Graduate Seminar Advanced Topology (S4D4)

Higher category theory

Tuesdays, 14:15-15:45

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Classical category theory is based on set theory, meaning that a 1-category has a collection of objects, a set of morphisms between each object, a choice of identity morphisms, and an associative and unital composition law. In higher category theory, we want to build some kind of category which has built homotopy theory into its foundations. An ∞-category is then a collection of objects, a *space* of morphisms between every pair of objects with a choice of identity morphisms, and a composition law which is only unital and associative *up to higher homotopy*. In particular, we do not ask that $h \circ (g \circ f)$ is equal to $(h \circ g) \circ f$ inside the associated mapping space, but that there is a choice of homotopy between them, so a path in this mapping space. The induced homotopies between 3-fold compositions then need to be glued together when iterated for 4-fold composition with 2-homotopies, so surfaces between different paths in this mapping space, and so on with n-homotopies for all $n \geqslant 1$. This is a lot more data than in classical category theory, but it is necessary to set up a general homotopy coherent category theory in order to make and prove statements in modern homotopy theory.

One approach to capture all of this data is to define ∞-categories inside simplicial sets, as pioneered by Joyal [Joy02] and Lurie [Lur09]. We will mostly follow the textbook of Land [Lan21]. The goal is to work through the basics of ∞-categories, from the fundamental definitions, to the Yoneda lemma and limits and colimits.

Each talk should be 90 minutes long, accounting for questions and comments, and so it is up to each presenter to chose exactly what should be presented from topic. We encourage you to organise a meeting with Jack Davies before your talk.

(08.04.2025, S. Hallajian) Simplicial sets as a model for topological spaces Define simplicial sets as well as the singular set and geometric realisation functor. Show these two functors are adjoint. Given some examples of simplicial sets. Define homotopy groups of simplicial sets. Show that the singular set-geometric realisation adjunction induces units and counits which are weak equivalences on nice objects. See [GJ99, Qui67].

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(15.04.2025, V. Pawig) From 1-categories to ∞-categories Construct the nerve functor $N\colon Cat \to sSet$, describe its essential image, as well as its adjoint $h\colon sSet \to Cat$. Define ∞-categories, Kan complexes, objects, morphisms, ∞-groupoid cores, subcategories, functors, and natural transformations. See [Lan21, 1.2.1-29 & 1.2.75-80].

(22.04.2025, D. Fierer) From simplicial categories to ∞-categories Define simplicial (and topological) categories and discuss how they model ∞-categories. Reconstruct the nerve functor N as being corepresented by the cosimplicial category $[\bullet]\colon \Delta \to \mathrm{Cat}$. Define the simplicial categories $\mathfrak{C}[\Delta^n]$ and define the simplicial nerve as the functor into simplicial sets corepresented by these simplicial categories. Define the ∞-category of spaces. See [Lan21, 1.2.30-71].

(29.04.2025, Y. Shu) Anodyne maps, fibrations, and the ∞-category of functors Define Kan, left, right, and inner fibrations and the associated anodyne maps. Define saturated closures and prove [Lan21, Corollary 1.3.16] that all anodyne maps are the saturated closures of the associated horn inclusions. Prove either [Lan21, Lemma 1.3.34] or one of the variants which further classify various anodyne morphisms in terms of saturated closures. Use this to prove [Lan21, Theorem 1.3.37 & Corollary 1.3.38] that gives a definition of the ∞-category of functors. Define mapping spaces and prove they are spaces [Lan21, Proposition 1.3.48].

(06.05.2025, L. Ngo) Joins, slices, and Joyal's theorem Introduce joins of simplicial sets, show that the join of two ∞-categories is an ∞-category, and use joins to define slice ∞-categories. Prove [Lan21, Theorem 1.4.23] which in turn follows from [Lan21, Lemma 1.4.22]. Use these results to prove Joyal's theorem [Lan21, Theorem 2.1.8 & Corollary 2.1.12] that ∞-groupoids are Kan complexes. Define the ∞-category of ∞-categories.

(20.05.2025, S. Viet) Equivalences of ∞-categories Define what it means for a functor of ∞-categories to be fully faithful or essentially surjective and prove [Lan21, Proposition 2.3.5] that Joyal equivalences are fully faithful and essentially surjective. Prove [Lan21, Theorem 2.3.20] that a fully faithful and essentially surjective functor of ∞-categories is a Joyal equivalence.

(27.05.2025, N. Ouagnar) Localisations of ∞-categories Introduce (Dwyer–Kan) localisations of ∞-categories and prove their uniqueness. Prove the existence of localisations [Lan21, Lemma 2.4.5 & Proposition 2.4.8]. Prove [Lan21, Lemma 2.4.9] and remark upon [Lan21, Theorem 2.4.10 & Corollary 3.2.26].

(03.06.2025, S. Westerlund) More on mapping spaces Define the fat join and fat slice ∞-categories. Prove [Lan21, Proposition 2.5.27] that the canonical map from the join to the fat join is a Joyal equivalence. Define the right and left mapping spaces and prove [Lan21, Lemma 2.5.30 & Corollary 2.5.31]. Deduce that the left and right mapping spaces agree with the mapping spaces previously seen [Lan21, Corollary 2.5.34]. Finish by proving that the functor from spaces to ∞-categories is fully faithful [Lan21, Corollary 2.5.38].

(01.07.2025, G. Rajkowsik) (co)Cartesian and fibrations and straightening–unstraightening Define (co)Cartesian fibrations and characterise them as in [Lan21, Lemma 3.1.2]. Prove [Lan21, Corollary 3.1.13]. Define (co)Cartesian fibrations, prove [Lan21, Proposition 3.1.23], and use this to prove that a Cartesian fibration is a right fibration if and only if it is conservative [Lan21, Corollary 3.1.25]. Discuss [Lan21, Theorem 3.3.10], some elements of its proof, and use it to prove [Lan21, Theorem 3.3.16 & Theorem 3.3.17]. [Lan21, §3.1-3].

(08.07.2025, M. Mancebo Mann) The Yoneda lemma Define the twisted arrow ∞-category and prove [Lan21, Proposition 4.2.4] to see that it actually defines an ∞-category. Use this to define the mapping space functor which is then adjoint to the Yoneda embedding (alternatively, give a construction of the Yoneda embedding as done in [Lan21, Remark 4.2.6]). Prove the Yoneda lemma [Lan21, Proposition 4.2.10] and that the Yoneda embedding is fully faithful [Lan21, Proposition 4.2.11].

(15.07.2025, O. Niezhenskyi) Limits and colimits Define terminal and initial objects in an ∞-category and prove [Lan21, Proposition 4.1.3]. Define colimits and colimit cones and prove [Lan21, Theorem 4.3.11]. Use this to conclude [Lan21, Lemma 4.3.13]. Define what it means for a functor to spaces to be representable and prove [Lan21, Proposition 4.3.19]. Discuss the examples of coproducts and pushouts, and discuss [Lan21, Theorem 4.3.28] that an ∞-category with coproducts and pushouts has all small colimits. Prove [Lan21, Theorem 4.3.37] that the ∞-categories of spaces and ∞-categories have all small limits and colimits.

References

[GJ99] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*, volume 174 of *Prog. Math.* Basel: Birkhäuser, 1999.

[Joy02] A. Joyal. Quasi-categories and Kan complexes. *J. Pure Appl. Algebra*, 175(1-3):207–222, 2002.

[Lan21] Markus Land. *Introduction to infinity-categories.* Compact Textb. Math. Cham: Birkhäuser, 2021.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170. Princeton, NJ: Princeton University Press, 2009.

[Qui67] D. G. Quillen. *Homotopical algebra.*, volume 43. Springer, Cham, 1967.