes a graded A -module as the direct sum $N \oplus M[1]$, and this A -action satisfies the Leibniz rule with respect to the mapping cone differential (3.2).

- (iii) Suppose that $f : A \rightarrow B$ is a homomorphism of differential graded rings, i.e., f is a multiplicative chain homomorphism. Then extension of scalars $M \mapsto M \otimes_A B$ is exact on cofibrant differential graded modules (since the underlying *graded* modules over the graded ring underlying A are projective), it takes cofibrant modules to cofibrant modules, and it preserves the chain homotopy relation. So extension of scalars induces an exact functor on the level of derived categories

$$\mathcal{D}(A) \begin{array}{c} \xrightarrow{-\otimes_A^L B} \\ \xrightleftharpoons[f^*]{} \\ \xrightarrow{\quad} \end{array} \mathcal{D}(B) \quad , \quad (3.18)$$

called the *left derived functor*. This derived functor has an exact right adjoint f^* induced by restriction of scalars along f . This is not completely obvious with our definition of the derived category, since a cofibrant differential graded B -module is usually *not* cofibrant when viewed as a DG module over A via f .

If $f : A \rightarrow B$ is a *quasi-isomorphism* of differential graded rings, then the derived functors of restriction and extension of scalars (3.18) are inverse *equivalences* of triangulated categories.

- (iv) Suppose A is a differential graded ring whose homology is concentrated in dimension zero. Then A is quasi-isomorphic, as a differential graded ring, to the zeroth homology ring H_0A . Indeed, a chain of two quasi-isomorphisms is given by

$$A \xleftarrow{\text{inclusion}} A_+ \xrightarrow{\text{projection}} H_0(A) .$$

Here A_+ is the differential graded sub-ring of A given by

$$(A_+)_n = \begin{cases} A_n & \text{for } n > 0 \\ \text{Ker}(d : A_0 \rightarrow A_{-1}) & \text{for } n = 0, \text{ and} \\ 0 & \text{for } n < 0. \end{cases}$$

Since the homology of A is trivial in negative dimensions, the inclusion $A_+ \rightarrow A$ is a quasi-isomorphism. Since A_+ is trivial in negative dimensions, the projection $A_+ \rightarrow H_0(A_+)$ is a homomorphism of differential graded rings, where the target is concentrated in dimension zero. This projection is also a quasi-isomorphism since the homology of A , and hence that of A_+ , is trivial in positive dimensions.

Homomorphism complexes. Let A be a DG ring and let M and N be DG modules over A , not necessarily cofibrant. We defined the *homomorphism complex* $\text{Hom}_A(M, N)$ as follows. In dimension $n \in \mathbb{Z}$, the chain group $\text{Hom}_A(M, N)_n$ is the group of graded A -module homomorphisms of degree n , i.e.,

$$\text{Hom}_A(M, N)_n = \text{Hom}_A(M[n], N) .$$

The differentials of M and N do not play any role in the definition of the chain groups, but they enter in the formula for the differential which makes $\text{Hom}_A(M, N)$ into a chain complex. This differential $d : \text{Hom}_A(M, N)_n \rightarrow \text{Hom}_A(M, N)_{n-1}$ is defined by

$$d(f) = d_N \circ f - (-1)^n f \circ d_M . \quad (3.19)$$

Here f is a graded A -module map of degree n and the composites $d_N \circ f$ and $f \circ d_M$ are then graded A -module maps of degree $n - 1$.

With this definition, the 0-cycles in $\mathrm{Hom}_A(M, N)$ are those graded A -module maps f which satisfy $d_N \circ f - f \circ d_M = 0$, so they are precisely the DG homomorphisms from M to N . Moreover, if $f, g : M \rightarrow N$ are two DG A -module maps, then the difference $f - g$ is a coboundary in the complex $\mathrm{Hom}_A(M, N)$ if and only if f is chain homotopic to g . So we have established a natural isomorphism

$$H_0(\mathrm{Hom}_A(M, N)) \cong [M, N]$$

between the zeroth homology of the complex $\mathrm{Hom}_A(M, N)$ and the chain homotopy classes of DG A -homomorphisms from M to N .

Now suppose that we have a third DG module L . Then the composition of graded A -module maps gives a bilinear pairing between the homomorphism complexes

$$\circ : \mathrm{Hom}_A(N, L)_m \times \mathrm{Hom}_A(M, N)_n \rightarrow \mathrm{Hom}_A(M, L)_{m+n} .$$

Moreover, composition and the differential (3.19) satisfy the Leibniz rule, i.e., for graded A -module maps $f : M \rightarrow N$ of degree n and $g : N \rightarrow L$ of degree m we have

$$d(g \circ f) = dg \circ f + (-1)^m g \circ df$$

as graded maps from M to L .

The following consequences are crucial for the remaining step in the tilting theorem:

- for every DG A -module M , the endomorphism complex $\mathrm{End}_A(M) = \mathrm{Hom}_A(M, M)$ is a differential graded ring under composition and M is a differential graded $\mathrm{End}_A(M)$ - A -bimodule;
- for every DG A -module N , the homomorphism complex $\mathrm{Hom}_A(M, N)$ is a differential graded module over $\mathrm{End}_A(M)$ under composition. Moreover the functor $\mathrm{Hom}_A(M, -) : \mathbf{Mod}\text{-}A \rightarrow \mathbf{Mod}\text{-}\mathrm{End}_A(M)$ is right adjoint to tensoring with the $\mathrm{End}_A(M)$ - A -bimodule M ;
- if M is cofibrant, then the functor $\mathrm{Hom}_A(M, -)$ is exact and takes quasi-isomorphisms of DG A -modules to quasi-isomorphisms. Moreover, its left adjoint $- \otimes_{\mathrm{End}_A(M)} M$ preserves cofibrant objects and chain homotopies. So there exists a derived functor on the level of derived categories

$$- \otimes_{\mathrm{End}_A(M)}^L M : \mathcal{D}(\mathrm{End}_A(M)) \rightarrow \mathcal{D}(A) ,$$

an exact functor which preserves infinite sums.

The following theorem is a special case of Lemma 6.1 in [Kel94a].

Theorem 3.20. *Let A be a DG ring and M a cofibrant A -module which is a small generator for the derived category $\mathcal{D}(A)$. Then the derived functor*

$$- \otimes_{\mathrm{End}_A(M)}^L M : \mathcal{D}(\mathrm{End}_A(M)) \rightarrow \mathcal{D}(A) \tag{3.21}$$

is an equivalence of triangulated categories.

Proof. The total left derived functor (3.21) is an exact functor between triangulated categories which preserves infinite sums. Moreover, it takes the free $\mathrm{End}_A(M)$ -module of rank one — which is a small generator for the derived category of $\mathrm{End}_A(M)$ — to the small generator M for $\mathcal{D}(A)$. The induced map of graded endomorphism rings

$$- \otimes_{\mathrm{End}_A(M)}^L M : \mathcal{D}(\mathrm{End}_A(M))(\mathrm{End}_A(M), \mathrm{End}_A(M))_* \rightarrow \mathcal{D}(A)(M, M)_*$$

is an isomorphism (both sides are isomorphic to the homology ring of $\mathrm{End}_A(M)$). So Proposition 3.10 shows that this derived functor is an equivalence of triangulated categories. \square

After all these preparations we can give the

Proof of the tilting theorem 3.13. Clearly, condition (3) implies condition (1). Now we assume condition (1) and we choose an exact equivalence F from the derived category $\mathcal{D}(R)$ to $\mathcal{D}(S)$. The defining properties of a tilting complex are preserved under exact equivalences of triangulated categories. Since $R[0]$, the free R -module of rank one, concentrated in dimension zero, is a tilting complex for the ring R , its image $T = F(R[0])$ is a tilting complex for S . Moreover, F restricts to a ring isomorphism

$$F : R \cong \mathcal{D}(R)(R[0], R[0]) \xrightarrow{\cong} \mathcal{D}(S)(T, T) .$$

Hence condition (2) holds.

For the implication (2) \implies (1) we are given a tilting complex T in $\mathcal{D}(S)$ and an isomorphism of rings $\mathcal{D}(S)(T, T) \cong R$. The complex T is naturally a differential graded $\text{End}_S(T)$ - S -bimodule, and by Theorem 3.20, the derived functor

$$- \otimes_{\text{End}_S(T)}^L T : \mathcal{D}(\text{End}_S(T)) \longrightarrow \mathcal{D}(S)$$

is an equivalence of triangulated categories. The isomorphism of graded rings

$$H_*(\text{End}_S(T)) \cong \mathcal{D}(S)(T, T)_*$$

and the defining property of a tilting complex show that the homology of $\text{End}_S(T)$ is concentrated in dimension zero. So there is a chain of two quasi-isomorphisms between $\text{End}_S(T)$ and the ring $H_0(\text{End}_S(T)) \cong \mathcal{D}(S)(T, T) \cong R$. Restriction and extension of scalars along these quasi-isomorphisms gives a chain of equivalences between the derived categories of the differential graded ring $\text{End}_S(T)$ and the derived category of the ordinary ring $\mathcal{D}(S)(T, T)$. Putting all of this together we end up with a chain of three equivalences of triangulated categories:

$$\mathcal{D}(R) \cong \mathcal{D}(\text{End}_S(T)_+) \cong \mathcal{D}(\text{End}_S(T)) \cong \mathcal{D}(S) .$$

It remains to prove the implication (2) \implies (3), assuming that R or S is flat. Let T be a tilting complex in $\mathcal{D}(S)$ and $f : \mathcal{D}(S)(T, T) \longrightarrow R$ an isomorphism of rings. The homology of $\text{End}_S(T)$ is isomorphic to the graded self maps of T in $\mathcal{D}(S)$, so it is concentrated in dimension 0. So the inclusion of the DG sub-ring $\text{End}_S(T)_+$ into the endomorphism DG ring $\text{End}_S(T)$ induces an isomorphism on homology, compare Remark 3.17 (iv). Since $\text{End}_S(T)_+$ is trivial in negative dimensions, there is a unique morphism of DG rings $\text{End}_S(T)_+ \longrightarrow R$ which realizes the isomorphism f on H_0 . We choose a *flat resolution* of $\text{End}_S(T)_+$, i.e., a DG ring E and a quasi-isomorphism of DG rings $E \xrightarrow{\sim} \text{End}_S(T)_+$, such that the functor $E \otimes -$ preserves quasi-isomorphisms between chain complexes of abelian groups (see for example [Kel99, 3.2 Lemma (a)]). We end up with a chain of two quasi-isomorphisms of DG rings

$$R \xleftarrow{\cong} E \xrightarrow{\sim} \text{End}_S(T) .$$

The complex T is naturally a DG $\text{End}_S(T)$ - S -bimodule, and we restrict the left action to E and view T as a DG E - S -bimodule. We choose a cofibrant replacement $T^c \xrightarrow{\sim} T$ as a DG E - S -bimodule. Then we obtain the desired complex of R - S -bimodules by

$$M = R \otimes_E T^c .$$

Tensoring with M over R has a total left derived functor

$$- \otimes_R^L M : \mathcal{D}(R) \longrightarrow \mathcal{D}(S) \tag{3.22}$$

(although this is not obvious with our definition since M need not be cofibrant as a complex of right S -modules, and then $- \otimes_R M$ does not takes values in cofibrant complexes). In

order to show that this derived functor is an exact equivalence we use that the diagram of triangulated categories

$$\begin{array}{ccc}
 \mathcal{D}(E) & \xrightarrow{-\otimes_E^L \text{End}_S(T)} & \mathcal{D}(\text{End}_S(T)) \\
 \downarrow -\otimes_E^L R & & \downarrow -\otimes_{\text{End}_S(T)}^L T \\
 \mathcal{D}(R) & \xrightarrow{-\otimes_R^L M} & \mathcal{D}(S)
 \end{array} \tag{3.23}$$

commutes up to natural isomorphism. Indeed, two ways around the square are given by derived tensor product with the E - S -bimodules M respectively T , so it suffices to find a chain of quasi-isomorphisms of DG bimodules between M and T .

Since E is cofibrant as a complex of abelian groups and the composite map $E \rightarrow \text{End}_S(T)_+ \rightarrow R$ is a quasi-isomorphism, $E \otimes S^{op}$ models the derived tensor product of R and S . If one of R or S are flat, then $R \otimes S^{op}$ also models the derived tensor product, so that the map

$$E \otimes S^{op} \longrightarrow R \otimes S^{op}$$

is a quasi-isomorphism of DG rings. Since T^c is cofibrant as an $E \otimes S^{op}$ -module, the induced map

$$T^c = (E \otimes S^{op})_{E \otimes S^{op}} T^c \longrightarrow (R \otimes S^{op})_{E \otimes S^{op}} T^c \cong R \otimes_E T^c = M$$

is a quasi-isomorphism. So we have a chain of two quasi-isomorphisms of E - S -bimodules

$$T \xleftarrow{\cong} T^c \xrightarrow{\cong} M.$$

The left and upper functors in the commutative square (3.23) are derived from extensions of scalars along quasi-isomorphisms of DG rings; thus they are exact equivalence of triangulated categories. The right vertical derived functor is an exact equivalence by Theorem 3.20. So we conclude that the lower horizontal functor (3.22) in the square (3.23) is also an exact equivalence of triangulated categories. This establishes condition (3). \square

3.3. Examples. Historically, tilting modules seem to have been the first examples of derived equivalences which are not Morita equivalences. I will not try to give an account of the history of tilting modules and rather refer to [KZ, Sec. 3.1] or [AGH].

Definition 3.24. Let R be a finite dimensional algebra over a field. A *tilting module* is a finitely generated R -module T with the following properties.

- (i) T has projective dimension 0 or 1,
- (ii) T has no self-extensions, i.e., $\text{Ext}_R^1(T, T) = 0$,
- (iii) there is an exact sequence of right R -modules

$$0 \longrightarrow R \longrightarrow T_1 \longrightarrow T_2 \longrightarrow 0$$

such that T_1 and T_2 are direct summands of a finite sum of copies of T .

Note that if the tilting module T is actually projective, then condition (ii) is automatic and the exact sequence required in (iii) splits. So then the free R -module of rank one is a summand of a finite sum of copies of T , and hence T is a finitely generated projective generator for $\text{Mod-}R$. So R is then Morita equivalent to the endomorphism ring of the tilting module T , by the Morita theorem 2.2. For a self-injective algebra, for example a group algebra over a field, the converse also holds; indeed, every module of finite projective

dimension over a self-injective algebra is already projective. So for these algebras, tilting is the same as Morita equivalence.

If the projective dimension of the tilting module T is 1, then we do not get an equivalence between the modules over R and $S = \text{End}_R(T)$, but we get a derived equivalence. Indeed, since R is noetherian, condition (i) implies that T has a 2-step resolution $P_1 \rightarrow P_0$ by two finitely generated projective R -modules; this resolution is a small object in the derived category $\mathcal{D}(R)$, and its graded self-maps in $\mathcal{D}(R)$ are concentrated in dimension 0 by condition (ii). Condition (iii) implies that the complex $R[0]$ is contained in the triangulated subcategory generated by the resolution, and the resolution is thus a generator for $\mathcal{D}(R)$, hence a tilting complex.

Example 3.25. For an example of a non-projective tilting module we fix a field k and we let A be the algebra of upper triangular 3×3 matrices over k ,

$$A = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} \mid x_{ij} \in k \right\} .$$

Up to isomorphism, there are three indecomposable projective right A -modules, namely the row vectors

$$P^1 = \{(y_1, y_2, y_3) \mid y_1, y_2, y_3 \in k\}$$

and its A -submodules

$$P^2 = \{(0, y_2, y_3) \mid y_2, y_3 \in k\} \quad \text{and} \quad P^3 = \{(0, 0, y_3) \mid y_3 \in k\} .$$

These projectives are the covers of three corresponding simple modules, namely

$$S^1 = P^1/P^2, \quad S^2 = P^2/P^3, \quad \text{and} \quad S^3 = P^3 .$$

In particular, S^3 is projective and S^1 and S^2 have projective dimension 1.

We define the tilting module T as the direct sum

$$T = P^1 \oplus P^2 \oplus S^2 .$$

The projective resolution

$$0 \longrightarrow P^1 \oplus P^2 \oplus P^3 \xrightarrow{\text{inclusion}} P^1 \oplus P^2 \oplus P^2 \longrightarrow S^2 \longrightarrow 0$$

can be used to calculate $\text{Ext}_A^1(T, T) = \text{Ext}_A^1(S^2, T) = 0$. Since $P^1 \oplus P^2 \oplus P^3$ is a free A -module of rank one, this short exact sequence verifies tilting condition (iii) in Definition 3.24 for the A -module T .

Altogether this shows that T is a tilting module for A of projective dimension one. So A ‘tilts’ to the endomorphism algebra of T ; this endomorphism algebra can be calculated directly, and it comes out to be another subalgebra of the 3×3 matrices over k , namely

$$\text{End}_A(T) \cong \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{pmatrix} \mid x_{ij} \in k \right\} .$$

(hint: the modules P^1 and S^2 do not map to each other nor to P^2 , and the remaining relevant morphism spaces are 1-dimensional over k .) The algebras A and $\text{End}_A(T)$ are *not* Morita equivalent. Indeed, both have exactly three isomorphism classes of indecomposable projective modules, but in one case these modules are ‘directed’ (i.e., linearly ordered under the existence of non-trivial homomorphisms), whereas in the other case two of these indecomposable projectives do not map to each other non-trivially.

The preceding example, and many other ones, are often described using representations of quivers. Indeed, the upper triangular matrices A and the tilted algebra $\text{End}_A(T)$ are isomorphic to the path algebras of the A_3 -quivers

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \quad \text{respectively} \quad \bullet \longleftarrow \bullet \longrightarrow \bullet .$$

Example 3.26. We obtain a tilting complex whose homology is concentrated in more than one dimension by ‘spreading out’ the free module of rank one. Let $R = R_1 \times R_2$ be the product of two rings. Let $P_1 = R_1 \times 0$ and $P_2 = 0 \times R_2$ be the two “blocks”, i.e., the projective R -bimodules corresponding to the central idempotents $(1, 0)$ and $(0, 1)$ in R . Then $R = P_1 \oplus P_2$ as an R -bimodule, and there are no non-trivial R -homomorphisms between P_1 and P_2 . Now take $T = P_1[0] \oplus P_2[n]$ for some number $n \neq 0$. This is a complex of R -modules with trivial differential whose homology is concentrated in two dimensions. Moreover, the complex T is a small generator for the derived category $\mathcal{D}(R)$. But the only non-trivial self-maps of T are of degree 0 since P_1 and P_2 don’t map to each other. Hence T is a tilting complex which is not (quasi-isomorphic to) a tilting module. The endomorphisms of T are again the ring R , so it is a non-trivial self-tilting complex of R . Under the equivalence $\mathcal{D}(R) \cong \mathcal{D}(R_1) \times \mathcal{D}(R_2)$, the self-equivalence induced by T is the identity on the first factor and the n -fold shift on the second factor.

More examples of tilting complexes can be found in Sections 4 and 5 of [Ric89b] or Chapter 5 of [KZ].

4. MORITA THEORY IN STABLE MODEL CATEGORIES

Now we carry the Morita philosophy one step further: we sketch Morita theory for ring spectra and for stable model categories. As a summary one can say that essentially everything which we have said for rings and differential graded rings works, suitably interpreted, for ring spectra as well.

First a few words about what we mean by a ring spectrum. The stable homotopy category of algebraic topology has a symmetric monoidal smash product; the monoids are homotopy-associative ring spectra, and they represent multiplicative cohomology theories. While the notion of a homotopy-associative ring spectrum is useful for many things, it does not have a good enough module theory for our present purpose. One can certainly consider spectra with a homotopy-associative action of a homotopy-associative ring spectrum; but the mapping cone of a homomorphism between such modules does not inherit a *natural* action of the ring spectrum, and the category of such modules does not form a triangulated category.

So in order to carry out the Morita-theory program we need a highly structured model for the category of spectra which admits a symmetric monoidal and homotopically well behaved smash product — before passing to the homotopy category ! The first examples of such categories were the S -modules [EKMM] and the symmetric spectra [HSS]; by now several more such categories have been constructed [Lyd98, MMSS]. All these spectra categories are *model categories* in the sense of Quillen [Qui67], and the appropriate notion of equivalence is that of a Quillen equivalence [Hov99, Def. 1.3.12] since these equivalences preserve the ‘homotopy theory’, not just the homotopy category; all known model categories of spectra are Quillen equivalent in a monoidal fashion.

For definiteness, we work in one specific category of spectra with nice smash product, namely the *symmetric spectra* based on simplicial sets, as introduced by Hovey, Shipley and Smith [HSS]. The monoids are called *symmetric ring spectra*, and I personally think that

they are the simplest kind of ring spectra; as far as their homotopy category is concerned, symmetric ring spectra are equivalent to the older notion of A_∞ -ring spectrum, and commutative symmetric ring spectra are equivalent to E_∞ -ring spectra. The good thing is that operads are not needed anymore.

However, using symmetric spectra is not essential and the results described in this section could also be developed in more or less the same way in any other of the known model categories of spectra with compatible smash product. Alternatively, we could have taken an axiomatic approach and use the term ‘spectra’ for any stable, monoidal model category in which the unit object ‘looks and feels’ like the sphere spectrum. Indeed, we are essentially only using the following properties of the category of symmetric spectra:

- (i) there is a symmetric monoidal *smash product*, which makes symmetric spectra into a *monoidal model category* ([Hov99, 4.2.6], [SS00]);
- (ii) the model structure is *stable* (Definition 4.1);
- (iii) the unit \mathbb{S} of the smash product is a small generator (Definition 3.7) of the homotopy category of spectra;
- (iv) the (derived) space of self maps of the unit object \mathbb{S} is weakly equivalent to $QS^0 = \text{hocolim}_n \Omega^n S^n$ and in the homotopy category, there are no maps of negative degree from \mathbb{S} to itself.

A large part of the material in this section is taken from a joint paper with Shipley [SS03]. Two other papers devoted to Morita theory in the context of ring spectra are [DGI] by Dwyer, Greenlees and Iyengar and [BL] by Baker and Lazarev.

4.1. Stable model categories. Recall from [Qui67, I.2] or [Hov99, 6.1] that the homotopy category of a pointed model category supports a suspension and a loop functor. In short, for any object X the map to the zero object can be factored

$$X \longrightarrow C \xrightarrow{\simeq} *$$

as a cofibration followed by a weak equivalence. The suspension of X is then defined as the quotient of the cofibration, $\Sigma X = C/X$. Dually, the loop object ΩX is the fiber of a fibration from a weakly contractible object to X . On the level of homotopy categories, the suspension and loop constructions become functorial, and Σ is left adjoint to Ω .

Definition 4.1. A *stable model category* is a pointed model category for which the functors Ω and Σ on the homotopy category are inverse equivalences.

The homotopy category of a stable model category has a large amount of extra structure, some of which is relevant for us. First of all, it is naturally a triangulated category, see [Hov99, 7.1.6] for a detailed proof. The rough outline is as follows: by definition of ‘stable’ the suspension functor is a self-equivalence of the homotopy category and it defines the shift functor. Since every object is a two-fold suspension, hence an abelian co-group object, the homotopy category of a stable model category is additive. Furthermore, by [Hov99, 7.1.11] the cofiber sequences and fiber sequences of [Qui67, 1.3] coincide up to sign in the stable case, and they define the distinguished triangles. The model categories which we consider have all limits and colimits, so the homotopy categories have infinite sums and products. Objects of a stable model category are called ‘generators’ or ‘small’ if they have this property as objects of the triangulated homotopy category, compare Definition 3.7.

A Quillen adjoint functor pair between stable model categories gives rise to total derived functors which are exact functors with respect to the triangulated structure; in other words both total derived functors commute with suspension and preserve distinguished triangles.

Examples 4.2.

(1) **Chain complexes.** In the previous section, we have already seen an important class of examples from algebra, namely the category of chain complexes over a ring R . This category actually has several different stable model structures: the *projective* model structure (see Remark 3.3 (iv)) and the *injective* model structure (see Remark 3.3 (v)) have as weak equivalences the quasi-isomorphisms. There is a clash of terminology here: the homotopy category in the sense of homotopical algebra is obtained by formally inverting the weak equivalences; so for the projective and injective model structures, this gives the unbounded derived category $\mathcal{D}(R)$. But the category of unbounded chain complexes admits another model structure in which the weak equivalences are the *chain homotopy equivalences*, see e.g. [CH, Ex. 3.4]. Thus for this model structure, the homotopy category is what is commonly called the homotopy category, often denoted by $K(R)$. The derived category $\mathcal{D}(R)$ is a quotient of the homotopy category $K(R)$; the derived category $\mathcal{D}(R)$ has a single small generator, but for example the homotopy category of chain complexes of abelian groups $K(\mathbb{Z})$ does not have a set of generators whatsoever, compare [Nee01, E.3.2]. The three stable model structures on chain complexes of modules have been generalized in various directions to chain complexes in abelian categories or to other differential graded objects, see [CH], [Bek00] and [Hov01a].

(2) **The stable module category of a Frobenius ring.** A different kind of algebraic example — not involving chain complexes — is formed by the stable module categories of Frobenius rings. A Frobenius ring A is defined by the property that the classes of projective and injective A -modules coincide. Important examples are finite dimensional self-injective algebras over a field, in particular finite dimensional Hopf-algebras, such as group algebras of finite groups. The *stable module category* has as objects the A -modules (*not* chain complexes of modules). Morphisms in the stable category are represented by module homomorphisms, but two homomorphisms are identified if their difference factors through a projective (= injective) A -module.

Fortunately the two different meanings of ‘stable’ fit together nicely: the stable *module* category is the homotopy category associated to a stable *model* category structure on the category of A -modules, see [Hov99, Sec. 2]. The cofibrations are the monomorphisms, the fibrations are the epimorphisms, and the weak equivalences are the maps which become isomorphisms in the stable category. Every finitely generated module is small when considered as an object of the stable module category. As in the case of chain complexes of modules, there is usually no point in making the model structure explicit since the cofibration, fibrations and weak equivalences coincide with certain well-known concepts.

Quillen equivalences between stable module categories arise under the name of *stable equivalences of Morita type* ([Bro94, Sec. 5], [KZ, Ch. 11]). For simplicity, suppose that A and B are two finite-dimensional self-injective algebras over a field k ; then A and B are in particular Frobenius rings. Consider an A - B -bimodule M (by which we mean a k -symmetric bimodule, also known as a right module over $A^{op} \otimes_k B$), which is projective as left A -module and as a right B -module separately. Then the adjoint functor pair

$$\text{Mod-}A \begin{array}{c} \xrightarrow{- \otimes_A M} \\ \xleftarrow{\text{Hom}_B(M, -)} \end{array} \text{Mod-}B \quad (4.3)$$

is Quillen adjoint pair with respect to the ‘stable’ model structures.

A stable equivalence of Morita type consists of an A - B -bimodule M and a B - A -bimodule N such that both M and N are projective as left and right modules separately, and such

that there are direct sum decompositions

$$N \otimes_A M \cong B \oplus X \quad \text{and} \quad M \otimes_B N \cong A \oplus Y$$

as bimodules, where Y is a projective A - A -bimodule and B is a projective B - B -bimodule. In this situation, the functors $-\otimes_A M$ and $-\otimes_B N$ induce inverse equivalences of the stable module categories. Moreover, the Quillen adjoint pair (4.3) is a Quillen equivalence.

Rickard observed [Ric89b] that a derived equivalence between self-injective, finite-dimensional algebras also gives rise to a stable equivalence of Morita type.

In the above algebraic examples, there is no real need for the language of model categories; moreover, ‘Morita theory’ is covered by Keller’s paper [Kel94a], which uses differential graded categories. A whole new world of stable model categories comes from homotopy theory, see the following list. The associated homotopy categories yield triangulated categories which are not immediately visible to the eyes of an algebraist, since they do not arise from abelian categories.

(3) **Spectra.** The prototypical example of a stable model category (which is not ‘algebraic’), is ‘the’ category of spectra. We review one model, the *symmetric spectra* of Hovey, Shipley and Smith [HSS] in more detail in Section 4.2. Many other model categories of spectra have been constructed, see for example [BF78, Rob87a, Jar97, EKMM, Lyd98, MMSS]. All known model categories of spectra Quillen equivalent (see e.g. [HSS, Thm. 4.2.5], [Sch01a] or [MMSS]), and their common homotopy category is referred to as *the* stable homotopy category. The sphere spectrum is a small generator for stable homotopy category.

(4) **Modules over ring spectra.** Modules over an S -algebra [EKMM, VII.1], over a symmetric ring spectrum [HSS, 5.4.2], or over an orthogonal ring spectrum [MMSS] form stable model categories. We recall symmetric ring spectra and their module spectra in Section 4.2. In each case a module is small if and only if it is weakly equivalent to a retract of a finite cell module. The free module of rank one is a small generator. More generally there are stable model categories of modules over ‘symmetric ring spectra with several objects’, or *spectral categories*, see [SS03, A.1].

(5) **Equivariant stable homotopy theory.** If G is a compact Lie group, there is a category of G -equivariant coordinate free spectra [LMS86] which is a stable model category. Modern versions of this model category are the G -equivariant orthogonal spectra of [MM02] and G -equivariant S -modules of [EKMM]. In this case the equivariant suspension spectra of the coset spaces G/H_+ for all closed subgroups $H \subseteq G$ form a set of small generators.

(6) **Presheaves of spectra.** For every Grothendieck site Jardine [Jar87] constructs a stable model category of presheaves of Bousfield-Friedlander type spectra; the weak equivalences are the maps which induce isomorphisms of the associated sheaves of stable homotopy groups. For a general site these stable model categories do not seem to have a set of small generators. A similar model structure for presheaves of symmetric spectra is developed in [Jar00a].

(7) **The stabilization of a model category.** Modulo technicalities, every pointed model category gives rise to an associated stable model category by ‘inverting’ the suspension functor, i.e., by passage to internal spectra. This has been carried out, under different hypotheses, in [Sch97] and [Hov01b].

(8) **Bousfield localization.** Following Bousfield [Bou75], localized model structures for modules over an S -algebra are constructed in [EKMM, VIII 1.1]. Hirschhorn [Hir03] shows that under quite general hypotheses the localization of a model category is again a model

category. The localization of a stable model category is stable and localization preserves generators. Smallness need not be preserved.

(9) **Motivic stable homotopy.** In [MV, Voe98] Morel and Voevodsky introduced the \mathbb{A}^1 -local model category structure for schemes over a base. An associated stable homotopy category of \mathbb{A}^1 -local T -spectra (where $T = \mathbb{A}^1/(\mathbb{A}^1 - 0)$ is the ‘Tate-sphere’) is an important tool in Voevodsky’s proof of the Milnor conjecture [Voe]. There are several stable model categories underlying this motivic stable homotopy category, see for example [Jar00b], [Hov01b], [Hu03] or [DØR].

4.2. Symmetric ring and module spectra. In this section we give a quick introduction to symmetric spectra and symmetric ring and module spectra. I recommend reading the original, self-contained paper by Hovey, Shipley and Smith [HSS]. At several points, our exposition differs from theirs; for example, we let the spheres act from the right.

Definition 4.4. [HSS] A symmetric spectrum consists of the following data:

- a sequence of pointed simplicial sets X_n for $n \geq 0$
- for each $n \geq 0$ a base-point preserving action of the symmetric group Σ_n on X_n
- pointed maps $\alpha_{p,q} : X_p \wedge S^q \longrightarrow X_{p+q}$ for $p, q \geq 0$ which are $\Sigma_p \times \Sigma_q$ -equivariant; here $S^1 = \Delta^1/\partial\Delta^1$, $S^q = (S^1)^{\wedge q}$ and Σ_q permutes the factors.

This data is subject to the following conditions:

- under the identification $X_n \cong X_n \wedge S^0$, the map $\alpha_{n,0} : X_n \wedge S^0 \longrightarrow X_n$ is the identity,
- for $p, q, r \geq 0$, the following square commutes

$$\begin{array}{ccc} X_p \wedge S^q \wedge S^r & \xrightarrow{\alpha_{p,q} \wedge \text{Id}} & X_{p+q} \wedge S^r \\ \cong \downarrow & & \downarrow \alpha_{p+q,r} \\ X_p \wedge S^{q+r} & \xrightarrow{\alpha_{p,q+r}} & X_{p+q+r} . \end{array} \quad (4.5)$$

A *morphism* $f : X \longrightarrow Y$ of symmetric spectra consists of Σ_n -equivariant pointed maps $f_n : X_n \longrightarrow Y_n$ for $n \geq 0$, which are compatible with the structure maps in the sense that $f_{p+q} \circ \alpha_{p,q} = \alpha_{p,q} \circ (f_p \wedge \text{Id}_{S^q})$ for all $p, q \geq 0$. The category of symmetric spectra is denoted by $\mathcal{S}p^\Sigma$.

The definition we have just given is somewhat redundant, and Hovey, Shipley and Smith use a more economical definition in [HSS, Def. 1.2.2]. Indeed, the commuting square (4.5), shows that all action maps $\alpha_{p,q}$ are given by composites of the maps $\alpha_{p,1} : X_p \wedge S^1 \longrightarrow X_{p+1}$ for varying p .

A first example is the symmetric *sphere spectrum* \mathbb{S} given by $\mathbb{S}_n = S^n$, where the symmetric group permutes the factors and $\alpha_{p,q} : S^p \wedge S^q \longrightarrow S^{p+q}$ is the canonical isomorphism. More generally, every pointed simplicial set K gives rise to a *suspension spectrum* $\Sigma^\infty K$ via

$$(\Sigma^\infty K)_n = K \wedge S^n ;$$

then we have $\mathbb{S} \cong \Sigma^\infty S^0$.

A symmetric spectrum is *cofibrant* if it has the left lifting property for levelwise acyclic fibrations. More precisely, A is cofibrant if the following holds: for every morphism $f : X \longrightarrow Y$ of symmetric spectra such that $f_n : X_n \longrightarrow Y_n$ is a weak equivalence and Kan fibration for all n , and for every morphism $\iota : A \longrightarrow X$, there exists a morphism $\bar{\iota} : A \longrightarrow Y$ such that $\iota = f\bar{\iota}$. Suspension spectra are examples of cofibrant symmetric spectra. An equivalent definition uses the *latching space* $L_n A$, a simplicial set which roughly is the ‘stuff coming

from dimensions below n' ; see [HSS, 5.2.1] for the precise definition. A symmetric spectrum A is cofibrant if and only if for all n , the map $L_n A \rightarrow A_n$ is injective and symmetric group Σ_n is freely on the complement of the image, see [HSS, Prop. 5.2.2]. An Ω -spectrum is defined by the properties that each simplicial set X_n is a Kan complex and all the maps $X_n \rightarrow \Omega(X_{n+1})$ adjoint to $\alpha_{n,1}$ are weak homotopy equivalences. The *stable homotopy category* has as objects the cofibrant symmetric Ω -spectra and as morphisms the homotopy classes of morphisms of symmetric spectra.

Although we just gave a perfectly good definition of the stable homotopy category, in order to work with it one needs an ambient model category structure. One such model structure is the *stable model structure* of [HSS, Thm. 3.4.4]. A morphism of symmetric spectra is a *stable equivalence* if it induces isomorphisms on all *cohomology theories* represented by (injective) Ω -spectra, see [HSS, Def. 3.1.3] for the precise statement. There is a notion of *cofibration* such that a symmetric spectrum X is cofibrant in the above sense if and only if the map from the trivial symmetric spectrum to X is a cofibration. The Ω -spectra then coincide with the stably fibrant symmetric spectra. There are other model structures for symmetric spectra with the same class of weak (=stable) equivalences, hence with the same homotopy category, for example the *S-model structure* which is hinted at in [HSS, 5.3.6].

Stable equivalences versus π_* -isomorphisms. One of the tricky points with symmetric spectra is the relationship between stable equivalences and π_* -isomorphisms. The stable equivalences are defined as the morphisms which induce isomorphisms on all *cohomology theories*; there is the strictly smaller class of morphisms which induce isomorphisms on stable homotopy groups. The k -th *stable homotopy group* of a symmetric spectrum X is defined as the colimit

$$\pi_k X = \operatorname{colim}_n \pi_{n+k} |X_n| ,$$

where $|X_n|$ denotes the geometric realization of the simplicial set X_n . The colimit is taken over the maps

$$\pi_{n+k} |X_n| \xrightarrow{-\wedge S^1} \pi_{n+k+1} (|X_n| \wedge S^1) \xrightarrow{(\alpha_{n,1})^*} \pi_{n+k+1} |X_{n+1}| . \quad (4.6)$$

While every π_* -isomorphism of symmetric spectra is a stable equivalence [HSS, Thm. 3.1.11], the converse is not true. The standard example is the following: consider the symmetric spectrum $F_1 S^1$ freely generated by the circle S^1 in dimension 1. Explicitly, $F_1 S^1$ is given by

$$(F_1 S^1)_n = \Sigma_n^+ \wedge_{\Sigma_{n-1}} S^{n-1} \wedge S^1 .$$

So $(F_1 S^1)_n$ is a wedge of n copies of S^n and in the stable range, i.e., up to roughly dimensions $2n$, the homotopy groups of $(F_1 S^1)_n$ are a direct sum of n copies of the homotopy groups of S^n . Moreover, in the stable range, the map in the colimit system (4.6) is a direct summand inclusion into $(n+1)$ copies of the homotopy groups of S^n . Thus in the colimit, the stable homotopy groups of the symmetric spectrum $F_1 S^1$ are a countably infinite direct sum of copies of the stable homotopy groups of spheres. Since $F_1 S^1$ is freely generated by the circle S^1 in dimension 1, it ought to be a desuspension of the suspension spectrum of the circle. However, the necessary symmetric group actions ‘blow up’ such free objects with the effect that the stable homotopy groups are larger than they should be. This example indicates that inverting only the π_* -isomorphisms would leave too many stable homotopy types, and the resulting category could not be equivalent to the usual stable homotopy category.

Smash product. One of the main features which distinguishes *symmetric* spectra from the more classical spectra is the internal smash product. The smash product of symmetric

spectra can be described via its universal property, analogous to universal property of the tensor product over a commutative ring. Indeed, if R is a commutative ring and M and N are right R -modules, then a bilinear map to another left R -module W is a map $b : M \times N \rightarrow W$ such that for each $m \in M$ the map $b(m, -) : N \rightarrow W$ and each $n \in N$ the map $b(-, n) : M \rightarrow W$ are R -linear. The tensor product $M \otimes_R N$ is the universal example of a right R -module together with a bilinear map from $M \times N$.

Let us define a *bilinear morphism* $b : (X, Y) \rightarrow Z$ from two symmetric spectra X and Y to a symmetric spectrum Z to consist of a collection of $\Sigma_p \times \Sigma_q$ -equivariant maps of pointed simplicial sets

$$b_{p,q} : X_p \wedge Y_q \rightarrow Z_{p+q}$$

for $p, q \geq 0$, such that for all $p, q, r \geq 0$, the following diagram commutes

$$\begin{array}{ccccc}
 & & X_p \wedge Y_q \wedge S^r & \xrightarrow{\text{Id} \wedge \text{twist}} & X_p \wedge S^r \wedge Y_q & \\
 & \swarrow \text{Id} \wedge \alpha_{q,r} & \downarrow b_{p,q} \wedge \text{Id} & & \downarrow \alpha_{p,r} \wedge \text{Id} & \\
 X_p \wedge Y_{q+r} & & Z_{p+q} \wedge S^r & & X_{p+r} \wedge Y_q & \\
 & \searrow b_{p,q+r} & \downarrow \alpha_{p+q,r} & & \downarrow b_{p+r,q} & \\
 & & Z_{p+q+r} & \xleftarrow{1 \times \chi_{r,q}} & Z_{p+r+q} &
 \end{array} \tag{4.7}$$

The automorphism $1 \times \chi_{r,q}$ of Z_{p+q+r} may look surprising at first sight. Here $1 \times \chi_{r,q} \in \Sigma_{p+r+q}$ denotes the block permutation which fixes the first p elements, and which moves the next q elements past the last r elements. This can be viewed as a topological version of the sign rule which says that when two symbols of degree q and r are permuted past each other, the sign $(-1)^{qr}$ should appear as well. The block permutation $\chi_{r,q}$ has sign $(-1)^{qr}$ and it compensates the upper vertical interchange of Y_q and S^r . A good way to keep track of such permutations is to carefully distinguish between indices such as $r+q$ and $q+r$. Of course these two numbers are equal, but the fact that one arises naturally instead of the other reminds us that a block permutation should be inserted.

The smash product $X \wedge Y$ is the universal example of a symmetric spectrum with a bilinear morphism from X and Y . In other words, it comes with a bilinear morphism $\iota : (X, Y) \rightarrow X \wedge Y$ such that for every symmetric spectrum Z the map

$$\mathcal{S}p^\Sigma(X \wedge Y, Z) \rightarrow \text{Bi-}\mathcal{S}p^\Sigma((X, Y), Z) \tag{4.8}$$

is bijective. If we suppose that such a universal object exist, this property characterizes the smash product and the maps $\iota_{p,q} : X_p \wedge Y_q \rightarrow (X \wedge Y)_{p+q}$ up to canonical isomorphism. An actual construction as a certain coequalizer is given in [HSS, Def. 2.2.3]; in this article, we will only use the universal property of the smash product.

We use the universal property to derive that the smash product is functorial and symmetric monoidal. For example, let $f : X \rightarrow Y$ and $f' : X' \rightarrow Y'$ be morphisms of symmetric spectra. Then the collection of maps of pointed simplicial sets

$$\left\{ X_p \wedge X'_q \xrightarrow{f_q \wedge f'_q} Y_p \wedge Y'_q \xrightarrow{\iota_{p,q}} (Y \wedge Y')_{p+q} \right\}_{p,q \geq 0}$$

form a bilinear morphism $(X, X') \rightarrow Y \wedge Y'$, so it corresponds to a unique morphism of symmetric spectra $f \wedge f' : X \wedge X' \rightarrow Y \wedge Y'$. The universal property implies functoriality

in both arguments. For the proof of the associativity of the smash product we notice that the family

$$\left\{ X_p \wedge Y_q \wedge Z_r \xrightarrow{\iota_{p,q} \wedge \text{Id}} (X \wedge Y)_{p+q} \wedge Z_r \xrightarrow{\iota_{p+q,r}} ((X \wedge Y) \wedge Z)_{p+q+r} \right\}_{p,q,r \geq 0}$$

and the family

$$\left\{ X_p \wedge Y_q \wedge Z_r \xrightarrow{\text{Id} \wedge \iota_{q,r}} X_p \wedge (Y \wedge Z)_{q+r} \xrightarrow{\iota_{p,q+r}} (X \wedge (Y \wedge Z))_{p+q+r} \right\}_{p,q,r \geq 0}$$

both have the universal property of a *tri-linear* morphism out of X , Y and Z . The uniqueness of universal objects gives a preferred isomorphism of symmetric spectra

$$(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z) .$$

The symmetry isomorphism $X \wedge Y \cong Y \wedge X$ corresponds to the bilinear morphism

$$\left\{ X_p \wedge Y_q \xrightarrow{\text{twist}} Y_q \wedge X_p \xrightarrow{\iota_{q,p}} (Y \wedge X)_{q+p} \xrightarrow{\chi_{q,p}} (Y \wedge X)_{p+q} \right\}_{p,q \geq 0} . \quad (4.9)$$

The block permutation $\chi_{q,p}$ is crucial here: without it we would not get a bilinear morphism in the sense of diagram (4.7). In much the same spirit, the universal properties can be used to provide unit isomorphisms $\mathbb{S} \wedge X \cong X \cong X \wedge \mathbb{S}$, to verify the coherence conditions of a symmetric monoidal structure, and to establish an isomorphism of suspension spectra

$$(\Sigma^\infty K) \wedge (\Sigma^\infty L) \cong \Sigma^\infty(K \wedge L) .$$

The symmetric monoidal structure given by the smash product of symmetric spectra is *closed* in the sense that internal function objects exist as well. For each pair of symmetric spectra X and Y there is a symmetric function spectrum $\text{Hom}(X, Y)$ [HSS, 2.2.9], and there are natural composition morphisms

$$\circ : \text{Hom}(Y, Z) \wedge \text{Hom}(X, Y) \longrightarrow \text{Hom}(X, Z)$$

which are associative and unital with respect to a unit map $\mathbb{S} \longrightarrow \text{Hom}(X, X)$. Moreover, the usual adjunction isomorphism

$$\mathcal{S}p^\Sigma(X \wedge Y, Z) \cong \mathcal{S}p^\Sigma(X, \text{Hom}(Y, Z))$$

relates the smash product and function spectra.

Ring and module spectra. The smash product of symmetric spectra leads to the concomitant concepts *symmetric ring spectra*, *module spectra* and *algebra spectra*.

Definition 4.10. A *symmetric ring spectrum* is a symmetric spectrum R together with morphisms of symmetric spectra

$$\eta : \mathbb{S} \longrightarrow R \quad \text{and} \quad \mu : R \wedge R \longrightarrow R ,$$

called the unit and multiplication map, which satisfy certain associativity and unit conditions (see [McL, VII.3]). A ring spectrum R is *commutative* if the multiplication map is unchanged when composed with the twist, or the symmetry isomorphism (4.9), of $R \wedge R$. A morphism of ring spectra is a morphism of spectra commuting with the multiplication and unit maps. If R is a symmetric ring spectrum, a *right R -module* is a spectrum N together with an action map $N \wedge R \longrightarrow N$ satisfying associativity and unit conditions (see again [McL, VII.4]). A morphism of right R -modules is a morphism of spectra commuting with the action of R . We denote the category of right R -modules by $\text{Mod-}R$.

With the universal property of the smash product we can make the structure of a symmetric ring spectrum more explicit. The multiplication map $\mu : R \wedge R \longrightarrow R$ corresponds to a family of pointed, $\Sigma_p \times \Sigma_q$ -equivariant maps

$$\mu_{p,q} : R_p \wedge R_q \longrightarrow R_{p+q}$$

for $p, q \geq 0$, which are bilinear in the sense of diagram (4.7). The maps are supposed to be associative and unital with respect to the maps $\eta_p : S^p \longrightarrow R_p$ which constitute the unit map $\eta : \mathbb{S} \longrightarrow R$.

The commutativity isomorphism of the smash product involves the block permutation $\chi_{q,p}$, see (4.9). So the multiplication of a symmetric ring spectrum is commutative if and only if the following diagrams commute for all $p, q \geq 0$

$$\begin{array}{ccc} R_p \wedge R_q & \xrightarrow{\mu_{p,q}} & R_{p+q} \\ \text{twist} \downarrow & & \downarrow \chi_{p,q} \\ R_q \wedge R_p & \xrightarrow{\mu_{q,p}} & R_{q+p} \end{array}$$

The block permutation $\chi_{p,q}$ has sign $(-1)^{pq}$, so this diagram is reminiscent of the sign rule in a graded ring which is commutative in the graded sense.

The unit \mathbb{S} of the smash product is a ring spectrum in a unique way, and \mathbb{S} -modules are the same as symmetric spectra. The smash product of two ring spectra is naturally a ring spectrum. For a ring spectrum R the opposite ring spectrum R^{op} is defined by composing the multiplication with the twist map $R \wedge R \longrightarrow R \wedge R$ (so in terms of the bilinear maps $\mu_{p,q} : R_p \wedge R_q \longrightarrow R_{p+q}$, a block permutation appears). The definitions of left modules and bimodules is hopefully clear; left R -modules and R - T -bimodule can also be defined as right modules over the opposite ring spectrum R^{op} , respectively right modules over the ring spectrum $R^{\text{op}} \wedge T$.

A formal consequence of having a closed symmetric monoidal smash product is that the category of R -modules inherits a smash product and function objects. The smash product $M \wedge_R N$ of a right R -module M and a left R -module N can be defined as the coequalizer, in the category of symmetric spectra, of the two maps

$$M \wedge R \wedge N \rightrightarrows M \wedge N$$

given by the action of R on M and N respectively. Alternatively, one can characterize $M \wedge_R N$ as the universal example of a symmetric spectrum which receives a bilinear map from M and N which is R -balanced, i.e., all the diagrams

$$\begin{array}{ccc} M_p \wedge R_q \wedge N_r & \xrightarrow{\text{Id} \wedge \alpha_{q,r}} & M_p \wedge N_{q+r} \\ \alpha_{p,q} \wedge \text{Id} \downarrow & & \downarrow \iota_{p,q+r} \\ M_{p+q} \wedge N_r & \xrightarrow{\iota_{p+q,r}} & (M \wedge N)_{p+q+r} \end{array} \quad (4.11)$$

commute. If M happens to be a T - R -bimodule and N an R - S -bimodule, then $M \wedge_R N$ is naturally a T - S -bimodule. If R is a commutative ring spectrum, the notions of left and right module coincide and agree with the notion of a symmetric bimodule. In this case \wedge_R is an internal symmetric monoidal smash product for R -modules. There are also internal function

spectra and function modules, enjoying the ‘usual’ adjointness properties with respect to the various smash products.

The modules over a symmetric ring spectrum R inherit a model category structure from symmetric spectra, see [HSS, Cor. 5.4.2] and [SS00, Thm. 4.1 (1)]. More precisely, a morphism of R -modules is called a weak equivalence (resp. fibration) if the underlying morphism of symmetric spectra is a stable equivalence (resp. stable fibration). The cofibrations are then determined by the left lifting property with respect to all acyclic fibrations in $\mathbf{Mod}\text{-}R$. This model structure is stable, so the homotopy category of modules over a ring spectrum is a triangulated category. The free module of rank one is a small generator.

For a map $R \rightarrow S$ of ring spectra, there is a Quillen adjoint functor pair analogous to restriction and extension of scalars: any S -module becomes an R -module if we let R act through the map. This functor has a left adjoint taking an R -module M to the S -module $M \wedge_R S$. If $R \rightarrow S$ is a stable equivalence, then the functors of restriction and extension of scalars are a Quillen equivalence between the categories of R -modules and S -modules, see [HSS, Thm. 5.4.5] and [SS00, Thm. 4.3].

Example 4.12 (Monoid ring spectra). If M is a simplicial monoid, and R is a symmetric ring spectrum, we define a symmetric spectrum $R[M]$ by

$$R[M]_n = R_n \wedge M^+,$$

where M^+ denotes the underlying simplicial set of M , with disjoint basepoint added. The unit map is the composite of the unit map of R and the wedge summand inclusion indexed by the unit of M ; the multiplication map $R[M] \wedge R[M] \rightarrow R[M]$ is induced from the bilinear morphism

$$(R_p \wedge M^+) \wedge (R_q \wedge M^+) \cong (R_p \wedge R_q) \wedge (M \times M)^+ \xrightarrow{\mu_{p,q} \wedge \text{mult.}} R_{p+q} \wedge M^+.$$

The construction of the monoid ring over \mathbb{S} is left adjoint to the functor which takes a symmetric ring spectrum R to the simplicial monoid R_0 .

Example 4.13 (Matrix ring spectra). Let R be a symmetric ring spectrum and consider the wedge (coproduct)

$$R \times n = \underbrace{R \vee \cdots \vee R}_n$$

of n copies of the free R -module of rank 1. In the usual stable model structure, the free module of rank 1 is cofibrant, hence so is $R \times n$. We choose a fibrant replacement $R \times n \xrightarrow{\sim} (R \times n)^f$. The ring spectrum of $n \times n$ matrices over R is defined as the endomorphism ring spectrum of $(R \times n)^f$,

$$M_n(R) = \text{End}_R((R \times n)^f).$$

The stable equivalence type of the matrix ring spectrum $M_n(R)$ is independent of the choice of fibrant replacement, see Corollary A.2.4 of [SS03]. Moreover, the underlying spectrum of $M_n(R)$ is isomorphic, in the stable homotopy category, to a sum of n^2 copies of R .

Example 4.14 (Eilenberg-Mac Lane spectra). For an abelian group A , the *Eilenberg-Mac Lane spectrum* HA is defined by

$$(HA)_n = A \otimes \mathbb{Z}[S^n],$$

i.e., the underlying simplicial set of the dimensionwise tensor product of A with the reduced free simplicial abelian generated by the simplicial n -sphere. The symmetric groups acts by permuting the smash factors of S^n . The homotopy groups of the symmetric spectrum HA are concentrated in dimension zero, where we have a natural isomorphism $\pi_0 HA \cong A$.

For two abelian groups A and B , a natural morphism of symmetric spectra

$$HA \wedge HB \longrightarrow H(A \otimes B)$$

is obtained, by the universal property (4.8), from the bilinear morphism

$$\begin{aligned} (HA)_n \wedge (HB)_m &= (A \otimes \mathbb{Z}[S^n]) \wedge (B \otimes \mathbb{Z}[S^m]) \\ &\longrightarrow (A \otimes B) \otimes \mathbb{Z}[S^{n+m}] = (H(A \otimes B))_{n+m} \end{aligned}$$

given by

$$\left(\sum_i a_i \cdot x_i \right) \wedge \left(\sum_j b_j \cdot x'_j \right) \longmapsto \sum_{i,j} (a_i \otimes b_j) \cdot (x_i \wedge x'_j) .$$

A unit map $\mathbb{S} \longrightarrow H\mathbb{Z}$ is given by the inclusion of generators. With respect to these maps, H becomes a lax symmetric monoidal functor from the category of abelian groups to the category of symmetric spectra. As a formal consequence, H turns a ring R into a symmetric ring spectrum with multiplication map

$$HR \wedge HR \longrightarrow H(R \otimes R) \longrightarrow HR .$$

Similarly, an R -module structure on A gives rise to an HR -module structure on HA .

The definition of the symmetric spectrum HA makes just as much sense when A is a *simplicial* abelian group; thus the Eilenberg-Mac Lane functor makes simplicial rings into symmetric ring spectra, respecting possible commutativity of the multiplications. With a little bit of extra care, the Eilenberg-Mac Lane construction can also be extended to a differential graded context, compare [SS03, App. B] and [Ri03].

For a fixed ring B , the modules over the Eilenberg-Mac Lane ring spectrum HB of a ring B have the same homotopy theory as complexes of B -modules. The first results of this kind were obtained by Robinson for A_∞ -ring spectra [Rob87b], and later for S -algebras in [EKMM, IV Thm. 2.4]; in both cases, equivalences of triangulated homotopy categories are constructed. But more is true: for any ring B , Theorem 5.1.6 of [SS03] provides a chain of two Quillen equivalences between the categories of unbounded chain complexes of B -modules and the HB -module spectra.

Example 4.15 (Cobordism spectra). We define a commutative symmetric ring spectrum MO whose stable homotopy groups are isomorphic to the ring of cobordism classes of smooth closed manifolds. We set

$$(MO)_n = EO(n)^+ \wedge_{O(n)} S^n .$$

Here $O(n)$ is the n -th orthogonal group consisting of Euclidean automorphisms of \mathbb{R}^n . The space $EO(n)$ is the geometric realization of the simplicial object of topological groups which in dimension k is the k -fold product of copies of $O(n)$, and where the face maps are projections. Thus $EO(n)$ is a topological group with a homomorphism $O(n) \longrightarrow EO(n)$ coming from the inclusion of 0-simplices. The underlying space of $EO(n)$ is contractible and has two commuting actions of $O(n)$ from the left and the right. The right $O(n)$ -action is used to form the orbit space $(MO)_n$, where we think of S^n as the one-point compactification of \mathbb{R}^n with its natural left $O(n)$ -action. Thus the space $(MO)_n$ still has a left $O(n)$ -action, which we restrict to an action of the symmetric group Σ_n , sitting inside $O(n)$ as the coordinate permutations. Topologically, $(MO)_n$ is nothing but the Thom space of the tautological bundle over the classifying space $BO(n)$.

The unit of the ring spectrum MO is given by the maps

$$S^n \cong O(n)^+ \wedge_{O(n)} S^n \longrightarrow EO(n)^+ \wedge_{O(n)} S^n = (MO)_n$$

using the ‘vertex map’ $O(n) \longrightarrow EO(n)$. There are multiplication maps

$$(MO)_p \wedge (MO)_q \longrightarrow (MO)_{p+q}$$

which are induced from the identification $S^p \wedge S^q \cong S^{p+q}$ which is equivariant with respect to the group $O(p) \times O(q)$, viewed as a subgroup of $O(p+q)$. The fact that these multiplication maps are associative and commutative uses that

- for topological groups G and H , the simplicial model of EG comes with a natural, associative and commutative isomorphism $E(G \times H) \cong EG \times EH$;
- the group monomorphisms $O(p) \times O(q) \longrightarrow O(p+q)$ are strictly associative, and the following diagram commutes

$$\begin{array}{ccc} O(p) \times O(q) & \longrightarrow & O(p+q) \\ \text{twist} \downarrow & & \downarrow \text{conj. by } \chi_{p,q} \\ O(q) \times O(p) & \longrightarrow & O(q+p) \end{array}$$

where the right vertical map is conjugation by the permutation matrix of the block permutation $\chi_{p,q}$.

In very much the same way we obtain commutative symmetric ring spectra model for the oriented cobordism spectrum MSO and the spin cobordism spectrum $MSpin$. The complex cobordism ring spectrum MU does not fit in here directly; one has to vary the notion of a symmetric spectrum slightly, and consider only symmetric spectra which are defined ‘in even dimensions’.

4.3. Characterizing module categories over ring spectra. Several of the examples of stable model categories in Section 4.1 already come as categories of modules over suitable rings or ring spectra. This is no coincidence. In fact, every stable model category with a single small generator has the same homotopy theory as the modules over a ring spectrum. This is an analog of Theorem 2.5, which characterizes module categories over a ring as the cocomplete abelian category with a small projective generator.

To an object P in a sufficiently nice stable model category \mathcal{C} we can associate a symmetric endomorphism ring spectrum $\text{End}_{\mathcal{C}}(P)$; among other things, this ring spectrum comes with an isomorphism of graded rings

$$\pi_* \text{End}_{\mathcal{C}}(P) \cong \text{Ho}(\mathcal{C})(P, P)_*$$

between the homotopy groups of $\text{End}_{\mathcal{C}}(P)$ and the morphism of P in the triangulated homotopy category of \mathcal{C} . The precise characterization is as follows:

Theorem 4.16. *Let \mathcal{C} be a stable model category which is simplicial, cofibrantly generated and proper. If \mathcal{C} has a small generator P , then there exists a chain of Quillen equivalences between \mathcal{C} and the model category of $\text{End}_{\mathcal{C}}(P)$ -modules,*

$$\mathcal{C} \simeq_Q \text{Mod-End}_{\mathcal{C}}(P) .$$

Unfortunately, the theorem is currently only known under the above technical hypothesis: the stable model category in question should be simplicial (see [Qui67, II.2], [Hov99, 4.2.18]), cofibrantly generated (see [Hov99, Sec. 2.1] or [SS00, Sec. 2]) and proper (see [BF78, Def. 1.2] or [HSS, Def. 5.5.2]). The conditions enter in the construction of ‘good’ endomorphism ring

spectra. I suspect however, that these hypothesis are not essential and can be eliminated with a clever use of framing techniques. For example, in [SS02, Sec. 6], we use framings to construct function spectra in a arbitrary stable model category; that construction does however not yield *symmetric* spectra, and there is no good composition pairing. The condition of being a *simplicial* model category can be removed by appealing to [RSS] or [Dug] where suitable model categories are replaced by Quillen equivalent simplicial model categories.

This theorem is a special case of the more general result which applies to stable model categories with a set of small generators (as opposed to a single small generator), see [SS03, Thm. 3.3.3].

Spectral model categories. In the algebraic situations which we considered in Sections 2 and 3, the key point is to have a good notion of endomorphism ring or endomorphism DG ring together with a ‘tautological’ functor

$$\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \longrightarrow \mathrm{Mod}\text{-}\mathrm{End}_{\mathcal{A}}(P) . \quad (4.17)$$

Then it is a matter of checking that when P is a small generator, the functor $\mathrm{Hom}(P, -)$ is either an equivalence of categories (in the context of abelian categories) or induces an equivalence of derived categories (in the context of DG categories). For abelian categories the situation is straightforward, and the ordinary endomorphism ring does the job. In the differential graded context already a little complication comes in because the categorical hom functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ need not preserve quasi-isomorphisms in general.

For stable model categories, the key construction is again to have an *endomorphism ring spectrum* $\mathrm{End}_{\mathcal{C}}(P)$ together with a homotopically well-behaved homomorphism functor (4.17) to modules over the endomorphism ring spectrum. This is easy for the following class of *spectral model categories* where composable function spectra are part of the data. A spectral model category is analogous to a simplicial model category [Qui67, II.2], but with the category of simplicial sets replaced by symmetric spectra. Roughly speaking, a spectral model category is a pointed model category which is compatibly enriched over the stable model category of spectra. In particular there are ‘tensors’ $K \wedge X$ and ‘cotensors’ X^K of an object X of \mathcal{C} and a symmetric spectrum K , and function symmetric spectra $\mathrm{Hom}_{\mathcal{C}}(A, Y)$ between two objects of \mathcal{C} . The compatibility is expressed by the following axiom which takes the place of [Qui67, II.2 SM7]; there are two equivalent ‘adjoint’ forms of this axiom, compare [Hov99, Lemma 4.2.2] or [SS03, 3.5].

(Pushout product axiom) For every cofibration $A \longrightarrow B$ in \mathcal{C} and every cofibration $K \longrightarrow L$ of symmetric spectra, the *pushout product map*

$$L \wedge A \cup_{K \wedge A} K \wedge B \longrightarrow L \wedge B$$

is a cofibration; the pushout product map is a weak equivalence if in addition $A \longrightarrow B$ is a weak equivalence in \mathcal{C} or $K \longrightarrow L$ is a stable equivalence of symmetric spectra.

A *spectral Quillen pair* is a Quillen adjoint functor pair $L : \mathcal{C} \longrightarrow \mathcal{D}$ and $R : \mathcal{D} \longrightarrow \mathcal{C}$ between spectral model categories together with a natural isomorphism of symmetric homomorphism spectra

$$\mathrm{Hom}_{\mathcal{C}}(A, RX) \cong \mathrm{Hom}_{\mathcal{D}}(LA, X)$$

which on the vertices of the 0-th level reduces to the adjunction isomorphism. A spectral Quillen pair is a *spectral Quillen equivalence* if the underlying Quillen functor pair is an ordinary Quillen equivalence.

A spectral model category is the same as a ‘ $\mathcal{S}p^{\Sigma}$ -model category’ in the sense of [Hov99, Def. 4.2.18]; Hovey’s condition 2 of [Hov99, 4.2.18] is automatic since the unit \mathbb{S} for the smash

product of symmetric spectra is cofibrant. Similarly, a spectral Quillen pair is a ‘ $\mathcal{S}p^\Sigma$ -Quillen functor’ in Hovey’s terminology. Examples of spectral model categories are module categories over a ring spectrum, and the category of symmetric spectra over a suitable simplicial model category [SS03, Thm. 3.8.2].

A spectral model category is in particular a *simplicial* and *stable* model category. Moreover, for X a cofibrant and Y a fibrant object of a spectral model category \mathcal{C} there is a natural isomorphism of graded abelian groups $\pi_*\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathrm{Ho}(\mathcal{C})(X, Y)_*$. These facts are discussed in Lemma 3.5.2 of [SS03].

For an object P in a spectral model category, the function spectrum $\mathrm{End}_{\mathcal{C}}(P) = \mathrm{Hom}_{\mathcal{C}}(P, P)$ is naturally a ring spectrum; the multiplication is a special case of the composition product

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \wedge \mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z) .$$

Also via the composition pairing, the function symmetric spectrum $\mathrm{Hom}_{\mathcal{C}}(P, X)$ becomes a right module over the symmetric ring spectrum $\mathrm{End}_{\mathcal{C}}(P)$ for any object X . In order for the endomorphism ring spectrum $\mathrm{End}_{\mathcal{C}}(P)$ to have the correct homotopy type, the object P should be both cofibrant and fibrant. In that case, the ring of homotopy groups $\pi_*\mathrm{End}_{\mathcal{C}}(P)$ is isomorphic to $\mathrm{Ho}(\mathcal{C})(P, P)_*$, the ring of graded self maps of P in the homotopy category of \mathcal{C} . Moreover, the homotopy type of the endomorphism ring spectrum then depends only on the weak equivalence type of the object (see [SS03, Cor. A.2.4]). Note that this is not completely obvious since taking endomorphisms is *not* a functor.

If P is a cofibrant object of a spectral model category, then the functor

$$\mathrm{Hom}_{\mathcal{C}}(P, -) : \mathcal{C} \longrightarrow \mathrm{Mod}\text{-}\mathrm{End}_{\mathcal{C}}(P)$$

is the right adjoint of a Quillen adjoint functor pair, see [SS03, 3.9.3 (i)]. The left adjoint is denoted

$$- \wedge_{\mathrm{End}_{\mathcal{C}}(P)} P : \mathrm{Mod}\text{-}\mathrm{End}_{\mathcal{C}}(P) \longrightarrow \mathcal{C} . \quad (4.18)$$

For spectral model categories, the proof of Theorem 4.16 is now straightforward, and very analogous to the proofs of Theorem 2.5 and Theorem 3.20; indeed, to obtain the following theorem, one applies Proposition 3.10 to the total left derived functor of the left Quillen functor (4.18).

Theorem 4.19. *Let \mathcal{C} be a spectral model category and P a cofibrant-fibrant object. If P is a small generator for \mathcal{C} , then the adjoint functor pair $\mathrm{Hom}_{\mathcal{C}}(P, -)$ and $- \wedge_{\mathrm{End}_{\mathcal{C}}(P)} P$ form a spectral Quillen equivalence.*

The remaining step is worked out in Theorem 3.8.2 of [SS03], which proves that every simplicial, cofibrantly generated, proper stable model category is Quillen equivalent to a spectral model category, namely the category $\mathcal{S}p(\mathcal{C})$ of *symmetric spectra over \mathcal{C}* . The proof is technical and we will not go into details here. Theorem 4.16 follows by combining [SS03, Theorem 3.8.2] and Theorem 4.19 to obtain a diagram of model categories and Quillen equivalences (the left adjoints are on top)

$$\mathcal{C} \begin{array}{c} \xrightarrow{\Sigma^\infty} \\ \xleftarrow{\mathrm{Ev}_0} \end{array} \mathcal{S}p(\mathcal{C}) \begin{array}{c} \xleftarrow{- \wedge_{\mathrm{End}_{\mathcal{C}}(P)} P} \\ \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(P, -)} \end{array} \mathrm{Mod}\text{-}\mathrm{End}_{\mathcal{C}}(P) .$$

4.4. Morita context for ring spectra. Now we come to ‘Morita theory for ring spectra’, by which we mean the question when two symmetric spectra have Quillen equivalent module categories. For ring spectra, there is a significant difference between a Quillen equivalence of the module categories and an equivalence of the homotopy categories. The former implies

the latter, but not conversely. The same kind of difference already exists for differential graded rings, but it is not visible for ordinary rings (see Example 4.5 (5)).

We call a symmetric spectrum X *flat* if the functor $X \wedge -$ preserves stable equivalences of symmetric spectra. If X is cofibrant, or more generally S -cofibrant in the sense of [HSS, 5.3.6], then X is flat, see [HSS, 5.3.10]. Every symmetric ring spectrum has a ‘flat resolution’: we may take a cofibrant approximation in the stable model structure of symmetric ring spectra [HSS, 5.4.3]; the underlying symmetric spectrum of the approximation is cofibrant, thus flat.

Theorem 4.20. (Morita context) *The following are equivalent for two symmetric ring spectra R and S .*

- (1) *There exists a chain of spectral Quillen equivalences between the categories of R -modules and S -modules.*
- (2) *There is a small, cofibrant and fibrant generator of the model category of S -modules whose endomorphism ring spectrum is stably equivalent to R .*

Both conditions are implied by the following condition.

- (3) *There exists an R - S -bimodule M such that the derived smash product functor*

$$- \wedge_R^L M : \mathrm{Ho}(\mathrm{Mod}\text{-}R) \longrightarrow \mathrm{Ho}(\mathrm{Mod}\text{-}S)$$

is an equivalence of categories.

If moreover R or S is flat as a symmetric spectrum, then all three conditions are equivalent.

Again there is a version of the Morita context 4.20 relative to a commutative symmetric ring spectrum k . In that case, R and S are k -algebras, condition (1) refers to k -linear spectral Quillen equivalences, condition (2) requires a stable equivalence of k -algebras, the bimodule M in (3) has to be k -symmetric and in the addendum, one of R or S has to be flat as a k -module.

Proof of Theorem 4.20. (2) \implies (1): Modules over a symmetric ring spectrum form a spectral model category; so this implication is a special case of Theorem 4.19, combined with the fact that stably equivalent ring spectra have Quillen equivalent module categories.

(1) \implies (2): To simplify things we suppose that there exists a single spectral Quillen equivalence

$$\mathrm{Mod}\text{-}R \begin{array}{c} \xrightarrow{\Lambda} \\ \xleftarrow{\Phi} \end{array} \mathrm{Mod}\text{-}S$$

with Λ the left adjoint. The general case of a chain of such Quillen equivalences is treated in [SS03, Thm. 4.1.2]. We choose a trivial cofibration $\iota : \Lambda(R) \longrightarrow \Lambda(R)^f$ of S -modules such that $M := \Lambda(R)^f$ is fibrant; since M is isomorphic in the homotopy category of S -modules to the image of the free R -module of rank one under the equivalence of homotopy categories, M is a small generator for the homotopy category of S -modules. It remains to show that the endomorphism ring spectrum of M is stably equivalent to R .

We define $\text{End}_S(\iota)$, the endomorphism ring spectrum of the S -module map $\iota : \Lambda(R) \longrightarrow M$, as the pullback in the diagram of symmetric spectra

$$\begin{array}{ccc} \text{End}_S(\iota) & \longrightarrow & \text{End}_S(M) \\ \downarrow & & \downarrow \iota^* \\ R & \xrightarrow{\iota_*} & \text{Hom}_S(\Lambda(R), M) \end{array} \quad (4.21)$$

The right vertical map ι^* is obtained by applying $\text{Hom}_S(-, M)$ to the acyclic cofibration ι ; since M is stably fibrant, ι^* is acyclic fibration of symmetric spectra. Since Λ and Φ form a Quillen equivalence, the adjoint $\hat{\iota} : R \longrightarrow \Phi(M)$ of ι is a stable equivalence of R -modules. The lower horizontal map ι_* is the composite stable equivalence

$$R \cong \text{Hom}_R(R, R) \xrightarrow{\text{Hom}_R(R, \hat{\iota})} \text{Hom}_R(R, \Phi(M)) \cong \text{Hom}_S(\Lambda(R), M) .$$

All maps in the pullback square (4.21) are thus stable equivalences and the morphism connecting $\text{End}_S(\iota)$ to R and $\text{End}_S(M)$ are homomorphisms of symmetric ring spectra (whereas the lower right corner $\text{Hom}_S(\Lambda(R), M)$ has no multiplication). So R is indeed stably equivalent, as a symmetric ring spectrum, to the endomorphisms of M .

(3) \implies (1): If M happens to be cofibrant as a right S -module, then smashing with M over R is a left Quillen equivalence from R -modules to S -modules. Since we did not assume that M is cofibrant over S , we have to be content with a chain of two Quillen equivalences, which we get as follows.

Let M be an R - S -bimodule as in condition (3). We choose a cofibrant approximation $\iota : R^c \longrightarrow R$ in the stable model structure of symmetric ring spectra and we view M as an R^c - S -bimodule by restriction of scalars. Then we choose a cofibrant approximation $M^c \longrightarrow M$ as R^c - S -bimodules. Since the underlying symmetric spectrum of R^c is cofibrant [SS00, 4.1 (3)], $R^c \wedge S$ is cofibrant as a right S -module, and thus every cofibrant R^c - S -bimodule is cofibrant as a right S -module. In particular, this holds for M^c , and so we have a chain of two spectral Quillen pairs

$$\text{Mod-}R \begin{array}{c} \xleftarrow{-\wedge_{R^c} R} \\ \xrightarrow{\iota^*} \end{array} \text{Mod-}R^c \begin{array}{c} \xleftarrow{-\wedge_{R^c} M^c} \\ \xrightarrow{\text{Hom}_S(M^c, -)} \end{array} \text{Mod-}S .$$

The left pair is a Quillen equivalence since the approximation map $\iota : R^c \longrightarrow R$ is a stable equivalence. For every cofibrant R^c -module X , the map

$$X \wedge_{R^c} M^c \longrightarrow X \wedge_{R^c} M \cong (X \wedge_{R^c} R) \wedge_R M$$

is a stable equivalence. This means that the diagram of homotopy categories and derived functors

$$\begin{array}{ccc} & \text{Ho}(\text{Mod-}R^c) & \\ \swarrow -\wedge_{R^c}^L R & & \searrow -\wedge_{R^c}^L M^c \\ \text{Ho}(\text{Mod-}R) & \xrightarrow{-\wedge_R^L M} & \text{Ho}(\text{Mod-}S) \end{array}$$

commutes up to natural isomorphism. Thus the right Quillen pair above induces an equivalence of homotopy categories, so it is a Quillen equivalence.

(2) \implies (3), assuming that R or S is flat: Let T be a cofibrant and fibrant small generator of $\text{Ho}(\text{Mod-}S)$ such that R is stably equivalent to the endomorphism ring spectrum

of T . We choose a cofibrant approximation $R^c \xrightarrow{\cong} R$ in the model category of symmetric ring spectra. Since T is cofibrant and fibrant, its endomorphism ring spectrum is fibrant. So any isomorphism between R and $\text{End}_S(T)$ in the homotopy category of symmetric ring spectra can be represented by a chain of two stable equivalences

$$R \xleftarrow{\cong} R^c \xrightarrow{\cong} \text{End}_S(T) .$$

The module T is naturally an $\text{End}_S(T)$ - S -bimodule, and we restrict the left action to R^c and view T as an R^c - S -bimodule. We choose a cofibrant replacement $T^c \xrightarrow{\cong} T$ as an R^c - S -bimodule. Then we set

$$M = R \wedge_{R^c} T^c ,$$

an R - S -bimodule. We have no reason to suppose that M is cofibrant as a right S -module, so we cannot assume that the functor $-\wedge_R M : \text{Mod-}R \rightarrow \text{Mod-}S$ is a left Quillen functor. Nevertheless, smashing with M over R takes stable equivalences between cofibrant R -modules to stable equivalences, so it has a total left derived functor

$$-\wedge_R^L M : \text{Ho}(\text{Mod-}R) \rightarrow \text{Ho}(\text{Mod-}S) ;$$

we claim that this functor is an equivalence.

Since R^c is cofibrant as a symmetric ring spectrum, it is also cofibrant as a symmetric spectrum [SS00, 4.1 (3)], so $R^c \wedge S^{op}$ models the derived smash product of R and S . If one of R or S is flat, then $R \wedge S^{op}$ also models the derived smash product, so that the map

$$R^c \wedge S^{op} \rightarrow R \wedge S^{op}$$

is a stable equivalence of symmetric ring spectra. Since T^c is cofibrant as an $R^c \wedge S^{op}$ -module, the induced map

$$T^c = (R^c \wedge S^{op})_{R^c \wedge S^{op}} T^c \rightarrow (R \wedge S^{op})_{R^c \wedge S^{op}} T^c \cong R \wedge_{R^c} T^c = M \quad (4.22)$$

is a stable equivalence. We smash the stable equivalence (4.22) from the left with an R^c -module X to get a natural map of S -modules

$$X \wedge_{R^c} T^c \rightarrow X \wedge_{R^c} M \cong (X \wedge_{R^c} R) \wedge_R M . \quad (4.23)$$

If X is cofibrant as an R^c -module, then $X \wedge_{R^c} -$ takes stable equivalences of left R^c -modules to stable equivalences, so in this case, the map (4.23) is a stable equivalence. Thus the diagram of triangulated categories and derived functors

$$\begin{array}{ccc} & \text{Ho}(\text{Mod-}R^c) & \\ \begin{array}{c} \swarrow \\ -\wedge_{R^c}^L R \end{array} & & \begin{array}{c} \searrow \\ -\wedge_{R^c}^L T^c \end{array} \\ \text{Ho}(\text{Mod-}R) & \xrightarrow{\quad -\wedge_R^L M \quad} & \text{Ho}(\text{Mod-}S) \end{array} \quad (4.24)$$

commutes up to natural isomorphism.

The left diagonal functor in the diagram (4.24) is derived from extensions of scalars along stable equivalences of ring spectra; such extension of scalars is a left Quillen equivalence, so the derived functor $-\wedge_{R^c}^L R$ is an exact equivalence of triangulated categories. We argued in the previous implication that any cofibrant $R^c \wedge S^{op}$ -module such as T^c has an underlying cofibrant right S -module. So smashing with T^c over R^c is a left Quillen functor. Since T^c is isomorphic to T in the homotopy category of S -modules, T^c is a small generator of $\text{Ho}(\text{Mod-}S)$. So the right diagonal derived functor in (4.24) is an exact equivalence by Proposition 3.10, applied to the free R^c -module of rank 1. So we conclude that the lower horizontal

functor in the diagram (4.24) is also an exact equivalence of triangulated categories. This establishes condition (3). \square

4.5. Examples. (1) **Matrix ring spectra.** As for classical rings (compare Example 2.3), matrix ring spectra give rise to the simplest kind of Morita equivalence. Indeed over any a ring spectrum R , the ‘free module of rank n ’, i.e., the wedge of n copies of R , is a small generator for the homotopy category of R -modules. The endomorphism ring spectrum of a (stably fibrant replacement) of $R \times n$ is the $n \times n$ matrix ring spectrum as we defined it in Example 4.13. So R and

$$M_n(R) = \text{End}_R((R \times n)^{\text{f}})$$

are Morita equivalent as ring spectra.

(2) **Upper triangular matrices.** In Example 3.25 we saw that the upper triangular 3×3 matrices over a field are derived equivalent, but not Morita equivalent, to its sub-algebra of matrices of the form

$$\left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{pmatrix} \mid x_{ij} \in k \right\} .$$

We cannot directly define the algebra of 3×3 matrices over a ring spectrum; the problem is that the usual basis of elementary matrices is closed under multiplication, but the unit matrix is a *sum* basis elements, not a single basis element.

The algebras of Example 3.25 can also be obtained as the path algebras of the two quivers

$$Q = \left\{ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \right\} \quad \text{respectively} \quad \left\{ 1 \longleftarrow 2 \longrightarrow 3 \right\} .$$

With this in mind we can now run Example 3.25 with the ground field replaced by a commutative symmetric ring spectrum R .

We construct the ‘path algebra’ indirectly via ‘representations of the quiver Q over R ’. A representation of Q over R is a collection

$$M = \left\{ M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \right\}$$

of three R -modules and two homomorphisms. A morphism $f : M \rightarrow N$ of representations consists of R -homomorphisms $f_i : M_i \rightarrow N_i$ for $i = 1, 2, 3$ satisfying $\alpha f_1 = f_2 \alpha$ and $\beta f_2 = f_3 \beta$. There is a stable model structure on the category of such representations in which a morphism $f : M \rightarrow N$ is a stable equivalence or fibration if and only if each R -homomorphism $f_i : M_i \rightarrow N_i$ is a stable equivalence or fibration for all $i = 1, 2, 3$. Moreover, f is a cofibration if and only if the morphisms

$$f_1 : M_1 \rightarrow N_1, \quad f_2 \cup \alpha : M_2 \cup_{M_1} N_1 \rightarrow N_2 \quad \text{and} \quad f_3 \cup \beta : M_3 \cup_{M_2} N_2 \rightarrow N_3$$

are cofibrations of R -modules.

We consider the ‘free’ or ‘projective’ representations P^i given by $P^1 = \{R \xrightarrow{=} R \xrightarrow{=} R\}$, $P^2 = \{* \rightarrow R \xrightarrow{=} R\}$ respectively $P^3 = \{* \rightarrow * \rightarrow R\}$. Then P^i represents the evaluation functor, i.e., we have a natural isomorphism

$$\text{Hom}_{Q\text{-rep}}(P^i, M) \cong M_i .$$

This implies that the wedge of these three representations is a small generator of the homotopy category of Q -representations. The three projective representations are cofibrant;

we let $M_3^\Delta(R)$ denote the endomorphism ring spectrum of a stably fibrant approximation of their wedge,

$$M_3^\Delta(R) = \text{End}_{Q\text{-rep}}((P^1 \vee P^2 \vee P^3)^f).$$

This symmetric ring spectrum deserves to be called the ‘upper triangular 3×3 matrices’ over R .

Now we find a different small generator for the model category of Q -representations which is the analog of the tilting module in Example 3.25. We note that the inclusion $P^3 \rightarrow P^2$ is not a cofibration, and the quotient $P^2/P^3 = \{ * \rightarrow R \rightarrow * \}$ is not cofibrant. A cofibrant approximation of this quotient is given by the representation $S^2 = \{ * \rightarrow R \rightarrow CR \}$ in which the second map is the inclusion of R into its cone. We form the representation

$$T = P^1 \vee P^2 \vee S^2.$$

Then S^2 represents the functor which sends a Q -representation M to the homotopy fiber of the morphism $\beta : M_2 \rightarrow M_3$, and thus

$$\text{Hom}_{Q\text{-rep}}(T, M) \cong M_1 \times M_2 \times \text{hofibre}(\beta : M_2 \rightarrow M_3).$$

This implies that T is also a small generator for the homotopy category of Q -representation over R .

We conclude that the upper triangular matrix algebra $M_3^\Delta(R)$ is Quillen equivalent to the endomorphism ring spectrum of a stably fibrant approximation of the representation T . Just as the endomorphisms of the generator $P^1 \vee P^2 \vee P^3$ should be thought of as upper triangular matrices, the endomorphisms of the generator $P^1 \vee P^2 \vee S^2$ are analogous to a certain algebra of 3×3 matrices over R , namely the ones of the form

$$\begin{pmatrix} R & R & R \\ * & R & * \\ * & * & R \end{pmatrix}.$$

Another example is obtained as follows. We consider the representation $S^1 = \{ R \rightarrow CR \xrightarrow{=} CR \}$ which is a cofibrant replacement of the representation P^1/P^2 and which represents the homotopy fiber of the morphism $\alpha : M_1 \rightarrow M_2$. Then

$$T' = S^1 \vee S^2 \vee P^3$$

is another small generator for the homotopy category of Q -representation over R . So $M_3^\Delta(R)$ is also Quillen equivalent to the derived endomorphism ring spectrum of T' , which is an algebra of 3×3 matrices of the form

$$\begin{pmatrix} R & \Omega R & * \\ * & R & \Omega R \\ * & * & R \end{pmatrix},$$

where each symbol ‘ $*$ ’ designates an entry in a stably contractible spectrum.

(3) **Uniqueness results for stable homotopy theory.** Theorem 4.16 characterizes module categories over ring spectra among stable model categories. This also yields a characterization of the model category of spectra: a stable model category is Quillen equivalent to the category of symmetric spectra if and only if it has a small generator P for which the unit map of ring spectra $\mathbb{S} \rightarrow \text{End}(P)$ is a stable equivalence. The technical conditions of being simplicial, cofibrantly generated and proper can be eliminated as in [SS02] with the use of *framings* [Hov99, Chpt. 5]. The paper [SS02] also gives other necessary and sufficient conditions for when a stable model category is Quillen equivalent to spectra – some of them

in terms of the homotopy category and the natural action of the stable homotopy groups of spheres. In [Sch01b], this result is extended to a uniqueness theorem showing that the 2-local stable homotopy category has only one underlying model category up to Quillen equivalence; the odd-primary version is work in progress. In another direction, the uniqueness result is extended to include the monoidal structure in [Sh2].

(4) **Chain complexes and Eilenberg-Mac Lane spectra.** For a ring R , the category of chain complexes of R -modules (under quasi-isomorphisms) is Quillen equivalent to the category of modules over the Eilenberg-Mac Lane ring spectrum HR . An explicit chain of two Quillen equivalences can be found in Theorem B.1.11 of [SS03]; I don't know if it is possible to compare the two categories by a single Quillen equivalence.

This result can also be viewed as an instance of Theorem 4.16: the free R -module of rank one, considered as a complex concentrated in dimension zero, is a small generator for the unbounded derived category of R . Since the homotopy groups of its endomorphism ring spectrum (as an object of the model category of chain complexes) are concentrated in dimension zero, the endomorphism ring spectrum is stably equivalent to the Eilenberg-Mac Lane ring spectrum for R (Proposition B.2.1 of [SS03]).

(5) **A generalized tilting theorem.** We interpret and generalize the tilting theory from the perspective of stable model categories. A *tilting object* in a stable model category \mathcal{C} as a small generator T such that the graded homomorphism group $\mathrm{Ho}(\mathcal{C})(T, T)_*$ in the homotopy category is concentrated in dimension zero. The following ‘generalized tilting theorem’ of [SS03, Thm. 5.1.1] then shows that the existence of a tilting object is necessary and sufficient for a stable model category to be Quillen equivalent or derived equivalent to the category of unbounded chain complexes over a ring.

Generalized tilting theorem. *Let \mathcal{C} be a stable model category which is simplicial, cofibrantly generated and proper, and let R be a ring. Then the following conditions are equivalent:*

- (1) *There is a chain of Quillen equivalences between \mathcal{C} and the model category of chain complexes of R -modules.*
- (2) *The homotopy category of \mathcal{C} is triangulated equivalent to the derived category $\mathcal{D}(R)$.*
- (3) *The model category \mathcal{C} has a tilting object whose endomorphism ring in $\mathrm{Ho}(\mathcal{C})$ is isomorphic to R .*

In the derived category of a ring, a tilting object is the same as a tilting complex, and the result reduces to Rickard's tilting theorem 3.13.

The generalized tilting situation enjoys one very special feature. In general, the notion of Quillen equivalence is considerably stronger than triangulated equivalence of homotopy categories; two examples are given in [Sch01b, 2.1 and 2.2]. Hence it is somewhat remarkable that for complexes of modules over rings, the two notions are in fact equivalent. In general the homotopy category determines the homotopy groups of the endomorphism ring spectrum, but not its homotopy type. The real reason behind the equivalences of conditions (1) and (2) above is the fact that in contrast to arbitrary ring spectra, Eilenberg-Mac Lane spectra are determined up to stable equivalence by their homotopy groups. We explain this in more detail in Section 5 of [SS03].

(6) **Frobenius rings.** As in Example 4.2 (2) we consider a Frobenius ring and assume that the stable module category has a small generator. Then we are in the situation of Theorem 4.16; however this example is completely algebraic, and there is no need to consider ring spectra to identify the stable module category as the derived category of a suitable ‘ring’.

In fact Keller shows [Kel94a, 4.3] that in such a situation there exists a differential graded algebra and an equivalence between the stable module category and the unbounded derived category of the differential graded algebra.

(7) **Smashing Bousfield localizations.** Let E be a spectrum and consider the E -local model category structure on some model category of spectra (see e.g. [EKMM, VIII 1.1]). This is another stable model category in which the localization of the sphere spectrum $L_E\mathbb{S}$ is a generator. This localized sphere is small if the localization is *smashing*, i.e., if a certain natural map $X \wedge L_E\mathbb{S} \rightarrow L_EX$ is a stable equivalence for all X . So for a smashing localization the E -local model category of spectra is Quillen equivalent to modules over the ring spectrum $L_E\mathbb{S}$ (which is the endomorphism ring spectrum of the localized sphere in the localized model structure).

(8) **Finite localization.** Suppose P is a small object of a triangulated category \mathcal{T} with infinite coproducts. Then there always exists an idempotent localization functor L_P on \mathcal{T} whose acyclics are precisely the objects of the localizing subcategory generated by P (compare [Mil92] or the proofs of [SS03, Lemma 2.2.1] or [HPS97, Prop. 2.3.17]). These localizations are often referred to as *finite Bousfield localizations* away from P .

This type of localization has a refinement to the model category level. Suppose \mathcal{C} is a stable model category and P a small object, and let L_P denote the associated localization functor on the homotopy category of \mathcal{C} . By 4.19, or rather the refined version [SS03, Thm. 3.9.3 (ii)], the acyclics for L_P are equivalent to the homotopy category of $\text{End}_{\mathcal{C}}(P)$ -modules, the equivalence arising from a Quillen adjoint functor pair. Furthermore the counit of the derived adjunction

$$\text{Hom}_{\mathcal{C}}(P, X) \wedge_{\text{End}_{\mathcal{C}}(P)}^L P \rightarrow X$$

is the acyclicization map and its cofiber is a model for the localization $L_P X$.

(9) **$K(n)$ -local spectra.** Even if a Bousfield localization is not smashing, Theorem 4.16 might be applicable. As an example we consider Bousfield localization with respect to the n -th Morava K-theory $K(n)$ at a fixed prime. The localization of the sphere is still a generator, but for $n > 0$ it is not small in the local category, see [HPS97, 3.5.2]. However the localization of any finite type n spectrum F is a small generator for the $K(n)$ -local category [HS99, 7.3]. Hence the $K(n)$ -local model category is Quillen equivalent to modules over the endomorphism ring spectrum of $L_{K(n)}F$.

I would like to conclude with a few words about **invariants** of ring spectra which are preserved under Quillen equivalence (but *not* in general under equivalences of triangulated homotopy categories). Such invariants include the algebraic K-theory [DuSh], topological Hochschild homology and topological cyclic homology.

In the classical framework, the center of a ring is invariant under Morita and derived equivalence. As a general philosophy for spectral algebra, definitions which use elements are not well-suited for generalization to ring spectra. So how do we define the ‘center’ of a ring spectrum, such that it only depends, up to stable equivalence, on the Quillen equivalence class of the module category? The center of an ordinary ring R is isomorphic to the endomorphism ring of R , considered as a bimodule over itself, via

$$\text{End}_{R \otimes R^{\text{op}}}(R) \rightarrow Z(R), \quad f \mapsto f(1).$$

So we define the *center* of a ring spectrum R as the endomorphism ring spectrum of a cofibrant-fibrant replacement of R , considered as a bimodule over itself,

$$Z(R) = \text{End}_{R \wedge R^{\text{op}}}(R^{\text{cf}}, R^{\text{cf}}). \quad (4.25)$$

In this definition, R should be flat as a symmetric spectrum, in order for the smash product $R \wedge R^{\text{op}}$ to have the ‘correct homotopy type’.

This kind of center of a ring spectrum is homotopy commutative; slightly more is true: the center (4.25) is often called the *topological Hochschild cohomology spectrum* of the ring spectrum R , and its multiplication extends to an action of an operad weakly equivalent to the operad of little discs, see [McCS, Thm. 3]. However, the above center is usually *not* stably equivalent to a *commutative* symmetric ring spectrum (or what is the same, E_∞ -homotopy commutative); the Gerstenhaber operations on the homotopy of $Z(R)$ are obstruction to higher order commutativity. So is it just a coincidence that the classical center of an ordinary ring is commutative? Or is there some ‘higher’, E_∞ -commutative, center of a ring spectrum, yet to be discovered?

REFERENCES

- [AF92] F. W. Anderson and K. R. Fuller, *Rings and categories of modules (second edition)*, Graduate Texts in Mathematics, **13**, Springer-Verlag, New York, 1992, viii+376 pp.
- [AGH] A. V. Arhangel’skii, K. R. Goodearl, and B. Huisgen-Zimmermann, *Kiiti Morita (1915–1995)*, Notices Amer. Math. Soc. **44** (1997), no. 6, 680–684.
- [Ba] H. Bass, *Algebraic K-theory*. W. A. Benjamin, Inc., New York-Amsterdam 1968 xx+762 pp.
- [BL] A. Baker, A. Lazarev, *Topological Hochschild cohomology and generalized Morita equivalence*, Preprint (2002). <http://arXiv.org/abs/math.AT/0209003>
- [BBD] A. A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*. Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque **100**, Soc. Math. France, Paris, 1982.
- [Bei78] A. A. Beilinson, *Coherent sheaves on P^n and problems in linear algebra*, Functional Anal. Appl. **12** (1978), no. 3, 214–216 (1979).
- [Bek00] T. Beke, *Sheafifiable homotopy model categories*. Math. Proc. Cambridge Philos. Soc. **129** (2000), 447–475.
- [Bou75] A. K. Bousfield, *The localization of spaces with respect to homology*, Topology **14** (1975), 133–150.
- [BF78] A. K. Bousfield and E. M. Friedlander, *Homotopy theory of Γ -spaces, spectra, and bisimplicial sets*, Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II (M. G. Barratt and M. E. Mahowald, eds.), Lecture Notes in Math., **658**, Springer, Berlin, 1978, pp. 80–130.
- [Bro94] M. Broué, *Equivalences of blocks of group algebras*. Finite-dimensional algebras and related topics (Ottawa, ON, 1992), 1–26, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **424**, Kluwer Acad. Publ., Dordrecht, 1994.
- [CH] D. Christensen, M. Hovey, *Quillen model structures for relative homological algebra*, Math. Proc. Cambridge Philos. Soc. **133** (2002), 261–293.
- [SGA 4 $\frac{1}{2}$] P. Deligne, *Cohomologie étale*. Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 $\frac{1}{2}$. Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier. Lecture Notes in Mathematics, Vol. 569. Springer-Verlag, Berlin-New York, 1977. iv+312pp.
- [Dug] D. Dugger, *Replacing model categories with simplicial ones*, Trans. Amer. Math. Soc. **353** (2001), 5003–5027.
- [DuSh] D. Dugger and B. Shipley, *K-theory and derived equivalences*, Preprint (2002) <http://arXiv.org/abs/math.KT/0209084>
- [DØR] B. Dundas, P. A. Østvær and O. Röndigs, *Enriched functors and motivic stable homotopy theory*. Preprint (2002) <http://www.math.ntnu.no/~dundas/>
- [DwSp95] W. G. Dwyer and J. Spalinski, *Homotopy theories and model categories*, Handbook of algebraic topology (Amsterdam), North-Holland, Amsterdam, 1995, pp. 73–126.
- [DG02] W. G. Dwyer, J. Greenlees, *Complete modules and torsion modules*, Amer. J. Math. **124** (2002), 199–220.
- [DGI] W. G. Dwyer, J. Greenlees, S. Iyengar. *Duality in algebra and topology*, Preprint (2002) <http://www.shef.ac.uk/~pm1jg/>
- [EKMM] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory. With an appendix by M. Cole*, Mathematical Surveys and Monographs, **47**, American Mathematical Society, Providence, RI, 1997, xii+249 pp.

- [GM] S. I. Gelfand, Y. I. Manin, *Methods of homological algebra*. Translated from the 1988 Russian original. Springer-Verlag, Berlin, 1996. xviii+372 pp.
- [Gr77] A. Grothendieck, *Groupes des classes des catégories abéliennes et triangulées, complexes parfaits*. Exposé VIII, Séminaire de Géométrie Algébrique du Bois-Marie SGA 5, Lecture Notes in Math., **589**, Springer, Berlin, 1977, pp. 351–371.
- [Hir03] P. S. Hirschhorn, *Model Categories and Their Localizations*, Mathematical Surveys and Monographs **99**, American Mathematical Society, 2003, 457 pp.
- [Hov99] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, **63**, American Mathematical Society, Providence, RI, 1999, xii+209 pp.
- [Hov01a] M. Hovey, *Model category structures on chain complexes of sheaves*, Trans. Amer. Math. Soc. **353** (2001), 2441–2457.
- [Hov01b] M. Hovey, *Spectra and symmetric spectra in general model categories*, J. Pure Appl. Algebra **165** (2001), 63–127.
- [HPS97] M. Hovey, J. H. Palmieri, and N. P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. **128** (1997), no. 610.
- [HSS] M. Hovey, B. Shipley, and J. Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), 149–208.
- [HS99] M. Hovey and N. P. Strickland, *Morava K-theories and localisation*, Mem. Amer. Math. Soc. **139** (1999)
- [Hu03] P. Hu, *S-modules in the category of schemes*. Mem. Amer. Math. Soc. **161** (2003), no. 767, viii+125 pp.
- [Jar87] J. F. Jardine, *Stable homotopy theory of simplicial presheaves*, Canad. J. Math. **39** (1987), 733–747.
- [Jar97] J. F. Jardine, *Generalized étale cohomology theories*, Progress in Mathematics **146**, Birkhäuser Verlag, Basel, 1997, x+317 pp.
- [Jar00a] J. F. Jardine, *Presheaves of symmetric spectra*, J. Pure Appl. Algebra **150** (2000), 137–154.
- [Jar00b] J. F. Jardine, *Motivic symmetric spectra*, Doc. Math. **5** (2000), 445–553.
- [Kel94a] B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), 63–102.
- [Kel94b] B. Keller, *A remark on the generalized smashing conjecture*, Manuscripta Math. **84** (1994), 193–198.
- [Kel96] B. Keller, *Invariance of cyclic homology under derived equivalence*. Representation theory of algebras (Cocoyoc, 1994), 353–361, CMS Conf. Proc., 18, Amer. Math. Soc., Providence, RI, 1996.
- [Kel98] B. Keller, *Invariance and localization for cyclic homology of DG algebras*. J. Pure Appl. Algebra **123** (1998), 223–273.
- [Kel99] B. Keller, *On the cyclic homology of exact categories*, J. Pure Appl. Algebra **136** (1999), 1–56.
- [Kel03] B. Keller, *Hochschild cohomology and derived Picard groups*, Preprint (2003)
<http://www.math.jussieu.fr/~keller/>
- [KZ] S. König, A. Zimmermann, *Derived equivalences for group rings*. With contributions by B. Keller, M. Linckelmann, J. Rickard and R. Rouquier. Lecture Notes in Mathematics, 1685. Springer-Verlag, Berlin, 1998. x+246 pp.
- [KM] I. Kriz and J. P. May, *Operads, algebras, modules and motives*. Astérisque No. **233**, (1995), iv+145pp.
- [Lam] T. Y. Lam, *Lectures on modules and rings*. Graduate Texts in Mathematics, 189. Springer-Verlag, New York, 1999. xxiv+557 pp.
- [LMS86] L. G. Lewis, Jr., J. P. May, and M. Steinberger, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, **1213**, Springer-Verlag, 1986.
- [Lyd98] M. Lydakis, *Simplicial functors and stable homotopy theory*, Preprint (1998)
<http://hopf.math.purdue.edu/>
- [McCS] J. McClure, J. Smith, *A solution of Deligne’s Hochschild cohomology conjecture*, Recent progress in homotopy theory (Baltimore, MD, 2000), 153–193, Contemp. Math. **293**, Amer. Math. Soc., Providence, RI, 2002.
- [McL] S. Mac Lane, *Categories for the working mathematician*. Graduate Texts in Mathematics, Vol. 5. Springer-Verlag, New York-Berlin, 1971. ix+262 pp.
- [MM02] M. A. Mandell and J. P. May, *Equivariant orthogonal spectra and S-modules*, Mem. Amer. Math. Soc. **159** (2002), x+108 pp.
- [MMSS] M. A. Mandell, J. P. May, S. Schwede and B. Shipley, *Model categories of diagram spectra*, Proc. London Math. Soc. **82** (2001), 441–512.

- [Mil92] H. Miller, *Finite localizations*, Bol. Soc. Mat. Mexicana (2) **37** (1992), (Papers in honor of José Adem), 383–389.
- [MV] F. Morel and V. Voevodsky, \mathbb{A}^1 -homotopy theory of schemes, Inst. Hautes Études Sci. Publ. Math. **90** (2001), 45–143.
- [Mo58] K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition*. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A 6 (1958), 83–142.
- [Nee92] A. Neeman, *The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel*. Ann. Sci. École Norm. Sup. (4) **25** (1992), 547–566.
- [Nee01] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies, 148. Princeton University Press, Princeton, NJ, 2001. viii+449 pp.
- [Qui67] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, **43**, Springer-Verlag, 1967.
- [Ren69] R. Rentschler, *Sur les modules M tels que $\text{Hom}(M, -)$ commute avec les sommes directes*, C. R. Acad. Sci. Paris Sér. A-B **268** (1969), A930–A933.
- [RSS] C. Rezk, S. Schwede and B. Shipley, *Simplicial structures on model categories and functors*, Amer. J. Math. **123** (2001), 551–575.
- [Ri03] B. Richter, *Symmetry properties of the Dold-Kan correspondence*, Math. Proc. Cambridge Philos. Soc. **134** (2003), 95–102.
- [Ric89a] J. Rickard, *Morita theory for derived categories*, J. London Math. Soc. (2) **39** (1989), 436–456.
- [Ric89b] J. Rickard, *Derived categories and stable equivalence*. J. Pure Appl. Algebra **61** (1989), 303–317.
- [Ric91] J. Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. (2) **43** (1991), 37–48.
- [Rob87a] A. Robinson, *Spectral sheaves: a model category for stable homotopy theory*, J. Pure Appl. Algebra **45** (1987), 171–200.
- [Rob87b] A. Robinson, *The extraordinary derived category*, Math. Z. **196** (1987), no.2, 231–238.
- [Sch97] S. Schwede, *Spectra in model categories and applications to the algebraic cotangent complex*, J. Pure Appl. Algebra **120** (1997), 77–104.
- [Sch01a] S. Schwede, *S-modules and symmetric spectra*, Math. Ann. **319** (2001), 517–532.
- [Sch01b] S. Schwede, *The stable homotopy category has a unique model at the prime 2*, Adv. Math. **164** (2001), 24–40.
- [SS00] S. Schwede and B. Shipley, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. **80** (2000), 491–511.
- [SS02] S. Schwede and B. Shipley, *A uniqueness theorem for stable homotopy theory*, Math. Z. **239** (2002), 803–828.
- [SS03] S. Schwede and B. Shipley, *Stable model categories are categories of modules*, Topology **42** (2003), 103–153.
- [Sh2] B. Shipley, *Monoidal uniqueness of stable homotopy theory*, Adv. in Math. **160** (2001), 217–240.
- [Spa88] N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), 121–154.
- [TT] R. W. Thomason, T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, 247–435, Progr. Math., 88, Birkhäuser Boston, Boston, MA, 1990.
- [Ver96] J.-L. Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque **239** (1997). With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis, xii+253 pp.
- [Voe] V. Voevodsky, *The Milnor Conjecture*, Preprint (1996)
- [Voe98] V. Voevodsky, \mathbb{A}^1 -homotopy theory, Doc. Math. ICM **I**, 1998, 417–442.
- [Wei94] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics **38**. Cambridge University Press, Cambridge, 1994. xiv+450 pp.

SFB 478 GEOMETRISCHE STRUKTUREN IN DER MATHEMATIK, WESTFÄLISCHE WILHELMS-UNIVERSITÄT MÜNSTER, GERMANY

E-mail address: sschwede@math.uni-muenster.de