A REAL-GLOBAL EQUIVARIANT SEGAL–BECKER SPLITTING, EXPLICIT BRAUER INDUCTION, AND GLOBAL ADAMS OPERATIONS

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Abstract. We prove a splitting result in global equivariant homotopy theory that is a simultaneous refinement of the Segal–Becker splitting and its 'Real' and equivariant generalizations, and of the explicit Brauer induction of Boltje and Symonds. We show that the morphism of ultra-commutative Real-global ring spectra from $\Sigma^\infty_+ B_{\mathrm{gl}}U(1)$ to the Real-global K-theory spectrum that classifies the tautological Real $U(1)$-representation admits a section on underlying Real-global infinite loop spaces. We prove that this global Segal–Becker splitting induces the Boltje–Symonds explicit Brauer induction on equivariant homotopy groups, and that it induces the classical Segal–Becker splittings on equivariant cohomology theories. As an application we rigidify the unstable Adams operations in Real-equivariant K-theory to global self-maps of the Real-global space $\mathbf{BUP}$.

Contents

Introduction

The purpose of this paper is to prove a splitting result in global equivariant homotopy theory that is a simultaneous refinement and generalization of the Segal–Becker splitting [6, 29], its 'Real' version [22], and its equivariant extension [10, 14], and of the explicit Brauer induction of Boltje [7] and Symonds [34]. The main player is a specific morphism of ultra-commutative Real-global ring spectra $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KR}$ from the unreduced suspension spectrum of the Real-global space $\mathbf{P}$, a multiplicative model of the global classifying space of $U(1)$ and a global refinement of $\mathbb{C}P^\infty$, to the Real-global K-theory spectrum. We construct a section to the morphism of underlying Real-global infinite loop spaces $\Omega^\bullet(\eta)\colon \Omega^\bullet(\Sigma^\infty_+\mathbf{P}) \longrightarrow \Omega^\bullet(\mathbf{KR})$ that is a Real-global σ-loop map. We prove that this *global Segal–Becker splitting* induces the classical Segal–Becker splittings on equivariant cohomology theories, and that it induces the Boltje–Symonds explicit Brauer induction on equivariant homotopy groups, As an application we rigidify the unstable Adams operations in equivariant K-theory to Real-global self-maps of the representing Real-global space $\mathbf{BUP}$.

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To put our results into perspective, we give a brief review of the history of the Segal–Becker splitting. Its origin is the statement that the morphism $\Sigma^\infty_+\mathbb{C}P^\infty \longrightarrow KU$ from the suspension spectrum of infinite complex projective space to the complex K-theory spectrum that classifies the tautological line bundle over $\mathbb{C}P^\infty$ has a section after passing to infinite loop spaces. The theorem implies in particular that the transformation of degree 0 cohomology theories on spaces induced by the morphism is a split epimorphism, and several papers on the subject state the splitting in this form. The original splitting theorem was proved by Segal in [29] for complex K-theory. Becker provided a different proof in [6] that also provides a splitting for real and symplectic K-theory. Segal's construction produces p-complete maps for every prime p that are then assembled via an arithmetic square; all other sources on the subject use transfers in some incarnation to construct the relevant splitting. It seems to be well-known that the section to $\Omega^\infty(\Sigma^\infty_+\mathbb{C}P^\infty) \longrightarrow \Omega^\infty KU \simeq \mathbb{Z}\times BU$ can be arranged as a loop map, but it does *not* deloop twice; we recall an argument in Remark 4.16.

In the paper [22], Nagata, Nishida and Toda prove a version of Segal's splitting for Real K-theory in the sense of Atiyah [2], i.e., the K-theory made from complex vector bundles over spaces with involutions, equipped with a fiberwise conjugate-linear involution. A different proof of the Real splitting was given by Kono [19]. On underlying non-equivariant spaces this recovers Segal splitting for complex K-theory; taking C_2-fixed points yields Becker's splitting for real K-theory. A version of the Segal–Becker splitting in motivic homotopy theory, for algebraic K-theory in place of topological K-theory, is provided in [16]. An equivariant generalization of the Segal–Becker splitting for finite groups G was obtained by Iriye and Kono [14, Theorem 1]. Iriye and Kono also claim in [14, Theorem 1'] that their method works in the same way for Real-equivariant K-theory, for finite groups with involution. However, I want to caution the reader about this claim, see Remark 5.5 for details.

Our Real-global splitting in Theorem B below implies all the above splitting result by specialization, i.e., by applying a suitable forgetful functor from Real-global to G-equivariant homotopy theory: the underlying non-equivariant splitting of Theorem B recovers Segal's and Becker's original result, and Theorem A even provides a delooping of the splitting. Keeping the Real direction and forgetting the equivariance gives the theorem of Nagata, Nishida and Toda [22] and Kono [19]. Forgetting the Real direction and passing to the unstable G-homotopy category for a finite group G yields the equivariant splitting of Iriye–Kono [14]. For general compact Lie groups, we obtain the equivariant Segal–Becker splitting in Crabb's paper [10]; and again, Theorem A provides a delooping of the equivariant splitting.

Now we give a more detailed outline of our own results. Throughout the paper we write $C = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ for the Galois group of $\mathbb{C}$ over $\mathbb{R}$, a group with two elements: the identity and complex conjugation. Our results live in the unstable and stable C-global homotopy categories. The non-equivariant morphism $\Sigma^\infty_+\mathbb{C}P^\infty \longrightarrow KU$ that classifies the tautological line bundle over $\mathbb{C}P^\infty$ has a particularly nice and prominent C-global refinement, a morphism of C-global spectra

$$\eta \ : \ \Sigma^\infty_+\mathbf{P} \ \longrightarrow \ \mathbf{KR} \ .$$

The global K-theory spectrum was introduced by Joachim [15], see also [24, Construction 6.4.9]; for every compact Lie group G, the underlying genuine G-spectrum of $\mathbf{KR}$ represents G-equivariant complex K-theory, see [15, Theorem 4.4] or [24, Corollary 6.4.23]. The complex-conjugation involution on $\mathbf{KR}$ was first investigated by Halladay and Kamel [12, Section 6], who show that that resulting genuine C-spectrum models Atiyah's Real K-theory spectrum. We show in Theorem B.59 that $\mathbf{KR}$ deserves to be called the *Real-global K-theory spectrum*: for every augmented Lie group $\alpha\colon G \longrightarrow C$, the genuine G-spectrum $\alpha^*(\mathbf{KR})$ represents α-equivariant Real K-theory KR_α.

The C-global space $\mathbf{P}$ is a specific global refinement of $\mathbb{C}P^\infty$, made from projective spaces of complex vector spaces, see Construction 4.1. The underlying G-equivariant homotopy type of $\mathbf{P}$ is that of the projective space of a complete complex G-universe, with involution by complex conjugation. It is a C-global classifying space, in the sense of [25, Construction A.4], for the extended circle group $\tilde{U}(1) = U(1) \rtimes C$ that we shall denote by $\tilde{T}$ throughout this paper; so the unreduced C-global suspension spectrum $\Sigma^\infty_+\mathbf{P}$ represents the functor $\pi_0^{\tilde{T}}$ on the C-global stable homotopy category, compare [25, Theorem A.17]. The morphism η is extremely highly structured, and has a range of marvelous properties. It is a morphism of ultra-commutative C-ring spectra that sends the universal element in $\pi_0^{\tilde{T}}(\Sigma^\infty_+\mathbf{P})$ to the class of the tautological Real $\tilde{T}$-representation in $\pi_0^{\tilde{T}}(\mathbf{KR}) \cong RR(\tilde{T})$. As a morphism of ultra-commutative ring spectra, the effect of η on equivariant homotopy groups is not only compatible with restriction, inflations and transfers, but also with products, multiplicative power operations and norms. In [27], I establish a global refinement and generalization of Snaith's celebrated theorem [31, 32], saying that KU can be obtained from $\Sigma^\infty_+\mathbb{C}P^\infty$ by 'inverting the Bott class': the morphism $\eta\colon \Sigma^\infty_+\mathbf{P} \longrightarrow \mathbf{KR}$ is initial in the ∞-category of ultra-commutative ring spectra among morphisms from $\Sigma^\infty_+\mathbf{P}$ that invert a specific family of representation-graded equivariant homotopy classes in $\pi^{U(n)}_{\nu_n}(\Sigma^\infty_+\mathbf{P})$, *pre-Bott classes*, for all $n \geq 1$.

The C-global space $\mathbf{U}$ is a specific global refinement of the infinite unitary group, made from the unitary groups of all hermitian inner product spaces, see Construction B.17. The underlying G-equivariant homotopy type of $\mathbf{U}$ is that of the unitary group of a complete complex G-universe, with involution by complex conjugation. The C-global space $\mathbf{U}$ features in a Real-global refinement of Bott periodicity, a C-global equivalence $\Omega^{1+\sigma}(\mathbf{U}) \sim \mathbf{U}$ that encodes equivariant Bott periodicity for all compact Lie groups at once, see Remark B.25. In Construction 4.7 we use the C-global stable splitting of $\Sigma^\infty_+\mathbf{U}$ from [25, Theorem 4.10] to construct a morphism

$$d \;:\; \mathbf{U} \;\longrightarrow\; \Omega^\bullet(\Sigma^\infty_+\mathbf{P} \wedge S^\sigma)$$

in the unstable C-global homotopy category, our deloop of the global Segal–Becker splitting. The splitting property is our first main result, to be proved as Theorem 4.9:

Theorem A. *The composite*

$$\mathbf{U} \xrightarrow{\ d\ } \Omega^\bullet(\Sigma^\infty_+\mathbf{P} \wedge S^\sigma) \xrightarrow{\Omega^\bullet(\eta\wedge S^\sigma)} \Omega^\bullet(\mathbf{KR} \wedge S^\sigma)$$

is a C-global equivalence.

We emphasize that the composite $\Omega^\bullet(\eta \wedge S^\sigma) \circ d$ of Theorem A is not just any C-global equivalence, but it coincides with the 'preferred infinite delooping' of $\mathbf{U}$, i.e., the C-global equivalence

$$\mathbf{U} \xrightarrow{\ \sim\ } \Omega^\bullet(\mathrm{sh}^\sigma\,\mathbf{KR})$$

established in Theorem B.57, up to a natural C-global equivalence $\mathbf{KR}\wedge S^\sigma \sim \mathrm{sh}^\sigma\,\mathbf{KR}$. In fact, all the work in proving Theorem A goes into showing precisely this. At the heart of the argument is a subtle connection between the global stable splitting and the preferred delooping of $\mathbf{U}$, two features that are a priori unrelated. As we show in Theorem 3.5, the adjoint $\Sigma^\infty\mathbf{U} \longrightarrow \mathrm{sh}^\sigma\,\mathbf{KR}$ of the preferred infinite delooping annihilates the higher terms of the stable global splitting (1.1).

The C-global space $\mathbf{BUP}$ is introduced in Construction B.3. It is a Real-global refinement of the space $\mathbb{Z} \times BU$ and its involution by complex conjugation. By Theorem B.12, $\mathbf{BUP}$ represents Real-equivariant K-theory. Just as $\mathbb{Z} \times BU$ is the infinite loop space of the topological K-theory spectrum KU, the global space $\mathbf{BUP}$ 'is' the Real-global infinite loop space of $\mathbf{KR}$, see Remark B.58. By looping the morphism $d\colon \mathbf{U} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P} \wedge S^\sigma)$ by the sign representation σ and composing with the Real-global Bott periodicity equivalence $\mathbf{BUP} \sim \Omega^\sigma\mathbf{U}$ from (B.27) we obtain another morphism in the unstable C-global homotopy category

$$c \;:\; \mathbf{BUP} \;\longrightarrow\; \Omega^\bullet(\Sigma^\infty_+\mathbf{P}) \;,$$

the *global Segal–Becker splitting*. By design, the morphism c is a Real-global σ-loop map. Its splitting property, to be proved as Corollary 4.15, is an easy consequence of Theorem A:

Theorem B. *The composite*

$$\mathbf{BUP} \xrightarrow{\ c\ } \Omega^\bullet(\Sigma^\infty_+\mathbf{P}) \xrightarrow{\Omega^\bullet(\eta)} \Omega^\bullet(\mathbf{KR})$$

is a C-global equivalence.

Morphisms that admit a section tend to admit many different sections. Our next result justifies that the global Segal–Becker splitting $c\colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P})$ is the 'correct' splitting to $\Omega^\bullet(\eta)$ by showing that it refines the classical equivariant Segal–Becker splittings, thereby also explaining the name. The latter are defined at the level of equivariant cohomology theories, and we review them in Construction 5.1. We show the following in Theorem 5.8:

Theorem C. *For all compact Lie groups G and all finite G-CW-complexes A, the composite*

$$KU_G(A) \cong [A, \mathbf{BUP}]^G \xrightarrow{[A,c]^G} [A, \Omega^\bullet(\Sigma^\infty_+\mathbf{P})]^G$$

coincides with the G-equivariant Segal–Becker splitting $\vartheta_{G,A}$ defined in (5.4).

The isomorphism between the G-equivariant complex K-group $KU_G(A)$ and the group $[A, \mathbf{BUP}]^G$ will be recalled in Construction B.6. The group $[A, \Omega^\bullet(\Sigma^\infty_+\mathbf{P})]^G$ is isomorphic to the group of morphisms from $\Sigma^\infty_+ A$ to $\Sigma^\infty_+\mathbf{P}(\mathcal{U}_G)$ is the G-equivariant stable homotopy category, where $\mathbf{P}(\mathcal{U}_G)$ is a classifying G-space for G-equivariant complex line bundles.

On equivariant homotopy groups, the global Segal–Becker splitting induces the so-called *explicit Brauer induction* of Boltje [7] and Symonds [34]. We review this result and the history of explicit Brauer induction in Remark 5.10. The following result will be proved as Corollary 5.15.

Theorem D. *For every compact Lie group G, the composite*

$$R(G) \;\cong\; \pi_0^G(\mathbf{BUP}) \xrightarrow{\pi_0^G(c)} \pi_0^G(\Sigma^\infty_+\mathbf{P}) \;\cong\; \mathbf{A}(T, G)$$

is the Boltje–Symonds explicit Brauer induction.

We alert the reader that Theorems C and D refer to equivariant complex K-theory of compact Lie groups, as opposed to more generally for Real-equivariant K-theory of augmented Lie groups. There are two reasons for this restriction. The first one is that the equivariant Segal–Becker splitting and explicit Brauer induction are usually discussed in this context in the classical literature. The notable exception is [14, Theorem 1'] where the Segal–Becker splitting is stated for Real-equivariant K-theory; however, I explain in Remark 5.5 why I am not entirely convinced of the correctness of this result. My doubts are related to the second reason for the restriction to trivially augmented compact Lie groups. As I explain in Remark 5.7, the direct extension of the Segal–Becker splitting via transfers to surjectively augmented Lie groups is *not additive* for the Whitney sum of Real-equivariant vector bundles. Since additivity fails, there is no obvious extension of the construction to virtual vector bundles, i.e., the Grothendieck group. This is consistent with the fact that also the map $[A, c]^\alpha\colon [A, \mathbf{BUP}]^\alpha \longrightarrow [A, \Omega^\bullet(\Sigma^\infty_+\mathbf{P})]^\alpha$ is *not additive* whenever the augmentation $\alpha\colon G \longrightarrow C$ is non-trivial, see Example 4.26. I believe that this non-additivity is a feature, and not a bug, of the Real-equivariant Segal–Becker splitting; it is intimately tied to the Segal–Becker splitting being a σ-loop map, but not an ordinary loop map, whenever there is an honest Real direction. For example, we can quantify the failure of additivity with the help of the 'exceptional unit' $\epsilon \in \pi_0^C(\mathbb{S})$ represented by the sign involution of S^σ, see Theorem 4.23 (ii).

As an application of the global Segal–Becker splitting, we construct global rigidifications of the unstable Adams operations on equivariant K-theory. In (6.5) we define the *n-th global Adams operation*

$$\Upsilon^n \;:\; \mathbf{BUP} \;\longrightarrow\; \mathbf{BUP}\;,$$

for $n \geq 1$. These global Adams operations are morphisms in the unstable C-global homotopy category that arise as global σ-loop maps, the deloopings being certain endomorphisms of $\mathbf{U}$. The following result, proved as Theorem 6.9, justifies the name of the global Adams operations:

Theorem E. *For every augmented Lie group $\alpha\colon G \longrightarrow C$ and every finite G-CW-complex A, the following square commutes:*

$$\begin{tikzcd} KR_G(A) \arrow[r, "\psi^n"] \arrow[d, "\cong"'] & KR_G(A) \arrow[d, "\cong"] \\ {[A, \mathbf{BUP}]^\alpha} \arrow[r, "{[A,\Upsilon^n]^\alpha}"'] & {[A, \mathbf{BUP}]^\alpha} \end{tikzcd}$$

The upper horizontal map in the diagram is the n-th classical Adams operation on Real-equivariant K-theory.

This paper concludes with two appendices. Appendix A develops some pieces of unstable C-global homotopy theory that are not covered in the existing sources [25, Appendix A] and [5]. In Appendix B we extend various results about global K-theory to the Real-global context.

Conventions. Throughout this paper we let $C = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ denote the Galois group of $\mathbb{C}$ over $\mathbb{R}$. We use the models of [25, Appendix A] and [5, Appendix A] to represent unstable and stable C-global homotopy types. So C-global spaces are represented by orthogonal C-spaces, relative to the notion of C-global equivalence introduced in [25, Definition A.2] and [5, Definition 3.2]. And C-global spectra are represented by orthogonal C-spectra, relative to the notion of C-global equivalence introduced in [25, Definition A.6]. Since the non-identity element of the group C always acts by some sort of complex conjugation, and the C-actions in this paper always encode conjugate-linear phenomena, we shall also use the term 'Real-global' as synonymous for 'C-global'. Our global Segal–Becker splitting and its deloop will thus be morphisms in the unstable Real-global homotopy category, i.e., the localization of the category of orthogonal C-spaces at the class of C-global equivalences.

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1. The C-global stable splitting of $\mathbf{U}$

A key player in this paper is the C-global ultra-commutative monoid $\mathbf{U}$ made from unitary groups, compare [24, Example 2.37]; we recall the definition in Construction B.17. The construction of our deloop of the global Segal–Becker splitting depends crucially on a C-global stable splitting of $\mathbf{U}$. In [25], I construct certain morphisms $s_k\colon \Sigma^\infty(\mathbf{Gr}_k)^{\mathrm{ad}(k)} \longrightarrow \Sigma^\infty_+\mathbf{U}$ in the C-global stable homotopy category such that the combined morphism

$$\sum s_k \ : \ \bigvee_{k\geq 0} \Sigma^\infty(\mathbf{Gr}_k)^{\mathrm{ad}(k)} \ \xrightarrow{\ \sim\ } \ \Sigma^\infty_+\mathbf{U} \tag{1.1}$$

is a C-global equivalence, see [25, Theorem 4.10]. The splitting (1.1) is a global-equivariant refinement of Miller's stable splitting [21] of the infinite unitary group. The orthogonal C-space $\mathbf{Gr}_k$ is made from Grassmannians of complex k-planes, with involution by complex conjugation; see Construction B.1. And $(\mathbf{Gr}_k)^{\mathrm{ad}(k)}$ denotes the global Thom space over $\mathbf{Gr}_k$ associated with the adjoint representation $\mathrm{ad}(k)$ of $U(k)$; see [25, Example 3.12]. In this section we review the construction of the splitting morphisms, and then translate the splitting (1.1) into an interpretation of the group $[\![\Sigma^\infty \mathbf{U}, X]\!]^C$ of stable C-global morphisms in terms of $\mathrm{ad}(k)$-graded $U(k)$-equivariant homotopy groups of X, see Theorem 1.13.

The C-global stable splitting morphism $s_k\colon (\mathbf{Gr}_k)^{\mathrm{ad}(k)} \longrightarrow \Sigma^\infty_+ \mathbf{U}$ ultimately stems from a specific $U(k)$-equivariant stable splitting of the 'top cell' in $U(k)^{\mathrm{ad}}$, the unitary group $U(k)$ acting on itself by conjugation. We review this splitting now, following Crabb's exposition in [9, page 39]. In [24] and [25] we use different conventions on whether the suspension coordinate in an orthogonal suspension spectrum is written on the left or right of the argument. In this paper, we adopt the convention of [24], with the suspension coordinate written on the left; this entails some minor changes to formulas in [25], moving some suspension coordinates to the other side.

Construction 1.2 (Splitting the top cell off $U(k)^{\mathrm{ad}}$)**.** The *extended unitary group*

$$\tilde{U}(k) \ = \ U(k) \rtimes C$$

is the semidirect product of the group C acting on the unitary group

$$U(k) \ = \ \{A \in M_k(\mathbb{C})\colon A \cdot \bar{A}^t = E_k\}$$

by coordinatewise complex conjugation. We write

$$\mathrm{ad}(k) \ = \ \{X \in M(k\times k;\mathbb{C})\colon X = -\bar{X}^t\}$$

for the $\mathbb{R}$-vector space of skew-hermitian complex $k\times k$ matrices. The unitary group $U(k)$ acts on $\mathrm{ad}(k)$ by conjugation, and the group C acts by coordinatewise complex conjugation. Together with the euclidean inner product $\langle X, Y\rangle = \mathrm{Tr}(\bar{X}^t\cdot Y) = -\mathrm{Tr}(X\cdot Y)$, these data make $\mathrm{ad}(k)$ into an orthogonal $\tilde{U}(k)$-representation; this action witnesses $\mathrm{ad}(k)$ as the adjoint representation of $\tilde{U}(k)$, whence the name.

The *Cayley transform* is the $\tilde{U}(k)$-equivariant open embedding

$$\mathrm{ad}(k) \ \longrightarrow \ U(k)^{\mathrm{ad}} \ , \quad X \ \longmapsto \ (X-1)(X+1)^{-1}$$

onto the subspace of $U(k)$ of those matrices that do not have $+1$ as an eigenvalue. The associated collapse map

$$U(k)^{\mathrm{ad}} \ \longrightarrow \ S^{\mathrm{ad}(k)}$$

admits a section in the stable homotopy category of genuine $\tilde{U}(k)$-spectra, as follows. We write

$$\mathrm{sa}(k) \ = \ \{Z \in M_k(\mathbb{C})\colon Z = \bar{Z}^t\}$$

for the $\mathbb{R}$-vector space of hermitian complex $k\times k$ matrices. Much like for $\mathrm{ad}(k)$, the unitary group $U(k)$ acts on $\mathrm{sa}(k)$ by conjugation, the group C acts by coordinatewise complex conjugation, and an invariant euclidean inner product is given by $\langle Z, Z'\rangle = \mathrm{Tr}(\bar{Z}^t\cdot Z') = \mathrm{Tr}(Z\cdot Z')$; these data make $\mathrm{sa}(k)$ into another orthogonal $\tilde{U}(k)$-representation. A basic linear algebra fact, sometimes referred to as 'polar decomposition', is that the $\tilde{U}(k)$-equivariant map

$$\phi_k \ : \ \mathrm{sa}(k)\times U(k)^{\mathrm{ad}} \ \longrightarrow \ M_k(\mathbb{C}) = \mathrm{sa}(k)\oplus\mathrm{ad}(k) \ , \quad (Z, A)\mapsto A\cdot\exp(-Z) \tag{1.3}$$

is an open embedding onto the general linear group $GL_k(\mathbb{C})$; for a proof, see for example [25, Proposition B.17]. This open embedding has an associated $\tilde{U}(k)$-equivariant collapse map

$$t_k \ : \ S^{\mathrm{sa}(k)\oplus\mathrm{ad}(k)} \ \longrightarrow \ S^{\mathrm{sa}(k)}\wedge U(k)^{\mathrm{ad}}_+ \tag{1.4}$$

that is a stable section to the previous collapse map $U(k)^{\mathrm{ad}} \longrightarrow S^{\mathrm{ad}(k)}$, see the argument after the proof of Theorem 1.8 in [9, page 39], or the proof of [25, Theorem 4.7].

Remark 1.5. The orthogonal $\tilde{U}(k)$-representations $\mathrm{sa}(k)$ and $\mathrm{ad}(k)$ of hermitian and skew-hermitian matrices are almost isomorphic. More precisely, the multiplication map

$$i\cdot - \ : \mathrm{sa}(k) \ \longrightarrow \ \mathrm{ad}(k) \ , \quad X \ \longmapsto \ i\cdot X$$

by the imaginary unit $i \in \mathbb{C}$ is an $\mathbb{R}$-linear and $U(k)$-equivariant isomorphism. It does *not* commute with complex conjugation, but rather satisfies

$$\overline{iX} \ = \ -i \cdot \bar{X} \ .$$

So multiplication by i is an isomorphism of orthogonal $\tilde{U}(k)$-representations

$$i \cdot - \ : \ \mathrm{sa}(k) \otimes \sigma \ \xrightarrow{\cong} \ \mathrm{ad}(k) \ ,$$

where σ denotes the 1-dimensional sign representation of $\tilde{U}(k)$, through the projection $U(k) \rtimes C \longrightarrow C$.

Next we recall how the $\tilde{U}(k)$-equivariant stable splitting (1.4) gives rise to a specific equivariant homotopy class τ_k in $\pi_{\mathrm{ad}(k)}^{\tilde{U}(k)}(\Sigma^\infty_+ \mathbf{U})$; this class in turn characterizes the C-global splitting morphism $s_k \colon \Sigma^\infty(\mathbf{Gr}_k)^{\mathrm{ad}(k)} \longrightarrow \Sigma^\infty_+ \mathbf{U}$ by the relation (1.10).

Throughout this paper, an *augmented Lie group* is a continuous homomorphism $\alpha \colon G \longrightarrow C$ from a compact Lie group G to $C = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, the Galois group of $\mathbb{C}$ over $\mathbb{R}$. A *Real α-representation* is a hermitian inner product space W equipped with a continuous $\mathbb{R}$-linear isometric G-action, such that for every $g \in G$, the translation map $l_g \colon W \longrightarrow W$ is $\alpha(g)$-linear. These Real representations of augmented Lie groups are the special case of Real-equivariant vector bundles, in the sense of Construction B.6, over one-point G-spaces.

Construction 1.6 (The C-global splitting morphism)**.** We let W be a Real representation of an augmented Lie group $\alpha \colon G \longrightarrow C$. We write uW for the underlying orthogonal G-representation, i.e., the underlying $\mathbb{R}$-vector space with the euclidean inner product $\langle v, w\rangle = \mathrm{Re}(v, w)$, the real part of the hermitian inner product. The map

$$\zeta^W \ : \ W \ \longrightarrow \ \mathbb{C} \otimes_{\mathbb{R}} (uW) = (uW)_{\mathbb{C}} \ , \quad w \longmapsto (1 \otimes w - i \otimes iw)/\sqrt{2} \tag{1.7}$$

is a G-equivariant $\mathbb{C}$-linear isometric embedding. So conjugation by ζ^W and extension by the identity on the orthogonal complement of its image is a continuous equivariant group monomorphism

$$\zeta^W_* \ : \ U(W) \ \longrightarrow \ U((uW)_{\mathbb{C}}) \ = \ \mathbf{U}(uW) \ .$$

We let G act on $\mathbf{U}(uW)$ via the $\mathbb{R}$-linear G-action on uW and functoriality of $\mathbf{U}$, and the involution on $\mathbf{U}$ through the augmentation $\alpha \colon G \longrightarrow C$. Since ζ^W is G-equivariant, the map ζ^W_* is G-equivariant for the conjugation action on the source. One should beware that ζ^W_* is *different* from the monomorphism that sends a unitary automorphism $\varphi \colon W \longrightarrow W$ to the unitary automorphism $(u\varphi)_{\mathbb{C}}$ of $(uW)_{\mathbb{C}}$.

A special case of this construction is the *tautological Real representation* ν_k of the extended unitary group $\tilde{U}(k)$, augmented by the projection $U(k) \rtimes C \longrightarrow C$. By definition, this is the vector space $\mathbb{C}^k$ with tautological $U(k)$-action, and with C acting by coordinatewise complex conjugation. In this case, ζ^W_* becomes a $\tilde{U}(k)$-equivariant monomorphism

$$\zeta^k_* = \zeta^{\nu_k}_* \ : \ U(k)^{\mathrm{ad}} = U(\nu_k) \ \longrightarrow \ \mathbf{U}(u(\nu_k)) \ . \tag{1.8}$$

The $\tilde{U}(k)$-equivariant collapse map t_k was defined in (1.4). In [25, Construction 4.4], we define a class

$$\tau_k \ \in \ \pi_{\mathrm{ad}(k)}^{\tilde{U}(k)}(\Sigma^\infty_+ \mathbf{U}) \tag{1.9}$$

(there denoted $\langle t_{k,0}\rangle$) as the one represented by the $\tilde{U}(k)$-map

$$\begin{aligned} S^{\nu_k \oplus \mathrm{sa}(k) \oplus \mathrm{ad}(k)} \ &\xrightarrow{S^{\nu_k} \wedge t_k} \ S^{\nu_k \oplus \mathrm{sa}(k)} \wedge U(k)^{\mathrm{ad}}_+ \ \xrightarrow{S^{\nu_k \oplus \mathrm{sa}(k)} \wedge \zeta^k_*} \ S^{\nu_k \oplus \mathrm{sa}(k)} \wedge \mathbf{U}(u(\nu_k))_+ \\ &\xrightarrow{S^{\nu_k \oplus \mathrm{sa}(k)} \wedge \mathbf{U}(i_1)_+} \ S^{\nu_k \oplus \mathrm{sa}(k)} \wedge \mathbf{U}(u(\nu_k) \oplus \mathrm{sa}(k))_+ = (\Sigma^\infty_+ \mathbf{U})(u(\nu_k) \oplus \mathrm{sa}(k)) \ . \end{aligned}$$

Here $i_1 \colon u(\nu_k) \longrightarrow u(\nu_k) \oplus \mathrm{sa}(k)$ is the embedding of the second summand. In [25], we consider τ_k as an equivariant stable homotopy class of the k-th stage of the eigenspace filtration of $\mathbf{U}$, but now we work in the ambient orthogonal space $\mathbf{U}$.

The *tautological class*

$$e_{\tilde{U}(k),\mathrm{ad}(k)} \in \pi^{\tilde{U}(k)}_{\mathrm{ad}(k)}(\Sigma^\infty(\mathbf{Gr}_k)^{\mathrm{ad}(k)})$$

is defined in [25, (A.16)]. By [25, Theorem A.17 (i)], the pair $(\Sigma^\infty(\mathbf{Gr}_k)^{\mathrm{ad}(k)}, e_{\tilde{U}(k),\mathrm{ad}(k)})$ represents the functor $\pi^{\tilde{U}(k)}_{\mathrm{ad}(k)}\colon \mathcal{GH}_C \longrightarrow \mathcal{A}b$ on the C-global stable homotopy category. The C-global splitting morphism $s_k\colon \Sigma^\infty(\mathbf{Gr}_k)^{\mathrm{ad}(k)} \longrightarrow \Sigma^\infty_+\mathbf{U}$ is defined in [25, (4.6)] by the property that it takes the tautological class to τ_k, i.e., by the relation

$$(s_k)_*(e_{\tilde{U}(k),\mathrm{ad}(k)}) = \tau_k . \tag{1.10}$$

In this paper, we shall mostly work with the reduced suspension spectrum $\Sigma^\infty\mathbf{U}$ (as opposed to the unreduced one), and with the 'reduced' version of the classes τ_k. We endow $\mathbf{U}$ with the intrinsic basepoint $1 \in \mathbf{U}$ consisting of the multiplicative units. We write $\mathbf{U}_+$ for $\mathbf{U}$ with an additional basepoint added. This comes with based maps $\mathbf{U}_+ \longrightarrow S^0$ and $\mathbf{U}_+ \longrightarrow \mathbf{U}$; the first of these takes $\mathbf{U}$ to the non-basepoint of S^0, and the second is the identity on $\mathbf{U}$ and maps the extra basepoint to the intrinsic basepoint 1. We write

$$\varrho : \Sigma^\infty_+\mathbf{U} \longrightarrow \Sigma^\infty S^0 = \mathbb{S} \qquad \text{and} \qquad q : \Sigma^\infty_+\mathbf{U} \longrightarrow \Sigma^\infty\mathbf{U}$$

for the morphisms induced on reduced suspension C-spectra. The combined morphism

$$(\varrho, q) : \Sigma^\infty_+\mathbf{U} \xrightarrow{\sim} \mathbb{S} \times (\Sigma^\infty\mathbf{U}) \tag{1.11}$$

is then a C-global equivalence. We set

$$\sigma_k = q_*(\tau_k) \in \pi^{\tilde{U}(k)}_{\mathrm{ad}(k)}(\Sigma^\infty\mathbf{U}) . \tag{1.12}$$

The following representability result is a fairly direct consequence of the stable splitting (1.1). We shall use it to construct C-global stable morphisms with source $\Sigma^\infty\mathbf{U}$, and to check commutativity of diagrams whose initial object is $\Sigma^\infty\mathbf{U}$. We let $[\![-,-]\!]^C$ denote the group of morphisms in the C-global stable homotopy category.

Theorem 1.13. *For every C-global spectrum X, the evaluation map*

$$[\![\Sigma^\infty\mathbf{U}, X]\!]^C \xrightarrow{\cong} \prod\nolimits_{k\geq 1} \pi^{\tilde{U}(k)}_{\mathrm{ad}(k)}(X) , \quad f \longmapsto (f_*(\sigma_k))_{k\geq 1}$$

is an isomorphism.

Proof. The splitting (1.1) proved in [25, Theorem 4.10] and the representability property of $(\Sigma^\infty(\mathbf{Gr}_k)^{\mathrm{ad}(k)}, e_{\tilde{U}(k),\mathrm{ad}(k)})$ together provide the natural isomorphism

$$[\![\Sigma^\infty_+\mathbf{U}, X]\!]^C \xrightarrow{\cong} \prod\nolimits_{k\geq 0} \pi^{\tilde{U}(k)}_{\mathrm{ad}(k)}(X) , \quad f \longmapsto (f_*(\tau_k))_{k\geq 0} . \tag{1.14}$$

The C-global equivalence (1.11) induces another isomorphism

$$[\![\mathbb{S}, X]\!]^C \times [\![\Sigma^\infty\mathbf{U}, X]\!]^C \xrightarrow{\cong} [\![\Sigma^\infty_+\mathbf{U}, X]\!]^C , \quad (a,b) \longmapsto a\circ\varrho + b\circ q .$$

The morphism $\varrho\colon \Sigma^\infty_+\mathbf{U} \longrightarrow \mathbb{S}$ sends the class τ_0 to $1 \in \pi_0(\mathbb{S})$, so the composite

$$[\![\mathbb{S}, X]\!]^C \xrightarrow{\varrho^*} [\![\Sigma^\infty_+\mathbf{U}, X]\!]^C \xrightarrow{f\mapsto f_*(\tau_0)} \pi_0^C(X)$$

is an isomorphism. Moreover, $q_*(\tau_0) = 0$ and $q_*(\tau_k) = \sigma_k$ for $k \geq 1$, so the isomorphism (1.14) restricts an isomorphism as in the statement of the theorem, where now the factor indexed by $k = 0$ is omitted. $\square$

2. Relations among equivariant stable homotopy classes in $\mathbf{U}$

The purpose of this section is to establishing some crucial relations between certain families of $\tilde{U}(k)$-equivariant homotopy classes of the global spectrum $\Sigma^\infty_+\mathbf{U}$, namely the classes τ_k defined in (1.9), their reduced cousins σ_k from (1.12), and certain classes u_k that we introduce in (2.16) below.

Throughout, we shall write σ for the *sign representation* of C, i.e., the 1-dimensional orthogonal representation on $\mathbb{R}$ with generator multiplying by -1. If $\alpha\colon G \longrightarrow C$ is an augmented Lie group, we abuse notation and also write σ for the orthogonal G-representation obtained by restriction along α, i.e., the elements in the kernel of α act by the identity, and all others act by -1.

The next proposition shows that for all $k \geq 1$, the class τ_k in $\pi^{\tilde{U}(k)}_{\mathrm{ad}(k)}(\Sigma^\infty_+\mathbf{U})$ defined in (1.9) lies in the augmentation ideal, i.e., the kernel of the ring homomorphism $\varrho_*\colon \pi^{\tilde{U}(k)}_\star(\Sigma^\infty_+\mathbf{U}) \longrightarrow \pi^{\tilde{U}(k)}_\star(\mathbb{S})$.

Proposition 2.1. *For $k \geq 1$, the class τ_k satisfies $\varrho_*(\tau_k) = 0$.*

Proof. Every matrix in $\mathrm{sa}(k)$ is unitarily diagonalizable with real eigenvalues. We write $\mathrm{sa}^{\geq}(k)$ for the closed $\tilde{U}(k)$-invariant subspace of $\mathrm{sa}(k)$ consisting of those hermitian matrices all of whose eigenvalues are greater or equal to 0. The continuous $\tilde{U}(k)$-equivariant map

$$\mathrm{sa}^{\geq 0}(k) \times [0,\infty) \longrightarrow \mathrm{sa}^{\geq 0}(k) , \quad (Z,t) \mapsto Z + t\cdot E_k \tag{2.2}$$

is proper. Indeed, if $Z \in \mathrm{sa}^{\geq}(k)$ has eigenvalues $x_1,\dots,x_k$, and $t \geq 0$, then $Z + tE_k$ also lies in $\mathrm{sa}^{\geq}(k)$ and has eigenvalues $x_1+t,\dots,x_k+t$. So

$$\|(Z,t)\| = \sqrt{x_1^2+\dots+x_k^2+t^2} \leq \sqrt{(x_1+t)^2+\dots+(x_k+t)^2} = \|Z+tE_k\| .$$

As a proper map, (2.2) extends continuously to the one-point compactifications

$$(\mathrm{sa}^{\geq 0}(k)\cup\{\infty\}) \wedge [0,\infty] \longrightarrow \mathrm{sa}^{\geq 0}(k)\cup\{\infty\} ;$$

this map is a $\tilde{U}(k)$-equivariant contracting homotopy of the space $\mathrm{sa}^{\geq 0}(k)\cup\{\infty\}$.

The singular value decomposition of complex matrices shows that every $X \in M_k(\mathbb{C})$ is of the form $X = A\cdot Z$ for some $A \in U(k)$ and some $Z \in \mathrm{sa}^{\geq 0}(k)$. The matrix Z in this decomposition is unique, namely the only matrix in $\mathrm{sa}^{\geq 0}(k)$ such that $Z^2 = \bar{X}^t \cdot X$. So the composite

$$\mathrm{sa}^{\geq 0}(k) \xrightarrow{\mathrm{incl}} M_k(\mathbb{C}) \longrightarrow U(k)\backslash M_k(\mathbb{C}) \tag{2.3}$$

is a continuous bijection, where the right hand side denotes the orbit space by the $U(k)$-action by left multiplication. The map

$$M_k(\mathbb{C}) \longrightarrow \mathrm{sa}^{\geq 0}(k) , \quad X \longmapsto \sqrt{\bar{X}^t\cdot X}$$

is continuous and invariant under left multiplication by unitary matrices. So it descends to a continuous map on $U(k)\backslash M_k(\mathbb{C})$, showing that the composite (2.3) is in fact a homeomorphism.

The conjugation action of $\tilde{U}(k)$ on $M_k(\mathbb{C})$ passes to a well-defined action on the orbit space $U(k)\backslash M_k(\mathbb{C})$ and the map (2.3) is $\tilde{U}(k)$-equivariant for this induced action on the target, and the conjugation action on the source. As an equivariant homeomorphism, the map (2.3) thus extends to a $\tilde{U}(k)$-equivariant homeomorphism between the one-point compactifications. We showed above that $\mathrm{sa}^{\geq 0}(k)\cup\{\infty\}$ is equivariantly contractible; the upshot is that the one-point compactification

$$(U(k)\backslash M_k(\mathbb{C}))\cup\{\infty\} = U(k)\backslash S^{M_k(\mathbb{C})}$$

is also $\tilde{U}(k)$-equivariantly contractible.

By inspection of definitions, the composite

$$S^{M_k(\mathbb{C})} \xrightarrow{t_k} S^{\mathrm{sa}(k)} \wedge U(k)_+ \xrightarrow{S^{\mathrm{sa}(k)}\wedge\varrho} S^{\mathrm{sa}(k)} \tag{2.4}$$

is given by

$$X \longmapsto \begin{cases} -\ln(\sqrt{\bar{X}^t \cdot X}) & \text{for } X \in GL_k(\mathbb{C}), \text{ and} \\ \infty & \text{otherwise.} \end{cases}$$

In particular, this composite is invariant under left translation by elements of $U(k)$, so it factors through the orbit space $U(k)\backslash S^{M_k(\mathbb{C})}$. Since the latter is equivariantly contractible, the composite (2.4) is $\tilde{U}(k)$-equivariantly nullhomotopic. Since the defining representative of the class $\varrho_*(\tau_k)$ factors through a suspension of the composite (2.4), this proves that $\varrho_*(\tau_k) = 0$. □

Construction 2.5 (Pre-Euler classes)**.** We will make frequent use of the *pre-Euler classes* associated to representations. If V is an orthogonal representation of a compact Lie group, we write

$$a_V \ \in \pi^G_{-V}(\mathbb{S}) \tag{2.6}$$

for the representation-graded G-equivariant homotopy class represented by the inclusion $\{0, \infty\} = S^0 \longrightarrow S^V$. We shall use various standard properties without further notice, such as the compactibility under restriction along continuous homomorphisms of compact Lie groups, the multiplicativity property $a_{V\oplus W} = a_V \cdot a_W$, or the fact that $a_V = 0$ whenever $V^G \neq 0$.

Construction 2.7. We define the diagonal embedding

$$\Delta \ : \ \mathbb{C}^k \ \longrightarrow \ M_k(\mathbb{C}) \qquad \text{by} \qquad \Delta(z_1, \dots, z_k) \ = \ \begin{pmatrix} z_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & z_k \end{pmatrix} .$$

We write $U(1, \dots, 1)$ for the diagonal subgroup of $U(k)$, i.e., the image of T^k under Δ. Whenever convenient, we use the diagonal embedding to identify T^k with its image $U(1, \dots, 1)$ in $U(k)$. We set

$$\tilde{U}(1, \dots, 1) \ = \ U(1, \dots, 1) \rtimes C \ \subset \ U(k) \rtimes C \ = \ \tilde{U}(k) \ ,$$

the semidirect product by the C-action by complex conjugation. We let

$$L \ = \ \mathrm{ad}(k) - \mathrm{ad}(k)^{U(1,\dots,1)}$$

denote the orthogonal complement of the $U(1, \dots, 1)$-fixed points on the adjoint representation. Concretely, L consists of the skew-hermitian matrices with 0s on the diagonal. Then

$$(k\sigma) \oplus L \ \xrightarrow{\cong} \mathrm{res}^{\tilde{U}(k)}_{\tilde{U}(1,\dots,1)}(\mathrm{ad}(k)) \ , \quad (y_1, \dots, y_k, X) \ \longmapsto \ \Delta(iy_1, \dots iy_k) \ + \ X$$

is an isomorphism of orthogonal $\tilde{U}(1, \dots, 1)$-representations. Then a_L and $a_{L\otimes\sigma}$ are the associated pre-Euler classes (2.6).

The ultra-commutative multiplication on $\mathbf{U}$ induces an ultra-commutative C-ring spectrum structure on the unreduced suspension spectrum $\Sigma^\infty_+\mathbf{U}$. This, in turn, induces a product structure on the equivariant homotopy groups of $\Sigma^\infty_+\mathbf{U}$. Given augmented Lie groups $\alpha\colon G \longrightarrow C$ and $\beta\colon K \longrightarrow C$, and orthogonal representations V and W of G and K, respectively, we write

$$\times \ : \ \pi^G_V(\Sigma^\infty_+\mathbf{U}) \times \pi^K_W(\Sigma^\infty_+\mathbf{U}) \ \longrightarrow \ \pi^{G\times_C K}_{V\oplus W}(\Sigma^\infty_+\mathbf{U})$$

for the biadditive pairing defined by

$$x \times y \ = \ p_1^*(x) \cdot p_2^*(y) \ ,$$

where $p_1\colon G \times_C K \longrightarrow G$ and $p_2\colon G \times_C K \longrightarrow K$ are the projections. The class τ_k in $\pi^{\tilde{U}(k)}_{\mathrm{ad}(k)}(\Sigma^\infty_+\mathbf{U})$ was defined in (1.9).

Theorem 2.8. *The relation*

$$a_{L\otimes\sigma} \cdot a_L \cdot (\mathrm{res}^{\tilde{U}(k)}_{\tilde{T}^k}(\tau_k)) \ = \ a_{L\otimes\sigma} \cdot (\tau_1 \times \cdots \times \tau_1)$$

holds in the group $\pi^{\tilde{T}^k}_{k\sigma - L\otimes\sigma}(\Sigma^\infty_+ \mathbf{U})$.

Proof. The map Δ takes T^k isomorphically onto $U(1,\dots,1)$, it identifies $\mathrm{ad}(1)^k$ with $\mathrm{ad}(k)^{U(1,\dots,1)}$, and it identifies $\mathrm{sa}(1)^k$ with $\mathrm{sa}(k)^{U(1,\dots,1)}$. We denote all these restrictions by Δ, too. The following diagram commutes:

$$\begin{tikzcd}
(\mathrm{sa}(1) \times U(1))^k \arrow[r, "(\phi_1)^k"] \arrow[d, "\mathrm{shuffle}"', "\cong"] & (\mathrm{sa}(1) \oplus \mathrm{ad}(1))^k \arrow[d, "\mathrm{shuffle}"] \\
\mathrm{sa}(1)^k \times U(1)^k \arrow[d, "\Delta\times\Delta"'] & \mathrm{sa}(1)^k \oplus \mathrm{ad}(1)^k \arrow[d, "\Delta\times\Delta"] \\
\mathrm{sa}(k) \times U(k)^{\mathrm{ad}} \arrow[r, "\phi_k"'] & \mathrm{sa}(k) \oplus \mathrm{ad}(k)
\end{tikzcd}$$

So the diagram of collapse maps associated to the horizontal open embeddings commutes, too. Hence also the left part of the following diagram of based continuous $\tilde{U}(1,\dots,1)$-maps commutes:

$$\begin{tikzcd}
S^{\nu_k} \wedge (S^{\mathrm{sa}(1)\oplus\mathrm{ad}(1)})^{\wedge k} \arrow[r, "S^{\nu_k}\wedge t_1^{\wedge k}"] \arrow[d, "\mathrm{shuffle}"', "\cong"] & S^{\nu_k} \wedge (S^{\mathrm{sa}(1)} \wedge U(1)_+)^{\wedge k} \arrow[r, "S^{\nu_k}\wedge(S^{\mathrm{sa}(1)}\wedge(\zeta^1_*)_+)^{\wedge k}"] \arrow[d, "\cong"', "\mathrm{shuffle}"] & S^{\nu_k} \wedge (S^{\mathrm{sa}(1)} \wedge \mathbf{U}(u(\nu_1))_+)^{\wedge k} \arrow[d, "\mathrm{shuffle}"] \\
S^{\nu_k \oplus \mathrm{sa}(1)^k \oplus \mathrm{ad}(1)^k} \arrow[d, "S^{\nu_k}\wedge\Delta\wedge\Delta"'] & S^{\nu_k} \wedge S^{\mathrm{sa}(1)^k} \wedge U(1)^k_+ \arrow[r, "S^{\nu_k}\wedge S^{\mathrm{sa}(1)^k}\wedge(\zeta^1_*)^k_+"] \arrow[d, "S^{\nu_k}\wedge\Delta\wedge\Delta_+"] & S^{\nu_k} \wedge S^{\mathrm{sa}(1)^k} \wedge (\mathbf{U}(u(\nu_1))^k)_+ \arrow[d, "S^{\nu_k}\wedge\Delta\wedge(\mu^{(k)})_+"] \\
S^{\nu_k \oplus \mathrm{sa}(k) \oplus \mathrm{ad}(k)} \arrow[r, "S^{\nu_k}\wedge t_k"] & S^{\nu_k \oplus \mathrm{sa}(k)} \wedge U(k)^{\mathrm{ad}}_+ \arrow[r, "S^{\nu_k\oplus\mathrm{sa}(k)}\wedge(\zeta^k_*)_+"'] \arrow[d, "S^{\nu_k\oplus\mathrm{sa}(k)}\wedge(\mathbf{U}(i_1)\circ\zeta^k_*)_+"'] & S^{\nu_k \oplus \mathrm{sa}(k)} \wedge \mathbf{U}(u(\nu_k))_+ \arrow[d, "S^{\nu_k\oplus\mathrm{sa}(k)}\wedge\mathbf{U}(i_1)_+"] \\
& (\Sigma^\infty_+ \mathbf{U})(u(\nu_k) \oplus \mathrm{sa}(k)) \arrow[r, equal] & S^{\nu_k \oplus \mathrm{sa}(k)} \wedge \mathbf{U}(u(\nu_k) \oplus \mathrm{sa}(k))_+
\end{tikzcd}$$

The iterated multiplication morphism

$$\mu^{(k)} \ : \ \mathbf{U}(u(\nu_1)) \times \cdots \times \mathbf{U}(u(\nu_1)) \ \longrightarrow \ \mathbf{U}(u(\nu_1) \oplus \cdots \oplus u(\nu_1)) = \mathbf{U}(u(\nu_k))$$

of $\mathbf{U}$ is given by orthogonal direct sum. So it participates in a commutative diagram:

$$\begin{tikzcd}
U(1) \times \cdots \times U(1) \arrow[r, "\zeta^1_*\times\cdots\times\zeta^1_*"] \arrow[d, "\Delta"'] & \mathbf{U}(u(\nu_1)) \times \cdots \times \mathbf{U}(u(\nu_1)) \arrow[d, "\mu^{(k)}"] \\
U(k)^{\mathrm{ad}} \arrow[r, "\zeta^k_*"'] & \mathbf{U}(u(\nu_k))
\end{tikzcd}$$

This implies the commutativity of the middle right part of the above diagram.

Now we can wrap up. Multiplication by $i \in \mathbb{C}$ is an isomorphism of orthogonal $\tilde{U}(k)$-representations between $\mathrm{ad}(k)$ and $\mathrm{sa}(k) \otimes \sigma$, see Remark 1.5. So the map $\Delta\colon S^k = S^{\mathrm{sa}(1)^k} \longrightarrow S^{\mathrm{sa}(k)}$ represents the pre-Euler class $a_{L\otimes\sigma}$. The clockwise composite in the above commutative diagram thus represents the class $a_{L\otimes\sigma} \cdot (\tau_1 \times \cdots \times \tau_1)$. And the counter clockwise composite represents the class $a_{L\otimes\sigma} \cdot a_L \cdot \tau_k$, so the diagram witnesses the desired relation. $\square$

For $k \geq 0$, the linear map $-\mathrm{Id}_{\nu_k}\colon \nu_k \longrightarrow \nu_k$ is a $\tilde{U}(k)$-fixed point of $U(k)^{\mathrm{ad}}$. Thus

$$\zeta_*^k(-\mathrm{Id}_{\nu_k}) \ \in \ \mathbf{U}(u(\nu_k))$$

is again a $\tilde{U}(k)$-fixed point, where $\zeta_*^k\colon U(k)^{\mathrm{ad}} \longrightarrow \mathbf{U}(u(\nu_k))$ was defined in (1.8). We write

$$v_k \ \in \pi_0^{\tilde{U}(k)}(\Sigma_+^\infty \mathbf{U}) \tag{2.9}$$

for the class represented by the $\tilde{U}(k)$-equivariant map

$$S^{\nu_k} \ \xrightarrow{-\wedge\zeta_*^k(-\mathrm{Id}_{\nu_k})} \ S^{\nu_k} \wedge \mathbf{U}(u(\nu_k))_+ \ = \ (\Sigma_+^\infty \mathbf{U})(u(\nu_k)) \ .$$

We recall that $a_{\mathrm{ad}(k)} \in \pi_{-\,\mathrm{ad}(k)}^{\tilde{U}(k)}(\mathbb{S})$ denotes the pre-Euler class (2.6) of the adjoint representation. For $0 \leq j \leq k$, we write $U(k-j,j)$ for the block subgroups of $U(k)$, consisting of the matrices of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ for $(A,B) \in U(k-j) \times U(j)$. And we write $\tilde{U}(k-j,j) = U(k-j,j) \rtimes C$ for the semidirect product with C acting by coordinatewise complex conjugation.

Theorem 2.10. *For every $k \geq 1$, the relation*

$$v_k \ = \ \sum_{0\leq j\leq k} \mathrm{tr}_{\tilde{U}(k-j,j)}^{\tilde{U}(k)}(1 \times (a_{\mathrm{ad}(j)} \cdot \tau_j))$$

holds in $\pi_0^{\tilde{U}(k)}(\Sigma_+^\infty \mathbf{U})$.

Proof. The map ϕ_k defined in (1.3) is an open embedding with image $GL_k(\mathbb{C})$. Every matrix Y in $\mathrm{sa}(k) \cap GL_k(\mathbb{C})$ is unitarily diagonalizable with non-zero real eigenvalues. If i is the number of positive eigenvalues of Y, counted with multiplicity, it is thus of the form

$$Y \ = \ B \cdot \begin{pmatrix} \exp(-Z) & 0 \\ 0 & -\exp(-Z') \end{pmatrix} \cdot B^{-1} \tag{2.11}$$

for some $B \in U(k)$ and $(Z,Z') \in \mathrm{sa}(i) \times \mathrm{sa}(k-i)$. Moreover, this representation is unique up to changing (B,Z,Z') to $(B\cdot(A\oplus A')^{-1}, {}^A Z, {}^{A'} Z')$ for some $(A,A') \in U(i) \times U(k-i)$. This shows that the map

$$\omega_i \ : \ \tilde{U}(k) \times_{\tilde{U}(i,k-i)} (\mathrm{sa}(i) \times \mathrm{sa}(k-i)) \ \longrightarrow \ \mathrm{sa}(k) \cap GL_k(\mathbb{C})$$

sending the equivalence class $[B; Z, Z']$ to (2.11) is a homeomorphism onto the open and closed subspace of matrices with exactly i positive eigenvalues. Altogether, this shows that the following commutative square of $\tilde{U}(k)$-equivariant maps is a pullback:

$$\begin{tikzcd}
\coprod_{0\leq i\leq k} \tilde{U}(k) \times_{\tilde{U}(i,k-i)} \mathrm{sa}(i) \times \mathrm{sa}(k-i) \arrow[rr, "\coprod \omega_i"] \arrow[d] & & \mathrm{sa}(k) \arrow[d, "{(-,0)}"] \\
\mathrm{sa}(k) \times U(k)^{\mathrm{ad}} \arrow[r, "\cong", "\phi_k"'] & GL_k(\mathbb{C}) \arrow[r] & \mathrm{sa}(k) \oplus \mathrm{ad}(k)
\end{tikzcd}$$

The left vertical map is given on the ith summand by the $\tilde{U}(k)$-equivariant extension of the map

$$\mathrm{sa}(i) \times \mathrm{sa}(k-i) \ \longrightarrow \ \mathrm{sa}(k) \times U(k)^{\mathrm{ad}} \ , \quad (Z,Z') \longmapsto \left(\left(\begin{smallmatrix} Z & 0 \\ 0 & Z' \end{smallmatrix}\right), \left(\begin{smallmatrix} E_i & 0 \\ 0 & -E_{k-i} \end{smallmatrix}\right) \right) \ .$$

The pullback yields a commutative diagram for the collapse maps associated to the horizontal open embeddings; after smashing with S^{ν_k}, this becomes the commutative left part of the following diagram:

$$\begin{tikzcd}
S^{\nu_k\oplus\mathrm{sa}(k)} \arrow[r, "S^{\nu_k}\wedge\sum\omega_i^\natural"] \arrow[d, "-\wedge 0"'] & S^{\nu_k}\wedge\left(\bigvee_{0\le i\le k}\tilde{U}(k)\ltimes_{\tilde{U}(i,k-i)} S^{\mathrm{sa}(i)\oplus\mathrm{sa}(k-i)}\right) \arrow[d] \arrow[dr, bend left] & \\
S^{\nu_k\oplus\mathrm{sa}(k)\oplus\mathrm{ad}(k)} \arrow[r, "S^{\nu_k}\wedge t_k"'] & S^{\nu_k\oplus\mathrm{sa}(k)}\wedge U(k)^{\mathrm{ad}}_+ \arrow[r, "S^{\nu_k\oplus\mathrm{sa}(k)}\wedge(\mathbf{U}(i_1)\circ\zeta^k_*)_+"'] & S^{\nu_k\oplus\mathrm{sa}(k)}\wedge\mathbf{U}(u(\nu_k)\oplus\mathrm{sa}(k))_+
\end{tikzcd}$$

The counter clockwise composite represents by the class $a_{\mathrm{ad}(k)}\cdot\tau_k$. So the diagram witnesses the relation

$$a_{\mathrm{ad}(k)}\cdot\tau_k \;=\; \sum_{0\le i\le k} f_i\ ,$$

where f_i is represented by the suspension by S^{ν_k} of the composite

$$S^{\mathrm{sa}(k)} \xrightarrow{\ \omega_i^\natural\ } \tilde{U}(k)\ltimes_{\tilde{U}(i,k-i)} S^{\mathrm{sa}(i)\oplus\mathrm{sa}(k-i)} \xrightarrow{\ \xi_i^\flat\ } S^{\mathrm{sa}(k)}\wedge\mathbf{U}(u(\nu_k)\oplus\mathrm{sa}(k))_+\ .$$

The second map $\xi_i^\flat$ is the $\tilde{U}(k)$-equivariant extension of the map

$$\begin{aligned}\xi_i\ :\ S^{\mathrm{sa}(i)\oplus\mathrm{sa}(k-i)} &\longrightarrow S^{\mathrm{sa}(k)}\wedge\mathbf{U}(u(\nu_k)\oplus\mathrm{sa}(k))_+\\ (Z,Z') &\longmapsto \begin{pmatrix}\exp(-Z) & 0\\ 0 & -\exp(-Z')\end{pmatrix}\wedge\zeta_*^{k-i}(-\mathrm{Id}_{\nu_{k-i}})\ .\end{aligned}$$

The collapse map $\omega_i^\natural$ associated to ω_i is closely related to, but different from, the collapse map that features in the definition of the transfer $\mathrm{tr}_{\tilde{U}(i,k-i)}^{\tilde{U}(k)}$. The latter collapse map, discussed for example in [24, (3.2.10)], is based on a different open embedding, namely the $\tilde{U}(k)$-equivariant extension of the map

$$\mathrm{sa}(i)\times\mathrm{sa}(k-i)\ \longrightarrow\ \mathrm{sa}(k)\ ,\quad (Z,Z')\ \longmapsto\ \begin{pmatrix}E_i+Z & 0\\ 0 & -E_{k-i}+Z'\end{pmatrix}$$

restricted to a small open disc around $(0,0)$. The differential of this second embedding at $(0,0)$ is the block embedding $\mathrm{sa}(i)\times\mathrm{sa}(k-i)\longrightarrow\mathrm{sa}(k)$, while the differential of the embedding ω_i at $(0,0)$ is the map

$$\mathrm{sa}(i)\times\mathrm{sa}(k-i)\ \longrightarrow\ \mathrm{sa}(k)\ ,\quad (Z,Z)\ \longmapsto \begin{pmatrix}-Z & 0\\ 0 & Z'\end{pmatrix}\ .$$

Since the differentials of these embeddings differ by the sign in the first block, the composite f_i differs from the transfer by multiplication by the equivariant homotopy class in $\pi_0^{\tilde{U}(i,k-i)}(\mathbb{S})$ of the involution

$$S^{\mathrm{sa}(i)\times\mathrm{sa}(k-i)}\ \longrightarrow\ S^{\mathrm{sa}(i)\times\mathrm{sa}(k-i)}\ ,\quad (Z,Z')\ \longmapsto\ (-Z,Z')\ .$$

We write $\epsilon_i\in\pi_0^{\tilde{U}(i)}(\mathbb{S})$ for the unit represented by the map $S^{\mathrm{sa}(i)}\longrightarrow S^{\mathrm{sa}(i)}$, $x\mapsto -x$; then the latter equivariant homotopy class is precisely $\epsilon_i\times 1$. This shows that

$$f_i\ =\ \mathrm{tr}_{\tilde{U}(i,k-i)}^{\tilde{U}(k)}(\epsilon_i\times v_{k-i})\ .$$

Altogether we have thus shown the relation

$$a_{\mathrm{ad}(k)}\cdot\tau_k\ =\ \sum_{0\le i\le k}\mathrm{tr}_{\tilde{U}(i,k-i)}^{\tilde{U}(k)}(\epsilon_i\times v_{k-i}) \tag{2.12}$$

in $\pi_0^{\tilde{U}(k)}(\Sigma^\infty_+\mathbf{U})$.

Now we consider the morphism of orthogonal C-spectra $\varrho\colon\Sigma^\infty_+\mathbf{U}\longrightarrow\mathbb{S}$ induced by the based map $\mathbf{U}_+\longrightarrow S^0$ that sends $\mathbf{U}$ to the non-basepoint. It satisfies $\varrho_*(v_{k-i})=1$ in $\pi_0^{\tilde{U}(i,k-i)}(\mathbb{S})$. So

$$\varrho_*(\epsilon_i\times v_{k-i})\ =\ \epsilon_i\times\varrho_*(v_{k-i})\ =\ \epsilon_i\times 1\ .$$

Because $\varrho_*(\tau_k)=0$ by Proposition 2.1, applying ϱ_* to (2.12) yields

$$\sum_{0\le i\le k}\mathrm{tr}_{\tilde U(i,k-i)}^{\tilde U(k)}(\epsilon_i\times 1)\ =\ \sum_{0\le i\le k}\mathrm{tr}_{\tilde U(i,k-i)}^{\tilde U(k)}\left(\varrho_*\left(\epsilon_i\times v_{k-i}\right)\right)\ =\ \varrho_*(a_{\mathrm{ad}(k)}\cdot\tau_k)\ =\ 0 \tag{2.13}$$

whenever $k\ge 1$. Now we deduce

$$\begin{aligned}\sum_{0\le j\le k}\mathrm{tr}_{\tilde U(k-j,j)}^{\tilde U(k)}(1\times(a_{\mathrm{ad}(j)}\cdot\tau_j))\ {}_{(2.12)}\!=&\ \sum_{0\le i\le j\le k}\mathrm{tr}_{\tilde U(k-j,j)}^{\tilde U(k)}(1\times\mathrm{tr}_{\tilde U(i,j-i)}^{\tilde U(j)}(\epsilon_i\times v_{j-i}))\\ =&\ \sum_{0\le i\le j\le k}\mathrm{tr}_{\tilde U(i,k-j,j-i)}^{\tilde U(k)}(\epsilon_i\times 1\times v_{j-i})\\ =&\ \sum_{0\le d\le k}\ \sum_{0\le i\le k-d}\mathrm{tr}_{\tilde U(k-d,d)}^{\tilde U(k)}(\mathrm{tr}_{\tilde U(i,k-d-i)}^{\tilde U(k-d)}(\epsilon_i\times 1)\times v_d)\\ {}_{(2.13)}\!=&\ v_k\end{aligned}$$

The third equation is the variable substitution $d=j-i$. □

Construction 2.14. We write

$$\mathfrak{c}\ :\ S^\sigma\ \xrightarrow{\cong}\ U(1)\ ,\quad \mathfrak{c}(x)=(x+i)(x-i)^{-1} \tag{2.15}$$

for the Cayley transform; it is C-equivariant for the sign action on the source, and complex conjugation on the target. For $k\ge 0$, we define a class

$$u_k\ \in\ \pi_\sigma^{\tilde U(k)}(\Sigma^\infty\mathbf{U}) \tag{2.16}$$

by stabilizing the class in $\pi_\sigma^{\tilde U(k)}(\mathbf{U},1)$ represented by the composite

$$\delta_k\ :\ S^\sigma\ \underset{\cong}{\xrightarrow{\mathfrak{c}}}\ U(1)\ \xrightarrow{\partial}\ U(k)^{\mathrm{ad}}\ \xrightarrow{\zeta_*^k}\ \mathbf{U}(u(\nu_k))\ .$$

The map $\partial\colon U(1)\longrightarrow U(k)$ is the diagonal map sending an element of $U(1)$ to the constant diagonal matrix in $U(k)$, and ζ_*^k was defined in (1.8). And 'stabilizing' means that u_k is represented by the composite

$$S^{\nu_k\oplus\sigma}\ \xrightarrow{S^{\nu_k}\wedge\delta_k}\ S^{\nu_k}\wedge\mathbf{U}(u(\nu_k))\ =\ (\Sigma_+^\infty\mathbf{U})(u(\nu_k))\ .$$

The unstable representative δ_k of u_k satisfies $\delta(0)=\zeta_*^k(-\mathrm{Id}_{\nu_k})$, which is the unstable representative for the class v_k defined in (2.9). So comparison of the definitions shows the relation

$$a_\sigma\cdot u_k\ =\ q_*(v_k) \tag{2.17}$$

in the group $\pi_0^{\tilde U(k)}(\Sigma^\infty\mathbf{U})$, where a_σ is the pre-Euler class (2.6) of the sign representation.

It will be convenient later to have a different representative for the class u_1. We have $\mathrm{sa}(1)=\mathbb{R}$ and $\mathrm{ad}(1)=i\cdot\mathbb{R}$, with trivial and sign action, respectively, by $\tilde T=\tilde U(1)$. So the collapse map $t_1\colon S^{\mathrm{sa}(1)\oplus\mathrm{ad}(1)}\longrightarrow S^{\mathrm{sa}(1)}\wedge U(1)_+$ is a map $t_1\colon S^{1\oplus\sigma}\longrightarrow S^1\wedge U(1)_+$.

Proposition 2.18. *The composite $(S^1\wedge q)\circ t_1\colon S^{1\oplus\sigma}\longrightarrow S^1\wedge U(1)$ is C-equivariantly homotopic to the suspension of the Cayley transform* (2.15). *In particular, $\sigma_1=u_1$ in $\pi_\sigma^{\tilde T}(\Sigma^\infty\mathbf{U})$.*

Proof. The Cayley transform is a homeomorphism, so we may show that the composite

$$S^{1\oplus\sigma}\ \xrightarrow{t_1}\ S^1\wedge U(1)_+\ \xrightarrow{S^1\wedge q}\ S^1\wedge U(1)\ \underset{\cong}{\xrightarrow{S^1\wedge\mathfrak{c}^{-1}}}\ S^{1\oplus\sigma} \tag{2.19}$$

is C-equivariantly homotopic to the identity. This is the case if and only if the underlying non-equivariant morphism and the restriction to C-fixed points are homotopic to the respective identity maps.

The restrictions to C-fixed points of (2.19) is the composite

$$S^1 \xrightarrow{(t_1)^C} S^1 \wedge \{\pm 1\}_+ \xrightarrow{S^1 \wedge q} S^1 \wedge \{\pm 1\} \xrightarrow[\cong]{S^1 \wedge (\mathfrak{c}^{-1})^C} S^1 \ . \tag{2.20}$$

Expanding all formulas shows that this composite collapses the contractible subset $[0,\infty]$ of S^1 to the basepoint and is given on $\mathbb{R}_{<0}$ by the formula

$$f \ : \ \mathbb{R}_{<0} \longrightarrow \mathbb{R} \ , \quad f(y) \ = \ -\ln(-y) \ .$$

So the composite (2.20) is indeed homotopic to the identity.

The map (2.19) itself collapses the contractible subset $[0,\infty] \times \{0\}$ of S^2 to the basepoint and factors through a homeomorphism

$$S^2/([0,\infty] \times \{0\}) \ \cong \ S^2 \ .$$

So the composite (2.19) is a non-equivariant homotopy equivalence. Expanding all formulas shows that the composite (2.19) is given on $\mathbb{R} \times \mathbb{R}_{>0}$ by

$$F \ : \ \mathbb{R} \times \mathbb{R}_{>0} \longrightarrow \mathbb{R}^2 \ , \quad F(x,y) \ = \ \left(-\ln(\sqrt{x^2+y^2}), \ x/y + \sqrt{(x/y)^2+1} \right) \ .$$

So (2.19) fixes the point $(0,1)$, and it is smooth near $(0,1)$ with differential $D_{(0,1)}F = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Since this differential has determinant 1, the composite (2.19) is indeed non-equivariantly homotopic to the identity. This concludes the proof of the claim that $(S^1 \wedge q) \circ t_1$ is C-equivariantly homotopic to $S^1 \wedge \mathfrak{c}$. The claim that $\sigma_1 = q_*(\tau_1)$ equals u_1 then follows by substituting the respective definitions. □

We write $\overline{\mathrm{ad}}(k)$ for the *reduced adjoint representation* of $\tilde{U}(k)$, i.e., the $\tilde{U}(k)$-subrepresentation consisting of those $X \in \mathrm{ad}(k)$ whose trace is 0. Similarly,

$$\overline{\mathrm{sa}}(k) \ = \ \{ Z \in \mathrm{sa}(k) \colon \mathrm{trc}(Z) = 0 \} \ .$$

The classes $a_{\overline{\mathrm{sa}}(k)}$ and $a_{\overline{\mathrm{ad}}(k)}$ are the associated pre-Euler classes (2.6).

Theorem 2.21. *For every $k \geq 1$, the relation*

$$a_{\overline{\mathrm{sa}}(k)} \cdot a_{\overline{\mathrm{ad}}(k)} \cdot \sigma_k \ = \ a_{\overline{\mathrm{sa}}(k)} \cdot u_k$$

holds in the group $\pi^{\tilde{U}(k)}_{\sigma - \overline{\mathrm{sa}}(k)}(\Sigma^\infty \mathbf{U})$.

Proof. The map $\partial \colon \mathbb{C} \longrightarrow M_k(\mathbb{C})$, $\partial(x) = \Delta(x,\dots,x)$ takes $U(1)$ isomorphically onto the center of $U(k)$, it identifies $\mathrm{ad}(1)$ with $\mathrm{ad}(k)^{U(k)}$, and it identifies $\mathrm{sa}(1)$ with $\mathrm{sa}(k)^{U(k)}$. We denote all these restrictions by ∂, too. The following diagram commutes:

$$\begin{tikzcd}
\mathrm{sa}(1) \times U(1) \arrow[r, "\phi_1"] \arrow[d, "\partial \times \partial"'] & \mathrm{sa}(1) \oplus \mathrm{ad}(1) \arrow[d, "\partial \times \partial"] \\
\mathrm{sa}(k) \times U(k)^{\mathrm{ad}} \arrow[r, "\phi_k"'] & \mathrm{sa}(k) \oplus \mathrm{ad}(k)
\end{tikzcd}$$

So the diagram of collapse maps associated to the horizontal open embeddings commutes, too. Thus the left part of the following diagram of based continuous $\tilde{U}(k)$-maps commutes:

$$\begin{tikzcd}
S^{\nu_k \oplus \mathrm{sa}(1) \oplus \mathrm{ad}(1)} \arrow[r, "S^{\nu_k} \wedge t_1"] \arrow[d, "S^{\nu_k} \wedge \partial \wedge \partial"'] & S^{\nu_k \oplus \mathrm{sa}(1)} \wedge U(1)_+ \arrow[r, "S^{\nu_k \oplus \mathrm{sa}(1)} \wedge (\zeta^k_* \circ \partial \circ q)"] \arrow[d, "S^{\nu_k} \wedge \partial \wedge \partial_+"] & S^{\nu_k \oplus \mathrm{sa}(1)} \wedge \mathbf{U}(u(\nu_k)) \arrow[d, "S^{\nu_k} \wedge \partial \wedge \mathbf{U}(u(\nu_k))"] \\
S^{\nu_k \oplus \mathrm{sa}(k) \oplus \mathrm{ad}(k)} \arrow[r, "S^{\nu_k} \wedge t_k"] & S^{\nu_k \oplus \mathrm{sa}(k)} \wedge U(k)^{\mathrm{ad}}_+ \arrow[r, "S^{\nu_k \oplus \mathrm{sa}(k)} \wedge (\zeta^k_* \circ q)"'] \arrow[d, "S^{\nu_k \oplus \mathrm{sa}(k)} \wedge (\mathbf{U}(i_1) \circ \zeta^k_* \circ q)"'] & S^{\nu_k \oplus \mathrm{sa}(k)} \wedge \mathbf{U}(u(\nu_k)) \arrow[d, "S^{\nu_k \oplus \mathrm{sa}(k)} \wedge \mathbf{U}(i_1)_+"] \\
& (\Sigma^\infty \mathbf{U})(u(\nu_k) \oplus \mathrm{sa}(k)) \arrow[r, equal] & S^{\nu_k \oplus \mathrm{sa}(k)} \wedge \mathbf{U}(u(\nu_k) \oplus \mathrm{sa}(k))
\end{tikzcd}$$

The map $\partial\colon S^{\mathrm{sa}(1)} \longrightarrow S^{\mathrm{sa}(k)}$ represents the pre-Euler class $a_{\overline{\mathrm{sa}}(k)}$. Proposition 2.18 shows that the composite $(S^1 \wedge q) \circ t_1 \colon S^{1 \oplus \sigma} \longrightarrow S^1 \wedge U(1)$ is C-equivariantly homotopic to the suspension of the Cayley transform (2.15). So the clockwise composite in the above commutative diagram represents the class $a_{\overline{\mathrm{sa}}(k)} \cdot u_k$. And the counter clockwise composite represents the class $a_{\overline{\mathrm{sa}}(k)} \cdot a_{\overline{\mathrm{ad}}(k)} \cdot \sigma_k$, so the diagram witnesses the desired relation. □

Proposition 2.22. *For all $k, l \geq 1$, the relation*

$$\mathrm{res}^{\tilde{U}(k+l)}_{U(k,l)}(u_{k+l}) \ = \ p_1^*(\mathrm{res}^{\tilde{U}(k)}_{U(k)}(u_k)) \ + \ p_2^*(\mathrm{res}^{\tilde{U}(l)}_{U(l)}(u_l))$$

holds in the group $\pi_1^{U(k,l)}(\Sigma^\infty \mathbf{U})$, where $p_1 \colon U(k,l) \longrightarrow U(k)$ and $p_2 \colon U(k,l) \longrightarrow U(l)$ are the projections to the blocks.

Proof. We first show an unstable precursor of the desired relation. We define $w_k \in \pi_1^{U(k)}(\mathbf{U}, 1)$ as the class of the continuous based map

$$S^1 \ \xrightarrow{\ \mathfrak{c}\ } \ U(1) \ \xrightarrow{\ \Delta\ } \ U(k) \ \xrightarrow{\ \zeta^k_*\ } \ \mathbf{U}(u(\nu_k))$$

that lands in the $U(k)$-fixed points of $\mathbf{U}(u(\nu_k))$. Then the map representing w_{k+l} is the product of the maps representing w_k and w_l, in the sense that it factors as the composite

$$S^1 \ \xrightarrow{\ \mathfrak{c}\ } \ U(1) \ \xrightarrow{(\zeta^k_* \circ \Delta, \zeta^l_* \circ \Delta)} \ \mathbf{U}(u(\nu_k)) \times \mathbf{U}(u(\nu_l)) \ \xrightarrow{\ \mu_{\nu_k, \nu_l}\ } \ \mathbf{U}(u(\nu_{k+l})) \ .$$

So the relation

$$\mathrm{res}^{\tilde{U}(k+l)}_{U(k,l)}(w_{k+l}) \ = \ p_1^*(\mathrm{res}^{\tilde{U}(k)}_{U(k)}(w_k)) \cdot p_2^*(\mathrm{res}^{\tilde{U}(l)}_{U(l)}(w_l))$$

holds in the group $\pi_1^{U(k,l)}(\mathbf{U}, 1)$, where the multiplication is formed under the group structure arising from the ultra-commutative multiplication of $\mathbf{U}$. The Eckmann–Hilton argument shows that this group structure on $\pi_1^{U(k,l)}(\mathbf{U}, 1)$ from the multiplication of $\mathbf{U}$ agrees with that as a fundamental group, i.e., by concatenation of loops. The stabilization map

$$\sigma^{U(k,l)} \ : \ \pi_1^{U(k,l)}(\mathbf{U}, 1) \ \longrightarrow \ \pi_1^{U(k,l)}(\Sigma^\infty \mathbf{U})$$

is a homomorphism for the concatenation of loops on the source and on the target, where it is the group structure coming from stability, i.e., the usual addition on equivariant stable homotopy groups. This proves the claim. □

It is important in Proposition 2.22 that we work in the ordinary, non-augmented unitary group $U(k,l)$, as opposed to the augmented unitary group $\tilde{U}(k,l)$, because the classes

$$\mathrm{res}^{\tilde{U}(k+l)}_{\tilde{U}(k,l)}(u_{k+l}) \qquad \text{and} \qquad p_1^*(u_k) \ + \ p_2^*(u_l)$$

are *different* in the group $\pi_\sigma^{\tilde{U}(k,l)}(\Sigma^\infty \mathbf{U})$. This feature is another manifestation of the non-additivity of certain pieces of structure for surjectively augmented Lie groups, such as the non-additivity in Example 4.26 or Remark 5.7.

Theorem 2.23. *For every $k \geq 1$, the relation*

$$u_k \ = \ \sum_{1\leq j\leq k} \mathrm{tr}^{\tilde{U}(k)}_{\tilde{U}(k-j,j)}(p_2^*(a_{\overline{\mathrm{ad}}(j)} \cdot \sigma_j))$$

holds in $\pi_\sigma^{\tilde{U}(k)}(\Sigma^\infty\mathbf{U})$, where $p_2\colon \tilde{U}(k) \longrightarrow \tilde{U}(k-j,j)$ is the projection to the second block.

Proof. For $k = 1$, we have $\overline{\mathrm{ad}}(1) = 0$, so the claim reduces to $u_1 = \sigma_1$, which holds by Proposition 2.18. For the rest of the proof we assume that $k \geq 2$.

In the next step we show that the desired relation holds after restriction to the subgroup $U(i, k-i)$ for every $1 \leq i \leq k-1$. All groups involved in this part of the argument augment trivially to C, so they act trivially on the sign representation σ. We abbreviate

$$\bar{u}_k \ = \ \mathrm{res}^{\tilde{U}(k)}_{U(k)}(u_k) \qquad \text{and} \qquad s_k \ = \ \mathrm{res}^{\tilde{U}(k)}_{U(k)}(a_{\overline{\mathrm{ad}}(k)} \cdot \sigma_k) \ ,$$

both classes lying in $\pi_1^{U(k)}(\Sigma^\infty\mathbf{U})$. For $1 \leq d \leq j-1$, we have $\overline{\mathrm{ad}}(j)^{U(d,j-d)} \neq 0$, thus

$$\mathrm{res}^{U(j)}_{U(d,j-d)}(s_j) \ = \ \mathrm{res}^{\tilde{U}(j)}_{U(d,j-d)}(a_{\overline{\mathrm{ad}}(j)}) \cdot \mathrm{res}^{\tilde{U}(j)}_{U(d,j-d)}(\sigma_j) \ = \ 0 \ .$$

The double coset formula for $\mathrm{res}^{U(k)}_{U(i,k-i)} \circ \mathrm{tr}^{U(k)}_{U(j,k-j)}$ established in [26, Proposition 1.3] yields

$$\begin{aligned}
&\sum_{1\leq j\leq k} \mathrm{res}^{U(k)}_{U(i,k-i)}(\mathrm{tr}^{U(k)}_{U(k-j,j)}(p_2^*(s_j)))\\
&\quad= \ \sum_{1\leq j\leq k}\ \sum_{0,i+j-k\leq d\leq j,i} \mathrm{tr}^{U(i,k-i)}_{U(d,i-d,j-d,k-j-i+d)}(\gamma_d^*(\mathrm{res}^{U(k-j,j)}_{U(i-d,k-j-i+d,d,j-d)}(p_2^*(s_j))))\\
&\quad= \ \sum_{1\leq j\leq i} \mathrm{tr}^{U(i,k-i)}_{U(i-j,j,k-i)}(\gamma_j^*(\mathrm{res}^{U(k-j,j)}_{U(i-j,k-i,j)}(p_2^*(s_j))))\\
&\qquad + \sum_{1\leq j\leq k-i} \mathrm{tr}^{U(i,k-i)}_{U(i,j,k-j-i)}(\gamma_0^*(\mathrm{res}^{U(k-j,j)}_{U(i,k-j-i,j)}(p_2^*(s_j))))\\
&\quad= \ \sum_{1\leq j\leq i} p_1^*(\mathrm{tr}^{U(i)}_{U(i-j,j)}(p_2^*(s_j))) \ + \sum_{1\leq j\leq k-i} p_2^*(\mathrm{tr}^{U(k-i)}_{U(k-i-j,j)}(p_2^*(s_j)))\\
&\quad= \ p_1^*(\bar{u}_i) \ + \ p_2^*(\bar{u}_{k-i}) \ = \ \mathrm{res}^{\tilde{U}(k)}_{U(i,k-i)}(u_k) \ .
\end{aligned}$$

The second equation uses that

$$\mathrm{res}^{U(k-j,j)}_{U(i-d,k-j-i+d,d,j-d)}(p_2^*(s_j)) \ = \ p_{3,4}^*(\mathrm{res}^{U(j)}_{U(d,j-d)}(s_j)) \ = \ 0$$

unless $d = j$ or $d = 0$, where $p_{3,4}\colon U(i-d, k-j-i+d, d, j-d) \longrightarrow U(d, j-d)$ is the projection to the last to blocks. The final equation is Proposition 2.22.

In the next step we show that the desired relation holds after restriction to the group $U(k)$. The space $U(k)^{\mathrm{ad}}$ is $U(k)$-equivariantly connected, in the sense that for every subgroup G of $U(k)$, the fixed point space $U(k)^G$, i.e., the centralizer of G in $U(k)$, is path connected. So by [24, Theorem 3.3.15 (i)], elements in $\pi_1^{U(k)}(\Sigma^\infty\mathbf{U})$ are detected by geometric fixed points for all closed subgroups of $U(k)$. In other words:

we may show the desired relation after applying Φ^G for all $G \leq U(k)$. If the subgroup G is subconjugate to $U(i, k-i)$ for some $1 \leq i \leq k-1$, then the relation holds after restriction to G by the first step, and hence also after taking G-geometric fixed points. If the subgroup G is not subconjugate to $U(i, k-i)$ for any $1 \leq i \leq k-1$, then the G-action on ν_k is irreducible, so the centralizer of G in $U(k)$ equals the center of $U(k)$. Then $\overline{\mathrm{sa}}(k)^G = \overline{\mathrm{ad}}(k)^G = 0$, and hence $\Phi^G(a_{\overline{\mathrm{sa}}(k)}) = \Phi^G(a_{\overline{\mathrm{ad}}(k)}) = 1$. Since G is not subconjugate to $U(j, k-j)$, we have $\Phi^G \circ \mathrm{tr}_{U(j,k-j)}^{U(k)} = 0$ for $1 \leq j \leq k-1$, see [24, Theorem 3.4.2 (ii)]. Thus

$$\begin{aligned} \Phi^G(u_k) \ &= \ \Phi^G(a_{\overline{\mathrm{ad}}(k)} \cdot u_k) \ = \ \Phi^G(a_{\overline{\mathrm{sa}}(k)} \cdot a_{\overline{\mathrm{ad}}(k)} \cdot \sigma_k) \\ &= \sum_{1 \leq j \leq k} \Phi^G(\mathrm{tr}_{\tilde{U}(k-j,j)}^{\tilde{U}(k)}(p_2^*(a_{\overline{\mathrm{ad}}(j)} \cdot \sigma_j))) \ . \end{aligned}$$

The second equation is Theorem 2.21. This concludes the proof that the relation holds after restriction to the group $U(k)$.

Now we complete the argument. The decomposition $\mathrm{ad}(k) \cong \overline{\mathrm{ad}}(k) \oplus \sigma$ as orthogonal $\tilde{U}(k)$-representations induces an isomorphism of C-global spaces

$$(\mathbf{Gr}_m)^{\mathrm{ad}(m)} \ \cong \ (\mathbf{Gr}_m)^{\overline{\mathrm{ad}}(m) \oplus \sigma} \ \cong \ (\mathbf{Gr}_m)^{\overline{\mathrm{ad}}(m)} \wedge S^\sigma \ .$$

So the global stable splitting (1.1) provides an isomorphism

$$\pi_\sigma^{\tilde{U}(k)}(\Sigma^\infty \mathbf{U}) \ \cong \ \bigoplus_{m \geq 1} \pi_\sigma^{\tilde{U}(k)}(\Sigma^\infty (\mathbf{Gr}_m)^{\mathrm{ad}(m)}) \ \cong \ \bigoplus_{m \geq 1} \pi_0^{\tilde{U}(k)}(\Sigma^\infty (\mathbf{Gr}_m)^{\overline{\mathrm{ad}}(m)}) \ .$$

So classes in $\pi_\sigma^{\tilde{U}(k)}(\Sigma^\infty \mathbf{U})$ are detected by geometric fixed points for all closed subgroups of $\tilde{U}(k)$. A caveat is that if $G \leq U(k)$, then $\sigma^G = \mathbb{R}$, and the geometric fixed points live in $\Phi_1^G(\Sigma^\infty \mathbf{U})$; and if G maps onto C, then $\sigma^G = 0$, and the geometric fixed points live in $\Phi_0^G(\Sigma^\infty \mathbf{U})$.

If the subgroup G is contained in $U(k)$, then the relations holds after restriction to G by the previous step, and hence also after taking G-geometric fixed points. If G is not contained in $U(k)$, then $\sigma^G = 0$, and thus $\Phi^G(a_\sigma) = 1$. Thus

$$\begin{aligned} \Phi^G(u_k) \ &= \ \Phi^G(a_\sigma \cdot u_k) \\ \text{(2.17)} \ &= \ \Phi^G(q_*(v_k)) \ = \ \sum_{1 \leq j \leq k} \Phi^G(\mathrm{tr}_{\tilde{U}(k-j,j)}^{\tilde{U}(k)}(a_\sigma \cdot p_2^*(a_{\overline{\mathrm{ad}}(j)} \cdot \sigma_j))) \\ &= \ \sum_{1 \leq j \leq k} \Phi^G(\mathrm{tr}_{\tilde{U}(k-j,j)}^{\tilde{U}(k)}(p_2^*(a_{\overline{\mathrm{ad}}(j)} \cdot \sigma_j))) \ . \end{aligned}$$

The third equation is Theorem 2.10, plus the fact that $q_*(\tau_0) = 0$, and

$$q_*(1 \times (a_{\mathrm{ad}(j)} \cdot \tau_j)) \ = \ q_*(p_2^*(a_{\mathrm{ad}(j)} \cdot \tau_j)) \ = \ p_2^*(a_{\mathrm{ad}(j)} \cdot q_*(\tau_j)) \ = \ a_\sigma \cdot p_2^*(a_{\overline{\mathrm{ad}}(j)} \cdot \sigma_j)$$

for all $j \geq 1$. □

3. The interplay of the global splitting and the eigenspace morphism

In this section we establish a subtle connection between two a priori unrelated features of the ultra-commutative monoid $\mathbf{U}$, namely its C-global stable splitting and its preferred infinite delooping. As we show in Theorem 3.5, the adjoint $\Sigma^\infty \mathbf{U} \longrightarrow \mathrm{sh}^\sigma \mathbf{KR}$ of the preferred infinite delooping $\mathbf{U} \sim \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{KR})$ from Theorem B.57 annihilates all the higher terms of the stable global splitting (1.1). This fact is fundamental for all other results in this paper.

The connective global K-theory spectrum $\mathbf{ku}$ is defined in [24, Construction 3.6.9], generalizing a configuration space model of Segal [30, Section 1] to the global equivariant context. It comes with a multiplicative involution ψ by complex conjugation that enhances it to the connective Real-global K-theory spectrum

$\mathbf{kr} = (\mathbf{ku}, \psi)$. We recall the definition in Construction B.34. The *eigenspace morphism* of based orthogonal C-spaces

$$\mathrm{eig} \ : \ \mathbf{U} \ \longrightarrow \ \Omega^\bullet(\mathrm{sh}^\sigma \, \mathbf{kr})$$

is defined in [24, (6.3.26)]; we recall the construction in (B.44). Here 'sh^σ' denotes the shift of an orthogonal C-spectrum by the sign representation, see [24, Construction 3.1.21]. Shifting by σ is Real-globally equivalent to suspending by S^σ, see [24, Proposition 3.1.25 (ii)]. And $\Omega^\bullet$ is the functor from orthogonal C-spectra to based orthogonal C-spaces that is right adjoint to the reduced suspension spectrum functor, see [24, Construction 4.1.6]; the orthogonal C-space $\Omega^\bullet X$ models the underlying 'Real-global infinite loop space' of a Real-global spectrum X. As the name suggest, the morphism (B.44) assigns to a unitary automorphism the configuration of eigenvalues and eigenspaces; the shift coordinate in $\mathrm{sh}^\sigma \, \mathbf{kr}$ is the place that stores the eigenvalues. We will mostly work with the adjoint

$$\mathrm{eig}^\natural \ : \ \Sigma^\infty \mathbf{U} \ \longrightarrow \ \mathrm{sh}^\sigma \, \mathbf{kr} \tag{3.1}$$

of the eigenspace morphism (B.44).

Corollary 3.3 below shows that for $k \geq 2$ and *after multiplication by certain pre-Euler classes*, the adjoint eigenspace morphism $\mathrm{eig}^\natural$ annihilates the restrictions $\mathrm{res}^{\tilde{U}(k)}_{\tilde{T}^k}(\sigma_k)$ of the classes defined in (1.12) that encode the global stable splitting of $\mathbf{U}$. We will argue later that the relevant pre-Euler classes are not zero divisors in the periodic theory $\mathbf{KR}$, which leads to the proof in Theorem 3.5 that the composite $\Omega^\bullet(\mathrm{sh}^\sigma j) \circ \mathrm{eig}^\natural \colon \Sigma^\infty \mathbf{U} \longrightarrow \mathrm{sh}^\sigma \, \mathbf{KR}$ annihilates the classes σ_k for $k \geq 2$.

We write $T^k = U(1, \dots, 1)$ for the diagonal maximal torus of $U(k)$, and we set $\tilde{T}^k = T^k \rtimes C$, augmented by the projection to C. We let $D \subset M_k(\mathbb{C})$ denote the $\tilde{T}^k$-invariant $\mathbb{C}$-subspace of lower subdiagonal matrices, i.e., those of the form

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ z_{2,1} & 0 & 0 & \dots & 0 \\ z_{3,1} & z_{3,2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ z_{k,1} & z_{k,2} & \dots & z_{k,k-1} & 0 \end{pmatrix} \tag{3.2}$$

Then D inherits the structure of a Real $\tilde{T}^k$-representation from $M_k(\mathbb{C})$. We let $a_D \in \pi^{\tilde{T}^k}_{-D}(\mathbb{S})$ denote the pre-Euler class of the lower subdiagonal representation (3.2), i.e., the equivariant homotopy class represented by the fixed point inclusion $S^0 \longrightarrow S^D$.

Corollary 3.3. *For every $k \geq 2$, the relation*

$$a_D^2 \cdot \mathrm{eig}^\natural_*(\mathrm{res}^{\tilde{U}(k)}_{\tilde{T}^k}(\sigma_k)) \ = \ 0$$

holds in the group $\pi^{\tilde{T}^k}_{k\sigma - D}(\mathrm{sh}^\sigma \, \mathbf{kr})$.

Proof. We recall that

$$\varrho \ : \ \Sigma^\infty_+ \mathbf{U} \ \longrightarrow \ \Sigma^\infty_+ * = \mathbb{S} \qquad \text{and} \qquad q \ : \ \Sigma^\infty_+ \mathbf{U} \ \longrightarrow \ \Sigma^\infty \mathbf{U}$$

denote the morphisms induced on reduced suspension C-spectra by the based maps $\mathbf{U}_+ \longrightarrow S^0$ and $\mathbf{U}_+ \longrightarrow \mathbf{U}$ that, respectively, map $\mathbf{U}$ to the non-basepoint of S^0, and are identity on $\mathbf{U}$. By Proposition 2.1, the class τ_1 in $\pi^{\tilde{T}}_\sigma(\Sigma^\infty_+ \mathbf{U})$ belongs to the augmentation ideal, i.e., $\varrho_*(\tau_1) = 0$. So the class $\tau_1 \times \dots \times \tau_1$ lies in the k-th power of the augmentation ideal, where k is the number of factors. The eigenspace morphism $\mathrm{eig} \colon \mathbf{U} \longrightarrow \Omega^\bullet(\mathrm{sh}^\sigma \, \mathbf{kr})$ is a C-global H-map by Proposition B.46. So Theorem B.47 shows that the map $(\mathrm{eig}^\natural \circ q)_*$ annihilates the square of the augmentation ideal of $\pi^G_\star(\Sigma^\infty_+ \mathbf{U})$. So

$$(\mathrm{eig}^\natural \circ q)_*(\tau_1 \times \dots \times \tau_1) \ = \ 0$$

for $k \geq 2$. Using Theorem 2.8 we obtain

$$\begin{aligned} a_{L\otimes\sigma} \cdot a_L \cdot \mathrm{eig}^\natural_*(\mathrm{res}^{\tilde{U}(k)}_{\tilde{T}^k}(\sigma_k)) \underset{(1.12)}{=} & \ (\mathrm{eig}^\natural \circ q)_*(a_{L\otimes\sigma} \cdot a_L \cdot \mathrm{res}^{\tilde{U}(k)}_{\tilde{T}^k}(\tau_k)) \\ = & \ (\mathrm{eig}^\natural \circ q)_*(a_{L\otimes\sigma} \cdot (\tau_1 \times \cdots \times \tau_1)) \\ = & \ a_{L\otimes\sigma} \cdot ((\mathrm{eig}^\natural \circ q)_*(\tau_1 \times \cdots \times \tau_1)) \ = \ 0 \end{aligned}$$

The map

$$k\sigma \oplus D \ \longrightarrow \ \mathrm{ad}(k) , \quad (y_1, \dots, y_k; A) \ \longmapsto \ \Delta(iy_1, \dots, iy_k) \ + \ A - \bar{A}^t$$

is an $\mathbb{R}$-linear and $\tilde{T}^k$-equivariant isomorphism; it identifies D with the subspace of off-diagonal matrices $L \subset \mathrm{ad}(k)$, as orthogonal $\tilde{T}^k$-representations. Because D is a Real $\tilde{T}^k$-representation, multiplication by $i \in \mathbb{C}$ is an isomorphism of orthogonal $\tilde{T}^k$-representation from D to $D\otimes\sigma$. So D is also isomorphic to $L\otimes\sigma$. Hence we can replace both $a_{L\otimes\sigma}$ and a_L by a_D, and deduce the desired relation $a_D^2 \cdot \mathrm{eig}^\natural_*(\mathrm{res}^{\tilde{U}(k)}_{\tilde{T}^k}(\sigma_k)) = 0$. ☐

For an augmented Lie group $\alpha\colon G \longrightarrow C$ and a compact G-space A, we write $KR_\alpha(A)$ for the Real α-equivariant K-group of A, the Grothendieck group of Real α-equivariant vector bundles over A, see Construction B.6. If A is endowed with a G-fixed basepoint, we write $\widetilde{KR}_\alpha(A)$ for the reduced K-group, the kernel of restriction to the basepoint $KR_\alpha(A) \longrightarrow KR_\alpha(*)$. The multiplicative structure by tensor product of Real-equivariant vector bundles makes the groups $KR_\alpha(A)$ and $\widetilde{KR}_\alpha(A)$ into modules over $KR_\alpha(*) = RR(\alpha)$, the Real representation ring of $\alpha\colon G \longrightarrow C$.

We call an $RR(\tilde{T}^k)$-module *Euler-torsion-free* if the Euler class of every irreducible Real $\tilde{T}^k$-representation with trivial T^k-fixed points acts injectively on the module.

Proposition 3.4. *For every $k \geq 1$ and $l \geq 0$, the $RR(\tilde{T}^k)$-module $\widetilde{KR}_{\tilde{T}^k}(S^{l\sigma})$ is Euler-torsion-free.*

Proof. We start by showing that for every abelian group M, the $RR(\tilde{T}^k)$-module $RR(\tilde{T}^k) \otimes M$ is Euler-torsion-free. The ring $RR(\tilde{T}^k)$ is a Laurent polynomial ring

$$RR(\tilde{T}^k) \ = \ \mathbb{Z}[x_1^\pm, \dots, x_k^\pm] \ ,$$

with $x_i = p_i^*(\nu_1)$ for $p_i\colon \tilde{T}^k \longrightarrow \tilde{T}$ the projection to the i-th factor. Indeed, the forgetful ring homomorphism

$$\mathrm{res}^{\tilde{T}^k}_{T^k} \ : \ RR(\tilde{T}^k) \ \longrightarrow \ RU(T^k) \ = \ \mathbb{Z}[x_1^\pm, \dots, x_k^\pm]$$

is injective by [4, page 13], and the ring $RU(T^k)$ is generated by the underlying unitary representations of the Laurent monomials in the x_i. So this restriction map is an isomorphism.

The irreducible Real $\tilde{T}^k$-representations λ correspond to the monomial units $x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}$ for $i_1, \dots, i_n \in \mathbb{Z}$; and λ has trivial T^k-fixed points precisely when not all i_j equal to 0. Moreover, the Euler class of λ is the element

$$e_\lambda \ = \ 1 - x_1^{i_1} \cdot \ldots \cdot x_n^{i_n} \ .$$

So for $\lambda^{T^k} = 0$, the underlying abelian group of $RR(\tilde{T}^k)/(e_\lambda)$ is free, and hence $\mathrm{Tor}(RR(\tilde{T}^k)/(e_\lambda), M) = 0$. This means that multiplication by e_λ is injective on $RR(\tilde{T}^k) \otimes M$, so $RR(\tilde{T}^k) \otimes M$ is Euler-torsion-free.

Now we turn to the proof of the proposition. All irreducible Real $\tilde{T}^k$-representations are 1-dimensional, and hence of real type, i.e., their only automorphisms are scalars from $\mathbb{R}$. So for every compact space A with trivial $\tilde{T}^k$-action, [4, Proposition 8.1] provides an isomorphism

$$RR(\tilde{T}^k) \otimes KO(A) \ \xrightarrow{\cong} \ KR_{\tilde{T}^k}(A) \ .$$

Applying this to $A = S^n$ and the restriction to its basepoint yields a commutative square

$$\begin{tikzcd} RR(\tilde{T}^k) \otimes KO(S^n) \arrow[r, "\cong"] \arrow[d] & KR_{\tilde{T}^k}(S^n) \arrow[d] \\ RR(\tilde{T}^k) \otimes KO(*) \arrow[r, "\cong"'] & KR_{\tilde{T}^k}(*) \end{tikzcd}$$

in which both horizontal maps are isomorphisms. So the upper map restricts to an isomorphism between the vertical kernels $RR(\tilde{T}^k) \otimes \widetilde{KO}(S^n)$ and $\widetilde{KR}_{\tilde{T}^k}(S^n)$.

Now we choose an $m \geq 0$ such that $l \leq 8m$. The orthogonal $\tilde{T}^k$-representation $1 \oplus \sigma$ underlies a Real $\tilde{T}^k$-representation (with trivial T^k-action). So Bott periodicity [3, Theorem (5.1)] for the Real $\tilde{T}^k$-representation $l \cdot (1 \oplus \sigma)$ and the 8-fold Bott periodicity of Real equivariant K-theory provide isomorphisms of $RR(\tilde{T}^k)$-modules

$$RR(\tilde{T}^k) \otimes \widetilde{KO}(S^{8m-l}) \;\cong\; \widetilde{KR}_{\tilde{T}^k}(S^{8m-l}) \;\cong\; \widetilde{KR}_{\tilde{T}^k}(S^{8m+l\sigma}) \;\cong\; \widetilde{KR}_{\tilde{T}^k}(S^{l\sigma}) \;.$$

We showed above that this left $RR(\tilde{T}^k)$-module is Euler-torsion-free, so this completes the proof. □

The global K-theory spectrum **KU** was introduced by Joachim [15, Definition 4.3] as a commutative orthogonal ring spectrum. Joachim showed in [15, Theorem 4.4] that the genuine G-spectrum underlying the global spectrum **KU** represents G-equivariant complex K-theory; another proof can be found in [24, Corollary 6.4.23]. Joachim's orthogonal ring spectrum can be enhanced to an orthogonal C-ring spectrum **KR** by suitably incorporating complex conjugation, see [12, Section 6]; this is the *periodic Real-global K-theory spectrum*. We review the definition in Construction B.51, and we show in Theorem B.59 that **KR** represents Real-equivariant K-theory for augmented Lie groups, justifying the name.

A morphism of commutative orthogonal C-ring spectra

$$j \;:\; \mathbf{kr} \;\longrightarrow\; \mathbf{KR}$$

from connective to periodic Real-global K-theory is defined in [24, Corollary 6.4.13]; we review the definition in Construction B.54.

Theorem 3.5. *For every $k \geq 2$, the morphism of orthogonal C-spectra $(\mathrm{sh}^\sigma j) \circ \mathrm{eig}^\natural \colon \Sigma^\infty \mathbf{U} \longrightarrow \mathrm{sh}^\sigma \mathbf{KR}$ annihilates the class $\sigma_k \in \pi^{\tilde{U}(k)}_{\mathrm{ad}(k)}(\Sigma^\infty \mathbf{U})$ defined in* (1.12).

Proof. We let $D \subset M_k(\mathbb{C})$ denote the Real $\tilde{T}^k$-representation of lower subdiagonal matrices (3.2). Real-equivariant Bott periodicity [3, Theorem (5.1)] provides an associated Bott class $\beta_D \in \pi^{\tilde{T}^k}_D(\mathbf{KR})$ with corresponding Euler class $e_D = \beta_D \cdot a_D$ in $\pi^{\tilde{T}^k}_0(\mathbf{KR})$. From Corollary 3.3 we deduce the relation

$$e_D^2 \cdot \mathrm{res}^{\tilde{U}(k)}_{\tilde{T}^k}(((\mathrm{sh}^\sigma j) \circ \mathrm{eig}^\natural)_*(\sigma_k)) \;=\; \beta_D^2 \cdot (\mathrm{sh}^\sigma j)_*(a_D^2 \cdot \mathrm{eig}^\natural_*(\mathrm{res}^{\tilde{U}(k)}_{\tilde{T}^k}(\sigma_k))) \;=\; 0$$

in $\pi^{\tilde{T}^k}_{k\sigma+D}(\mathrm{sh}^\sigma \mathbf{KR})$. Real-equivariant Bott periodicity for D also provides an isomorphism of $\pi^{\tilde{T}^k}_0(\mathbf{KR})$-modules

$$\pi^{\tilde{T}^k}_{k\sigma+D}(\mathrm{sh}^\sigma \mathbf{KR}) \;\cong\; \pi^{\tilde{T}^k}_{k\sigma}(\mathrm{sh}^\sigma \mathbf{KR}) \;\cong\; \pi^{\tilde{T}^k}_{(k-1)\sigma}(\mathbf{KR}) \;.$$

By Theorem B.59, the ring $\pi^{\tilde{T}^k}_0(\mathbf{KR})$ is isomorphic to the Real representation ring $RR(\tilde{T}^k)$, in a way that identifies the $\pi^{\tilde{T}^k}_0(\mathbf{KR})$-module $\pi^{\tilde{T}^k}_{(k-1)\sigma}(\mathbf{KR})$ with the $RR(\tilde{T}^k)$-module $\widetilde{KR}_{\tilde{T}^k}(S^{(k-1)\sigma})$. This $RR(\tilde{T}^k)$-module is Euler-torsion-free by Proposition 3.4. The $\tilde{T}^k$-representation D has trivial T^k-fixed points, and thus multiplication by the Euler class e_D is injective on $\pi^{\tilde{T}^k}_{(k-1)\sigma}(\mathbf{KR})$. So we deduce that

$$\mathrm{res}^{\tilde{U}(k)}_{\tilde{T}^k}(((\mathrm{sh}^\sigma j) \circ \mathrm{eig}^\natural)_*(\sigma_k)) \;=\; 0$$

in $\pi^{\tilde{T}^k}_{k\sigma+D}(\mathrm{sh}^\sigma \mathbf{KR}) \cong \pi^{\tilde{T}^k}_{(k-1)\sigma+D}(\mathbf{KR})$. We let $\overline{\mathrm{ad}}(k)$ denote the reduced adjoint representation of $\tilde{U}(k)$, i.e., the subrepresentation of $\mathrm{ad}(k)$ of matrices with trivial trace. Any isomorphism $\mathrm{ad}(k) \cong \sigma \oplus \overline{\mathrm{ad}}(k)$ provides an identification

$$\pi^{\tilde{U}(k)}_{\mathrm{ad}(k)}(\mathrm{sh}^\sigma \mathbf{KR}) \ \cong\ \pi^{\tilde{U}(k)}_{\sigma\oplus\overline{\mathrm{ad}}(k)}(\mathrm{sh}^\sigma \mathbf{KR}) \ \cong\ \pi^{\tilde{U}(k)}_{\overline{\mathrm{ad}}(k)}(\mathbf{KR}) \ .$$

The restriction homomorphism

$$\mathrm{res}^{\tilde{U}(k)}_{\tilde{T}^k} \ :\ \widetilde{KR}_{\tilde{U}(k)}(S^{\overline{\mathrm{ad}}(k)}) \ \longrightarrow\ \widetilde{KR}_{\tilde{T}^k}(S^{\overline{\mathrm{ad}}(k)})$$

is split injective, see for example [3, Proposition (5.2)]. So also the restriction homomorphism

$$\mathrm{res}^{\tilde{U}(k)}_{\tilde{T}^k} \ :\ \pi^{\tilde{U}(k)}_{\overline{\mathrm{ad}}(k)}(\mathbf{KR}) \ \longrightarrow\ \pi^{\tilde{T}^k}_{\overline{\mathrm{ad}}(k)}(\mathbf{KR}) = \pi^{\tilde{T}^k}_{(k-1)\sigma+D}(\mathbf{KR})$$

is injective. Hence $((\mathrm{sh}^\sigma j)\circ \mathrm{eig}^\natural)_*(\sigma_k) = 0$ in $\pi^{\tilde{U}(k)}_{\mathrm{ad}(k)}(\mathrm{sh}^\sigma \mathbf{KR}) \cong \pi^{\tilde{U}(k)}_{\overline{\mathrm{ad}}(k)}(\mathbf{KR})$. □

The fact that the morphism $(\mathrm{sh}^\sigma j)\circ \mathrm{eig}^\natural \colon \Sigma^\infty \mathbf{U} \longrightarrow \mathrm{sh}^\sigma \mathbf{KR}$ annihilates the class σ_k for all $k \geq 2$ is crucial for all further results in this paper. We alert the reader that the eigenspace morphism $\mathrm{eig}^\natural \colon \Sigma^\infty \mathbf{U} \longrightarrow \mathrm{sh}^\sigma \mathbf{kr}$ itself does *not* annihilate the class σ_k for any $k \geq 1$, see Remark 3.6 below. At this point one might want to recall that the name 'connective global K-theory' has to be taken with a grain of salt, in that the morphism $j \colon \mathbf{kr} \longrightarrow \mathbf{KR}$ is an equivariant connective cover for *finite* groups by Theorems 6.3.27 and 6.4.21 of [24], but not generally for compact Lie groups of positive dimension.

Remark 3.6. The composite

$$\Sigma^\infty \mathbf{U} \ \xrightarrow{\mathrm{eig}^\natural}\ \mathrm{sh}^\sigma \mathbf{kr} \ \xrightarrow{\mathrm{sh}^\sigma \dim}\ \mathrm{sh}^\sigma(Sp^\infty)$$

does not annihilate σ_k for $k \geq 2$. We show this for $k = 2$, the other cases being similar but slightly more involved. Here Sp^∞ denotes the orthogonal spectrum, with trivial C-action, made from infinite symmetric products of spheres, compare [24, Example 5.3.10]. And $\dim \colon \mathbf{kr} \longrightarrow Sp^\infty$ is the dimension homomorphism to the infinite symmetric product spectrum, defined in [24, Example 6.3.36]. The spectrum Sp^∞ is $\mathcal{F}in$-globally equivalent to the Eilenberg–MacLane spectrum for the constant global functor $\underline{\mathbb{Z}}$, see Propositions 5.3.9 and 5.3.12 of [24]. However, for compact Lie groups G of positive dimension, the groups $\pi^G_*(Sp^\infty)$ are typically not concentrated in dimension 0, and the ring $\pi^G_0(Sp^\infty)$ need not be isomorphic to $\mathbb{Z}$. For example, $\pi^T_1(Sp^\infty) \cong \mathbb{Q}$, see [24, Theorem 5.3.16], and the abelian group $\pi^{SU(2)}_0(Sp^\infty)$ has rank 2, see [23, Example 4.16]. For an orthogonal representation V of a *connected* compact Lie group G, the map

$$Sp^\infty(S^{V^G}) \ \longrightarrow\ (Sp^\infty(S^V))^G$$

induced by the fixed point inclusion $V^G \longrightarrow V$ is a homeomorphism by [24, Proposition B.42]. So the G-geometric fixed point spectrum of Sp^∞ is an Eilenberg–MacLane spectrum for $\mathbb{Z}$. In particular, the ring $\Phi^G_0(Sp^\infty)$ is isomorphic to $\mathbb{Z}$ whenever G is connected.

The class u_k from (2.16) is represented by the suspension by S^{ν_k} of the composite

$$\zeta^k_* \circ \partial \circ \mathfrak{c} \ :\ S^\sigma \ \longrightarrow\ \mathbf{U}(u(\nu_k)) \ .$$

This map sends $x \in S^1$ to the unitary automorphism of $u(\nu_k)_\mathbb{C}$ that is multiplication by $\mathfrak{c}(x)$ on the image of the $\mathbb{C}$-linear monomorphism $\zeta^k \colon \nu_k \longrightarrow u(\nu_k)_\mathbb{C}$, and the identity on its orthogonal complement. The morphism $\mathrm{eig}^\natural \colon \Sigma^\infty \mathbf{U} \longrightarrow \mathrm{sh}^\sigma \mathbf{kr}$ extracts eigenvalues and eigenspaces, and turns the eigenvalues into a suspension coordinate via $\mathfrak{c}^{-1}$; and as the name suggests, the morphism $\dim \colon \mathbf{kr} \longrightarrow Sp^\infty$ takes a configuration of vector spaces to the configuration of the dimensions. So the class $((\mathrm{sh}^\sigma \dim)\circ \mathrm{eig}^\natural)_*(u_k)$ is represented by the map

$$S^{\nu_k\oplus\sigma} \ \longrightarrow\ Sp^\infty(S^{\nu_k\oplus\sigma}) \ =\ \mathrm{sh}^\sigma(Sp^\infty)(u(\nu_k)) \ , \quad x \ \longmapsto\ k\cdot x \ ,$$

the point $x \in S^{\nu_k \oplus \sigma}$ with multiplicity k. This map represents $\mathrm{sh}^\sigma(k \cdot 1)$, the k-fold multiple of the shifted multiplicative unit in $\pi_\sigma^{\tilde{U}(k)}(\mathrm{sh}^\sigma(Sp^\infty))$, and thus

$$((\mathrm{sh}^\sigma \dim) \circ \mathrm{eig}^\natural)_*(u_k) \;=\; \mathrm{sh}^\sigma(k \cdot 1)\ .$$

Theorem 2.23 for $k = 2$ and the fact that $\sigma_1 = u_1$ provide the relation

$$a_{\overline{\mathrm{ad}}(2)} \cdot \sigma_2 \;=\; u_2 - \mathrm{tr}_{\tilde{U}(1,1)}^{\tilde{U}(2)}(p_2^*(u_1))\ .$$

Thus

$$\begin{aligned} a_{\overline{\mathrm{ad}}(2)} \cdot ((\mathrm{sh}^\sigma \dim) \circ \mathrm{eig}^\natural)_*(\sigma_2) \;&=\; ((\mathrm{sh}^\sigma \dim) \circ \mathrm{eig}^\natural)_*(u_2) \;-\; \mathrm{tr}_{\tilde{U}(1,1)}^{\tilde{U}(2)}(p_2^*(((\mathrm{sh}^\sigma \dim) \circ \mathrm{eig}^\natural)_*(u_1))) \\ &=\; \mathrm{sh}^\sigma\left(2 - \mathrm{tr}_{\tilde{U}(1,1)}^{\tilde{U}(2)}(1)\right)\ . \end{aligned}$$

By the remark immediately before [23, Example 4.16], the classes 1 and $\mathrm{tr}_{U(1,1)}^{U(2)}(1)$ are linearly independent in the group $\pi_0^{U(2)}(Sp^\infty)$. Hence the classes 1 and $\mathrm{tr}_{\tilde{U}(1,1)}^{\tilde{U}(2)}(1)$ are linearly independent in the group $\pi_0^{\tilde{U}(2)}(Sp^\infty)$. Thus the class $((\mathrm{sh}^\sigma \dim) \circ \mathrm{eig}^\natural)_*(\sigma_2)$ is non-zero.

4. The global Segal–Becker splitting

In this section we construct the global Segal–Becker splitting $c\colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma_+^\infty \mathbf{P})$ in the unstable Real-global homotopy category, see (4.13). This morphism comes into existence as a C-global σ-loop map, the delooping being the morphism $d\colon \mathbf{U} \longrightarrow \Omega^\bullet(\Sigma_+^\infty \mathbf{P} \wedge S^\sigma)$ defined in (4.8). The fact that the morphisms d and c are indeed sections to $\Omega^\bullet(\eta \wedge S^\sigma)\colon \Omega^\bullet(\Sigma_+^\infty \mathbf{P} \wedge S^\sigma) \longrightarrow \Omega^\bullet(\mathbf{KR} \wedge S^\sigma)$ and to $\Omega^\bullet(\eta)\colon \Omega^\bullet(\Sigma_+^\infty \mathbf{P}) \longrightarrow \Omega^\bullet(\mathbf{KR})$, respectively, are proved in Theorem 4.9 and Corollary 4.15. In Corollary 4.19 we show that the composite $c \circ h\colon \mathbf{P} \longrightarrow \Omega^\bullet(\Sigma_+^\infty \mathbf{P})$ of the global Segal–Becker splitting with the morphism $h\colon \mathbf{P} \longrightarrow \mathbf{BUP}$ that represent the inclusion of line bundles into virtual vector bundles is the unit of the adjunction $(\Sigma_+^\infty, \Omega^\bullet)$.

Construction 4.1 (The Real-global ultra-commutative monoid $\mathbf{P}$)**.** We recall the orthogonal C-space $\mathbf{P}$ made from complex projective spaces, compare [24, (2.3.20)], a specific ultra-commutative model for the Real-global classifying space of the extended circle group $\tilde{T} = \tilde{U}(1) = U(1) \rtimes C$. In [24], we use the notation $\mathbf{P}^{\mathbb{C}}$ to distinguish this version made from complex projective spaces from the real version. In this paper, the real version plays no role, so we simplify notation and drop the superscript '$\mathbb{C}$'. The value of $\mathbf{P}$ at the inner product space V is

$$\mathbf{P}(V) \;=\; P(\mathrm{Sym}(V_{\mathbb{C}}))\ ,$$

the complex projective space of the symmetric algebra of the complexification. The structure map $\mathbf{P}(\varphi)\colon \mathbf{P}(V) \longrightarrow \mathbf{P}(W)$ induced by a linear isometric embedding $\varphi\colon V \longrightarrow W$ takes a complex line to its image under $\mathrm{Sym}(\varphi_{\mathbb{C}})\colon \mathrm{Sym}(V_{\mathbb{C}}) \longrightarrow \mathrm{Sym}(W_{\mathbb{C}})$. The involution $\psi(V)\colon \mathbf{P}(V) \longrightarrow \mathbf{P}(V)$ is induced by complex conjugation on $\mathrm{Sym}(V_{\mathbb{C}})$, exploiting that also conjugate-linear maps take $\mathbb{C}$-subspaces to $\mathbb{C}$-subspaces.

The inclusions $V_{\mathbb{C}} \longrightarrow \mathrm{Sym}(V_{\mathbb{C}})$ as the linear summands induce maps of projective spaces

$$\ell(V)\ :\ \mathbf{Gr}_1(V) \;=\; P(V_{\mathbb{C}}) \;\longrightarrow\; P(\mathrm{Sym}(V_{\mathbb{C}})) \;=\; \mathbf{P}(V)\ .$$

As V varies, these maps form a morphism of orthogonal C-spaces

$$\ell\ :\ \mathbf{Gr}_1 \xrightarrow{\ \sim\ } \mathbf{P} \tag{4.2}$$

that is a C-global equivalence by [5, Lemma 3.8]. The reason for using the 'bigger' model $\mathbf{P}$ in the first place is that the tensor product of complex lines makes $\mathbf{P}$ into an ultra-commutative monoid, see [24, Example 2.3.8]. The ultra-commutative multiplication of $\mathbf{P}$ does *not* restrict to a multiplication on $\mathbf{Gr}_1$.

The orthogonal C-spaces $\mathbf{Gr}_1$ and $\mathbf{P}$ are Real-global classifying spaces, in the sense of [25, Construction A.4], for the extended circle group $\tilde{T}$, by [25, Theorem A.33]. Hence they represent the functor $\pi_0^{\tilde{T}}$ on the unstable Real-global homotopy category. We will later need to refer to the universal elements, the tautological classes, so we recall them here. The line

$$\mathbb{L} \;=\; \mathbb{C}\cdot(1\otimes 1 - i\otimes i) \;=\; \mathrm{im}(\zeta^1\colon \nu_1 \longrightarrow u(\nu_1)_{\mathbb{C}})$$

is a $\tilde{T}$-fixed point of $P(u(\nu_1)_{\mathbb{C}}) = \mathbf{Gr}_1(u(\nu_1))$, where $\zeta^1 = \zeta^{\nu_1}$ is defined in (1.7). The *unstable tautological class* is its homotopy class $[\mathbb{L}] \in \pi_0^{\tilde{T}}(\mathbf{Gr}_1)$. The *stable tautological class*

$$\tilde{e}_{\tilde{T}} \;\in\; \pi_0^{\tilde{T}}(\Sigma^\infty_+ \mathbf{Gr}_1)$$

is the class represented by the $\tilde{T}$-equivariant map

$$S^{\nu_1} \xrightarrow{-\wedge\mathbb{L}} S^{\nu_1}\wedge P(u(\nu_1)_{\mathbb{C}})_+ \;=\; (\Sigma^\infty_+\mathbf{Gr}_1)(u(\nu_1))\ .$$

We also set

$$(4.3)\qquad u_{\tilde{T}} \;=\; \ell_*[\mathbb{L}] \;\in\; \pi_0^{\tilde{T}}(\mathbf{P}) \qquad\text{and}\qquad e_{\tilde{T}} \;=\; (\Sigma^\infty_+\ell)_*(\tilde{e}_{\tilde{T}}) \;\in\; \pi_0^{\tilde{T}}(\Sigma^\infty_+\mathbf{P})$$

for the images of the tautological classes under the C-global equivalence (4.2). Then both the pairs $(\mathbf{Gr}_1, [\mathbb{L}])$ and $(\mathbf{P}, u_{\tilde{T}})$ represent the functor $\pi_0^{\tilde{T}}$ on the unstable Real-global homotopy category. And both the pairs $(\Sigma^\infty_+\mathbf{Gr}_1, \tilde{e}_{\tilde{T}})$ and $(\Sigma^\infty_+\mathbf{P}, e_{\tilde{T}})$ represents the functor $\pi_0^{\tilde{T}}$ on the Real-global stable homotopy category.

Theorem 1.13 lets us define C-global morphisms from $\Sigma^\infty\mathbf{U}$ by specifying their values on the classes σ_k in $\pi^{\tilde{U}(k)}_{\mathrm{ad}(k)}(\Sigma^\infty\mathbf{U})$ defined in (1.12). So we let

$$(4.4)\qquad d^\flat \;:\; \Sigma^\infty\mathbf{U} \longrightarrow \Sigma^\infty_+\mathbf{P}\wedge S^\sigma$$

denote the unique morphism in the C-global stable homotopy category such that

$$d^\flat_*(\sigma_k) \;=\; \begin{cases} e_{\tilde{T}}\wedge S^\sigma & \text{for } k=1, \text{ and}\\ \quad 0 & \text{for } k\geq 2.\end{cases}$$

In the case $k=1$ we have implicitly identified the sign representation σ with $\mathrm{ad}(1)$ by sending $x\in\sigma$ to $i\cdot x\in\mathrm{ad}(1)$.

We define a morphism of based orthogonal C-spaces

$$b \;:\; (\mathbf{Gr}_1)_+\wedge S^\sigma \;\longrightarrow\; \mathbf{U}$$

at an inner product space V as the map

$$b(V) \;:\; ((\mathbf{Gr}_1)_+\wedge S^\sigma)(V) = P(V_{\mathbb{C}})_+\wedge S^\sigma \;\longrightarrow\; U(V_{\mathbb{C}}) = \mathbf{U}(V)$$

that sends $L\wedge x$ to the unitary automorphism $b(V)(L\wedge x)$ of $V_{\mathbb{C}}$ that is multiplication by $\mathfrak{c}(x)\in U(1)$ on the complex line L, and the identity on the orthogonal complement of L. The C-global equivalence $\ell\colon \mathbf{Gr}_1 \longrightarrow \mathbf{P}$ was defined in (4.2).

Proposition 4.5.

(i) *The relation* $(\Sigma^\infty b)_*(\tilde{e}_{\tilde{T}}\wedge S^\sigma) = \sigma_1$ *holds in the group* $\pi^{\tilde{T}}_\sigma(\Sigma^\infty\mathbf{U})$.

(ii) *The composite*

$$\Sigma^\infty_+\mathbf{Gr}_1\wedge S^\sigma \xrightarrow{\Sigma^\infty b} \Sigma^\infty\mathbf{U} \xrightarrow{d^\flat} \Sigma^\infty_+\mathbf{P}\wedge S^\sigma$$

in the Real-global stable homotopy category equals the morphism $\Sigma^\infty_+\ell\wedge S^\sigma$.

(iii) *The relation*

$$d^\flat_*(u_k) \;=\; \mathrm{tr}^{\tilde{U}(k)}_{\tilde{U}(k-1,1)}(1\times e_{\tilde{T}})\wedge S^\sigma\ .$$

holds in the group $\pi^{\tilde{U}(k)}_\sigma(\Sigma^\infty_+\mathbf{P}\wedge S^\sigma)$.

Proof. (i) By inspection of definitions, the following diagram commutes:

$$\begin{tikzcd}
S^\sigma \arrow[rr, "\mathfrak{c}", "(2.15)"'] \arrow[d, "\mathbb{L}\wedge -"'] & & U(1) \arrow[d, "\zeta^1_*"] \\
P(u(\nu_1)_\mathbb{C})_+ \wedge S^\sigma \arrow[r, equal] & ((\mathbf{Gr}_1)_+ \wedge S^\sigma)(u(\nu_1)) \arrow[r, "b(u(\nu_1))"'] & \mathbf{U}(u(\nu_1))
\end{tikzcd}$$

Here $\mathbb{L} = \mathbb{C} \cdot (1 \otimes 1 - i \otimes i)$ is the $\tilde{T}$-invariant line that defines the unstable tautological class in $\pi_0^{\tilde{T}}(\mathbf{Gr}_1)$. Smashing with S^{ν_1}, passing to homotopy classes and exploiting Proposition 2.18 proves the first relation

$$(\Sigma^\infty b)_*(\tilde{e}_{\tilde{T}} \wedge S^\sigma) \ = \ u_1 \ = \ \sigma_1 \ .$$

(ii) By part (i), the relation

$$(d^\flat \circ \Sigma^\infty b)_*(\tilde{e}_{\tilde{T}} \wedge S^\sigma) \ = \ d^\flat_*(\sigma_1) \ = \ e_{\tilde{T}} \wedge S^\sigma \ = \ (\Sigma^\infty_+ \ell \wedge S^\sigma)_*(\tilde{e}_{\tilde{T}} \wedge S^\sigma)$$

holds in the group $\pi_\sigma^{\tilde{T}}(\Sigma^\infty_+ \mathbf{P} \wedge S^\sigma)$. Since the pair $(\Sigma^\infty_+ \mathbf{Gr}_1 \wedge S^\sigma, \tilde{e}_{\tilde{T}} \wedge S^\sigma)$ represents the functor $\pi_\sigma^{\tilde{T}}$ on the C-global stable homotopy category, this proves that $d^\flat \circ \Sigma^\infty b = \Sigma^\infty_+ \ell \wedge S^\sigma$.

(iii) The defining property of the morphism $d^\flat \colon \Sigma^\infty \mathbf{U} \longrightarrow \Sigma^\infty_+ \mathbf{P} \wedge S^\sigma$ and the relation provided by Theorem 2.23 yield

$$\begin{aligned}
d^\flat_*(u_k) \ &= \ \sum_{1 \le j \le k} d^\flat_*(\mathrm{tr}_{\tilde{U}(k-j,j)}^{\tilde{U}(k)}(p_2^*(a_{\overline{\mathrm{ad}(j)}} \cdot \sigma_j)) \\
&= \ \sum_{1 \le j \le k} \mathrm{tr}_{\tilde{U}(k-j,j)}^{\tilde{U}(k)}(p_2^*(a_{\overline{\mathrm{ad}(j)}} \cdot d^\flat_*(\sigma_j)) \\
&= \ \mathrm{tr}_{\tilde{U}(k-1,1)}^{\tilde{U}(k)}(p_2^*(e_{\tilde{T}} \wedge S^\sigma)) \ = \ \mathrm{tr}_{\tilde{U}(k-1,1)}^{\tilde{U}(k)}(1 \times e_{\tilde{T}}) \wedge S^\sigma \ .
\end{aligned}$$

□

Construction 4.6 (The morphism $\eta \colon \Sigma^\infty_+ \mathbf{P} \longrightarrow \mathbf{KR}$)**.** The morphism of non-equivariant spectra from $\Sigma^\infty_+ \mathbb{C}P^\infty$ to KU that classifies the tautological complex line bundle over $\mathbb{C}P^\infty$ has a particularly nice and prominent Real-global refinement, a morphism of ultra-commutative C-ring spectra

$$\eta \ : \ \Sigma^\infty_+ \mathbf{P} \ \longrightarrow \ \mathbf{KR} \ .$$

The morphism η is defined as a composite of two morphisms of ultra-commutative ring spectra

$$\Sigma^\infty_+ \mathbf{P} \ \xrightarrow{\ \mu\ } \ \mathbf{kr} \ \xrightarrow{\ j\ } \ \mathbf{KR} \ .$$

The first morphism μ is the inclusion of the 'rank 1' part in the rank filtration, compare [24, Construction 6.3.40]; its value at an inner product space V is the C-equivariant map

$$\mu(V) \ : \ (\Sigma^\infty_+ \mathbf{P})(V) = S^V \wedge P(\mathrm{Sym}(V_\mathbb{C}))_+ \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_\mathbb{C}), S^V) = \mathbf{kr}(V) \ , \quad v \wedge L \longmapsto [L; v] \ .$$

We review the definition of the morphism $j \colon \mathbf{kr} \longrightarrow \mathbf{KR}$ in Construction B.54.

The underlying morphism of global spectra of η classifies the tautological $\tilde{T}$-representation ν_1, in the following sense. As mentioned earlier, the pair $(\Sigma^\infty_+ \mathbf{P}, e_{\tilde{T}})$ represents the functor $\pi_0^{\tilde{T}}$, where $e_{\tilde{T}}$ is the stable tautological class (4.3). Under the preferred identification of $\pi_0^{\tilde{T}}(\mathbf{KR})$ with the Real representation ring $RR(\tilde{T})$ given by Theorem B.59, the element $e_{\tilde{T}}$ maps to the class of the tautological $\tilde{T}$-representation, i.e.,

$$\eta_*(e_{\tilde{T}}) \ = \ [\nu_1] \quad \text{in } \pi_0^{\tilde{T}}(\mathbf{KR}) \cong RR(\tilde{T}).$$

The morphism η is extremely highly structured, and has a range of marvelous properties. Because η is a morphism of ultra-commutative ring spectra, its effect on equivariant homotopy groups is not only compatible with restriction, inflations and transfers, but also with multiplicative power operations and norms. Moreover, in [27], the author establishes a global generalization of Snaith's celebrated theorem [31, 32], saying that KU can be obtained from $\Sigma^\infty_+\mathbb{C}P^\infty$ by 'inverting the Bott class'.

Construction 4.7 (The σ-deloop of the global Segal–Becker splitting)**.** As a Quillen adjoint functor pair for the C-global model structures, the pair $(\Sigma^\infty,\Omega^\bullet)$ derives to an adjoint functor pair at the level of C-global homotopy categories. The stable morphism $d^\flat$ from (4.4) is thus adjoint to an unstable morphism in the homotopy category of based C-global spaces

$$d\ :\ \mathbf{U}\ \longrightarrow\ \Omega^\bullet(\Sigma^\infty_+\mathbf{P}\wedge S^\sigma)\ . \tag{4.8}$$

This morphism is our deloop of the global Segal–Becker splitting.

We can now prove Theorem A from the introduction:

Theorem 4.9. *The composite*

$$\mathbf{U}\ \xrightarrow{\ d\ }\ \Omega^\bullet(\Sigma^\infty_+\mathbf{P}\wedge S^\sigma)\ \xrightarrow{\Omega^\bullet(\eta\wedge S^\sigma)}\ \Omega^\bullet(\mathbf{KR}\wedge S^\sigma)$$

is a Real-global equivalence and a C-global H-map.

Proof. Shifting and suspending an equivariant orthogonal spectrum by a representation are naturally equivalent, see [24, Proposition 3.1.25]. The relevant case for our purposes is the sign representation of the group C, and the natural C-global equivalence $\lambda^\sigma_X\colon X\wedge S^\sigma\longrightarrow \mathrm{sh}^\sigma X$ defined in [24, (3.1.23)]. We claim that the following diagram commutes in the C-global stable homotopy category:

$$\begin{tikzcd} \Sigma^\infty\mathbf{U} \arrow[r, "d^\flat", "(4.4)"'] \arrow[d, "\mathrm{eig}^\natural"', "(3.1)"] & \Sigma^\infty_+\mathbf{P}\wedge S^\sigma \arrow[r, "\eta\wedge S^\sigma"] & \mathbf{KR}\wedge S^\sigma \arrow[d, "\sim"', "\lambda^\sigma_{\mathbf{KR}}"] \\ \mathrm{sh}^\sigma\,\mathbf{kr} \arrow[rr, "\mathrm{sh}^\sigma j"'] & & \mathrm{sh}^\sigma\,\mathbf{KR} \end{tikzcd} \tag{4.10}$$

By Theorem 1.13, it suffices to show that both composites agree on the classes σ_k for all $k\geq 1$. For $k\geq 2$, we have $d^\flat_*(\sigma_k)=0$ by the definition (4.4) of the morphism a, and $((\mathrm{sh}^\sigma j)\circ\mathrm{eig}^\natural)_*(\sigma_k)=0$ by Theorem 3.5. So both composites in the diagram (4.10) annihilate the class σ_k.

For $k=1$ we consider the following diagram of morphisms of orthogonal C-spectra:

$$\begin{tikzcd} \Sigma^\infty_+\mathbf{Gr}_1\wedge S^\sigma \arrow[r, "\Sigma^\infty_+\ell\wedge S^\sigma", "\sim"'] \arrow[d, "\Sigma^\infty b"'] & \Sigma^\infty_+\mathbf{P}\wedge S^\sigma \arrow[r, "\mu\wedge S^\sigma"'] \arrow[rr, bend left, "\eta\wedge S^\sigma"] & \mathbf{kr}\wedge S^\sigma \arrow[r, "j\wedge S^\sigma"'] \arrow[d, "\sim"', "\lambda^\sigma_{\mathbf{kr}}"] & \mathbf{KR}\wedge S^\sigma \arrow[d, "\sim"', "\lambda^\sigma_{\mathbf{KR}}"] \\ \Sigma^\infty\mathbf{U} \arrow[rr, "\mathrm{eig}^\natural"'] & & \mathrm{sh}^\sigma\,\mathbf{kr} \arrow[r, "\mathrm{sh}^\sigma j"'] & \mathrm{sh}^\sigma\,\mathbf{KR} \end{tikzcd}$$

The right square commutes by naturality of the λ-morphisms. We claim that the left part also commutes. Indeed, expanding definitions shows that both composites send an element

$$v\wedge L\wedge x\ \in\ S^V\wedge P(V_{\mathbb{C}})_+\wedge S^\sigma=(\Sigma^\infty_+\mathbf{Gr}_1\wedge S^\sigma)(V)$$

to the one-element configuration $[L,(v,x)]$ in $\mathscr{C}(\mathrm{Sym}((V\oplus\sigma)_{\mathbb{C}}),S^{V\oplus\sigma})=(\mathrm{sh}^\sigma\,\mathbf{kr})(V)$ of the point $(v,x)\in S^{V\oplus\sigma}$ labeled by the line L, embedded via $V_{\mathbb{C}}\longrightarrow(V\oplus\sigma)_{\mathbb{C}}\longrightarrow\mathrm{Sym}((V\oplus\sigma)_{\mathbb{C}})$.

Given the commutativity of the previous diagram, we thus obtain:

$$\begin{aligned} ((\mathrm{sh}^\sigma j)\circ\mathrm{eig}^\natural)_*(\sigma_1)\ &=\ ((\mathrm{sh}^\sigma j)\circ\mathrm{eig}^\natural\circ(\Sigma^\infty b))_*(\tilde e_{\tilde T}\wedge S^\sigma)\\ &=\ (\lambda^\sigma_{\mathbf{KR}}\circ(\eta\wedge S^\sigma)\circ(\Sigma^\infty_+\ell\wedge S^\sigma))_*(\tilde e_{\tilde T}\wedge S^\sigma)\\ &\overset{(4.3)}{=}\ (\lambda^\sigma_{\mathbf{KR}}\circ(\eta\wedge S^\sigma))_*(e_{\tilde T}\wedge S^\sigma)\ =\ (\lambda^\sigma_{\mathbf{KR}}\circ(\eta\wedge S^\sigma)\circ d^\flat)_*(\sigma_1) \end{aligned}$$

The first equation is Proposition 4.5. The final equation is part of the definition (4.4) of the morphism a. This concludes the proof that the diagram (4.10) commutes.

We pass to adjoints for the adjunction $(\Sigma^\infty, \Omega^\bullet)$, which turns (4.10) into a commutative diagram in the unstable C-global homotopy category:

$$\begin{tikzcd}
\mathbf{U} \arrow[r, "d"] \arrow[d, "\mathrm{eig}"'] & \Omega^\bullet(\Sigma^\infty_+ \mathbf{P} \wedge S^\sigma) \arrow[r, "\Omega^\bullet(\eta \wedge S^\sigma)"] & \Omega^\bullet(\mathbf{KR} \wedge S^\sigma) \arrow[d, "\Omega^\bullet(\lambda^\sigma_{\mathbf{KR}})", "\sim"'] \\
\Omega^\bullet(\mathrm{sh}^\sigma\, \mathbf{kr}) \arrow[rr, "\Omega^\bullet(\mathrm{sh}^\sigma\, j)"'] & & \Omega^\bullet(\mathrm{sh}^\sigma\, \mathbf{KR})
\end{tikzcd}$$

The composite $\Omega^\bullet(\mathrm{sh}^\sigma j) \circ \mathrm{eig}\colon \mathbf{U} \longrightarrow \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{KR})$ is a C-global equivalence by Theorem B.57. The morphism $\lambda^\sigma_{\mathbf{KR}}\colon \mathbf{KR} \wedge S^\sigma \longrightarrow \mathrm{sh}^\sigma \mathbf{KR}$ is a C-global equivalence by [24, Proposition 3.1.25]; hence the right vertical morphism $\Omega^\bullet(\lambda^\sigma_{\mathbf{KR}})$ is a C-global equivalence of orthogonal C-spaces. Thus $\Omega^\bullet(\eta \wedge S^\sigma) \circ d\colon \mathbf{U} \longrightarrow \Omega^\bullet(\mathbf{KR} \wedge S^\sigma)$ is a C-global equivalence, as claimed.

We show in Proposition B.46 that the eigenspace morphism $\mathrm{eig}\colon \mathbf{U} \longrightarrow \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{kr})$ is a C-global H-map. All unstable C-global morphisms arising via the functor $\Omega^\bullet$ are C-global H-maps for the 'stable' H-space structures of Construction B.39. In particular, $\Omega^\bullet(\mathrm{sh}^\sigma j)$ is a C-global H-map, and hence so is $\Omega^\bullet(\mathrm{sh}^\sigma j)\circ\mathrm{eig}$. The morphism $\Omega^\bullet(\lambda^\sigma_{\mathbf{KR}})$ is simultaneously a C-global equivalence and a C-global H-map. So the previous commutative diagram shows that $\Omega^\bullet(\eta \wedge S^\sigma) \circ d$ is a C-global H-map. ☐

Now we construct the actual global Segal–Becker splitting $c\colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})$ by looping the morphism $d\colon \mathbf{U} \longrightarrow \Omega^\bullet(\Sigma^\infty_+ \mathbf{P} \wedge S^\sigma)$ by the sign representation σ, and exploiting the Real-global Bott periodicity equivalence $\mathbf{BUP} \sim \Omega^\sigma \mathbf{U}$.

Construction 4.11 (The global Segal–Becker splitting)**.** The ultra-commutative C-monoid $\mathbf{BUP}$ is defined in [24, Example 2.4.33], and we recall the construction in B.3. In Theorem B.24 we establish a Real-global form of Bott periodicity, summarized by the Real-global equivalence of ultra-commutative C-monoids

$$\gamma \;:\; \mathbf{BUP} \xrightarrow{\sim} \Omega^\sigma \mathbf{U}$$

defined in (B.27). This equivalence refines the global Bott periodicity theorem [24, Theorem 2.5.41] to the Real-global context. Another ingredient is a natural C-global equivalence of orthogonal C-spaces

$$\xi_X \;:\; \Omega^\bullet X \xrightarrow{\sim} \Omega^\sigma(\Omega^\bullet(X \wedge S^\sigma)) \;, \tag{4.12}$$

where X is an orthogonal C-spectrum. We define it as the composite of the C-global equivalence

$$\Omega^\bullet(\eta^\sigma_X) \;:\; \Omega^\bullet(X) \xrightarrow{\sim} \Omega^\bullet(\Omega^\sigma(X \wedge S^\sigma))$$

obtained from the unit $\eta^\sigma_X\colon X \longrightarrow \Omega^\sigma(X \wedge S^\sigma)$ of the adjunction $(- \wedge S^\sigma, \Omega^\sigma)$, followed by the isomorphism $\Omega^\bullet(\Omega^\sigma(X \wedge S^\sigma)) \cong \Omega^\sigma(\Omega^\bullet(X \wedge S^\sigma))$ that swaps the order in which the loops are taken. The morphism η^σ_X is a C-global equivalence by [24, Proposition 3.1.25 (ii)], hence so is the morphism ξ_X.

We then define the *global Segal–Becker splitting*

$$c \;:\; \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+ \mathbf{P}) \tag{4.13}$$

as the unique morphism in the unstable C-global homotopy category that makes the following diagram commute:

$$\begin{tikzcd}
\mathbf{BUP} \arrow[r, "c"] \arrow[d, "\gamma"', "\sim"] & \Omega^\bullet(\Sigma^\infty_+ \mathbf{P}) \arrow[d, "\xi_{\Sigma^\infty_+ \mathbf{P}}", "\sim"'] \\
\Omega^\sigma \mathbf{U} \arrow[r, "\Omega^\sigma d"'] & \Omega^\sigma(\Omega^\bullet(\Sigma^\infty_+ \mathbf{P} \wedge S^\sigma))
\end{tikzcd} \tag{4.14}$$

The next corollary verifies that the global Segal–Becker splitting is indeed a splitting of the morphism $\Omega^\bullet(\eta)\colon \Omega^\bullet(\Sigma^\infty_+\mathbf{P}) \longrightarrow \Omega^\bullet(\mathbf{KR})$. In Theorem 5.8 will we show that the underlying global Segal–Becker realizes the classical equivariant Segal–Becker splittings at the level of equivariant cohomology theories, thereby justifying its name.

We alert the reader that the global Segal–Becker splitting is *not* a C-global H-map, and that the induced map $[A,c]^\alpha\colon [A,\mathbf{BUP}]^\alpha \longrightarrow [A,\Omega^\bullet(\Sigma^\infty_+\mathbf{P})]^\alpha$ is *not* additive for surjectively augmented Lie groups, see Example 4.26. Additivity does hold for compact Lie groups with trivial augmentation, though. Indeed, the 'non-Real' global morphisms underlying c is a global loop map by construction, and hence a global H-map. So for compact Lie groups G with trivial augmentation, the underlying G-map of c is a loop map, and hence induces additive maps on $[A,-]^G$.

Corollary 4.15. *The composite*

$$\mathbf{BUP} \xrightarrow{\ c\ } \Omega^\bullet(\Sigma^\infty_+\mathbf{P}) \xrightarrow{\Omega^\bullet(\eta)} \Omega^\bullet(\mathbf{KR})$$

is a Real-global equivalence and a C-global H-map.

Proof. We consider the commutative diagram in the homotopy category of based C-global spaces:

$$\begin{tikzcd}
\mathbf{BUP} \arrow[r, "c"] \arrow[d, "\gamma"', "\sim"] & \Omega^\bullet(\Sigma^\infty_+\mathbf{P}) \arrow[r, "\Omega^\bullet(\eta)"] \arrow[d, "\xi_{\Sigma^\infty_+\mathbf{P}}"', "\sim"] & \Omega^\bullet(\mathbf{KR}) \arrow[d, "\sim"', "\xi_{\mathbf{KR}}"] \\
\Omega^\sigma\mathbf{U} \arrow[r, "\Omega^\sigma d"'] & \Omega^\sigma(\Omega^\bullet(\Sigma^\infty_+\mathbf{P}\wedge S^\sigma)) \arrow[r, "\Omega^\sigma(\Omega^\bullet(\eta\wedge S^\sigma))"'] & \Omega^\sigma(\Omega^\bullet(\mathbf{KR}\wedge S^\sigma))
\end{tikzcd}$$

The morphism $\Omega^\bullet(\eta\wedge S^\sigma)\circ d$ is a C-global equivalence and a C-global H-map by Theorem 4.9. Hence the lower horizontal composite is also both a C-global equivalence and a C-global H-map. The morphism γ is a C-global equivalence of ultra-commutative C-monoids, hence in particular a C-global H-map. And also the morphism $\xi_{\mathbf{KR}}$ is simultaneously a C-global equivalence and a C-global H-map. Since the left and right vertical morphisms and the lower horizontal composite are Real-global equivalence and C-global H-maps, so is the upper horizontal composite. $\square$

Remark 4.16 (No section deloops twice)**.** Since $\Omega^\bullet(\eta)\colon \Omega^\bullet(\Sigma^\infty_+\mathbf{P}) \longrightarrow \Omega^\bullet(\mathbf{KR})$ is a Real-global infinite loop map with an unstable section, one can wonder how often one can deloop an unstable section. Our construction of the section $c\colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P})$ presents it as a C-global σ-loop map, the deloop being the morphism $d\colon \mathbf{U} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P}\wedge S^\sigma)$. If we forget the C-global action and pass to underlying global morphisms, these data witness the global Segal–Becker splitting as global loop map. However, one cannot do better than this, not even non-equivariantly, as we now recall.

The H-space structure on the infinite unitary group U coming from Bott periodicity coincides with the one from the group structure of U. Under the Pontryagin product, the mod 2 homology $H_*(U;\mathbb{F}_2)$ is an exterior $\mathbb{F}_2$-algebra on classes $a_i \in H_{2i+1}(U;\mathbb{F}_2)$ for $i\geq 0$. In contrast, $H_*(\Omega^\infty(\Sigma^\infty_+\mathbb{C}P^\infty\wedge S^1);\mathbb{F}_2)$ is a polynomial $\mathbb{F}_2$-algebra on the iterated Kudo–Araki operations on a basis of $\tilde H_*(\mathbb{C}P^\infty_+\wedge S^1;\mathbb{F}_2)$, see for example [11, Theorem 5.1]. So the epimorphism of commutative graded $\mathbb{F}_2$-algebras

$$(\Omega^\infty(\eta\wedge S^1))_*\ :\ H_*(\Omega^\infty(\Sigma^\infty_+\mathbb{C}P^\infty\wedge S^1);\mathbb{F}_2)\ \longrightarrow\ H_*(\Omega^\infty(KU\wedge S^1);\mathbb{F}_2)\ \cong\ H_*(U;\mathbb{F}_2)$$

does not admit a multiplicative section. Hence the map $\Omega^\infty(\eta\wedge S^1)\colon \Omega^\infty(\Sigma^\infty_+\mathbb{C}P^\infty\wedge S^1) \longrightarrow \Omega^\bullet(KU\wedge S^1)$ does not have a section that is an H-map, much less a loop map.

Construction 4.17. The C-global equivalence $\ell\colon \mathbf{Gr}_1 \xrightarrow{\sim} \mathbf{P}$ was defined in (4.2). The morphism of ultra-commutative C-monoids $i\colon \mathbf{Gr} \longrightarrow \mathbf{BUP}$ was introduced in [24, page 215], see also (B.4); it is a Real-global group completion by [24, Theorem 2.5.33] and its Real-global generalization. By Theorem B.12, the morphism i represents the inclusion of Real-equivariant vector bundles into virtual Real-equivariant vector bundles. We define

$$h \ : \ \mathbf{P} \longrightarrow \mathbf{BUP}$$

as the unique morphism in the unstable C-global homotopy category that makes the following diagram commute:

$$\begin{tikzcd} \mathbf{Gr}_1 \arrow[r, "\ell", "\sim"'] \arrow[d, "\mathrm{incl}"'] & \mathbf{P} \arrow[d, "h"] \\ \mathbf{Gr} \arrow[r, "i"'] & \mathbf{BUP} \end{tikzcd} \tag{4.18}$$

So the morphism h represents the inclusion of line bundles into virtual vector bundles.

Corollary 4.19. *The composite*

$$\mathbf{P}_+ \xrightarrow{\ h\ } \mathbf{BUP} \xrightarrow{\ c\ } \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})$$

is the unit of the adjunction $(\Sigma^\infty_+, \Omega^\bullet)$.

Proof. We write $u\colon \mathbf{P} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P})$ for the unit of the adjunction. Proposition 4.5 shows that

$$d^\flat \circ (\Sigma^\infty b) \ = \ \Sigma^\infty_+ \ell \wedge S^\sigma \ : \ \Sigma^\infty_+ \mathbf{Gr}_1 \wedge S^\sigma \ \longrightarrow \ \Sigma^\infty_+ \mathbf{P} \wedge S^\sigma \ .$$

The morphism $d\colon \mathbf{U} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P} \wedge S^\sigma)$ is adjoint to $d^\flat$. So passing to adjoints for the adjunction $(\Sigma^\infty, \Omega^\bullet)$ yields

$$d \circ b \ = \ u \circ (\ell_+ \wedge S^\sigma) \ : \ (\mathbf{Gr}_1)_+ \wedge S^\sigma \ \longrightarrow \ \Omega^\bullet(\Sigma^\infty_+ \mathbf{P} \wedge S^\sigma) \ .$$

The morphism $b\colon (\mathbf{Gr}_1)_+ \wedge S^\sigma \longrightarrow \mathbf{U}$ is adjoint to the restriction of $\beta\colon \mathbf{Gr} \longrightarrow \Omega^\sigma \mathbf{U}$ defined in (B.28) to the summand $\mathbf{Gr}_1 \subset \mathbf{Gr}$. So passing to adjoints for the adjunction $(- \wedge S^\sigma, \Omega^\sigma)$ yields

$$(\Omega^\sigma d) \circ \beta \circ \mathrm{incl} \ = \ \xi_{\Sigma^\infty_+ \mathbf{P}} \circ u \circ \ell_+ \ : \ (\mathbf{Gr}_1)_+ \ \longrightarrow \ \Omega^\sigma(\Omega^\bullet(\Sigma^\infty_+ \mathbf{P} \wedge S^\sigma)) \ .$$

So

$$\xi_{\Sigma^\infty_+\mathbf{P}} \circ c \circ h \circ \ell_+ \ =_{(4.14)} \ (\Omega^\sigma d) \circ \gamma \circ i \circ \mathrm{incl} \ =_{(B.29)} \ (\Omega^\sigma d) \circ \beta \circ \mathrm{incl} \ = \ \xi_{\Sigma^\infty_+\mathbf{P}} \circ u \circ \ell_+ \ .$$

Since ℓ and $\xi_{\Sigma^\infty_+\mathbf{P}}$ are C-global equivalences, we can cancel them and deduce the desired relation $c \circ h = u$. □

We end this section with a discussion of the additivity, or rather the failure thereof, of the global Segal–Becker splitting. If we forget the Real direction and pass to underlying 'non-Real' global spaces, then the sign action on S^σ disappears, and σ-loops become ordinary loops. By the Eckmann–Hilton argument, the loop addition then coincides with the abelian monoid structure from any ultra-commutative multiplication, and loop maps are automatically additive. In particular, the morphism of global spaces underlying the Real-global Segal–Becker splitting is a global loop map, and so the induced map $[A, c]^G_*\colon [A, \mathbf{BUP}]^G \longrightarrow [A, \Omega^\bullet(\Sigma^\infty_+\mathbf{P})]^G$ is additive whenever G is a compact Lie group with trivial augmentation to C.

However, the map $[A, c]^\alpha_*$ induced by the Real-global Segal–Becker splitting is *not* generally additive for surjective augmentations $\alpha\colon G \longrightarrow C$. In fact, additivity already fails for $G = C$ with identity augmentation, and for $A = *$, see Example 4.26. The rest of this section aims to explain this more carefully, including a quantification of the failure of additivity by the action of the unit $\epsilon \in \pi_0^C(\mathbb{S})$ represented by the sign involution of S^σ, see Theorem 4.23 (ii).

Our book keeping device to quantify the deviation from additivity will be a certain piece of natural structure on σ-loop objects.

Construction 4.20 (A binary operation on $\Omega^\sigma X$)**.** We let X be a based orthogonal C-space. We shall define a specific binary operation (4.21) on the σ-loop space $\Omega^\sigma X$. We consider the C-equivariant based map

$$m' \ : \ U(1) \ \longrightarrow \ U(1)\vee U(1)$$
$$m'(z) \ = \ \begin{cases} (\ \ z^4,1) & \text{if } \mathrm{Re}(z)\geq 0, \text{ and} \\ (-z^2,2) & \text{if } \mathrm{Re}(z)\leq 0. \end{cases}$$

The second coordinate in the formula for $m'(z)$ specifies in which of the two wedge summands of $U(1)\vee U(1)$ the respective point lies. The map m' is clearly C-equivariant for the action by complex conjugation on all instances of $U(1)$. We define a C-equivariant map $m\colon S^\sigma \longrightarrow S^\sigma\vee S^\sigma$ by conjugating m' with the Cayley transform (2.15), i.e., as the composite

$$S^\sigma \ \xrightarrow[\cong]{\mathfrak{c}} U(1) \ \xrightarrow{m'} \ U(1)\vee U(1) \ \xrightarrow[\cong]{\mathfrak{c}^{-1}\vee c^{-1}} \ S^\sigma\vee S^\sigma \ .$$

Now we let X be a based orthogonal C-space. Precomposition with $m\colon S^\sigma \longrightarrow S^\sigma\vee S^\sigma$ yields a morphism of based orthogonal C-spaces

$$(4.21) \qquad m^* \ : \ (\Omega^\sigma X)\times(\Omega^\sigma X) \ \cong \ \mathrm{map}_*(S^\sigma\vee S^\sigma, X) \ \xrightarrow{\mathrm{map}_*(m,X)} \ \mathrm{map}_*(S^\sigma,X)=\Omega^\sigma X \ .$$

In contrast to $\pi_1(X,x)=[S^1,X]_*$, the set $[S^\sigma,X]^C_*$ has no group structure that is natural for C-equivariant maps in X. Equivalently, the σ-sphere does not admit an equivariant 'pinch map' $S^\sigma \longrightarrow S^\sigma\vee S^\sigma$, i.e., such that the composite with each of the two projections is equivariantly homotopic to the identity. The map $m\colon S^\sigma\longrightarrow S^\sigma\vee S^\sigma$ is a partial remedy of this defect. If we forget the C-action and look at the underlying non-equivariant homotopy class of m, it represents the element $xy^{-1}x$ in the free group $\pi_1(S^1\vee S^1,*)$, where x and y denote the classes of the left and right summand inclusions.

We write $\epsilon\colon S^\sigma\longrightarrow S^\sigma$ for the sign involution, i.e., $\epsilon(x)=-x$. For every based orthogonal C-space X, it induces an involution $\epsilon^*\colon \Omega^\sigma X\longrightarrow\Omega^\sigma X$ by precomposition.

Proposition 4.22. *Let X be a based orthogonal C-space. Then the composite*

$$\Omega^\sigma X \ \xrightarrow{\Delta} \ (\Omega^\sigma X)\times(\Omega^\sigma X) \ \xrightarrow{m^*} \ \Omega^\sigma X$$

is equivariantly homotopic to the identity, and the composite

$$\Omega^\sigma X \ \xrightarrow{(*,\mathrm{Id})} \ (\Omega^\sigma X)\times(\Omega^\sigma X) \ \xrightarrow{m^*} \ \Omega^\sigma X$$

is equivariantly homotopic to $\epsilon^*\colon\Omega^\sigma X\longrightarrow\Omega^\sigma X$.

Proof. We exploit that C-equivariant selfmaps of $U(1)$, for the action by complex conjugation, are characterized up to equivariant based homotopy by their value on the fixed point -1 and by the degree of the underlying non-equivariant map. The composite

$$U(1) \ \xrightarrow{m'} \ U(1)\vee U(1) \ \xrightarrow{\nabla} \ U(1)$$

fixes -1 and has underlying degree $+1$, so it is equivariantly based homotopic to the identity, where ∇ denotes the fold map. The composite

$$U(1) \ \xrightarrow{m'} \ U(1)\vee U(1) \ \xrightarrow{p_2} \ U(1)$$

fixes -1 and has underlying degree -1, so it is equivariantly based homotopic to complex conjugation, where p_2 denotes the projection to the second wedge summand. After conjugation with the Cayley transform, these properties become that facts that the two composites

$$S^\sigma \ \xrightarrow{m} \ S^\sigma\vee S^\sigma \ \xrightarrow{\nabla} \ S^\sigma \qquad \text{and} \qquad S^\sigma \ \xrightarrow{m} \ S^\sigma\vee S^\sigma \ \xrightarrow{p_2} \ S^\sigma$$

are equivariantly based homotopic to the identity and to the sign involution $\epsilon\colon S^\sigma \longrightarrow S^\sigma$, respectively. The claim follows by applying $\mathrm{map}_*(-,X)$ to the maps and homotopies. □

We abuse notation and also write $\epsilon \in \pi_0^C(\mathbb{S})$ for the C-equivariant stable homotopy class of the sign involution $\epsilon\colon S^\sigma \longrightarrow S^\sigma$. This stable homotopy element satisfies $\epsilon^2 = 1$ and is related to the transfer by $\epsilon = 1 - \mathrm{tr}_{\{1\}}^C(1)$. Part (ii) of the next theorem refers to the module structure of the group

$$[A, \Omega^\sigma(\Omega^\bullet Y)]^\alpha \;\cong\; \pi_0^G(\mathrm{map}(A_+ \wedge S^\sigma, \alpha^*(Y)))$$

over the ring $\pi_0^G(\mathbb{S})$.

The following theorem is straightforward for trivially augmented Lie groups. Indeed, in this case the group G acts trivially on the loop coordinate in S^σ, so the G-map underlying $c\colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P})$ is a loop map, and thus induces an additive map $c_*\colon [A, \mathbf{BUP}]^G \longrightarrow [A, \Omega^\bullet(\Sigma^\infty_+\mathbf{P})]^G$. Moreover, $\mathrm{res}^C_{\{1\}}(\epsilon) = -1$, so if $\alpha\colon G \longrightarrow C$ is the trivial homomorphism, then $\alpha^*(\epsilon) = -1$. The formula of part (ii) of the following theorem thus becomes $c_*(2x+y) = 2c_*(x) + c_*(y)$. In contrast, if $\alpha\colon G \longrightarrow C$ is surjective, then $\alpha^*(\epsilon) \neq -1$ in $\pi_0^G(\mathbb{S})$, and $c_*\colon [A, \mathbf{BUP}]^\alpha \longrightarrow [A, \Omega^\bullet(\Sigma^\infty_+\mathbf{P})]^\alpha$ need not be additive, see Example 4.26.

Theorem 4.23. *Let $\alpha\colon G \longrightarrow C$ be an augmented Lie group, and let A be a finite G-CW-complex.*

(i) *Let $\psi\colon \mathbf{U} \longrightarrow \mathbf{U}$ be a morphism in the based C-global homotopy category. Then the map*

$$(\Omega^\sigma\psi)_* \;:\; [A, \Omega^\sigma\mathbf{U}]^\alpha \;\longrightarrow\; [A, \Omega^\sigma\mathbf{U}]^\alpha$$

satisfies the relation

$$(\Omega^\sigma\psi)_*(2x - y) \;=\; 2\cdot(\Omega^\sigma\psi)_*(x) - (\Omega^\sigma\psi)_*(y)$$

for all $x, y \in [A, \Omega^\sigma\mathbf{U}]^\alpha$.

(ii) *The map*

$$c_* \;:\; [A, \mathbf{BUP}]^\alpha \;\longrightarrow\; [A, \Omega^\bullet(\Sigma^\infty_+\mathbf{P})]^\alpha \;=\; (\Sigma^\infty_+\mathbf{P})^0_\alpha(A)$$

satisfies the relation

$$c_*(2x+y) \;=\; (1 - \alpha^*(\epsilon))\cdot c_*(x) \;+\; c_*(y)$$

for all $x, y \in [A, \mathbf{BUP}]^\alpha$.

Proof. We start with a preliminary observation. We let M be abelian group endowed with a group homomorphism $m\colon M \times M \longrightarrow M$ such that $m(x,x) = x$ for all $x \in M$. Then

$$m(x,y) \;=\; m(x+0, 0+y) \;=\; m(x,0) + m(0,y)$$

for all $x, y \in M$, by the homomorphism property. Hence

$$x \;=\; m(x,x) \;=\; m(x,0) + m(0,x)\ ,$$

and both relations together yield

$$m(x,y) \;=\; x - m(0,x) + m(0,y)\ . \tag{4.24}$$

(i) The binary operation m^* on $\Omega^\sigma X$ defined in (4.21) is natural for morphisms of Real-global spaces in X. In particular, it is natural for the multiplication morphism $\mathbf{U} \boxtimes \mathbf{U} \longrightarrow \mathbf{U}$. Hence the map

$$m^* \;=\; [A, m^*]^\alpha \;:\; [A, \Omega^\sigma\mathbf{U}]^\alpha \times [A, \Omega^\sigma\mathbf{U}]^\alpha \;\longrightarrow\; [A, \Omega^\sigma\mathbf{U}]^\alpha$$

is a homomorphism of abelian groups. Proposition 4.22 shows that $m^*(x,x) = x$ and $m^*(0,x) = [A, \epsilon^*]^\alpha(x)$. Relation (4.24) and Proposition B.33 thus yield the relation

$$m^*(x,y) \;=\; x - m^*(0,x) + m^*(0,y) \;=\; x - [A,\epsilon^*]^\alpha(x) + [A,\epsilon^*]^\alpha(y) \;=\; 2x - y$$

for all $x, y \in [A, \Omega^\sigma \mathbf{U}]^\alpha$. The map $(\Omega^\sigma\psi)_*$ is compatible with all operations that are natural for C-global σ-loop spaces. This includes in particular the binary operation m^*. So

$$\begin{aligned}(\Omega^\sigma\psi)_*(2x-y) \ &= \ (\Omega^\sigma\psi)_*(m^*(x,y)) \\ &= \ m^*((\Omega^\sigma\psi)_*(x), (\Omega^\sigma\psi)_*(y)) \ = \ 2\cdot(\Omega^\sigma\psi)_*(x) - (\Omega^\sigma\psi)_*(y)\end{aligned}$$

for all $x, y \in [A, \Omega^\sigma \mathbf{U}]^\alpha$.

(ii) We claim that for every orthogonal C-spectrum Y, the map $[A, \epsilon^*]^\alpha \colon [A, \Omega^\sigma(\Omega^\bullet Y)]^\alpha \longrightarrow [A, \Omega^\sigma(\Omega^\bullet Y)]^\alpha$ is multiplication by the class $\alpha^*(\epsilon) \in \pi_0^G(\mathbb{S})$. This is almost a tautology, and a special case of a much more general fact: for every orthogonal representation V of a compact Lie group G, and every continuous based G-map $f\colon S^V \longrightarrow S^V$, precomposition with f and multiplication by $[f] \in \pi_0^G(\mathbb{S})$ coincide on $\pi_V^G(X)$ for every orthogonal G-spectrum X.

Also for every orthogonal C-spectrum Y, the map

$$m^* \ = \ [A, m^*]^\alpha \ : \ [A, \Omega^\sigma(\Omega^\bullet Y)]^\alpha \times [A, \Omega^\sigma(\Omega^\bullet Y)]^\alpha \ \longrightarrow \ [A, \Omega^\sigma(\Omega^\bullet Y)]^\alpha$$

is additive for the abelian group structure arising from stability. Relation (4.24) and the claim above thus yield the relation

$$m^*(x,y) \ = \ x - m^*(0,x) + m^*(0,y) \ = \ x - [A,\epsilon^*]^\alpha(x) + [A,\epsilon^*]^\alpha(y) \ = \ x - \alpha^*(\epsilon)\cdot x + \alpha^*(\epsilon)\cdot y$$

for all $x, y \in [A, \Omega^\sigma(\Omega^\bullet Y)]^\alpha$. The map

$$(\Omega^\sigma d)_* = [A, \Omega^\sigma d]^\alpha \ : \ [A, \Omega^\sigma \mathbf{U}]^\alpha \ \longrightarrow \ [A, \Omega^\sigma(\Omega^\bullet(\Sigma^\infty_+ \mathbf{P} \wedge S^\sigma))]^\alpha$$

is compatible with all operations that are natural for C-global σ-loop spaces. This includes in particular the binary operation m^*. So

$$\begin{aligned}(\Omega^\sigma d)_*(2x-y) \ &= \ (\Omega^\sigma d)_*(m^*(x,y)) \\ &= \ m^*((\Omega^\sigma d)_*(x), (\Omega^\sigma d)_*(y)) \\ &= \ (\Omega^\sigma d)_*(x) - \alpha^*(\epsilon)\cdot(\Omega^\sigma d)_*(x) + \alpha^*(\epsilon)\cdot(\Omega^\sigma d)_*(y)\end{aligned} \tag{4.25}$$

for all $x, y \in [A, \Omega^\sigma \mathbf{U}]^\alpha$.

The global Segal–Becker splitting c was defined by the commutative square (4.14). It induces a commutative diagram

$$\begin{tikzcd}
{[A, \mathbf{BUP}]^\alpha} \arrow[r, "{[A,c]^\alpha}"] \arrow[d, "{[A,\gamma]^\alpha}"', "\cong"] & {[A, \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})]^\alpha} \arrow[d, "\cong"', "{[A, \xi_{\Sigma^\infty_+ \mathbf{P}}]^\alpha}"] \\
{[A, \Omega^\sigma \mathbf{U}]^\alpha} \arrow[r, "{[A,\Omega^\sigma d]^\alpha}"'] & {[A, \Omega^\sigma(\Omega^\bullet(\Sigma^\infty_+ \mathbf{P} \wedge S^\sigma))]^\alpha}
\end{tikzcd}$$

in which all objects are abelian groups, but the horizontal maps are *not* generally additive. The morphism γ is a Real-global equivalence of ultra-commutative C-monoids, so the induced bijection is additive. The C-global equivalence $\xi_{\Sigma^\infty_+ \mathbf{P}}$ arises from a C-global stable map, so the induced bijection is additive and compatible with multiplication by the class $\alpha^*(\epsilon)$. So relation (4.25) for the lower horizontal map implies the analogous relation for the upper horizontal map $c_* = [A, c]^\alpha$:

$$c_*(2x-y) \ = \ c_*(x) - \alpha^*(\epsilon)\cdot c_*(x) + \alpha^*(\epsilon)\cdot c_*(y) \ .$$

Setting $x = 0$ yields $c_*(-y) = \alpha^*(\epsilon)\cdot c_*(y)$, and thus

$$\begin{aligned}c_*(2x+y) \ &= \ c_*(2x-(-y)) \\ &= \ c_*(x) - \alpha^*(\epsilon)\cdot c_*(x) + \alpha^*(\epsilon)\cdot c_*(-y) \ = \ (1-\alpha^*(\epsilon))\cdot c_*(x) + c_*(y) \ . \qquad \square\end{aligned}$$

Example 4.26 (c_* is not additive)**.** In the special case of the group C augmented by the identity, and for $A = *$, the global Segal–Becker splitting becomes a map

$$c_* \ : \ \pi_0^C(\mathbf{BUP}) \ \longrightarrow \ \pi_0^C(\Sigma^\infty_+\mathbf{P}) \ .$$

In this case the group $\pi_0^C(\mathbf{BUP}) \cong RR(C)$ is infinite cyclic, generated by the class $x = h_*(\mathrm{res}^{\tilde{T}}_C(u_{\tilde{T}}))$, where $u_{\tilde{T}} \in \pi_0^{\tilde{T}}(\mathbf{P})$ is the unstable tautological class (4.3), and $h\colon \mathbf{P} \longrightarrow \mathbf{BUP}$ is the 'inclusion of line bundles' defined in Construction 4.17. Moreover, $\pi_0^C(\Sigma^\infty_+\mathbf{P})$ is a free module of rank 1 over the ring $\pi_0^C(\mathbb{S}) = \mathbb{Z}[\epsilon]/(\epsilon^2-1)$. We have $c_*(0) = 0$ and

$$c_*(x) \ = \ c_*(h_*(\mathrm{res}^{\tilde{T}}_C(u_{\tilde{T}}))) \ = \ \mathrm{res}^{\tilde{T}}_C(c_*(h_*(u_{\tilde{T}}))) \ = \ \mathrm{res}^{\tilde{T}}_C(e_{\tilde{T}}) \ = \ 1 \ .$$

The third equation uses that $c \circ h\colon \mathbf{P} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P})$ is the adjunction unit, see Corollary 4.19. Theorem 4.23 (ii) provides the relation

$$c_*(2x) \ = \ (1-\epsilon)\cdot c_*(x) \ = \ 1-\epsilon \ \neq \ 2 \ = \ 2\cdot c_*(x) \ .$$

In particular, the map c_* is *not* additive.

5. The G-equivariant Segal–Becker splitting and explicit Brauer induction

In this section we show that our global Segal–Becker splitting $c\colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P})$ rigidifies and globalizes the classical equivariant Segal–Becker splittings at the level of equivariant cohomology theories, and that it induces the Boltje–Symonds 'explicit Brauer induction' on equivariant homotopy groups, The first fact is Theorem C of the introduction, and Theorem 5.8 below; the second fact is Theorem D of the introduction, and Corollary 5.15 below.

In most of this section we restrict to trivially augmented Lie groups; or, equivalently, we only look at the effect of the underlying of 'non-Real' global phenomena, after forgetting the C-action. The reason for this is twofold. Firstly, the equivariant Segal–Becker splitting and the explicit Brauer induction have previously almost only been considered for equivariant complex K-theory over compact Lie groups, without any augmentation to $C = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and without any conjugate-linear phenomena involved. The notable exception is [14, Theorem 1'], which, however, has to be taken with a grain of salt, see Remark 5.5.

Secondly, forgetting the C-action removes the equivariant twist from the procedures of shifting, suspending and looping by the sign representation σ. In particular, forgetting the C-action reveals the global Segal–Becker splitting $c\colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P})$ as a global loop map (as opposed to a Real-global σ-loop map); consequently, the effect of this global loop map on functors such as $[A,-]^G$ is an *additive* map. For non-trivial augmentations $\alpha\colon G \longrightarrow C$, the map $[A,c]^\alpha\colon [A,\mathbf{BUP}]^\alpha \longrightarrow [A,\Omega^\bullet(\Sigma^\infty_+\mathbf{P})]^\alpha$ is typically not additive, see Example 4.26.

Construction 5.1 (The equivariant Segal–Becker splitting)**.** We let $\alpha\colon G \longrightarrow C$ be an augmented Lie group, and we let A be a finite G-CW-complex. We recall the equivariant Segal–Becker splitting via equivariant transfers due to Iriye–Kono [14, §3], following Crabb's presentation [10]. Crabb only discusses the construction for complex vector bundles, which corresponds to the special case where the augmentation α is trivial, and hence need not be mentioned. A large part of the construction generalizes to general augmented Lie groups. However, as we explain in more detail below, an obstruction appears when one attempts to extend the construction from vector bundles to virtual vector bundles.

We let $\xi\colon E \longrightarrow A$ be a Real α-vector bundle. We denote by

$$P\xi \ : \ PE \ \longrightarrow \ A \tag{5.2}$$

the projectivized bundle; its fiber over $a \in A$ is the projective space of the complex vector space $E_a = \xi^{-1}(\{a\})$. The total space PE inherits a continuous G-action, and the projection to A is G-equivariant. The projection (5.2) thus has an associated transfer, a morphism in the homotopy category of genuine

G-spectra from $\Sigma^\infty_+ A$ to $\Sigma^\infty_+ PE$. Since A is a finite G-CW-complex, this transfer is represented by a continuous based G-map

$$\tau(P\xi) \ : \ S^V \wedge A_+ \ \longrightarrow \ S^V \wedge (PE)_+ \ ,$$

for some orthogonal G-representation V. The tautological G-equivariant line bundle over PE is a Real α-line bundle, and thus classified by a continuous G-map

$$\kappa \ : \ PE \ \longrightarrow \ \mathbf{P}(W)$$

for some sufficiently large orthogonal G-subrepresentation W of $\mathcal{U}_G$. By enlarging, if necessary, we may assume that $W = V$. We write

$$\vartheta(\xi) \ \in \ [A, \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})]^\alpha$$

for the class of the adjoint to the composite

$$S^V \wedge A_+ \ \xrightarrow{\tau(P\xi)} \ S^V \wedge (PE)_+ \ \xrightarrow{S^V \wedge \kappa_+} \ S^V \wedge \mathbf{P}(V)_+ \ .$$

In the classical sources one finds a verification that the class $\vartheta(\xi)$ only depends on the isomorphism class of the G-vector bundle ξ. So the construction provides a well-defined map

$$\vartheta_{\alpha,A} \ : \ \mathrm{Vect}^R_\alpha(A) \ \longrightarrow \ [A, \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})]^\alpha \tag{5.3}$$

that is natural for continuous G-maps in A, and for restriction along morphisms of augmented Lie groups.

At this point the paths for trivially and surjectively augmented Lie groups diverge. If the augmentation $\alpha\colon G \longrightarrow C$ is trivial, then Real α-vector bundles are just complex G-vector bundles, and we are in the context discussed by Crabb [10]. In this case the map (5.3) is additive for the Whitney sum of vector bundles, see [10, Lemma 2.6]. The map thus extends uniquely to an additive map

$$\vartheta_{G,A} \ : \ KU_G(A) \ \longrightarrow \ [A, \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})]^G \tag{5.4}$$

on the group completion $KU_G(A)$, the complex G-equivariant K-group of A. These equivariant Segal–Becker splittings (5.4) are again natural for continuous G-maps in A, and for restriction along continuous group homomorphisms between compact Lie groups.

As we show in Remark 5.7 below, the equivariant Segal–Becker splitting (5.3) is *not* generally additive if the augmentation $\alpha\colon G \longrightarrow C$ is non-trivial. Consequently, one cannot appeal to the group completion property to extend (5.3) from $\mathrm{Vect}^R_\alpha(A)$ it its group completion $KR_\alpha(A)$. Our global Segal–Becker splitting provides a section

$$KR_\alpha(A) \ \cong_{(\mathrm{B}.10)} \ [A, \mathbf{BUP}]^\alpha \ \xrightarrow{[A,c]^\alpha} \ [A, \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})]^\alpha \ ;$$

this section is also not additive for non-trivial augmentations, see Example 4.26.

Remark 5.5. A Real Lie group in the sense of [3, Section 5] is an augmented Lie group that arises as the semidirect product $\tilde{G} = G \rtimes_\tau C$ by a multiplicative involution τ of a compact Lie group G. Kono and Iriye state in [14, Theorem 1'] that for compact $\tilde{G}$-spaces A 'there exists a split epimorphism'

$$\lambda_* \ : \ [A, \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})]^{\tilde{G}} \ \longrightarrow \ KR_{\tilde{G}}(A) \ .$$

As for a proof, Kono and Iriye say that 'a parallel argument' as in the trivially augmented case applies.

I would like to explain why I find this claim suspicious. Kono and Iriye do not specify the map λ_* that they claim to be a split epimorphism; however, the context and the earlier 'non-Real' part of their paper suggests that the map they have in mind should be

$$[A, \Omega^\bullet(\eta)]^{\tilde{G}} \;:\; [A, \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})]^{\tilde{G}} \;\longrightarrow\; [A, \Omega^\bullet(\mathbf{KR})]^{\tilde{G}} \cong KR_{\tilde{G}}(A) \;,$$

with the isomorphism from Theorem B.59. Moreover, the transfer techniques employed by Kono and Iriye in §2 of their paper suggest that the construction of the splitting they had in mind specializes to the map (5.3) when applied to actual (as opposed to virtual) Real-equivariant vector bundles. The usual meaning of 'split epimorphism' as one with an *additive* section is not consistent with this interpretation, as the map (5.3) is provably not additive, see Remark 5.7 below. This apparent inconsistency would disappear if Kono and Iriye had a different splitting in mind, or if 'split epimorphism' simply means 'surjective homomorphism'.

The universal case of a Real-equivariant vector bundle of dimension k is the tautological $\tilde{U}(k)$-representation ν_k, considered as a Real $\tilde{U}(k)$-equivariant vector bundle over a point. So we need to understand the equivariant Segal–Becker splitting for this.

Proposition 5.6. *Let $[\nu_k] \in KR_{\tilde{U}(k)}(*)$ denote the class of the tautological Real $\tilde{U}(k)$-representation on $\mathbb{C}^k$, considered as a Real $\tilde{U}(k)$-equivariant vector bundle over a point. Then*

$$\vartheta_{\tilde{U}(k),*}[\nu_k] \;=\; \mathrm{tr}^{\tilde{U}(k)}_{\tilde{U}(k-1,1)}(1 \times e_{\tilde{T}})$$

in $[*, \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})]^{\tilde{U}(k)} = \pi_0^{\tilde{U}(k)}(\Sigma^\infty_+ \mathbf{P})$, where $e_{\tilde{T}} \in \pi_0^{\tilde{T}}(\Sigma^\infty_+ \mathbf{P})$ is the stable tautological class (4.3).

Proof. The projective space $P(\nu_k)$ of the tautological representation is a homogeneous space: the group $\tilde{U}(k)$ acts transitively on $P(\nu_k)$, and the complex line

$$l \;=\; \mathbb{C} \cdot (0, \ldots, 0, 1)$$

stabilizer group $\tilde{U}(k-1,1)$. So the equivariant transfer

$$\tau(P(\nu_k)) \;:\; \mathbb{S} \;\longrightarrow\; \Sigma^\infty_+ P(\nu_k)$$

associated with the unique $\tilde{U}(k)$-map $P(\nu_k) \longrightarrow *$ sends $1 \in \pi_0^{\tilde{U}(k)}(\mathbb{S})$ to the class

$$\mathrm{tr}^{\tilde{U}(k)}_{\tilde{U}(k-1,1)}(\sigma^{\tilde{U}(k-1,1)}[l]) \;\in \pi_0^{\tilde{U}(k)}(\Sigma^\infty_+ P(\nu_k)) \;,$$

where $[l] \in \pi_0(P(\nu_k)^{\tilde{U}(k-1,1)})$ is the class represented by the $\tilde{U}(k-1,1)$-fixed point l, and

$$\sigma^{\tilde{U}(k-1,1)} \;:\; \pi_0(P(\nu_k)^{\tilde{U}(k-1,1)}) \;\longrightarrow\; \pi_0^{\tilde{U}(k-1,1)}(\Sigma^\infty_+ P(\nu_k))$$

is the stabilization map [24, (3.3.12)].

The stabilizer group $\tilde{U}(k-1,1)$ acts on the invariant line l through the homomorphism $q\colon \tilde{U}(k-1,1) \longrightarrow \tilde{T}$ that projects to the last block; so the classifying $\tilde{U}(k)$-map

$$\kappa \;:\; P(\nu_k) \;\longrightarrow\; \mathbf{P}(\mathcal{U}_{U(k)})$$

for the tautological line bundle satisfies

$$\kappa_*[l] \;=\; q^*(u_{\tilde{T}}) \quad \text{in } \pi_0^{\tilde{U}(k-1,1)}(\mathbf{P}) \;,$$

where $u_{\tilde{T}} \in \pi_0^{\tilde{T}}(\mathbf{P})$ is the unstable tautological class (4.3). Thus

$$\begin{aligned}(\Sigma^\infty_+ \kappa)_*(\sigma^{\tilde{U}(k-1,1)}[l]) \;&=\; \sigma^{\tilde{U}(k-1,1)}(\kappa_*[l]) \\ &=\; \sigma^{\tilde{U}(k-1,1)}(q^*(u_{\tilde{T}})) \;=\; q^*(\sigma^{\tilde{T}}(u_{\tilde{T}})) \;=\; 1 \times e_{\tilde{T}} \;.\end{aligned}$$

Combining these observations yields

$$\begin{aligned}\vartheta_{\tilde U(k),*}[\nu_k] \ &= \ ((\Sigma^\infty_+\kappa)\circ\tau(P(\nu_k)))_*(1) \ = \ (\Sigma^\infty_+\kappa)_*(\mathrm{tr}_{\tilde U(k-1,1)}^{\tilde U(k)}(\sigma^{\tilde U(k-1,1)}[l]))\\ &= \ \mathrm{tr}_{\tilde U(k-1,1)}^{\tilde U(k)}((\Sigma^\infty_+\kappa)_*(\sigma^{\tilde U(k-1,1)}[l])) \ = \ \mathrm{tr}_{\tilde U(k-1,1)}^{\tilde U(k)}(1\times e_{\tilde T}) \ .\end{aligned}$$

□

Remark 5.7 (Beware non-additivity)**.** We can now show that the map $\vartheta_{\alpha,A}$ from (5.3) is *not* generally additive for augmented Lie groups with non-trivial augmentation. This is in contrast to trivially augmented Lie groups, where the map is additive by [10, Lemma 2.6]. We need the double coset formula for $\mathrm{res}^{\tilde U(2)}_C\circ\mathrm{tr}^{\tilde U(2)}_{\tilde U(1,1)}$, where C sits inside $\tilde U(2)$ as complex conjugation. The general double coset formula for $\mathrm{res}^G_K\circ\mathrm{tr}^G_H$ can be found in [20, IV Section 6] or [24, Theorem 3.4.9], and we need to specialize it. The homogeneous space $\tilde U(2)/\tilde U(1,1)$ is homeomorphic to the projective space $P(\mathbb{C}^2)$, and the C-action on $\tilde U(2)/\tilde U(1,1)$ by left translation corresponds to the action on $P(\mathbb{C}^2)$ by complex conjugation. The double coset space $C\backslash\tilde U(2)/\tilde U(1,1)$ is thus homeomorphic to a 2-disc D^2, stratified by its boundary (coming from the C-fixed points) and the interior (corresponding to the points of $\tilde U(2)/\tilde U(1,1)$ on which C acts freely). The C-fixed point $(\tilde U(2)/\tilde U(1,1))^C\cong P(\mathbb{C}^2)^C$ are homeomorphic to S^1, thus have Euler characteristic 0, and so do not contribute to the double coset formula. The non-singular part is an open 2-disc, with internal Euler characteristic

$$\chi^\sharp(C\backslash P(\mathbb{C}^2)_{\mathrm{free}}) \ = \ \chi(D^2)-\chi(\partial D^2) \ = \ 1 \ .$$

So the double coset formula becomes

$$\mathrm{res}^{\tilde U(2)}_C\circ\mathrm{tr}^{\tilde U(2)}_{\tilde U(1,1)} \ = \ \mathrm{tr}^C_{\{1\}}\circ\mathrm{res}^{\tilde U(1,1)}_{\{1\}} \ .$$

Now we consider the group C, augmented over itself by the identity. We show that the map $\vartheta_{C,*}\colon \mathrm{Vect}^R_C(*)\longrightarrow\pi_0^C(\Sigma^\infty_+\mathbf{P})$ is not additive. The ring $\pi_0^C(\Sigma^\infty_+\mathbf{P})$ is isomorphic to $\mathbb{Z}[\epsilon]/(\epsilon^2-1)$, for ϵ the class of the sign involution of S^σ. The monoid $\mathrm{Vect}^R_C(*)$ of isomorphism classes of Real C-representations is isomorphic to $\mathbb{N}$, generated by the class of $\mathbb{C}$ with C-action by complex conjugation; another name for this generator is $\mathrm{res}^{\tilde T}_C[\nu_1]$, with C embedded in $\tilde T$ as complex conjugation. Proposition 5.6 thus yields

$$\vartheta_{C,*}[\mathbb{C}] \ = \ \vartheta_{C,*}(\mathrm{res}^{\tilde T}_C[\nu_1]) \ = \ \mathrm{res}^{\tilde T}_C(\vartheta_{\tilde T,*}[\nu_1]) \ = \ \mathrm{res}^{\tilde T}_C(e_{\tilde T}) \ = \ 1 \ .$$

We have

$$[\mathbb{C}]\oplus[\mathbb{C}] \ \cong \ \mathrm{res}^{\tilde U(2)}_C(\nu_2)$$

as Real C-representations. So by Proposition 5.6 and the double coset formula,

$$\begin{aligned}\vartheta_{C,*}(2\cdot[\mathbb{C}]) \ &= \ \vartheta_{C,*}(\mathrm{res}^{\tilde U(2)}_C[\nu_2]) \ = \ \mathrm{res}^{\tilde U(2)}_C(\vartheta_{\tilde U(2),*}[\nu_2])\\ &= \ \mathrm{res}^{\tilde U(2)}_C(\mathrm{tr}^{\tilde U(2)}_{\tilde U(1,1)}(1\times e_{\tilde T})) \ = \ \mathrm{tr}^C_{\{1\}}(\mathrm{res}^{\tilde U(1,1)}_{\{1\}}(1\times e_{\tilde T}))\\ &= \ \mathrm{tr}^C_{\{1\}}(1) \ = \ 1-\epsilon \ \neq \ 2 \ = \ 2\cdot\vartheta_{C,*}[\mathbb{C}] \ .\end{aligned}$$

So the map $\vartheta_{C,*}$ is not additive.

Now we proceed to prove Theorem C of the introduction, saying that our global Segal–Becker splitting induces the classical equivariant Segal–Becker at the level of equivariant cohomology theories. In Theorem B.12 we exhibit a natural isomorphism of abelian monoids

$$\langle-\rangle \ : \ [A,\mathbf{Gr}]^\alpha \ \xrightarrow{\cong} \ \mathrm{Vect}^R_\alpha(A)$$

for every augmented Lie group $\alpha\colon G\longrightarrow C$ and every finite G-CW-complex A. In [24, page 215] we define a morphism of ultra-commutative C-monoids $i\colon \mathbf{Gr}\longrightarrow\mathbf{BUP}$ that is a Real-global group completion, see also (B.4).

Theorem 5.8. *Let $\alpha\colon G\longrightarrow C$ be an augmented Lie group, and let A be a finite G-CW-complex.*

(i) *The following diagram commutes:*

$$\begin{tikzcd}
{[A, \mathbf{Gr}]^\alpha} \arrow[r, "\langle - \rangle", "\cong\ (B.9)"'] \arrow[d, "{[A,i]^\alpha}"'] & \mathrm{Vect}^R_\alpha(A) \arrow[d, "\vartheta_{G,A}"] \\
{[A, \mathbf{BUP}]^\alpha} \arrow[r, "{[A,c]^\alpha}"'] & {[A, \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})]^\alpha}
\end{tikzcd}$$

(ii) *If the augmentation α is trivial, then the composite*

$$KU_G(A) \ \cong_{(B.10)} \ \ [A, \mathbf{BUP}]^G \ \xrightarrow{[A,c]^G} \ \ [A, \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})]^G$$

coincides with the G-equivariant Segal–Becker splitting $\vartheta_{G,A}$ defined in (5.4).

Proof. (i) For every augmented Lie group $\alpha\colon G \longrightarrow C$, the following diagram commutes by the definition of the morphism $c\colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})$ from the morphism $d\colon \mathbf{U} \longrightarrow \Omega^\bullet(\Sigma^\infty_+ \mathbf{P} \wedge S^\sigma)$, which in turn was defined as the adjoint of $d^\flat\colon \Sigma^\infty \mathbf{U} \longrightarrow \Sigma^\infty_+ \mathbf{P} \wedge S^\sigma$:

$$\begin{tikzcd}
\pi_0^\alpha(\mathbf{Gr}) \arrow[r, "i_*"] \arrow[dr, bend right, "\beta_*"'] & \pi_0^\alpha(\mathbf{BUP}) \arrow[rrr, "c_*"] \arrow[d, "\gamma_*", "\cong"'] & & & \pi_0^\alpha(\Sigma^\infty_+ \mathbf{P}) \arrow[d, "- \wedge S^\sigma", "\cong"'] \\
& \pi_0^\alpha(\Omega^\sigma \mathbf{U}) \arrow[r, equal] & \pi_\sigma^\alpha(\mathbf{U}) \arrow[r, "\sigma^\alpha"] \arrow[rr, bend right, "d_*"'] & \pi_\sigma^\alpha(\Sigma^\infty \mathbf{U}) \arrow[r, "d^\flat_*"] & \pi_\sigma^\alpha(\Sigma^\infty_+ \mathbf{P} \wedge S^\sigma)
\end{tikzcd}$$

The map $\sigma^\alpha\colon \pi_\sigma^\alpha(\mathbf{U}) \longrightarrow \pi_\sigma^\alpha(\Sigma^\infty \mathbf{U})$ is the stabilization map [24, (3.3.12)].

The $\tilde{U}(k)$-equivariant linear embedding $\zeta^k = \zeta^{\nu_k}\colon \nu_k \longrightarrow u(\nu_k)_\mathbb{C}$ was defined in (1.7). Its image is a $\tilde{U}(k)$-invariant linear subspace of dimension k, and thus a $\tilde{U}(k)$-fixed point of $Gr_k^\mathbb{C}(u(\nu_k)_\mathbb{C}) = \mathbf{Gr}_k(\nu_k)$. We write

$$\{\nu_k\} \ = \ [\mathrm{im}(\zeta^k)] \ \in \ \pi_0^{\tilde{U}(k)}(\mathbf{Gr}_k) \subset \pi_0^{\tilde{U}(k)}(\mathbf{Gr})$$

for its homotopy class. By inspection of definitions, the map $\beta_*\colon \pi_0^{\tilde{U}(k)}(\mathbf{Gr}) \longrightarrow \pi_0^{\tilde{U}(k)}(\Omega^\sigma \mathbf{U})$ defined in (B.28) sends $\{\nu_k\}$ to the homotopy class of the map

$$S^\sigma \ \xrightarrow{\partial \circ \mathfrak{c}} \ U(k) \ \xrightarrow{\zeta^k_*} \ \mathbf{U}(u(\nu_k)) \ .$$

Since u_k is represented by the suspension by S^{ν_k} of the exact same map, this proves that

$$\sigma^{\tilde{U}(k)}(\beta_*\{\nu_k\}) \ = \ u_k \ .$$

The commutative diagram thus shows the relation

$$(c \circ i)_*\{\nu_k\} \wedge S^\sigma \ = \ d^\flat_*(u_k) = \ \mathrm{tr}_{\tilde{U}(k-1,1)}^{\tilde{U}(k)}(1 \times e_{\tilde{T}}) \wedge S^\sigma$$

in the group $\pi_\sigma^{\tilde{U}(k)}(\Sigma^\infty_+ \mathbf{P} \wedge S^\sigma)$; the second equation is Proposition 4.5 (iii). Since suspension by S^σ is bijective, this proves that

$$(5.9) \qquad (c \circ i)_*\{\nu_k\} \ = \ \mathrm{tr}_{\tilde{U}(k-1,1)}^{\tilde{U}(k)}(1 \times e_{\tilde{T}}) \ = \ \vartheta_{\tilde{U}(k),*}[\nu_k] \ .$$

The second equation is Proposition 5.6. The map (B.9) takes $\{\nu_k\}$ to the isomorphism class of ν_k, considered as a Real $\tilde{U}(k)$-vector bundle over a point. So equation (5.9) shows that the square of part (i) commutes for $G = \tilde{U}(k)$ and $A = *$ on the class $\{\nu_k\}$.

Now we can prove part (i). We compose both composites in the diagram with the map $[A, \mathbf{Gr}_k]^\alpha \longrightarrow [A, \mathbf{Gr}]^\alpha$ induced by the inclusion $\mathbf{Gr}_k \longrightarrow \mathbf{Gr}$. Letting α and A vary yields two C-global transformations from $\mathbf{Gr}_k$ to $\Omega^\bullet(\Sigma^\infty_+ \mathbf{P})$ in the sense of Definition A.8. The orthogonal C-space $\mathbf{Gr}_k$ is a global classifying space for the augmented Lie group $\tilde{U}(k)$, and the class $\{\nu_k\}$ is the tautological class $u_{\tilde{U}(k),*}$ in the sense of (A.10). Since the C-global transformations coincide for $\alpha = \tilde{U}(k)$ and $A = *$ on the class of ν_k, the two C-global transformations commute altogether by the uniqueness part of Theorem A.11. In other words: the diagram commutes for all classes in the image of $[A, \mathbf{Gr}_k]^\alpha$ for some $k \geq 0$.

Now we suppose that A is 'G-connected' in the sense that the group $\pi_0(G)$ acts transitively on $\pi_0(A)$. This ensures that every Real α-vector bundle over A has constant rank, and every class in $[A, \mathbf{Gr}]^\alpha$ is in the image of $[A, \mathbf{Gr}_k]^\alpha$ for some $k \geq 0$. Hence the diagram (i) commutes for such A.

Finally we let A be any finite G-CW-complex. Then $A = A_1 \amalg \ldots \amalg A_m$ is a disjoint union of G-connected finite G-CW-complexes, indexed by the $\pi_0(G)$-orbits of $\pi_0(A)$. We write $\iota_j \colon A_j \longrightarrow A$ for the inclusion of the j-th summand. Then the map

$$(\iota_1^*, \ldots, \iota_m^*) \ : \ [A, \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})]^\alpha \ \longrightarrow \ [A_1, \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})]^\alpha \times \cdots \times [A_m, \Omega^\bullet(\Sigma^\infty_+ \mathbf{P})]^\alpha$$

is bijective. Hence it suffices to show that the diagram commutes after postcomposition with ι_j^* for each $1 \leq j \leq m$. But this is the case by naturality for $\iota_j \colon A_j \longrightarrow A$, and because the diagram commutes for A_j by the previous case.

(ii) Part (i) shows that maps $\vartheta_{G,A}$ and $[A, c]^G \circ (\text{B.10})^{-1}$ coincide on all classes in $KU_G(A)$ that are represented by an equivariant vector bundle. Since the morphism c is a global loop map after forgetting the C-action, the induced map $[A, c]^G$ is additive for trivially augmented Lie groups. The map $\vartheta_{G,A}$ is additive by [10, Lemma 2.6]. Since vector bundles of constant rank generate $KU_G(A)$ as an abelian group, this proves the theorem. $\square$

We conclude this section with the proof of Theorem D from the introduction, saying that the effect of the global Segal–Becker splitting $c \colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_* \mathbf{P})$ on equivariant homotopy groups is the *explicit Brauer induction* of Boltje and Symonds.

Remark 5.10 (Explicit Brauer induction)**.** Brauer showed in [8, Theorem I] that the complex representation ring of a finite group is generated, as an abelian group, by representations that are induced from 1-dimensional representations of subgroups. Segal generalized this result to compact Lie groups in [28, Proposition 3.11 (ii)], where 'induction' refers to smooth induction. We write $\mathbf{A}(T, G)$ for the free abelian group with a basis the symbols $[H, \chi]$, where H runs over all conjugacy classes of closed subgroup of G with finite Weyl group, and $\chi \colon H \longrightarrow T = U(1)$ runs over all characters of H. The Brauer–Segal theorem can then be paraphrased as the fact that the map

$$\mathbf{A}(T, G) \ \longrightarrow \ R(G) \tag{5.11}$$

that sends $[H, \chi]$ to $\operatorname{tr}_H^G(\chi^*[\nu_1])$ is surjective, for every compact Lie group G, where $[\nu_1] \in R(T)$ is the class of the tautological T-representation on $\mathbb{C}$. Informally speaking, an 'explicit Brauer induction' is a collection of sections to the maps (5.11) that are specified by a direct recipe, for example an explicit formula, and with naturality properties as the group G varies. So such maps give an 'explicit and natural' way to write virtual representations as sums of induced representations of 1-dimensional representations. What qualifies as 'explicit' is, of course, in the eye of the beholder.

The first explicit Brauer induction was Snaith's formula [33, Theorem 2.16]; however, Snaith's maps are not additive and not compatible with restriction to subgroups. Boltje [7] specified a different explicit Brauer induction formula for finite groups by purely algebraic means; Symonds [34] gave a topological interpretation of the same section in the context of compact Lie groups. Symonds' construction [34, §4] of the sections

$$b_G \ : \ R(G) \ \longrightarrow \ \mathbf{A}(T, G) \tag{5.12}$$

is designed so that

$$b_{U(k)}[\nu_k] \ = \ [U(k-1, 1), q] \qquad \text{in} \quad \mathbf{A}(T, U(k)) \tag{5.13}$$

for all $k \geq 1$, where $q\colon U(k-1,1) \longrightarrow T$ is the projection to the second block. The Boltje–Symonds maps are additive and natural for restriction along continuous group homomorphisms; and the value of b_G at a 1-dimensional representation with character $\chi\colon G \longrightarrow T$ is given by

$$b_G[\chi] \ = \ [G, \chi] \ \in \ \mathbf{A}(T, G) \ .$$

The Boltje–Symonds maps (5.12) are not (and in fact cannot be) in general compatible with transfers.

Since the orthogonal space $\mathbf{P}$ is a global classifying space for the circle group T, the global functor $\underline{\pi}_0(\Sigma^\infty_+\mathbf{P})$ is represented by T, see [24, Proposition 4.2.5]. In more down-to-earth terms, this means that the abelian group $\pi_0^G(\Sigma^\infty_+\mathbf{P})$ is free, with a basis given by the classes $\mathrm{tr}_H^G(\chi^*(e_T))$, for (H,χ) ranging over all conjugacy classes of closed subgroups H of G with finite Weyl group, and all continuous homomorphisms $\chi\colon H \longrightarrow T$; see [24, Corollary 4.1.13]. The group $\mathbf{A}(T,G)$ was defined as a free abelian group with a corresponding basis, so the map

$$\mathbf{A}(T,G) \ \longrightarrow \ \pi_0^G(\Sigma^\infty_+\mathbf{P}) \ , \quad [H,\chi] \ \longmapsto \ \mathrm{tr}_H^G(\chi^*(e_T)) \tag{5.14}$$

is an isomorphism of abelian groups.

The Boltje–Symonds map $b_G\colon R(G) \longrightarrow \mathbf{A}(T,G)$ is the special case of the equivariant Segal–Becker splitting (5.4) for $A = *$, in the sense that the composite

$$R(G) = KU_G(*) \ \xrightarrow{\vartheta_{G,*}} \ [*, \Omega^\bullet(\Sigma^\infty_+\mathbf{P})]^G \ = \ \pi_0^G(\Sigma^\infty_+\mathbf{P}) \ \cong_{(5.14)} \ \mathbf{A}(T,G)$$

agrees with (5.12). Indeed, b_G and $\vartheta_{G,*}$ coincide for $G = U(k)$ on the class of the tautological representation ν_k, by (5.13) and Proposition 5.6. So they agree on arbitrary unitary representations by naturality, and on virtual representations by additivity. So the next theorem becomes a special case of Theorem 5.8 for $A = *$.

Corollary 5.15. *For every compact Lie group G, the map $\pi_0^G(c)\colon \pi_0^G(\mathbf{BUP}) \longrightarrow \pi_0^G(\Sigma^\infty_+\mathbf{P})$ equals the composite*

$$\pi_0^G(\mathbf{BUP}) \ \cong_{(B.10)} \ R(G) \ \xrightarrow[(5.12)]{b_G} \ \mathbf{A}(T,G) \ \cong_{(5.14)} \ \pi_0^G(\Sigma^\infty_+\mathbf{P}) \ .$$

6. Global Adams operations

In this section we give an application of the global Segal–Becker splitting: we construct Real-global rigidifications (6.5) of the unstable Adams operations in equivariant K-theory.

Construction 6.1 (Adams operations in equivariant K-theory)**.** We let $\alpha\colon G \longrightarrow C$ be an augmented Lie group. We recall the construction of the λ-operations and Adams operations on the Grothendieck ring $KR_\alpha(A)$ of Real α-equivariant vector bundles over a compact G-space A. For all Real α-vector bundles ξ and ζ over the same base, the Real α-vector bundle $\Lambda^n(\xi \oplus \zeta)$ is isomorphic to $\bigoplus_{i=0}^n \Lambda^i(\xi) \otimes_{\mathbb{C}} \Lambda^{n-i}(\zeta)$. So the map

$$\Lambda \ : \ \mathrm{Vect}^R_\alpha(A) \ \longrightarrow \ KR_\alpha(A)[\![t]\!] \ , \quad [\xi] \ \longmapsto \ \sum\nolimits_{n\geq 0} [\Lambda^n(\xi)] \cdot t^n$$

takes addition in the abelian monoid of isomorphism classes of α-vector bundles to multiplication in the power series ring $KR_\alpha(A)[\![t]\!]$. All these power series moreover have constant term $\Lambda^0(\xi) = 1$, and are thus invertible. So Λ defines a monoid homomorphism from $\mathrm{Vect}^R_\alpha(A)$ to the multiplicative group of the ring $KR_\alpha(A)[\![t]\!]$. The universal property of the Grothendieck construction thus yields an extension to a group homomorphism

$$\Lambda \ : \ KR_\alpha(A) \ \longrightarrow \ (KR_\alpha(A)[\![t]\!])^\times \ .$$

The λ-operations $\lambda^i\colon KR_\alpha(A) \longrightarrow KR_\alpha(A)$ are then defined by

$$\Lambda(x) \ = \ \sum\nolimits_{i\geq 0} \lambda^i(x) \cdot t^i \ .$$

By design, these operations extend the exterior powers on classes of actual Real vector bundles. The λ-operations make the ring $KR_\alpha(A)$ into a special λ-ring, see [1, Theorem 1.5 (i)]. Every special λ-ring supports *Adams operations*, i.e., ring homomorphisms $\psi^n\colon R \longrightarrow R$ for $n \geq 1$ that satisfy $\psi^n \circ \psi^m = \psi^{nm}$ for all $n, m \geq 1$, as well as the congruence $\psi^p(x) \equiv x^p$ modulo (p) for every prime p, see [1, §5]. We are particularly interested in these Adams operation

$$\psi^n \ : \ KR_\alpha(A) \ \longrightarrow \ KR_\alpha(A)$$

in the case of Real-equivariant K-theory. One key property is that on the class of a Real line bundle ξ, the Adams operation is given by

$$\psi^n[\xi] \ = \ [\xi^{\otimes n}] \ .$$

Taking exterior power of vector bundles is natural both for G-maps in A, and for restriction along continuous homomorphisms of augmented Lie groups. Hence the λ-operations and the Adams operations inherit both kinds of naturalities.

Example 6.2. The second Adams operation $\psi^2\colon KR_\alpha(A) \longrightarrow KR_\alpha(A)$ is given on the class of a Real α-vector bundle ξ by the formula

$$\psi^2[\xi] \ = \ [\mathrm{Sym}^2(\xi)] \ - \ [\Lambda^2(\xi)] \ ,$$

the formal difference of the second symmetric and exterior power of ξ. Indeed, this formula has the correct behavior on line bundles, is additive for Whitney sum in ξ, and natural in (α, A). So the various naturality properties force ψ^2 to be given by this formula. In general, ψ^n can be described on vector bundles by certain alternating sums of certain polynomial functors, but the general formula is not as simple. The formula for ψ^2 shows that the Adams operations do not generally send vector bundles (other than line bundles) to vector bundles, but rather to virtual vector bundles.

We shall now use the power endomorphisms of the ultra-commutative monoid $\mathbf{P}$ to define the global Adams operations on $\mathbf{BUP}$, by employing our splitting to 'retract' them off the induced endomorphisms of $\Sigma^\infty_+\mathbf{P}$.

Construction 6.3 (Global Adams operations)**.** For $n \geq 1$, we write

$$\mu_n \ : \ T \ \longrightarrow \ T \ , \quad \mu_n(\lambda) = \lambda^n$$

for the n-th power homomorphism, and we write

$$\tilde{\mu}_n = \mu_n \rtimes C \ : \ \tilde{T} \ \longrightarrow \ \tilde{T}$$

for its extension to the extended circle group $\tilde{T} = T \rtimes C$. Since the pair $(\mathbf{P}, u_{\tilde{T}})$ represents the functor $\pi_0^{\tilde{T}}$, we can define a morphism $\phi^n\colon \mathbf{P} \longrightarrow \mathbf{P}$ in the unstable C-global homotopy category by the requirement that

$$\phi^n_*(u_{\tilde{T}}) \ = \ \tilde{\mu}_n^*(u_{\tilde{T}})$$

in $\pi_0^{\tilde{T}}(\mathbf{P})$. The morphism ϕ^n then represents raising a line bundle to its n-th power. We define

$$\kappa^n \ : \ \mathbf{U} \ \longrightarrow \ \mathbf{U} \tag{6.4}$$

as the unique morphism in the based unstable C-global homotopy category making the following diagram commute:

$$\begin{tikzcd}
\mathbf{U} \arrow[rr, "\kappa^n"] \arrow[d, "d"'] & & \mathbf{U} \arrow[d, "\sim"', "\Omega^\bullet(\eta\wedge S^\sigma)\circ d"] \\
\Omega^\bullet(\Sigma^\infty_+\mathbf{P} \wedge S^\sigma) \arrow[r, "\Omega^\bullet(\Sigma^\infty_+\phi^n\wedge S^\sigma)"'] & \Omega^\bullet(\Sigma^\infty_+\mathbf{P} \wedge S^\sigma) \arrow[r, "\Omega^\bullet(\eta\wedge S^\sigma)"'] & \Omega^\bullet(\mathbf{KR} \wedge S^\sigma)
\end{tikzcd}$$

The C-global equivalence $\gamma\colon \mathbf{BUP} \xrightarrow{\sim} \Omega^\sigma\mathbf{U}$ is defined in (B.27). We define the *n-th global Adams operation*

$$\Upsilon^n \ : \ \mathbf{BUP} \ \longrightarrow \ \mathbf{BUP} \tag{6.5}$$

as the unique morphism in the unstable C-global homotopy category making the following diagram commute:

$$\begin{tikzcd} \mathbf{BUP} \arrow[r, "\Upsilon^n"] \arrow[d, "\gamma"', "\sim"] & \mathbf{BUP} \arrow[d, "\gamma", "\sim"'] \\ \Omega^\sigma \mathbf{U} \arrow[r, "\Omega^\sigma(\kappa^n)"'] & \Omega^\sigma \mathbf{U} \end{tikzcd} \tag{6.6}$$

Expanding the definition (6.4) of κ^n, and the definition (4.13) of the Segal–Becker splitting c shows that the following diagram commutes in the unstable C-global homotopy category:

$$\begin{tikzcd} \mathbf{BUP} \arrow[rr, "\Upsilon^n"] \arrow[d, "c"'] & & \mathbf{BUP} \arrow[d, "\Omega^\bullet(\eta)\circ c", "\sim"'] \\ \Omega^\bullet(\Sigma^\infty_+ \mathbf{P}) \arrow[r, "\Omega^\bullet(\Sigma^\infty_+ \phi^n)"'] & \Omega^\bullet(\Sigma^\infty_+ \mathbf{P}) \arrow[r, "\Omega^\bullet(\eta)"'] & \Omega^\bullet(\mathbf{KR}) \end{tikzcd} \tag{6.7}$$

Clearly, the morphism ϕ^1 is the identity of $\mathbf{P}$, κ^1 is the identity of $\mathbf{U}$, and thus Υ^1 is the identity of $\mathbf{BUP}$.

The next proposition verifies a globally-coherent version of the design criterion for Adams operations, namely that on line bundles, ψ^n is the n-th tensor power. The morphism of C-global spaces $h\colon \mathbf{P} \longrightarrow \mathbf{BUP}$ was defined in Construction 4.17; it represents the inclusion of line bundles into virtual vector bundles.

Proposition 6.8. *For every $n \geq 1$, the following diagram commutes in the unstable C-global homotopy category:*

$$\begin{tikzcd} \mathbf{P} \arrow[r, "\phi^n"] \arrow[d, "h"'] & \mathbf{P} \arrow[d, "h"] \\ \mathbf{BUP} \arrow[r, "\Upsilon^n"'] & \mathbf{BUP} \end{tikzcd}$$

Proof. The following diagram commutes by Corollary 4.19 and naturality of the adjunction unit:

$$\begin{tikzcd} & \mathbf{P} \arrow[rr, "\phi^n"] \arrow[dl, "h"'] \arrow[dd, "\text{unit}"] & & \mathbf{P} \arrow[dd, "\text{unit}"'] \arrow[dr, "h"] & \\ \mathbf{BUP} \arrow[dr, "c"'] & & & & \mathbf{BUP} \arrow[dl, "c"'] \arrow[d, "\Omega^\bullet(\eta)\circ c", "\sim"'] \\ & \Omega^\bullet(\Sigma^\infty_+ \mathbf{P}) \arrow[rr, "\Omega^\bullet(\Sigma^\infty_+ \phi^n)"'] & & \Omega^\bullet(\Sigma^\infty_+ \mathbf{P}) \arrow[r, "\Omega^\bullet(\eta)"'] & \Omega^\bullet(\mathbf{KR}) \end{tikzcd}$$

The commutative diagram (6.7) then proves the claim. $\square$

The next theorem justifies the name 'global Adams operation' for the morphism $\Upsilon^n\colon \mathbf{BUP} \longrightarrow \mathbf{BUP}$.

Theorem 6.9. *For every augmented Lie group $\alpha\colon G \longrightarrow C$, every finite G-CW-complex A and every $n \geq 1$, the following square commutes:*

$$\begin{tikzcd} {[A, \mathbf{BUP}]^\alpha} \arrow[r, "\Upsilon^n_*"] \arrow[d, "(B.10)"', "\cong"] & {[A, \mathbf{BUP}]^\alpha} \arrow[d, "(B.10)", "\cong"'] \\ KR_\alpha(A) \arrow[r, "\psi^n"'] & KR_\alpha(A) \end{tikzcd}$$

Proof. We define a map

$$\delta_{\alpha,A} \;:\; [A,\mathbf{BUP}]^\alpha \;\longrightarrow\; KR_\alpha(A) \qquad \text{by} \qquad \delta_{\alpha,A}(x) \;=\; \psi^n\langle x\rangle - \langle \Upsilon^n_*(x)\rangle \ ,$$

the difference of the two composites around the diagram in question. We need to show that $\delta_{\alpha,A}=0$ for all (α,A).

We claim that these maps satisfy:

(a) The maps $\delta_{\alpha,A}$ are natural for continuous G-maps in A, and for morphisms of augmented Lie groups.
(b) The map $\delta_{\alpha,A}$ satisfies $\delta_{\alpha,A}(2x-y)=2\cdot\delta_{\alpha,A}(x)-\delta_{\alpha,A}(y)$ for all $x,y\in[A,\mathbf{BUP}]^\alpha$.
(c) The map $\delta_{\alpha,A}$ is additive whenever the augmentation α is trivial.

Property (a) is clear because the Adams operations and the isomorphisms (B.10) are natural in A and α, and so are the maps $[A,\Upsilon^n]^\alpha$, since they arise from a morphism of Real-global spaces.

(b) Applying $[A,-]^\alpha$ to the defining diagram (6.6) for Υ^n yields a commutative diagram

$$\begin{tikzcd}
{[A,\mathbf{BUP}]^\alpha} \arrow[r, "\Upsilon^n_*"] \arrow[d, "\gamma_*"', "\cong"] & {[A,\mathbf{BUP}]^\alpha} \arrow[d, "\cong"', "\gamma_*"] \\
{[A,\Omega^\sigma\mathbf{U}]^\alpha} \arrow[r, "(\Omega^\sigma(\kappa^n))_*"'] & {[A,\Omega^\sigma\mathbf{U}]^\alpha}
\end{tikzcd}$$

in which all objects are abelian groups, but the horizontal maps are not a priori additive. The vertical maps are isomorphisms of abelian groups because γ is a C-global equivalence of ultra-commutative C-monoids. As induced by a σ-loop map, the map $(\Omega^\sigma(\kappa^n))_*$ satisfies

$$(\Omega^\sigma(\kappa^n))_*(2x-y) \;=\; 2\cdot(\Omega^\sigma(\kappa^n))_*(x) \;-\; (\Omega^\sigma(\kappa^n))_*(y)$$

for all $x,y\in[A,\Omega^\sigma\mathbf{U}]^\alpha$, see Theorem 4.23 (i). So the upper horizontal map Υ^n_* satisfies the analogous relation

$$\Upsilon^n_*(2x-y) \;=\; 2\cdot\Upsilon^n_*(x)-\Upsilon^n_*(y)$$

for all $x,y\in[A,\mathbf{BUP}]^\alpha$. The Adams operation ψ^n is additive, so the difference $\delta_{\alpha,A}$ still has the weak additivity property (b).

(c) The 'non-Real' global morphism underlying Υ^n is a global loop map, by construction. So for all compact Lie groups G with trivial augmentation, the map $\Upsilon^n_*\colon[A,\mathbf{BUP}]^G\longrightarrow[A,\mathbf{BUP}]^G$ is additive. The Adams operation ψ^n is additive, too, hence so is the difference $\delta_{\alpha,A}$.

Now we show in five steps that the maps $\delta_{\alpha,A}$ vanish. The first step is $(\alpha,A)=(\tilde T,*)$ and the class of the tautological Real $\tilde T$-representation. The commutative diagram (4.18) that defines the morphism h yields the relation

$$i_*\{\nu_1\} \;=\; h_*(\ell_*[\mathbb{L}]) \;=_{(4.3)}\; h_*(u_{\tilde T})$$

in $\pi_0^{\tilde T}(\mathbf{BUP})$. We observe that

$$\begin{aligned}\langle\Upsilon^n_*(i_*\{\nu_1\})\rangle &=\; \langle\Upsilon^n_*(h_*(u_{\tilde T}))\rangle \;=\; \langle h_*(\phi^n_*(u_{\tilde T}))\rangle \;=\; \langle h_*(\tilde\mu_n^*(u_{\tilde T}))\rangle\\
&=\; \langle\tilde\mu_n^*(h_*(u_{\tilde T}))\rangle \;=\; \langle\tilde\mu_n^*(i_*\{\nu_1\})\rangle \;=\; \tilde\mu_n^*\langle i_*\{\nu_1\}\rangle \;=\; \tilde\mu_n^*[\nu_1] \;=\; \psi^n[\nu_1] \;=\; \psi^n\langle i_*\{\nu_1\}\rangle \ .\end{aligned}$$

The second equation is Proposition 6.8. This relation precisely means that $\delta_{\tilde T,*}(i_*\{\nu_1\})=0$.

The second step deals with the tautological Real $\tilde U(k)$-representation ν_k. For $1\le i\le k$, we let $p_i\colon T^k\longrightarrow\tilde T$ denote the projection to the i-th factor, followed by the inclusion $T\longrightarrow\tilde T$ into the extended circle group. Then

$$\begin{aligned}\mathrm{res}^{\tilde U(k)}_{T^k}(\delta_{\tilde U(k),*}(i_*\{\nu_k\})) \;&=\; \delta_{T^k,*}(\mathrm{res}^{\tilde U(k)}_{T^k}(i_*\{\nu_k\})) \;=\; \delta_{T^k,*}(p_1^*(i_*\{\nu_1\})+\dots+p_k^*(i_*\{\nu_1\}))\\
&=\; \sum_{1\le i\le k}\delta_{T^k,*}(p_i^*(i_*\{\nu_1\})) \;=\; \sum_{1\le i\le k}p_i^*(\mathrm{res}^{\tilde T}_T(\delta_{\tilde T,*}(i_*\{\nu_1\}))) \;=\; 0\ .\end{aligned}$$

The first and forth equation are the naturality property (a) in the group. The third equation uses that the map $\delta_{T^k,*}$ is additive by (c), because T^k is trivially augmented. The restriction homomorphism $\mathrm{res}^{\tilde{U}(k)}_{U(k)}\colon RR(\tilde{U}(k)) \longrightarrow R(U(k))$ is injective, see for example [4, page 13]. And the restriction homomorphism $\mathrm{res}^{U(k)}_{T^k}\colon R(U(k)) \longrightarrow R(T^k)$ is injective, too. So this proves that $\delta_{\tilde{U}(k),*}(i_*\{\nu_k\}) = 0$ in $RR(\tilde{U}(k))$, the Real representation ring of $\tilde{U}(k)$.

In the third step we fix $k \geq 0$, and we write $j^k_{\alpha,A}$ for the composite

$$[A, \mathbf{Gr}_k]^\alpha \xrightarrow{i^k_*} [A, \mathbf{BUP}]^\alpha \xrightarrow{\delta_{\alpha,A}} KR_\alpha(A) \xrightarrow{(\mathrm{B.10})^{-1}} [A, \mathbf{BUP}]^\alpha \ .$$

Here $i^k\colon \mathbf{Gr}_k \longrightarrow \mathbf{BUP}$ is the restriction of the morphism $i\colon \mathbf{Gr} \longrightarrow \mathbf{BUP}$ to the k-th summand. Letting α and A vary yields a C-global transformation j^k from $\mathbf{Gr}_k$ to $\mathbf{BUP}$ in the sense of Definition A.8. The orthogonal C-space $\mathbf{Gr}_k$ is a global classifying space for the augmented Lie group $\tilde{U}(k)$, and the class $\{\nu_k\}$ is the tautological class $u_{\tilde{U}(k),*}$ in the sense of (A.10). We showed in the previous step that this C-global transformation vanishes on the tautological class; so $j^k_{\alpha,A}$ vanishes on all elements of $[A, \mathbf{Gr}_k]^\alpha$, for all augmented Lie groups and over all finite equivariant CW-complexes, by the uniqueness part of Theorem A.11.

In the fourth step we assume that A is 'G-connected' in the sense that the group $\pi_0(G)$ acts transitively on $\pi_0(A)$. This ensures that every Real α-vector bundle over A has constant rank. By the isomorphism of Theorem B.12 (iii), every element of $[A, \mathbf{BUP}]^\alpha$ is thus of the form $j^k_{\alpha,A}(y) - j^l_{\alpha,A}(z)$ for some $k, l \geq 0$, some $y \in [A, \mathbf{Gr}_k]^\alpha$ and some $z \in [A, \mathbf{Gr}_l]^\alpha$. Then

$$\begin{aligned} \delta_{\alpha,A}(j^k_{\alpha,A}(y) - j^l_{\alpha,A}(z)) \ &= \ \delta_{\alpha,A}(2 \cdot j^k_{\alpha,A}(y) - j^{k+l}_{\alpha,A}(y+z)) \\ \overset{\text{(b)}}{=} \ & \ 2 \cdot \delta_{\alpha,A}(j^k_{\alpha,A}(y)) \ - \ \delta_{\alpha,A}(j^{k+l}_{\alpha,A}(y+z)) \ = \ 0 \ . \end{aligned}$$

The third equality is step three.

In the fifth and final step we let A be any finite G-CW-complex. Then $A = A_1 \amalg \ldots \amalg A_m$ is a disjoint union of G-connected finite G-CW-complexes, indexed by the $\pi_0(G)$-orbits of $\pi_0(A)$. We write $\iota_j\colon A_j \longrightarrow A$ for the inclusion of the j-th summand. Then

$$\iota_j^*(\delta_{\alpha,A}(x)) \ = \ \delta_{\alpha,A_j}(\iota_j^*(x)) \ = \ 0$$

for all $1 \leq j \leq m$ and every $x \in [A, \mathbf{BUP}]^\alpha$ by the fourth step, because A_j is G-connected. Real-equivariant K-theory takes disjoint unions to products, so the map

$$(\iota_1^*, \ldots, \iota_m^*) \ : \ KR_\alpha(A) \ \longrightarrow \ KR_\alpha(A_1) \times \cdots \times KR_\alpha(A_m)$$

is an isomorphism of rings. Hence $\delta_{\alpha,A}(x) = 0$, which concludes the proof. □

The Adams operations in Real-equivariant K-theory satisfy the relation $\psi^m \circ \psi^n = \psi^{mn}$ for all $m, n \geq 1$. So by Theorem 6.9, the morphisms of Real-global spaces

$$\Upsilon^m \circ \Upsilon^n, \ \Upsilon^{mn} \ : \ \mathbf{BUP} \ \longrightarrow \ \mathbf{BUP}$$

induce the same map on $[A, \mathbf{BUP}]^\alpha$ for all augmented Lie groups and all finite equivariant CW-complexes. I do not know if in fact $\Upsilon^m \circ \Upsilon^n = \Upsilon^{mn}$ as endomorphisms of $\mathbf{BUP}$ in the unstable C-global homotopy category. Or even better, if the σ-deloopings (6.4) of the global Adams operations satisfy $\kappa^m \circ \kappa^n = \kappa^{mn}\colon \mathbf{U} \longrightarrow \mathbf{U}$. Also, Υ^n induces additive maps upon applying $[A, -]^\alpha$ for all augmented Lie groups and all finite equivariant CW-complexes; a natural question is thus whether Υ^n is a C-global H-map. I expect this to be the case, but our techniques do suffice to show it.

Appendix A. Some C-global homotopy theory

In this appendix we develop some basics about C-global homotopy theory that we need in this paper. In the remainder of the article, the group C will be the Galois group of $\mathbb{C}$ over $\mathbb{R}$; the arguments in this appendix work more generally, and here we let C be any compact Lie group. The main results of this appendix are the classification of C-global transformations with source a C-global classifying space in Theorem A.11, and the criterion of Proposition A.16 to recognize coinduced C-spaces. Two other references that develop general C-global homotopy theory are [25, Appendix A] and [5].

An *inner product space* is a finite-dimensional real vector space equipped with a scalar product, i.e., a positive-definite symmetric bilinear form. We denote by $\mathbf{L}$ the category with objects the inner product spaces and morphisms the linear isometric embeddings. The category $\mathbf{L}$ is a topological category, with morphism spaces topologized as Stiefel manifolds.

Definition A.1. Let C be a compact Lie group. An *orthogonal C-space* is a continuous functor from the linear isometries category $\mathbf{L}$ to the category of C-spaces.

The notion of *C-global equivalence* for morphisms of orthogonal C-spaces is defined in [25, Definition A.2] and [5, Definition 3.2]. It generalizes that of global equivalences of orthogonal spaces from [24, Definition 1.1.2], to which it reduces when C is the trivial group. The C-global equivalences are part of the *C-global model structure* on the category of orthogonal C-spaces established in [5, Theorem A.20]. When C is trivial group, this specializes to the global model structure on orthogonal spaces from [24, Theorem 1.2.21].

Construction A.2 (Equivariant homotopy sets)**.** We introduce the equivariant homotopy sets defined by orthogonal C-spaces, for a compact Lie group C. When C is the trivial group, these are discussed in more detail in [24, Section 1.5].

For every compact Lie group G, we choose a complete G-universe, i.e., an orthogonal G-representation $\mathcal{U}_G$ of countably infinite dimension such that every finite-dimensional G-representation embeds into $\mathcal{U}_G$ by an $\mathbb{R}$-linear G-equivariant isometric embedding. We write $s(\mathcal{U}_G)$ for the poset, under inclusion, of finite-dimensional G-subrepresentations of $\mathcal{U}_G$. We let E be an orthogonal C-space, we let $\alpha\colon G \longrightarrow C$ be a continuous homomorphism of compact Lie groups. For every $V \in s(\mathcal{U}_G)$, the space $E(V)$ becomes a $(C \times G)$-space via the C-action on E, and the G-action on V through the functoriality of E. We write $\alpha^\flat(E(V))$ for the G-space obtained by restriction of scalars along $(\alpha, \mathrm{Id})\colon G \longrightarrow C \times G$. Said differently, G acts diagonally on $\alpha^\flat(E(V))$, via the G-action on E through α, and on V. For a G-space, we set

$$[A, E]^\alpha \;=\; \operatorname{colim}_{V \in s(\mathcal{U}_G)} [A, \alpha^\flat(E(V))]^G \ .$$

Here $[-,-]^G$ is the set of G-equivariant homotopy classes of equivariant maps. The colimit is taken along the maps $E(V) \longrightarrow E(W)$ induced by the inclusions for $V \subseteq W$ in $s(\mathcal{U}_G)$.

The sets $[A, E]^\alpha$ are contravariantly functorial for continuous G-maps in A by precomposition, and covariantly functorial for morphisms of orthogonal C-spaces in E. Another functoriality in $\alpha\colon G \longrightarrow C$ will be discussed in Construction A.4 below.

When C is the trivial group, the following proposition specializes to [24, Proposition 1.5.3]. The proof in the present C-global context is almost literally the same, so we omit it.

Proposition A.3. *Let $\alpha\colon G \longrightarrow C$ be a continuous homomorphism of compact Lie groups, let E be an orthogonal C-space, and let A be a G-space.*

(i) *Suppose that the G-space A is compact and the orthogonal space underlying E is* closed*, i.e., all structure maps $E(\psi)\colon E(V) \longrightarrow E(W)$ are closed embeddings. Then the canonical map*

$$[A, E]^\alpha \;\longrightarrow\; [A, \alpha^\flat(E(\mathcal{U}_G))]^G$$

is bijective, where $E(\mathcal{U}_G) = \operatorname{colim}_{V \in s(\mathcal{U}_G)} E(V)$.

(ii) *If A is a finite G-CW-complex, then the assignment $E \mapsto [A,E]^\alpha$ sends C-global equivalences of orthogonal C-spaces to bijections.*

(iii) *If F is another orthogonal C-space, then the map*

$$([A,p_E],[A,p_F]) \ : \ [A,E\times F]^\alpha \ \longrightarrow \ [A,E]^\alpha \times [A,F]^\alpha$$

is bijective, where p_E and p_F are the projections.

Construction A.4 (Functoriality in the group)**.** Let $\alpha\colon G \longrightarrow C$ be a continuous homomorphism of compact Lie groups, and let E be an orthogonal C-space. A continuous homomorphism $\beta\colon K \longrightarrow G$ of compact Lie groups induces a restriction map

$$\beta^* \ : \ [A,E]^\alpha \ \longrightarrow \ [\beta^*(A),E]^{\alpha\beta} \tag{A.5}$$

by restriction of actions along β, as follows. Given $V \in s(\mathcal{U}_G)$ and a continuous G-equivariant map $f\colon A \longrightarrow \alpha^\flat(E(V))$, restriction of actions along β yields a K-equivariant map

$$\beta^*(f) \ : \ \beta^*(A) \ \longrightarrow \ \beta^*(\alpha^\flat(E(V))) \ = \ (\alpha\beta)^\flat(E(\beta^*(V))) \ .$$

We choose a K-equivariant linear isometric embedding $j\colon \beta^*(V) \longrightarrow \mathcal{U}_K$ into the chosen K-universe. Then $W = j(\beta^*(V))$ is a finite-dimensional K-subrepresentation, and thus an element of the poset $s(\mathcal{U}_K)$; and the embedding j restricts to an isomorphism of K-representations $j\colon \beta^*(V) \cong W$. The restriction map (A.5) sends the class represented by f in $[A,E]^\alpha$ to the class in $[\beta^*(A),E]^{\alpha\beta}$ represented by the composite K-equivariant map

$$\beta^*(A) \ \xrightarrow{\beta^*(f)} \ (\alpha\beta)^\flat(E(\beta^*(V))) \ \xrightarrow[\cong]{E(j)} \ (\alpha\beta)^\flat(E(W)) \ .$$

The analogous argument as in the special case $C = \{1\}$ and $A = *$ in [24, Proposition 1.5.8] shows that the resulting class in $[\beta^*(A),E]^{\alpha\beta}$ does not depend on the choice of embedding $j\colon \beta^*(V) \longrightarrow \mathcal{U}_K$, and so the construction is well-defined. Given well-definedness, the construction is clearly contravariantly functorial in the groups augmented to C: given another continuous homomorphism $\gamma\colon L \longrightarrow K$, we have

$$\gamma^* \circ \beta^* \ = \ (\beta\circ\gamma)^* \ : \ [A,E]^\alpha \ \longrightarrow \ [(\beta\gamma)^*(A),E]^{\alpha\beta\gamma} \ .$$

Construction A.6 (Induction isomorphisms)**.** We let $\alpha\colon G \longrightarrow C$ be a continuous homomorphism of compact Lie groups, we let E be an orthogonal C-space, and we let H be a closed subgroup of G. For an H-space B, we write

$$[1,-] \ : \ B \ \longrightarrow \ G\times_H B \ , \quad y \ \longmapsto \ [1,y]$$

for the H-equivariant unit of the adjunction $(G\times_H -, \mathrm{res}^G_H)$. The adjunction bijections

$$[G\times_H B, \alpha^\flat(E(V))]^G \ \cong \ [B,(\alpha|_H)^\flat(E(V))]^H \ , \quad [f] \ \longmapsto \ [\mathrm{res}^H_G(f)\circ[1,-]]$$

for $V \in s(\mathcal{U}_G)$, and the fact that the underlying H-universe of $\mathcal{U}_G$ is a complete H-universe provide an *induction isomorphism*: the composite

$$[G\times_H B,E]^\alpha \ \xrightarrow{\mathrm{res}^G_H} \ [G\times_H B,E]^{\alpha|_H} \ \xrightarrow{[1,-]^*} \ [B,E]^{\alpha|_H} \tag{A.7}$$

is bijective, where res^G_H is short hand for the restriction homomorphism (A.5) associated to the inclusion $H \longrightarrow G$.

In the body of this paper, we verify that the global Segal–Becker splitting $c\colon \mathbf{BUP} \longrightarrow \Omega^\bullet(\Sigma^\infty_+\mathbf{P})$ induces the classical equivariant Segal–Becker splittings on equivariant cohomology theories; and we verify that the global Adams operation $\Upsilon^n\colon \mathbf{BUP} \longrightarrow \mathbf{BUP}$ induces the classical Adams operation on Real-equivariant K-groups. In both cases we are dealing with 'global' natural transformations from the functor of isomorphism classes of Real vector bundles of some fixed rank.

If H is a closed subgroup of smaller dimension in a compact Lie group G, then the underlying H-space of a G-CW-complex need not admit an H-CW-structure; an example is given in [13, Section 2]. Nevertheless, the underlying H-space of a finite G-CW-complex is always H-homotopy equivalent to a finite H-CW-complex, see [13, Corollary B]. Consequently, for every continuous homomorphism $\beta\colon K \longrightarrow G$ of compact Lie groups, the restriction functor β^* takes G-spaces of the G-homotopy type of a finite G-CW-complex to K-spaces of the K-homotopy type of a finite K-CW-complex.

Definition A.8 (C-global transformations)**.** Let C be a compact Lie group, and let E and F be orthogonal C-spaces. A *C-global transformation* τ from F to E consists of maps

$$\tau_{\alpha,A} \ : \ [A,F]^\alpha \ \longrightarrow \ [A,E]^\alpha$$

for all continuous homomorphism $\alpha\colon G \longrightarrow C$ of compact Lie groups and all G-spaces A of the G-homotopy type of a finite G-CW-complex that are natural in the following sense:

- For every continuous G-map $f\colon B \longrightarrow A$, we have
$$\tau_\alpha \circ [f,F]^\alpha \ = \ [f,E]^\alpha \circ \tau_{\alpha,A} \ : \ [A,F]^\alpha \ \longrightarrow \ [A,E]^\alpha \ .$$
- For every continuous homomorphism $\beta\colon K \longrightarrow G$ of compact Lie groups, we have
$$\tau_{\alpha\beta,\beta^*(A)} \circ \beta^* \ = \ \beta^* \circ \tau_{\alpha,A} \ : \ [A,F]^\alpha \ \longrightarrow \ [\beta^*(A),E]^{\alpha\beta} \ .$$

For a continuous homomorphism $\alpha\colon G \longrightarrow C$ and an orthogonal C-space E, we shall also use the notation

$$\pi_0^\alpha(E) \ = \ \mathrm{colim}_{V\in s(\mathcal{U}_G)} \ \pi_0(\alpha^\flat(E(V))^G) \ = \ \mathrm{colim}_{V\in s(\mathcal{U}_G)} \ \pi_0(E(V)^{\Gamma(\alpha)}).$$

Here $\Gamma(\alpha) = \{(\alpha(g),g)\colon g\in G\} \subset C\times G$ is the graph of α. Evaluation at the unique point is a natural bijection $[*,\alpha^\flat(E(V))]^G \cong \pi_0(\alpha^\flat(E(V))^G)$; in the colimit over the poset $s(\mathcal{U}_G)$, this becomes a natural bijection

$$[*,E]^\alpha \ \cong \ \pi_0^\alpha(E) \ .$$

In the following we shall routinely identify $[*,E]^\alpha$ and $\pi_0^\alpha(E)$ in this way without further notice.

Construction A.9 (Global classifying spaces)**.** We let $\beta\colon K \longrightarrow C$ be a continuous homomorphism of compact Lie groups. We choose a faithful K-representation V and define the *C-global classifying space* $B_{\mathrm{gl}}\beta$ as the orthogonal C-space with values

$$(B_{\mathrm{gl}}\beta)(V) \ = \ C\times_\beta \mathbf{L}(V,-) \ .$$

In more detail, $B_{\mathrm{gl}}\beta$ is the quotient of the orthogonal C-space $C\times \mathbf{L}(V,-)$ by the equivalence relation

$$(c,\varphi) \ \sim \ (c\cdot\beta(k),\varphi\circ l_k)$$

for all $(c,k)\in C\times K$ and all linear isometric embeddings $\varphi\colon V\longrightarrow W$, where $l_k\colon V\longrightarrow V$ is the action of $k\in K$. The notation $B_{\mathrm{gl}}\beta$ is slightly abusive in that we do not record the choice of faithful K-representation. This abuse is justified by [25, Proposition A.5], which shows that $B_{\mathrm{gl}}\beta$ is independent of the choice of faithful representation up to a preferred zigzag of C-global equivalences.

The equivalence class

$$[1,\mathrm{Id}_V] \ \in \ C\times_\beta \mathbf{L}(V,V) \ = \ (B_{\mathrm{gl}}\beta)(V)$$

is a K-fixed point of $\beta^\flat((B_{\mathrm{gl}}\beta)(V))$, so it represents a class

$$u_\beta \ \in \ \pi_0^\beta(B_{\mathrm{gl}}\beta) \ , \tag{A.10}$$

the *tautological class.*

Theorem A.11. *Let $\beta\colon K\longrightarrow C$ be a continuous homomorphism of compact Lie groups, and let E be an orthogonal C-space. For every class y in $\pi_0^\beta(E)$, there is a unique C-global transformation τ from $B_{\mathrm{gl}}\beta$ to E such that the map*

$$\tau_{\beta,*} \ : \ \pi_0^\beta(B_{\mathrm{gl}}\beta) \ \longrightarrow \ \pi_0^\beta(E)$$

sends the tautological class u_β to y.

Proof. To construct τ we represent y by a K-fixed point $\tilde{y} \in (\beta^\flat(E(W))^K$, for some K-representation W. By enlarging W, if necessary, we can assume that there is a K-equivariant linear isometric embedding $\varphi\colon V \longrightarrow W$ from the faithful K-representation that is implicit in the construction of $B_{\mathrm{gl}}\beta$. By [25, Proposition A.5], the morphism of orthogonal C-spaces

$$\varphi^* = C \times_\beta \mathbf{L}(\varphi, -) \ : \ Y = C \times_\beta \mathbf{L}(W, -) \ \longrightarrow \ C \times_\beta \mathbf{L}(V, -) \ = \ B_{\mathrm{gl}}\beta$$

is a C-global equivalence. Moreover, if we let $[\![1, \mathrm{Id}_W]\!] \in \pi_0^\beta(Y)$ denote the class represented by the K-fixed point $[1, \mathrm{Id}_W]$ of $\beta^\flat(C \times_\beta \mathbf{L}(W, W)) = \beta^\flat(Y(W))$, then $\pi_0^\beta(\varphi^*)[\![1, \mathrm{Id}_W]\!] = u_\beta$.

The fixed point $\tilde{y}$ is represented by a unique morphism of orthogonal C-spaces $f\colon Y \longrightarrow E$ such that the map

$$f(W) \ : \ C \times_\beta \mathbf{L}(W, W) = Y(W) \ \longrightarrow \ E(W)$$

takes $[1, \mathrm{Id}_W]$ to $\tilde{y}$. The global equivalence φ^* and the morphism f induce a C-global transformation τ from $B_{\mathrm{gl}}\beta$ to E with constituents

$$\tau_{\alpha,A} \ : \ [A, B_{\mathrm{gl}}\beta]^\alpha \ \xrightarrow[\cong]{([A,\varphi^*]^\alpha)^{-1}} \ [A, Y]^\alpha \ \xrightarrow{[A,f]^\alpha} \ [A, E]^\alpha \ .$$

This C-global transformation satisfies

$$\tau_{\beta,*}(u_\beta) \ = \ \pi_0^\beta(f)(\pi_0^\beta(\varphi^*)^{-1}(u_\beta)) \ = \ \pi_0^\beta(f)[\![1, \mathrm{Id}_W]\!] \ = \ [\tilde{y}] \ = \ y \ .$$

The proof of the uniqueness clause is more involved. We let V be the faithful K-representation that is implicit in the definition of $B_{\mathrm{gl}}\beta$. We let W be an inner product space. The group $C \times O(W)$ acts on $(B_{\mathrm{gl}}\beta)(W) = C \times_\beta \mathbf{L}(V, W)$ by

$$(c, A) \cdot [d, \varphi] \ = \ [cd, A \circ \varphi] \ .$$

This action is transitive; and for a linear isometric embedding $\varphi\colon V \longrightarrow W$, the stabilizer of the point $[1, \varphi] \in C \times_\beta \mathbf{L}(V, W)$ is the subgroup

$$\mathcal{S}[\varphi] \ = \ \{(c, A) \in C \times O(W) \colon \text{there is } k \in K \text{ such that } c = \beta(k) \text{ and } A\varphi = \varphi l_k\} \ .$$

Since the K-action on V is faithful, for every $(c, A) \in \mathcal{S}[\varphi]$, the $k \in K$ such that $c = \beta(k)$ and $A\varphi = \varphi l_k$ is unique, and we can define a continuous homomorphism $\gamma\colon \mathcal{S}[\varphi] \longrightarrow K$ by letting $\gamma(c, A)$ be the unique element of K such that $A\varphi = \varphi l_k$. We let $\Pi\colon C \times O(W) \longrightarrow C$ denote the projection to the first factor. Then the square of continuous group homomorphisms

$$\begin{tikzcd}
\mathcal{S}[\varphi] \arrow[r, "\mathrm{incl}"] \arrow[d, "\gamma"'] & C \times O(W) \arrow[d, "\Pi"] \\
K \arrow[r, "\beta"'] & C
\end{tikzcd}$$

commutes by design. Since the map

$$(C \times O(W))/\mathcal{S}[\varphi] \ \longrightarrow \ C \times_\beta \mathbf{L}(V, W) = (B_{\mathrm{gl}}\beta)(W) \ , \quad [c, A] \ \longmapsto \ [c, A\varphi]$$

is an $(C \times O(W))$-equivariant homeomorphism, the induction isomorphism (A.7) shows that the composite

$$[C \times_\beta \mathbf{L}(V, W), E]^\Pi \ \xrightarrow{\mathrm{res}^{C \times O(W)}_{\mathcal{S}[\varphi]}} \ [C \times_\beta \mathbf{L}(V, W), E]^{\beta\gamma} \ \xrightarrow{[1,\varphi]^*} \ \pi_0^{\beta\gamma}(E)$$

is bijective for every C-orthogonal space E. The identity of $C \times_\beta \mathbf{L}(V, W)$ represents a tautological class

$$[\mathrm{Id}_{C \times_\beta \mathbf{L}(V,W)}] \ \in \ [C \times_\beta \mathbf{L}(V, W), B_{\mathrm{gl}}\beta]^\Pi$$

that satisfies

$$[1, \varphi]^*(\mathrm{res}^{C \times O(W)}_{\mathcal{S}[\varphi]}[\mathrm{Id}_{C \times_\beta \mathbf{L}(V,W)}]) \ = \ [\![1, \varphi]\!] \ = \ \gamma^*(u_\beta)$$

in $\pi_0^{\beta\gamma}(B_{\mathrm{gl}}\beta)$.

Now we let τ be any C-global transformation from $B_{\mathrm{gl}}\beta$ to E. Its naturality properties yield the relations

$$\begin{aligned}[1,\varphi]^*(\mathrm{res}^{C\times O(W)}_{\mathcal{S}[\varphi]}(\tau_{\Pi,C\times_\beta \mathbf{L}(V,W)}[\mathrm{Id}_{C\times_\beta \mathbf{L}(V,W)}]))& \\ =\ \tau_{\beta\gamma,*}([1,\varphi]^*(\mathrm{res}^{C\times O(W)}_{\mathcal{S}[\varphi]}[\mathrm{Id}_{C\times_\beta \mathbf{L}(V,W)}]))& \\ =\ \tau_{\beta\gamma,*}(\gamma^*(u_\beta))\ =\ \gamma^*(\tau_{\beta,*}(u_\beta))\ .&\end{aligned}$$

Since the composite $[1,\varphi]^*\circ \mathrm{res}^{C\times O(W)}_{\mathcal{S}[\varphi]}$ is bijective by the induction isomorphism (A.7), this relation shows that—and how— the class $\tau_{\Pi,C\times_\beta \mathbf{L}(V,W)}[\mathrm{Id}_{C\times_\beta \mathbf{L}(V,W)}]$ is determined by the class $\tau_{\beta,*}(u_\beta)$.

Now we consider another compact Lie group G, a continuous homomorphism $\alpha\colon G\longrightarrow C$, a finite G-CW-complex A, and a class $x\in[A,B_{\mathrm{gl}}\beta]^\alpha$. We represent x by a continuous G-map $f\colon A\longrightarrow \alpha^\flat((B_{\mathrm{gl}}\beta)(W))$ for some G-representation W. If $A=\emptyset$, then $[A,E]^\alpha$ has only one element, and there is nothing to show. If A is nonempty, then also $(B_{\mathrm{gl}}\beta)(W)$ is nonempty, and there exists a linear isometric embedding $\varphi\colon V\longrightarrow W$.

We let $\rho\colon G\longrightarrow O(W)$ classify the G-action on W. Then $(\alpha,\rho)\colon G\longrightarrow C\times O(W)$ is a morphism over C, in the sense that $\Pi\circ(\alpha,\rho)=\alpha$. The G-action on $\alpha^\flat(C\times_\beta \mathbf{L}(V,W))=\alpha^\flat((B_{\mathrm{gl}}\beta)(W))$ is obtained from the $(C\times O(W))$-action by restriction along $(\alpha,\rho)\colon G\longrightarrow (C\times O(W))$. So

$$f\ :\ A\ \longrightarrow\ (\alpha,\rho)^*(C\times_\beta \mathbf{L}(V,W))$$

is G-equivariant. Moreover, the tautological relation

$$x\ =\ [f]\ =\ f^*((\alpha,\rho)^*[\mathrm{Id}_{C\times_\beta \mathbf{L}(V,W)}])\ .$$

holds, i.e., x is the image of the tautological class under the composite

$$[C\times_\beta \mathbf{L}(V,W),B_{\mathrm{gl}}\beta]^\Pi\ \xrightarrow{(\alpha,\rho)^*}\ [(\alpha,\rho)^*(C\times_\beta \mathbf{L}(V,W)),B_{\mathrm{gl}}\beta]^\alpha\ \xrightarrow{f^*}\ [A,B_{\mathrm{gl}}\beta]^\alpha\ .$$

The naturality properties of the C-global transformation τ yield

$$\begin{aligned}\tau_{\alpha,A}(x)\ &=\ \tau_{\alpha,A}(f^*((\alpha,\rho)^*[\mathrm{Id}_{C\times_\beta \mathbf{L}(V,W)}]))\\ &=\ f^*((\alpha,\rho)^*(\tau_{\Pi,C\times_\beta \mathbf{L}(V,W)}[\mathrm{Id}_{C\times_\beta \mathbf{L}(V,W)}]))\ .\end{aligned}$$

We argued above that the class $\tau_{\Pi,C\times_\beta \mathbf{L}(V,W)}[\mathrm{Id}_{C\times_\beta \mathbf{L}(V,W)}]$ is determined by the class $\tau_{\beta,*}(u_\beta)$. So also $\tau_{\alpha,A}(x)$ is determined by the class $\tau_{\beta,*}(u_\beta)$. This completes the proof of uniqueness. □

In the rest of this appendix, we investigate the class of *coinduced* C-global spaces. For these objects, the C-global information is determined, in the precise sense of Proposition A.14, in global information of the underlying global space, i.e., after forgetting the C-action. This class of objects is relevant for our purposes because for $C=\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, the C-global spaces $\mathbf{BUP}$, $\mathbf{U}$ and $\Omega^\bullet(\mathbf{KR})$ are coinduced, see Theorems B.22 and B.56.

In the following we shall write $[\![-,-]\!]^C$ for the set of morphisms in the C-global homotopy category, i.e., the localization of the category of orthogonal C-spaces at the class of C-global equivalences. We let EC be a universal free C-space, i.e., a free C-CW-complex whose underlying space is contractible. And we shall write $p\colon EC\longrightarrow *$ for the unique map.

Definition A.12. Let C be a compact Lie group. An orthogonal C-space X is *coinduced* if for every orthogonal C-space A the map

$$[\![A\times p,X]\!]^C\ :\ [\![A,X]\!]^C\ \longrightarrow\ [\![A\times EC,X]\!]^C$$

induced by $A\times p\colon A\times EC\longrightarrow A$ is bijective.

Proposition A.13. *Let C be a compact Lie group. Let $\phi\colon A\longrightarrow B$ be a morphism of orthogonal C-spaces that is a global equivalence of underlying orthogonal spaces after forgetting the C-action. Then for every free C-space E, the morphism $\phi\times E\colon A\times E\longrightarrow B\times E$ is a C-global equivalence.*

Proof. Product with any topological space preserves global equivalences of orthogonal spaces by [24, Proposition 1.1.9 (vii)]. Since the underlying morphism of ϕ is a global equivalence, so is the underlying morphism of $\phi \times E$. To show that $\phi \times E$ is a C-global equivalence, it thus remains to solve the lifting property from the defining property [25, Definition A.2] of C-global equivalences for all non-trivial continuous homomorphisms $\alpha\colon G \longrightarrow C$. But if α is non-trivial, then $E^{\alpha(G)} = \emptyset$ because C acts freely on E. So for every G-representation V, the space

$$((B \times E)(V))^{\Gamma(\alpha)} \;=\; B(V)^{\Gamma(\alpha)} \times E^{\alpha(G)}$$

is empty, where $\Gamma(\alpha)$ is the graph of α. Hence there are no lifting problems to solve for α, which completes the proof that $\phi \times E$ is a C-global equivalence. □

Proposition A.14. *Let X and Y be coinduced orthogonal C-spaces.*

(i) *For every morphism of orthogonal C-spaces $\phi\colon A \longrightarrow B$ that is a global equivalence of underlying orthogonal spaces after forgetting the C-action, the map*

$$[\![\phi, X]\!]^C \;:\; [\![B, X]\!]^C \;\longrightarrow\; [\![A, X]\!]^C$$

is bijective.

(ii) *Let $f\colon X \longrightarrow Y$ be a morphism that is a global equivalence of underlying orthogonal spaces after forgetting the C-action. Then f is a C-global equivalence.*

Proof. (i) Because the underlying morphism of ϕ is a global equivalence and the C-action on EC is free, the morphism $\phi \times EC\colon A \times EC \longrightarrow B \times EC$ is a C-global equivalence by Proposition A.13. So the lower horizontal map in the following commutative diagram is bijective:

$$\begin{tikzcd}
{[\![B, X]\!]^C} \arrow[r, "{[\![\phi, X]\!]^C}"] \arrow[d, "{[\![B\times p, X]\!]^C}"', "\cong"] & {[\![A, X]\!]^C} \arrow[d, "{[\![A\times p, X]\!]^C}", "\cong"'] \\
{[\![B \times EC, X]\!]^C} \arrow[r, "\cong", "{[\![\phi\times EC, X]\!]^C}"'] & {[\![A \times EC, X]\!]^C}
\end{tikzcd}$$

The vertical maps are bijective because X is coinduced, so also $[\![\phi, X]\!]^C$ is bijective, as claimed.

(ii) Since X is coinduced, the map $[\![f, X]\!]^C\colon [\![Y, X]\!]^C \longrightarrow [\![X, X]\!]^C$ is bijective by part (i). So there is a morphism $g\colon Y \longrightarrow X$ in the C-global homotopy category such that $gf = \mathrm{Id}_X$. Then

$$[\![f, Y]\!](fg) \;=\; fgf \;=\; f \;=\; [\![f, Y]\!](\mathrm{Id}_Y)\ .$$

Since Y is coinduced, the map $[\![f, Y]\!]^C\colon [\![Y, Y]\!]^C \longrightarrow [\![X, Y]\!]^C$ is bijective, again by part (i), so $fg = \mathrm{Id}_Y$. Hence f is an isomorphism in the C-global homotopy category, and thus a C-global equivalence. □

An orthogonal C-space X is fibrant in the C-global model structure of [5, Theorem A.20] if and only if the following condition holds, see [5, Definition A.13]: for every continuous homomorphism of compact Lie groups $\alpha\colon G \longrightarrow C$ and every linear isometric embedding of G-representations $\varphi\colon V \longrightarrow W$ such that G acts faithfully on V, the map

$$X(\varphi)^{\Gamma(\alpha)} \;:\; X(V)^{\Gamma(\alpha)} \;\longrightarrow\; X(W)^{\Gamma(\alpha)}$$

is a weak equivalence, where $\Gamma(\alpha) = \{(\alpha(g), g)\colon g \in G\}$ is the graph of α. We will call such orthogonal C-spaces *C-fibrant.*

Proposition A.15. *A C-fibrant orthogonal C-space X is coinduced if and only if the morphism of orthogonal C-spaces*

$$\mathrm{map}(p, X) \;:\; X \;\longrightarrow\; \mathrm{map}(EC, X)$$

is a C-global equivalence.

Proof. The adjoint functors $(-\times EC, \mathrm{map}(EC,-))$ are a Quillen functor pair for the C-global model structure, so they induce an adjoint pair of derived functors at the level of the C-global homotopy category. The left adjoint $-\times EC$ is fully homotopical, i.e., it preserves arbitrary C-global equivalences, for example by Proposition A.13; so it descends to the C-global homotopy category, and the descended functor is the derived left adjoint. Since X is fibrant, the orthogonal C-space $\mathrm{map}(EC,X)$ models the total right derived functor of $\mathrm{map}(EC,-)$ on X. The following diagram commutes:

$$\begin{tikzcd} \llbracket A, X\rrbracket^C \arrow[d, "{\llbracket A,\mathrm{map}(p,X)\rrbracket^C}"'] \arrow[dr, bend left, "{\llbracket A\times p,X\rrbracket^C}"] & \\ \llbracket A,\mathrm{map}(EC,X)\rrbracket^C \arrow[r, "\cong", "\text{adjunction}"'] & \llbracket A\times EC, X\rrbracket^C \end{tikzcd}$$

So X is coinduced if and only if for every orthogonal C-space A, the left vertical map $\llbracket A,\mathrm{map}(p,X)\rrbracket^C$ is bijective. This happens if and only if $\mathrm{map}(p,X)$ is an isomorphism in the C-global homotopy category, which is equivalent to $\mathrm{map}(p,X)$ being a C-global equivalence. □

If X is an orthogonal space and G a compact Lie group, the *underlying G-space* of X is the G-space

$$X(\mathcal{U}_G) \ = \ \mathrm{colim}_{V\in s(\mathcal{U}_G)}\, X(V) \ ;$$

the colimit is formed over the poset $s(\mathcal{U}_G)$, under inclusion, of finite-dimensional G-subrepresentations of the chosen complete G-universe $\mathcal{U}_G$. If X is an orthogonal C-space, then the C-action on X induces a continuous C-action on $X(\mathcal{U}_G)$ that commutes with the G-action, so $X(\mathcal{U}_G)$ becomes a $(C\times G)$-space.

By [5, Proposition 3.5], a morphism $f\colon X\longrightarrow Y$ between closed orthogonal C-spaces is a C-global equivalence if and only if for every continuous homomorphism $\alpha\colon G\longrightarrow C$ of compact Lie groups, the map

$$f(\mathcal{U}_G)^{\Gamma(\alpha)} \ : \ X(\mathcal{U}_G)^{\Gamma(\alpha)} \ \longrightarrow \ Y(\mathcal{U}_G)^{\Gamma(\alpha)}$$

is a weak equivalence, where as before $\Gamma(\alpha)$ is the graph of α.

Proposition A.16. *Let X be an orthogonal C-space whose underlying orthogonal space is closed. Then the following conditions are equivalent.*

(i) *The orthogonal C-space X is coinduced.*
(ii) *For every continuous homomorphism $\alpha\colon G\longrightarrow C$ of compact Lie groups, the map*

$$\mathrm{map}^{\Gamma(\alpha)}(p, X(\mathcal{U}_G)) \ : \ X(\mathcal{U}_G)^{\Gamma(\alpha)} \ \longrightarrow \ \mathrm{map}^{\Gamma(\alpha)}(EC, X(\mathcal{U}_G))$$

is a weak equivalence.

Proof. We start with a preliminary reduction step. By choosing an acyclic cofibration $X\longrightarrow X'$ with fibrant target in the C-global model structure, we can assume without loss of generality in both parts that X is not only closed, but also C-fibrant. We let $\alpha\colon G\longrightarrow C$ be a continuous homomorphism, and we let V be a faithful G-representation. Because X is C-fibrant and closed, the left vertical map in the commutative diagram

(A.17)
$$\begin{tikzcd} X(V)^{\Gamma(\alpha)} \arrow[r, "{\mathrm{map}^{\Gamma(\alpha)}(p,X(V))}", "\sim"'] \arrow[d, "\sim"'] & \mathrm{map}^{\Gamma(\alpha)}(EC, X(V)) \arrow[d, "\sim"] \\ X(\mathcal{U}_G)^{\Gamma(\alpha)} \arrow[r, "{\mathrm{map}^{\Gamma(\alpha)}(p,X(\mathcal{U}_G))}"'] & \mathrm{map}^{\Gamma(\alpha)}(EC, X(\mathcal{U}_G)) \end{tikzcd}$$

is a weak equivalence. Because the functor $\mathrm{map}(EC,-)$ preserves the class of $(C\times G)$-maps that are graph subgroup equivalences, the right vertical map is a weak equivalence, too.

(i)$\Longrightarrow$(ii) Because X is coinduced and C-fibrant, Proposition A.15 shows that the morphism $\mathrm{map}(p, X)\colon X \longrightarrow \mathrm{map}(EC, X)$ is a C-global equivalence. Because X is C-fibrant, so is $\mathrm{map}(EC, X)$. As a C-global equivalence between C-fibrant objects, the morphism $\mathrm{map}(p, X)$ is a C-level equivalence, see [5, Lemma A.19]. So the upper horizontal map in the diagram (A.17) is a weak equivalence. Hence the lower horizontal map in (A.17) is a weak equivalence, too, proving condition (ii).

(ii)$\Longrightarrow$(i) We turn the previous argument around. Because the lower horizontal map in the diagram (A.17) is a weak equivalence, so is the upper horizontal map, for every faithful G-representation V. So the morphism $\mathrm{map}(p, X)\colon X \longrightarrow \mathrm{map}(EC, X)$ is a C-level equivalence, and hence a C-global equivalence. Because X is C-fibrant, Proposition A.15 shows that X is coinduced. □

Proposition A.18. *Let X be a pointed orthogonal C-space whose underlying orthogonal C-space is coinduced. Then for every finite based C-CW-complex E, the orthogonal C-space $\mathrm{map}_*(E, X)$ has a coinduced underlying orthogonal C-spaces.*

Proof. Since the functor $\mathrm{map}_*(E, -)$ preserves C-global equivalences between based orthogonal C-spaces, we can assume without loss of generality that X is C-fibrant. Then the morphism $p^*\colon X \longrightarrow \mathrm{map}(EC, X)$ is a C-global equivalence by Proposition A.15. Since $\mathrm{map}_*(E, -)$ preserves C-global equivalences, also the morphism

$$\mathrm{map}_*(E, p^*) \ : \ \mathrm{map}_*(E, X) \ \longrightarrow \ \mathrm{map}_*(E, \mathrm{map}(EC, X))$$

is a C-global equivalence. The morphism $p^*\colon \mathrm{map}_*(E, X) \longrightarrow \mathrm{map}(EC, \mathrm{map}_*(E, X))$ is isomorphic to $\mathrm{map}_*(E, p^*)$, and hence a C-global equivalence. Since X is C-fibrant, so is $\mathrm{map}_*(E, X)$, so another application of Proposition A.15 shows that $\mathrm{map}_*(E, X)$ is coinduced. □

Appendix B. Real-global K-theory

In this appendix we extend various features of global K-theory to the Real-global context. In particular, we provide Real-global generalizations of many results in Sections 2.5, 6.3 and 6.4 of [24]. Some sample results in this appendix are as follows. In Theorem B.12 we show that the Real-global space **BUP** represent Real-global K-theory. In Theorem B.24, we establish Real-global Bott periodicity, in the form of an equivalence of Real-global ultra-commutative monoids between **BUP** and $\Omega^\sigma \mathbf{U}$,

We show in Theorem B.59 that the *Real-global K-theory spectrum* **KR** deserves its name: for every augmented Lie group $\alpha\colon G \longrightarrow C = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, the genuine G-spectrum $\alpha^*(\mathbf{KR})$ represents α-equivariant Real K-theory KR_α. Theorem B.57 shows that (and how) the Real-global space **U** is the Real-global infinite loop space of $\mathbf{KR} \wedge S^\sigma$. Consequently, the Real-global space $\mathbf{BUP} \sim \Omega^\sigma \mathbf{U}$ 'is' the Real-global infinite loop space underlying **KR**, see Remark B.58.

B.1. **Unstable Real-global K-theory: BUP and U.** In this subsection we extend various unstable features of global K-theory to the Real-global context. The main results here are that the Real-global spaces **Gr** and **BUP** represent Real-global vector bundles and Real-global K-theory, respectively, see Theorem B.12; and we establish Real-global Bott periodicity, proving an equivalence of Real-global ultra-commutative monoids between **BUP** and $\Omega^\sigma \mathbf{U}$, see Theorem B.24. Along the way, we show that the orthogonal C-spaces **BUP** and **U** are coinduced, see Theorem B.22.

Construction B.1 ($\mathbf{Gr}_k$)**.** The *extended unitary group* $\tilde{U}(k)$ is the augmented Lie group consisting of the semidirect product $U(k) \rtimes C$ of the complex conjugation action of C on $U(k)$, augmented by the projection $U(k) \rtimes C \longrightarrow C$. We recall the Grassmannian model $\mathbf{Gr}_k$ for the Real-global classifying space of $\tilde{U}(k)$. We deviate slightly from the notation of [24, Section 2.3], where **Gr** without a superscript is used for the real version of the additive Grassmannian, and where the complex version is denoted $\mathbf{Gr}^{\mathbb{C}}$; also, the homogeneous summand $\mathbf{Gr}_k$ is written $\mathbf{Gr}^{\mathbb{C},[k]}$. In the present paper, we shall almost exclusively work with complex Grassmannians, so they will be referred to by the simpler name without superscript. The value of $\mathbf{Gr}_k$ at a euclidean inner product space V is

$$\mathbf{Gr}_k(V) \ = \ Gr_k^{\mathbb{C}}(V_{\mathbb{C}}) \ ,$$

the Grassmannian of complex k-planes in the complexification $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$. The structure map $\mathbf{Gr}_k(\varphi) \colon \mathbf{Gr}_k(V) \longrightarrow \mathbf{Gr}_k(W)$ induced by a linear isometric embedding $\varphi \colon V \longrightarrow W$ takes takes the images under the complexified linear isometric embedding $\varphi_{\mathbb{C}} \colon V_{\mathbb{C}} \longrightarrow W_{\mathbb{C}}$. The involution

$$\psi(V) \ : \ \mathbf{Gr}_k(V) \ \longrightarrow \ \mathbf{Gr}_k(V)$$

that makes it an orthogonal C-space is complex conjugation. Here we exploit the fact that the complexification of an $\mathbb{R}$-vector space V comes with a preferred $\mathbb{C}$-semilinear involution

$$\psi_V \ : \ V_{\mathbb{C}} \ \longrightarrow \ V_{\mathbb{C}} \ , \quad \lambda \otimes v \ \longmapsto \ \bar{\lambda} \otimes v \ . \tag{B.2}$$

The involution $\psi(V)$ takes a $\mathbb{C}$-subspace $L \subset V_{\mathbb{C}}$ to the conjugate subspace $\bar{L} = \psi_V(L)$.

By [24, Proposition A.31], for every augmented compact Lie group $\beta \colon K \longrightarrow C$ and every complete K-universe $\mathcal{U}_K$, the complex Stiefel manifold $\mathbf{L}^{\mathbb{C}}(\nu_k, \mathcal{U}_K^{\mathbb{C}})$ with its $(K \times_C \tilde{U}(k))$-action by

$$(k, A) \cdot \varphi \ = \ l_k \circ \varphi \circ l_A^{-1}$$

is a universal $(K \times_C \tilde{U}(k))$-space for the family of those closed subgroups that intersect $1 \times U(k)$ trivially. A consequence spelled out in [24, Theorem A.33 (i)] is that $\mathbf{Gr}_k$ receives a C-global equivalence from

$$B_{\mathrm{gl}}\tilde{U}(k) \ = \ C \times_{\tilde{U}(k)} \mathbf{L}(u(\nu_k), -) \ ,$$

which is a C-global classifying space, in the sense of [24, Construction A.4], of the augmented Lie group $\tilde{U}(k)$.

Construction B.3 (**Gr** and **BUP**)**.** The orthogonal C-space

$$\mathbf{Gr} \ = \ \coprod\nolimits_{k \geq 0} \mathbf{Gr}_k$$

is the disjoint union of the Grassmannians from Construction B.1. So the value of **Gr** at an inner product space V is the disjoint union of all complex Grassmannians in the complexification $V_{\mathbb{C}}$. Direct sum of subspaces, plus identification along the isomorphism $V_{\mathbb{C}} \oplus W_{\mathbb{C}} \cong (V \oplus W)_{\mathbb{C}}$ provides an ultra-commutative multiplication on **Gr**, compatible with the involution by complex conjugation. This structure makes **Gr** into an ultra-commutative C-monoid space.

The ultra-commutative monoid **BUP** is the Real-global analog of the ultra-commutative monoid **BOP** introduced in [24, Example 2.4.1], and its underlying global space is the complex periodic Grassmannian with the same name from [24, Example 2.4.33]. The values of **BUP** are

$$\mathbf{BUP}(V) \ = \ \coprod\nolimits_{n \geq 0} Gr_n^{\mathbb{C}}(V_{\mathbb{C}}^2) \ ,$$

the full Grassmannian of complex subspaces of $V_{\mathbb{C}}^2$. The structure map $\mathbf{BUP}(\varphi) \colon \mathbf{BUP}(V) \longrightarrow \mathbf{BUP}(W)$ associated with a linear isometric embedding $\varphi \colon V \longrightarrow W$ is given by

$$\mathbf{BUP}(\varphi)(L) \ = \ \varphi_{\mathbb{C}}^2(L) \ + \ ((W - \varphi(V))_{\mathbb{C}} \oplus 0) \ .$$

It is important that while the C-spaces $\mathbf{BUP}(V)$ and $\mathbf{Gr}(V^2)$ are equal, their structure maps are different, making them distinct Real-global homotopy types.

An ultra-commutative multiplication of **BUP** is given by

$$\mu_{V,W} \ : \ \mathbf{BUP}(V) \times \mathbf{BUP}(W) \ \longrightarrow \mathbf{BUP}(V \oplus W) \ , \quad \mu_{V,W}(L, L') \ = \ \kappa_{V,W}(L \oplus L') \ ,$$

where $\kappa_{V,W} \colon V_{\mathbb{C}}^2 \oplus W_{\mathbb{C}}^2 \cong (V \oplus W)_{\mathbb{C}}^2$ is the shuffle isomorphism $\kappa_{V,W}(v, v', w, w') = (v, w, v', w')$. An involution

$$\psi(V) \ : \ \mathbf{BUP}(V) \ \longrightarrow \ \mathbf{BUP}(V)$$

is defined in the same way as for **Gr** by applying the complex conjugation involution $\psi^2_V : V^2_{\mathbb{C}} \longrightarrow V^2_{\mathbb{C}}$ to complex subspaces. All these data makes **BUP** into an ultra-commutative C-monoid.

We define a morphism of ultra-commutative C-monoids

$$i \ : \ \mathbf{Gr} \longrightarrow \mathbf{BUP} \tag{B.4}$$

at a euclidean inner product space V by

$$\mathbf{Gr}(V) \ = \ \coprod_{m\geq 0} Gr^{\mathbb{C}}_m(V_{\mathbb{C}}) \ \longrightarrow \ \coprod_{n\geq 0} Gr^{\mathbb{C}}_n(V^2_{\mathbb{C}}) = \mathbf{BUP}(V) \ , \quad L \ \longmapsto \ V_{\mathbb{C}} \oplus L \ .$$

In [24, Proposition 2.4.5], we show in the real (with small 'r') context, that for every compact Lie group G and every G-space A, the analogous homomorphism $[A, i^{\mathbb{R}}]^G : [A, \mathbf{Gr}^{\mathbb{R}}]^G \longrightarrow [A, \mathbf{BOP}]^G$ is a group completion of abelian monoids. The same arguments also show that the 'non-Real' morphism $i^{\mathbb{C}} : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{BUP}$ underlying $i : \mathbf{Gr} \longrightarrow \mathbf{BUP}$ induces a group completion of abelian monoids $[A, i^{\mathbb{C}}]^G : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow [A, \mathbf{BUP}]^G$. All arguments in the proof of [24, Proposition 2.4.5] carry over almost literally to our presents Real-global context, and thereby show the following result:

Proposition B.5. *For every augmented Lie group $\alpha : G \longrightarrow C$ and every G-space A, the homomorphism*

$$[A, i]^\alpha \ : \ [A, \mathbf{Gr}]^\alpha \ \longrightarrow \ [A, \mathbf{BUP}]^\alpha$$

is a group completion of abelian monoids.

Construction B.6 (Real-equivariant vector bundles)**.** We recall the notion of 'Real-equivariant vector bundles' for an augmented Lie group $\alpha : G \longrightarrow C$. This concept encompasses real and complex equivariant vector bundles, and Atiyah's Real vector bundles [2]. A *Real α-vector bundle* over a G-space A is the data of

- a complex vector bundle $\xi : E \longrightarrow A$,
- a continuous G-action on the total space E,

such that the projection ξ is G-equivariant, and for all $(g, a) \in G \times A$, the translation map $g\cdot - : E_a \longrightarrow E_{ga}$ is $\alpha(g)$-linear. In other words, translation by g is $\mathbb{C}$-linear if $\alpha(g) = 1$; and translation by g is conjugate-linear if $\alpha(g) \neq 1$.

A key example is the tautological vector bundle γ_V over the Grassmannian $\mathbf{Gr}(V) = \amalg_{n\geq 0} Gr^{\mathbb{C}}_n(V_{\mathbb{C}})$ for an orthogonal G-representation V. Here G acts diagonally on $\mathbf{Gr}(V)$, through the complexification of the given action on V, and through complex conjugation along the augmentation $\alpha : G \longrightarrow C$. In other words, for a complex subspace L of $V_{\mathbb{C}}$, we set

$$g \cdot L \ = \ \begin{cases} l^{\mathbb{C}}_g(L) & \text{for } \alpha(g) = 1, \text{ and} \\ \psi_V(l^{\mathbb{C}}_g(L)) = l^{\mathbb{C}}_g(\psi_V(L)) & \text{for } \alpha(g) \neq 1. \end{cases}$$

Here $l_g : V \longrightarrow V$ is translation by $g \in G$, and $l^{\mathbb{C}}_g : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}$ is its complexification.

The total space of the tautological bundle γ_V over $\mathbf{Gr}(V)$ is

$$\{(w, L) \ \in V_{\mathbb{C}} \times \mathbf{Gr}(V) : w \in L\} \ ;$$

this bundle does *not* have constant rank. The group G acts on the total space by

$$g \cdot (w, L) \ = \ \begin{cases} (l^{\mathbb{C}}_g(w), l^{\mathbb{C}}_g(L)) & \text{for } \alpha(g) = 1, \text{ and} \\ (\psi_V(l^{\mathbb{C}}_g(w)), \psi_V(l^{\mathbb{C}}_g(L))) & \text{for } \alpha(g) \neq 1. \end{cases}$$

These data make γ_V into a Real α-vector bundle over $\mathbf{Gr}(V)$.

We recall in Examples B.13 and B.14 how Real-equivariant vector bundles specialize to real and complex equivariant vector bundles for trivial and product augmentations, respectively. For the group C augmented by the identity, Real-equivariant vector bundles specialize to Atiyah's Real vector bundles from [2]. In [3, Section 5] and [4, Section 6], Atiyah and Segal consider what they call 'Real Lie groups', i.e., Lie groups

G equipped with a multiplicative involution $\tau\colon G \longrightarrow G$; this corresponds to case of a semidirect product $G \rtimes_\tau C$, augmented by the projection to the second factor. The case of general augmented Lie groups is considered by Karoubi in [17, 18].

Now we explain in which sense the Real-global space $\mathbf{Gr}$ represents Real-equivariant vector bundles, and in which sense the Real-global space $\mathbf{BUP}$ represents Real-equivariant K-theory. The constructions and theorems are adaptations of results from [24, Section 2.4] from the global to the Real-global context.

Construction B.7. We let $\alpha\colon G \longrightarrow C$ be an augmented Lie group, and $k \geq 0$. We write $\mathrm{Vect}^{k,R}_\alpha(A)$ for the set of isomorphism classes of Real α-vector bundles of rank k over a G-space A. We define a map

$$\langle - \rangle \ : \ [A, \mathbf{Gr}_k]^\alpha \ = \ \mathrm{colim}_{V \in s(\mathcal{U}_G)} \, [A, \alpha^\flat(\mathbf{Gr}_k(V))]^G \ \longrightarrow \ \mathrm{Vect}^{k,R}_\alpha(A) \tag{B.8}$$

as follows. We let $f\colon A \longrightarrow \alpha^\flat(\mathbf{Gr}_k(V))$ be a continuous G-map, for some orthogonal G-representation V. We pull back the tautological Real α-vector bundle γ^k_V over $\mathbf{Gr}_k(V)$ and obtain a Real α-vector bundle $f^*(\gamma^k_V)\colon E \longrightarrow A$ of rank k. Since the base $\mathbf{Gr}_k(V)$ of the tautological bundle is compact, the isomorphism class of the bundle $f^*(\gamma^k_V)$ depends only on the G-homotopy class of f. So the construction yields a well-defined map

$$[A, \alpha^\flat(\mathbf{Gr}_k(V))]^G \ \longrightarrow \ \mathrm{Vect}^{k,R}_\alpha(A) \ , \quad [f] \ \longmapsto \ [f^*(\gamma^k_V)] \ .$$

If $\varphi\colon V \longrightarrow W$ is a linear isometric embedding of orthogonal G-representations, then the restriction along $\mathbf{Gr}_k(\varphi)\colon \mathbf{Gr}_k(V) \longrightarrow \mathbf{Gr}_k(W)$ of the tautological Real α-vector bundle γ_W over $\mathbf{Gr}_k(W)$ is isomorphic to the tautological Real α-vector bundle γ_V over $\mathbf{Gr}_k(V)$. So the two α-vector bundles $f^*(\gamma_V)$ and $(\mathbf{Gr}_k(\varphi) \circ f)^*(\gamma_W)$ over A are isomorphic. We can thus pass to the colimit over the poset $s(\mathcal{U}_G)$ of finite-dimensional G-subrepresentations of $\mathcal{U}_G$, and get a well-defined map (B.8).

We let $\mathrm{Vect}^R_\alpha(A)$ denote the commutative monoid, under Whitney sum, of isomorphism classes of Real α-vector bundles over A. We define a monoid homomorphism

$$\langle - \rangle \ : \ [A, \mathbf{Gr}]^\alpha \ \longrightarrow \ \mathrm{Vect}^R_\alpha(A) \tag{B.9}$$

in much the same way as the map (B.8), with the main difference that now the vector bundles need not have constant rank. The map (B.9) is a monoid homomorphism because all additions in sight arise from direct sum of inner product spaces.

Now we 'group complete' the picture. We denote by $KR_\alpha(A)$ the α-equivariant Real K-group of A, i.e., the group completion (Grothendieck group) of the abelian monoid $\mathrm{Vect}^R_\alpha(A)$. In some other references, the Real K-group $KR_\alpha(A)$ is denoted $KR_G(A)$, i.e., only the group G, but not the augmentation $\alpha\colon G \longrightarrow C$, is recorded in the notation. The composite

$$[A, \mathbf{Gr}]^\alpha \ \xrightarrow[\text{(B.9)}]{\langle - \rangle} \ \mathrm{Vect}^R_\alpha(A) \ \longrightarrow \ \ KR_\alpha(A)$$

is a monoid homomorphism to an abelian group. The morphism $[A,i]^\alpha\colon [A, \mathbf{Gr}]^\alpha \longrightarrow [A, \mathbf{BUP}]^\alpha$ is a group completion of abelian monoids by Proposition B.5, where $i\colon \mathbf{Gr} \longrightarrow \mathbf{BUP}$ was defined in (B.4). So there is a unique homomorphism of abelian groups

$$\langle - \rangle \ : \ [A, \mathbf{BUP}]^\alpha \ \longrightarrow \ KR_\alpha(A) \tag{B.10}$$

such that the following square commutes:

$$\begin{tikzcd}
{[A, \mathbf{Gr}]^\alpha} \arrow[r, "\langle - \rangle"] \arrow[d, "{[A,i]^\alpha}"'] & \mathrm{Vect}^R_\alpha(A) \arrow[d] \\
{[A, \mathbf{BUP}]^\alpha} \arrow[r, "\langle - \rangle"'] & KR_\alpha(A)
\end{tikzcd} \tag{B.11}$$

The next theorem generalizes Proposition 2.4.5 and Theorem 2.4.10 of [24] from the orthogonal and unitary to the Real-equivariant context.

Theorem B.12. *Let $\alpha\colon G \longrightarrow C$ be an augmented Lie group, and let A be a compact G-space.*

(i) *For every $k \geq 0$, the map* (B.8)

$$\langle - \rangle \ : \ [A, \mathbf{Gr}_k]^\alpha \ \longrightarrow \ \mathrm{Vect}_\alpha^{k,R}(A)$$

is bijective.

(ii) *The monoid homomorphism* (B.9)

$$\langle - \rangle \ : \ [A, \mathbf{Gr}]^\alpha \ \longrightarrow \ \mathrm{Vect}_\alpha^{R}(A)$$

is an isomorphism.

(iii) *The group homomorphism* (B.10)

$$\langle - \rangle \ : \ [A, \mathbf{BUP}]^\alpha \ \longrightarrow \ KR_\alpha(A)$$

is an isomorphism.

Proof. The first statement is proved by essentially the same arguments as its real (with small 'r') predecessor in [24, Theorem 2.4.10], mutatis mutandis. The argument uses that the complex Stiefel manifold $\mathbf{L}^{\mathbb{C}}(\nu_k, \mathcal{U}_G^{\mathbb{C}})$ with its $(G \times_C \tilde{U}(k))$-action by

$$(g, A) \cdot \varphi \ = \ l_g \circ \varphi \circ l_A^{-1}$$

is a universal $(G \times_C \tilde{U}(k))$-space for the family of those closed subgroups that intersect $1 \times U(k)$ trivially, see [24, Proposition A.31]. This, in turn, implies that the G-space $\alpha^\flat(\mathbf{Gr}_k(\mathcal{U}_G^{\mathbb{C}})) = \mathbf{L}^{\mathbb{C}}(\nu_k, \mathcal{U}_G^{\mathbb{C}})/U(k)$ is a classifying space for rank k Real α-equivariant vector bundles over compact G-spaces.

(ii) We suppose first that A is G-connected, i.e., the group $\pi_0(G)$ acts transitively on $\pi_0(A)$. This ensures that every Real α-vector bundle over A has constant rank, and that every continuous G-map $A \longrightarrow \alpha^\flat(\mathbf{Gr}(V))$ factors through $\alpha^\flat(\mathbf{Gr}_k(V))$ for some $k \geq 0$. Hence both vertical maps in the following commutative square, induced by the inclusions $\mathbf{Gr}_k \longrightarrow \mathbf{Gr}$, are bijective:

$$\begin{tikzcd}
\coprod_{k\geq 0} [A, \mathbf{Gr}_k]^\alpha \arrow[r, "\langle - \rangle", "(B.8)"'] \arrow[d] & \coprod_{k\geq 0} \mathrm{Vect}_\alpha^{k,R}(A) \arrow[d] \\
{[A, \mathbf{Gr}]^\alpha} \arrow[r, "(B.9)", "\langle - \rangle"'] & \mathrm{Vect}_\alpha^{R}(A)
\end{tikzcd}$$

The upper horizontal map is bijective by (i), hence the lower horizontal map is bijective.

In the general case we decompose $A = A_1 \amalg \ldots \amalg A_m$ is a disjoint union of G-connected finite G-CW-complexes, indexed by the $\pi_0(G)$-orbits of $\pi_0(A)$. Since both functors $[-, \mathbf{Gr}]^\alpha$ and Vect_α^R take finite disjoint unions to products, the general case follows.

Part (iii) follows from (ii) because in the commutative square (B.11) both vertical maps are group completions of abelian monoids, by Proposition B.5 and by definition, respectively. $\square$

As we shall explain in the next two examples, the isomorphism $[A, \mathbf{BUP}]^\alpha \cong KR_\alpha(A)$ generalizes analogous isomorphisms for real K-groups and for complex K-groups that are already discussed in [24, Section 2.4].

Example B.13 (Trivially augmented Lie groups)**.** For a compact Lie group G, we write G^{tr} for the trivially augmented Lie group, i.e., G endowed with the trivial homomorphism to C. Then Real G^{tr}-vector bundles are nothing but complex vector bundles, and thus

$$KR_{G^{\mathrm{tr}}}(A) \ = \ KU_G(A) \ .$$

The underlying orthogonal space of $\mathbf{BUP}$ is the complex periodic Grassmannian with the same name from [24, Example 2.4.33]. For trivially augmented Lie groups, Theorem B.12 thus specializes to an isomorphism

$$\langle - \rangle \ : \ [A, \mathbf{BUP}]^{G^{\mathrm{tr}}} \ \longrightarrow \ KU_G(A) \ ,$$

the complex analog of the isomorphism from [24, Theorem 2.4.10].

Example B.14 (Product augmented Lie groups)**.** We augment the product of a compact Lie group G with the Galois group C by the projection $G \times C \longrightarrow C$ to the second factor. In the following, we shall leave the projection implicit and simply write $G \times C$ for this augmented Lie group.

We consider a G-space A, and we let the group C act trivially on A. Then a Real $(G \times C)$-vector bundle over A is nothing but a real G-vector bundle (with small 'r'). More precisely, the action of the element $(1, \psi) \in G \times C$ on the fiber $E_a = \xi^{-1}(a)$ of a Real $(G \times C)$-vector bundle is a conjugate linear involution on the complex vector space E_a, i.e., a real structure. The $\mathbb{R}$-subspaces $(E_a)^\psi$ fixed by this involution form a real G-vector subbundle

$$\xi^\psi \ = \ \ker(\mathrm{Id} - (1,\psi) \cdot - \colon \xi \longrightarrow \xi)$$

of ξ, and ξ is naturally isomorphic to the complexification of ξ^ψ. This construction implements an equivalence of categories between Real $(G \times C)$-vector bundles and real G-vector bundles over any G-space with trivial C-action. The equivalence induces an isomorphism

$$KR_{G \times C}(A) \ \cong \ KO_G(A) \ . \tag{B.15}$$

The orthogonal subspace of $\mathbf{BUP}$ fixed by the involution 'is' the periodic Grassmannian $\mathbf{BOP}$ from [24, Example 2.4.1]. More precisely, the complexification maps

$$\mathbf{BOP}(V) \ = \ \amalg_{n \geq 0} Gr_n^{\mathbb{R}}(V^2) \ \longrightarrow \ \amalg_{n \geq 0} Gr_n^{\mathbb{C}}(V_{\mathbb{C}}^2) \ = \ \mathbf{BUP}(V) \ , \quad L \ \longmapsto \ \mathbb{C} \otimes_{\mathbb{R}} L$$

form an isomorphism of ultra-commutative monoids to the ψ-fixed subobject $\mathbf{BUP}^\psi$. For any G-space A endowed with trivial C-action, it thus induces an isomorphism

$$[A, \mathbf{BOP}]^G \ \xrightarrow{\cong} \ [A, \mathbf{BUP}]^{G \times C} \ . \tag{B.16}$$

Under the identifications (B.15) and (B.16), Theorem B.12 thus specializes to the isomorphism

$$\langle - \rangle \ : \ [A, \mathbf{BOP}]^G \ \longrightarrow \ KO_G(A)$$

from [24, Theorem 2.4.10].

Construction B.17 (The ultra-commutative monoid $\mathbf{U}$)**.** We recall the ultra-commutative monoid $\mathbf{U}$ made from unitary groups, compare [24, Example 2.37]. The euclidean inner product $\langle -, - \rangle$ on V induces a hermitian inner product $(-,-)$ on the complexification $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$, defined as the unique sesquilinear form that satisfies $(1 \otimes v, 1 \otimes w) = \langle v, w \rangle$ for all $v, w \in V$. The value of the orthogonal space $\mathbf{U}$ on V is

$$\mathbf{U}(V) \ = \ U(V_{\mathbb{C}}) \ ,$$

the unitary group of the complexification of V. The complexification of every $\mathbb{R}$-linear isometric embedding $\varphi \colon V \longrightarrow W$ preserves the hermitian inner products, so we can define a continuous group homomorphism

$$\mathbf{U}(\varphi) \ : \ \mathbf{U}(V) \ \longrightarrow \ \mathbf{U}(W)$$

by conjugation with $\varphi_{\mathbb{C}} \colon V_{\mathbb{C}} \longrightarrow W_{\mathbb{C}}$ and the identity on the orthogonal complement of the image of $\varphi_{\mathbb{C}}$. The commutative multiplication of $\mathbf{U}$ is given by the direct sum of unitary automorphisms

$$\mathbf{U}(V) \times \mathbf{U}(W) \ \longrightarrow \ \mathbf{U}(V \oplus W) \ , \quad (A, B) \mapsto \ A \oplus B \ ,$$

where we implicitly used the preferred complex isometry $V_{\mathbb{C}} \oplus W_{\mathbb{C}} \cong (V \oplus W)_{\mathbb{C}}$. If G is a compact Lie group, then the underlying G-space of $\mathbf{U}$ is the unitary group of a complete complex G-universe. Since unitary G-representations break up into isotypical summand, its G-fixed points decompose as a weak product, indexed by the isomorphism classes of irreducible unitary G-representations, of infinite unitary groups. We refer to [24, Example 2.37] for more details.

The orthogonal space $\mathbf{U}$ comes with an involution

$$\psi \ : \ \mathbf{U} \longrightarrow \mathbf{U}$$

that is an automorphism of ultra-commutative monoids. The value of ψ at V is the map

$$\psi(V) \ : \ U(V_{\mathbb{C}}) \longrightarrow U(V_{\mathbb{C}}) \ , \quad A \longmapsto \psi_V \circ A \circ \psi_V \ ,$$

where $\psi_V \colon V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}}$ is the canonical $\mathbb{C}$-semilinear conjugation involution (B.2). This involution makes $\mathbf{U}$ into a orthogonal C-space, representing an unstable Real-global homotopy type.

We write $\mathbf{BU}$ for the orthogonal C-subspace of $\mathbf{BUP}$ with values

$$\mathbf{BU}(V) \ = \ Gr^{\mathbb{C}}_{|V|}(V^2_{\mathbb{C}}) \ ,$$

where $|V| = \dim_{\mathbb{R}}(V)$. This subobject is the homogeneous summand of degree 0 in the $\mathbb{Z}$-grading on $\mathbf{BUP}$, and closed under the ultra-commutative multiplication. Our next aim is to show that the ultra-commutative C-monoid $\mathbf{BU}$ is a C-global deloop of $\mathbf{U}$, as the notation suggests. The delooping will be witnessed by a zigzag of two Real-global equivalences.

Construction B.18. We define an ultra-commutative C-monoid F and morphisms of ultra-commutative monoids

$$\mathbf{U} \xrightarrow{\ g\ } F \xleftarrow{\ h\ } \Omega(\mathbf{BU}) \ .$$

To this end we introduce an auxiliary ultra-commutative C-monoid $\mathcal{L}$. Its value at an inner product space is the Stiefel manifold

$$\mathcal{L}(V) \ = \ \mathbf{L}^{\mathbb{C}}(V_{\mathbb{C}}, V^2_{\mathbb{C}})$$

of $\mathbb{C}$-linear isometric embeddings of $V_{\mathbb{C}}$ into $V^2_{\mathbb{C}}$. The groups $O(V)$ and C act by conjugation, i.e., by

$${}^A\varphi \ = \ A^2_{\mathbb{C}} \circ \varphi \circ A^{-1}_{\mathbb{C}} \qquad \text{and} \qquad {}^\psi\varphi \ = \ \psi^2_V \circ \varphi \circ \psi_V$$

for $(A, \varphi) \in O(V) \times \mathbf{L}^{\mathbb{C}}(V_{\mathbb{C}}, V^2_{\mathbb{C}})$. An ultra-commutative multiplication is defined by

$$\mathcal{L}(V) \times \mathcal{L}(W) \longrightarrow \mathcal{L}(V \oplus W) \ , \quad (\varphi, \phi) \longmapsto \kappa_{V,W} \circ (\varphi \oplus \phi) \ ,$$

where $\kappa_{V,W} \colon V^2_{\mathbb{C}} \oplus W^2_{\mathbb{C}} \longrightarrow (V \oplus W)^2_{\mathbb{C}}$ is $\kappa_{V,W}(v, v', w, w') = (v, w, v', w')$. The unit is $i_1 \colon V_{\mathbb{C}} \longrightarrow V^2_{\mathbb{C}}$, $i_1(v) = (v, 0)$, which is $(O(V) \times C)$-fixed and multiplicative.

A morphism of ultra-commutative C-monoids $\mathrm{im} \colon \mathcal{L} \longrightarrow \mathbf{BU}(V)$ sends $\varphi \in \mathbf{L}(V_{\mathbb{C}}, V^2_{\mathbb{C}})$ to its image $\mathrm{im}(\varphi) \in Gr^{\mathbb{C}}_{|V|}(V^2_{\mathbb{C}}) = \mathbf{BU}(V)$. We define the ultra-commutative monoid F as the homotopy fiber of the morphism $\mathrm{im} \colon \mathcal{L} \longrightarrow \mathbf{BU}(V)$ over the unit of $\mathbf{BU}$, i.e., by a pullback diagram in the category of ultra-commutative C-monoids:

(B.19)
$$\begin{tikzcd} F \arrow[r, "p"] \arrow[d, "q"'] & \mathcal{L} \arrow[d, "\mathrm{im}"] \\ \{0\} \times_{\mathbf{BU}} \mathbf{BU}^{[0,1]} \arrow[r, "\mathrm{ev}_1"'] & \mathbf{BU} \end{tikzcd}$$

Explicitly, $P(V)$ is the space of all pairs $(\omega, \varphi) \in \mathbf{BU}(V)^{[0,1]} \times \mathcal{L}(V)$ consisting of a path $\omega \colon [0,1] \longrightarrow \mathbf{BU}(V) = Gr_{|V|}(V^2_{\mathbb{C}})$ and a linear isometric embedding $\varphi \colon V_{\mathbb{C}} \longrightarrow V^2_{\mathbb{C}}$ such that

$$\omega(0) \ = \ V_{\mathbb{C}} \oplus 0 \quad \text{and} \quad \omega(1) \ = \ \mathrm{im}(\varphi) \ .$$

As a pullback, the $O(V)$-action, structure maps, involution, multiplication and unit are all inherited from $\mathbf{BU}$ and $\mathcal{L}$.

A morphism of ultra-commutative C-monoids $f \colon \mathbf{U} \longrightarrow F$ is given at V by

$$f(V) \ : \ \mathbf{U}(V) = U(V_{\mathbb{C}}) \longrightarrow F(V) \ , \quad A \longmapsto (\mathrm{const}_{V_{\mathbb{C}} \oplus 0}, i_1 \circ A) \ .$$

Here $\mathrm{const}_{V_{\mathbb{C}}\oplus 0}\colon [0,1] \longrightarrow \mathbf{BU}(V)$ is the constant path at $V_{\mathbb{C}} \oplus 0$, and $i_1\colon V_{\mathbb{C}} \longrightarrow V^2_{\mathbb{C}}$ is the embedding as the first summand. A morphism of ultra-commutative monoids $g\colon \Omega(\mathbf{BU}) \longrightarrow F$ is given at V by

$$g(V) \ : \ \mathrm{map}_*(S^1, \mathbf{BU}(V)) \ \longrightarrow \ F(V) \ , \quad \omega \ \longmapsto \ (\omega \circ t, i_1) \ ,$$

where $t\colon [0,1] \longrightarrow S^1$ is the continuous map $t(x) = (2x-1)/(x(1-x))$ that factors through a homeomorphism $[0,1]/\{0,1\} \cong S^1$.

Proposition B.20. *The morphisms*

$$f \ : \ \mathbf{U} \longrightarrow F \qquad \text{and} \qquad g \ : \ \Omega(\mathbf{BU}) \ \longrightarrow \ F$$

are C-global equivalences of ultra-commutative C-monoids.

Proof. We start with the morphism f, which we show is even a C-level equivalence. We let $\alpha\colon G \longrightarrow C$ be an augmented Lie group, with graph $\Gamma = \{(\alpha(\gamma), \gamma)\colon \gamma \in G\}$. We let V be an orthogonal G-representation. The map

$$\text{(B.21)} \qquad \mathrm{im}(V)^\Gamma \ : \ \mathcal{L}(V)^\Gamma \ = \ (\mathbf{L}^{\mathbb{C}}(V_{\mathbb{C}}, V^2_{\mathbb{C}}))^\Gamma \ \longrightarrow \ (Gr^{\mathbb{C}}_{|V|}(V^2_{\mathbb{C}}))^\Gamma \ = \ \mathbf{BU}(V)^\Gamma$$

is a disjoint union of projections from Stiefel manifolds to the associated Grassmannian manifolds. Hence this map is a locally trivial fiber bundle, and thus a Serre fibration. So the map from the strict fiber of (B.21) over $V_{\mathbb{C}} \oplus 0$ to the homotopy fiber is a weak homotopy equivalence. The right action of $\mathbf{U}(V)^\Gamma = U(V_{\mathbb{C}})^\Gamma$ on $i_1\colon V_{\mathbb{C}} \longrightarrow V^2_{\mathbb{C}}$ by precomposition identifies $\mathbf{U}(V)^\Gamma$ with the strict fiber of (B.21). So the map

$$f(V)^\Gamma \ : \ \mathbf{U}(V)^\Gamma \ \longrightarrow \ \{V_{\mathbb{C}} \oplus 0\} \times_{\mathbf{BU}(V)^\Gamma} (\mathbf{BU}(V)^\Gamma)^{[0,1]} \times_{\mathbf{BU}(V)^\Gamma} \mathcal{L}(V)^\Gamma \ = \ F(V)^\Gamma$$

is a weak homotopy equivalence. This shows that the morphism $f\colon \mathbf{U} \longrightarrow F$ is a C-level equivalence in the sense of [5, Theorem A.2], and hence in particular a C-global equivalence.

To deal with the morphism g we show first that the orthogonal C-space $\mathcal{L}$ is C-globally trivial. For every inner product space V, the homotopy

$$\begin{aligned} H \ : \ \mathbf{L}^{\mathbb{C}}(V_{\mathbb{C}}, V^2_{\mathbb{C}}) \times [0,1] \ &\longrightarrow \ \mathbf{L}^{\mathbb{C}}(V^2_{\mathbb{C}}, V^2_{\mathbb{C}} \oplus V^2_{\mathbb{C}}) \\ H(\varphi, t)(w, w') \ &= \ (\cos(t\pi/2) \cdot \varphi(w), (w', \sin(t\pi/2) \cdot w)) \end{aligned}$$

witnesses that the map

$$- \oplus i_1 \ : \ \mathcal{L}(V) = \mathbf{L}^{\mathbb{C}}(V_{\mathbb{C}}, V^2_{\mathbb{C}}) \ \longrightarrow \ \mathbf{L}^{\mathbb{C}}(V^2_{\mathbb{C}}, V^2_{\mathbb{C}} \oplus V^2_{\mathbb{C}})$$

is $(O(V) \times C)$-equivariantly homotopic to a constant map. The structure map

$$\mathcal{L}(i_1) \ : \ \mathcal{L}(V) \ \longrightarrow \ \mathcal{L}(V^2)$$

is the composite of $- \oplus i_1$ and the homeomorphism induced by $\kappa_{V,V}\colon V^2_{\mathbb{C}} \oplus V^2_{\mathbb{C}} \cong (V \oplus V)^2_{\mathbb{C}}$, so the structure map $\mathcal{L}(i_1)$ is also $(O(V) \times C)$-equivariantly nullhomotopic. So the underlying orthogonal C-space of $\mathcal{L}$ is C-globally contractible.

The lower horizontal evaluation morphism in the defining pullback square (B.19) is a C-level fibration. So also its base change $p\colon F \longrightarrow \mathcal{L}$ is a C-level fibration. Moreover, the right vertical morphism in the pullback square

$$\begin{tikzcd} \Omega(\mathbf{BU}) \arrow[r] \arrow[d, "g"'] & * \arrow[d, "\sim"] \\ F \arrow[r, "p"'] & \mathcal{L} \end{tikzcd}$$

is a C-global equivalence by the above. The pullback of a C-global equivalence along a C-level fibration is again a C-global equivalence, see [5, Lemma A.18]. So $g\colon \Omega(\mathbf{BU}) \longrightarrow F$ is a C-global equivalence. $\square$

Coinduced C-global spaces were introduced in Definition A.12. As we shall now show, the representing C-space **BUP** for Real-global K-theory is an example. This is useful for our purposes because C-global equivalences between coinduced objects can be detected on underlying global spaces, see Proposition A.14.

Theorem B.22. *The orthogonal C-spaces* **BUP**, **U** *and* $\Omega^\sigma\mathbf{U}$ *are coinduced.*

Proof. For showing that **BUP** is coinduced we will exploit that it represents Real-equivariant K-theory, by Theorem B.12 (iii). We abuse notation and simply write C for augmented Lie group $\mathrm{Id}_C\colon C\longrightarrow C$. We let $\beta\in\widetilde{KR}_C(S^{1+\sigma})$ denote the Bott class associated with the 1-dimensional Real C-representation on $\mathbb{C}$, see for example [2, Theorem 2.3] or [3, Theorem 5.1]. Then $i^*(\beta)$ is a class in $\widetilde{KR}_C(S^1)$, where $i\colon S^1\longrightarrow S^{1+\sigma}$ is the inclusion of the C-fixed points. Hence $(i^*(\beta))^3$ lies in the group $\widetilde{KR}_C(S^3)$, which is isomorphic to $\widetilde{KO}(S^3)\cong\pi_3(\mathbf{KO})$, and hence trivial.

Now we let $\alpha\colon G\longrightarrow C$ be an augmented Lie group. We abuse notation and also write σ for $\alpha^*(\sigma)$, the restriction of the sign C-representation along the augmentation. Inflation along $\alpha\colon G\longrightarrow C$ is compatible with products in Real-equivariant K-theory, so $\alpha^*(i^*(\beta)^3)=(i^*(\beta_{\alpha^*(\mathbb{C})}))^3=0$ in the group $\widetilde{KR}_\alpha(S^3)$, where $\beta_{\alpha^*(\mathbb{C})}\in\widetilde{KR}_\alpha(S^{1+\sigma})$ is the Bott class of the Real α-representation $\alpha^*(\mathbb{C})$. So external multiplication by the class $\alpha^*(i^*(\beta)^3)$ is trivial on reduced KR_α-cohomology of every based finite G-CW-complex. An instance of such a multiplication is the composite

$$\widetilde{KR}_\alpha(A\wedge S^{3\sigma})\ \xrightarrow{i_3^*}\ \widetilde{KR}_\alpha(A)\ \xrightarrow{\beta^3_{\alpha^*(\mathbb{C})}\cdot}\ \widetilde{KR}_\alpha(A\wedge S^{3(1+\sigma)})\ ,$$

where $i_3\colon S^0\longrightarrow S^{3\sigma}$ is the inclusion of the points $\{0,\infty\}$. Multiplication by $\beta_{\alpha^*(\mathbb{C})}$ is an isomorphism, see [3, Theorem (5.1)], so the restriction map i_3^* above is trivial. Thus for every $n\geq 3$, the long exact sequence in KR_α-cohomology of the cofiber sequence

$$A\wedge S(n\sigma)_+\ \xrightarrow{A\wedge p_+}\ A\ \xrightarrow{A\wedge i_n}\ A\wedge S^{n\sigma}$$

splits off a short exact sequence:

$$0\ \longrightarrow\ \widetilde{KR}_\alpha(A)\ \xrightarrow{(A\wedge p)^*}\widetilde{KR}_\alpha(A\wedge S(n\sigma)_+)\xrightarrow{\partial}\ \widetilde{KR}^1_\alpha(A\wedge S^{n\sigma})\ \longrightarrow\ 0$$

These exact sequences for n and $n+3$ participate in a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{KR}_\alpha(A) \arrow[r, "(A\wedge p_+)^*"] \arrow[d, equal] & \widetilde{KR}_\alpha(A\wedge S((n+3)\sigma)_+) \arrow[r, "\partial"] \arrow[d, "(A\wedge j_+)^*"] & \widetilde{KR}^1_\alpha(A\wedge S^{(n+3)\sigma}) \arrow[r] \arrow[d, "(A\wedge S^{n\sigma}\wedge i_3)^*"] & 0 \\
0 \arrow[r] & \widetilde{KR}_\alpha(A) \arrow[r, "(A\wedge p_+)^*"'] & \widetilde{KR}_\alpha(A\wedge S(n\sigma)_+) \arrow[r, "\partial"'] & \widetilde{KR}^1_\alpha(A\wedge S^{n\sigma}) \arrow[r] & 0
\end{tikzcd}$$

Here $j\colon S(n\sigma)\longrightarrow S((n+3)\sigma)$ is $j(x_1,\dots,x_n)=(x_1,\dots,x_n,0,0,0)$. The right vertical map is another instance of a restriction map i_3^*, and hence trivial. So the middle vertical map $(A\wedge j_+)^*$ factors through the lower left horizontal map $(A\wedge p_+)^*$. This shows that the maps $(A\wedge p_+)^*$ form a pro-isomorphism from the constant tower with value $\widetilde{KR}_\alpha(A)$ to the tower

$$\dots\longrightarrow\widetilde{KR}_\alpha(A\wedge S((n+1)\sigma)_+)\longrightarrow\widetilde{KR}_\alpha(A\wedge S(n\sigma)_+)\longrightarrow\dots\longrightarrow\widetilde{KR}_\alpha(A\wedge S(\sigma)_+)\ .$$

The natural isomorphism $\langle-\rangle\colon[A,\mathbf{BUP}]^\alpha\cong KR_\alpha(A)$ from Theorem B.12 passes to reduced groups, and provides a natural isomorphism

$$\widetilde{KR}_\alpha(A)\ \cong\ [A,\mathbf{BUP}]^\alpha_*\ ,$$

where A is any finite based G-CW-complex. So the maps $(A\wedge p_+)^*\colon[A,\mathbf{BUP}]^\alpha_*\longrightarrow[A\wedge S(n\sigma)_+,\mathbf{BUP}]^\alpha_*$ form a pro-isomorphism from the constant tower with value $[A,\mathbf{BUP}]^\alpha_*$ to the tower

$$\dots\longrightarrow[A\wedge S((n+1)\sigma)_+,\mathbf{BUP}]^\alpha_*\longrightarrow[A\wedge S(n\sigma)_+,\mathbf{BUP}]^\alpha_*\longrightarrow\dots\longrightarrow[A\wedge S(\sigma)_+,\mathbf{BUP}]^\alpha_*\ .$$

In particular, $[A, \mathbf{BUP}]^\alpha_*$ maps isomorphically to the inverse limit

$$[A, \mathbf{BUP}]^\alpha_* \xrightarrow{\cong} \lim_n \, [A \wedge S(n\sigma)_+, \mathbf{BUP}]^\alpha_* \ ,$$

and the derived inverse limit of the tower vanishes. The G-space $S(\infty\sigma) = \bigcup_{n\geq 0} S(n\sigma)$ is a model for the universal free C-space EC. So the group $[A \wedge EC_+, \alpha^\flat(\mathbf{BUP}(\mathcal{U}_G))]^G_*$ participates in a Milnor short exact sequence with the inverse limit and vanishing derived inverse limit. We conclude that the map

$$(A \wedge p_+)^* \ : \ [A, \mathbf{BUP}(\mathcal{U}_G)]^\Gamma_* \ \longrightarrow \ [A \wedge EC_+, \mathbf{BUP}(\mathcal{U}_G)]^\Gamma_* \cong [A, \mathrm{map}(EC, \mathbf{BUP}(\mathcal{U}_G))]^\Gamma_*$$

is an isomorphism for every finite based G-CW-complex A, where Γ is the graph of $\alpha\colon G \longrightarrow C$. Taking $A = S^k$, for $k \geq 0$ and with trivial G-action shows that the map

$$(p^*)^\Gamma \ : \ \mathbf{BUP}(\mathcal{U}_G)^\Gamma \ \longrightarrow \ \mathrm{map}^\Gamma(EC, \mathbf{BUP}(\mathcal{U}_G))$$

induces a bijection on π_0 and isomorphisms of homotopy groups at the distinguished basepoint. Since $\mathbf{BUP}(\mathcal{U}_G)$ is a group-like G-E_∞-space, $(p^*)^\Gamma$ is a map of non-equivariant group-like E_∞-spaces, and thus a weak homotopy equivalence. Proposition A.16 shows that the orthogonal C-space $\mathbf{BUP}$ is coinduced.

Since $\mathbf{BUP}$ is coinduced, so is $\Omega(\mathbf{BUP}) = \Omega(\mathbf{BU})$ by Proposition A.18. Since $\mathbf{U}$ is C-globally equivalent to $\Omega(\mathbf{BU})$ by Proposition B.20, it is coinduced. Then $\Omega^\sigma\mathbf{U}$ is coinduced by Proposition A.18. □

In [24, Theorem 2.5.41], we establish a global equivariant form of Bott periodicity by exhibiting a zigzag of two global equivalences of ultra-commutative monoids between $\mathbf{BUP}$ and $\Omega\mathbf{U}$. We shall now lift this to a Real-global equivalence between $\mathbf{BUP}$ and $\Omega^\sigma\mathbf{U}$.

Construction B.23. The ultra-commutative monoid $\mathbf{U}$ comes with an involution by complex conjugation. We endow $\Omega^\sigma\mathbf{U} = \mathrm{map}_*(S^\sigma, \mathbf{U})$ with the involution that is diagonal from the sign action on S^σ and the complex conjugation involution of $\mathbf{U}$. This becomes an ultra-commutative C-monoid via the pointwise multiplication inherited from $\mathbf{U}$.

We let $\mathrm{sh}_\otimes \mathbf{U} = \mathrm{sh}^{\mathbb{R}^2}_\otimes \mathbf{U}$ denote the 'multiplicative shift' of $\mathbf{U}$ by $\mathbb{R}^2$ in the sense of [24, Example 1.1.11]. The values of this orthogonal space are thus given by

$$(\mathrm{sh}_\otimes \mathbf{U})(V) \ = \ \mathbf{U}(V \otimes \mathbb{R}^2) \ = \ U((V \otimes \mathbb{R}^2)_{\mathbb{C}}) \ .$$

The structure maps, ultra-commutative multiplication and an involution are inherited from $\mathbf{U}$, making $\mathrm{sh}_\otimes \mathbf{U}$ into an ultra-commutative C-monoid. The embeddings

$$j(V) \ = \ (- \otimes (1,0)) \ : \ V \ \longrightarrow \ V \otimes \mathbb{R}^2$$

as the first summand induce a morphism of ultra-commutative monoids

$$\mathbf{U} \circ j \ : \ \mathbf{U} \ \longrightarrow \ \mathrm{sh}_\otimes \mathbf{U} \ .$$

On [24, page 224] we define a morphism of ultra-commutative monoids $\bar\beta\colon \mathbf{BUP} \longrightarrow \Omega(\mathrm{sh}_\otimes \mathbf{U})$. At that point we were not taking any involutions into account. However, inspection of all definitions shows that this morphism is C-equivariant with respect to the complex conjugation involutions on $\mathbf{BUP}$ and $\mathbf{U}$, and the sign involution in the loop coordinate in the target; the sign involution arises because the Cayley transform $\mathfrak{c}\colon S^1 \longrightarrow U(1)$ is C-equivariant for the sign involution on the source and complex conjugation on the target. So $\bar\beta$ is also a morphism of ultra-commutative C-monoids

$$\bar\beta \ : \ \mathbf{BUP} \ \longrightarrow \ \Omega^\sigma(\mathrm{sh}_\otimes \mathbf{U}) \ .$$

The following result generalizes [24, Theorem 2.5.41] to the Real-global context.

Theorem B.24 (Real-global Bott periodicity)**.** *The morphisms of ultra-commutative C-monoids*

$$\mathbf{BUP} \ \xrightarrow{\bar\beta} \ \Omega^\sigma(\mathrm{sh}_\otimes \mathbf{U}) \ \xleftarrow{\Omega^\sigma(\mathbf{U}\circ j)} \ \Omega^\sigma\mathbf{U}$$

are Real-global equivalences.

Proof. The morphism of ultra-commutative C-monoids $\mathbf{U} \circ j \colon \mathbf{U} \longrightarrow \mathrm{sh}_\otimes \mathbf{U}$ is a Real-global equivalence by [5, Lemma 3.8]. Hence $\Omega^\sigma(\mathbf{U} \circ j) \colon \Omega^\sigma \mathbf{U} \longrightarrow \Omega^\sigma(\mathrm{sh}_\otimes \mathbf{U})$ is a Real-global equivalence, too. The orthogonal C-spaces $\mathbf{BUP}$ and $\Omega^\sigma \mathbf{U}$ are coinduced by Theorem B.22. The morphism $\Omega^\sigma(\mathbf{U} \circ j)$ is a Real-global equivalence, so $\Omega^\sigma(\mathrm{sh}_\otimes \mathbf{U})$ is also coinduced. The morphism $\bar\beta \colon \mathbf{BUP} \longrightarrow \Omega^\sigma(\mathrm{sh}_\otimes \mathbf{U})$ is a global equivalence of underlying non-Real orthogonal spaces by [24, Theorem 2.5.41]. So $\bar\beta$ is a Real-global equivalence by Proposition A.14. □

Remark B.25. Proposition B.20 and Theorem B.24 provide Real-global equivalences of ultra-commutative C-monoids

$$\Omega^{\sigma+1}(\mathbf{BUP}) \ = \ \Omega^\sigma(\Omega(\mathbf{BUP})) \ = \ \Omega^\sigma(\Omega(\mathbf{BU})) \ \sim \ \Omega^\sigma(\mathbf{U}) \ \sim \ \mathbf{BUP} \ .$$

These Real-global equivalences implement a highly structured refinement of the $(\sigma+1)$-periodicity, also called $(2,1)$-periodicity, of Real-equivariant K-theory.

Construction B.26. We denote by

$$\gamma \ : \ \mathbf{BUP} \ \xrightarrow{\ \sim\ } \ \Omega^\sigma \mathbf{U} \tag{B.27}$$

the unique morphism in the Real-global homotopy category of ultra-commutative C-monoids such that $\Omega^\sigma(\mathbf{U} \circ j) \circ \gamma = \bar\beta$. A morphism of ultra-commutative monoids $\beta \colon \mathbf{Gr} \longrightarrow \Omega^\sigma \mathbf{U}$ is defined in [24, (2.5.38)]. In the body of the paper we need to access the definition of this morphism, so we recall it here. The value of β at an inner product space V

$$\beta(V) \ : \ \mathbf{Gr}(V) = Gr(V_{\mathbb{C}}) \ \longrightarrow \ \Omega^\sigma(U(V_{\mathbb{C}})) = (\Omega^\sigma \mathbf{U})(V) \tag{B.28}$$

sends a complex subspace $L \subset V_{\mathbb{C}}$ to the loop $\beta(V)(L) \colon S^\sigma \longrightarrow U(V_{\mathbb{C}})$ such that $\beta(V)(L)(x)$ is multiplication by $\mathfrak{c}(x) \in U(1)$ on L, and the identity on the orthogonal complement of L. The morphism β is C-equivariant with respect to the complex conjugation involutions on $\mathbf{Gr}$ and $\mathbf{U}$, and the sign involution in the loop coordinate in the target, one more time because the Cayley transform $\mathfrak{c} \colon S^1 \longrightarrow U(1)$ is C-equivariant for the sign involution on the source and complex conjugation on the target. So β is also a morphism of ultra-commutative C-monoids

$$\beta \ : \ \mathbf{Gr} \ \longrightarrow \ \Omega^\sigma \mathbf{U} \ .$$

Inspection of definitions shows that the following square commutes:

$$\begin{tikzcd}
\mathbf{Gr} \arrow[r, "\beta"] \arrow[d, "i"'] & \Omega^\sigma \mathbf{U} \arrow[d, "\Omega^\sigma(\mathbf{U}\circ j)", "\sim"'] \\
\mathbf{BUP} \arrow[r, "\sim", "\bar\beta"'] & \Omega^\sigma(\mathrm{sh}_\otimes \mathbf{U})
\end{tikzcd}$$

So

$$\Omega^\sigma(\mathbf{U} \circ j) \circ \gamma \circ i \ = \ \bar\beta \circ \gamma \circ i \ = \ \Omega^\sigma(\mathbf{U} \circ j) \circ \beta \ .$$

Since $\Omega^\sigma(\mathbf{U} \circ j)$ is a Real-global equivalence, this implies the relation

$$\gamma \circ i \ = \ \beta \ : \ \mathbf{Gr} \ \longrightarrow \ \Omega^\sigma \mathbf{U} \tag{B.29}$$

as morphisms in the homotopy category of Real-global ultra-commutative monoids.

Construction B.30 (The inverse of $\mathbf{U}$)**.** The inverse maps

$$\Im(V) \ : \ \mathbf{U}(V) = U(V_{\mathbb{C}}) \ \longrightarrow \ U(V_{\mathbb{C}}) = \mathbf{U}(V) \ , \quad \Im(V)(A) = A^{-1}$$

are compatible with complex conjugation and make the following diagrams commutes:

$$\begin{tikzcd}
\mathbf{U}(V) \times \mathbf{U}(W) \arrow[r, "\Im(V)\times\Im(W)"] \arrow[d, "\oplus"'] & \mathbf{U}(V) \times \mathbf{U}(W) \arrow[d, "\oplus"] \\
\mathbf{U}(V \oplus W) \arrow[r, "\Im(V\oplus W)"'] & \mathbf{U}(V \oplus W)
\end{tikzcd}$$

So they form a morphism of ultra-commutative C-monoids $\Im\colon \mathbf{U} \longrightarrow \mathbf{U}$ that models the inverse map in the ultra-commutative addition. Indeed, composition of unitary automorphism is a morphism of ultra-commutative C-monoids $\circ\colon \mathbf{U} \times \mathbf{U} \longrightarrow \mathbf{U}$ that makes the following diagram commutes:

(B.31)
$$\begin{tikzcd}
& \mathbf{U} \boxtimes \mathbf{U} \arrow[d, "(\rho_1,\rho_2)"', "\sim"] \arrow[dr, bend left, "\mu^{\mathbf{U}}"] & \\
\mathbf{U} \arrow[r, "(\mathrm{Id},\Im)"'] & \mathbf{U} \times \mathbf{U} \arrow[r, "\circ"'] & \mathbf{U}
\end{tikzcd}$$

Here $\rho_1, \rho_2\colon \mathbf{U}\boxtimes\mathbf{U} \longrightarrow \mathbf{U}$ are the projections to the two factors; the combined morphism $(\rho_1, \rho_2)\colon \mathbf{U}\boxtimes\mathbf{U} \longrightarrow \mathbf{U} \times \mathbf{U}$ is a C-global equivalence by [5, Proposition 3.9]. The lower horizontal composite is constant with value the identity automorphism, and hence the zero morphism in the ultra-commutative multiplication. So this diagram witnesses that $\Im$ is the inverse morphism.

As before, we write $\epsilon\colon S^\sigma \longrightarrow S^\sigma$ for the sign involution, $\epsilon(x) = -x$. Precomposition with ϵ is an involution $\epsilon^*\colon \Omega^\sigma\mathbf{U} \longrightarrow \Omega^\sigma\mathbf{U}$. Under the Cayley transform (2.15), the sign involution ϵ becomes complex conjugation on $U(1)$. So the unitary automorphisms $\beta(V)(L)(x) \in \mathbf{U}(V)$ and $\beta(V)(L)(\epsilon(x))$ have the same eigenspace L, but inverse eigenvalues. Hence $\beta(V)(L)(x)$ and $\beta(V)(L)(\epsilon(x))$ are inverse to each other. This shows that the following diagrams of morphisms of ultra-commutative C-monoids commutes:

(B.32)
$$\begin{tikzcd}
\mathbf{Gr} \arrow[r, "\beta"] \arrow[d, "\beta"'] & \Omega^\sigma\mathbf{U} \arrow[d, "\epsilon^*"] \\
\Omega^\sigma\mathbf{U} \arrow[r, "\Omega^\sigma\Im"'] & \Omega^\sigma\mathbf{U}
\end{tikzcd}$$

Proposition B.33. *Let $\alpha\colon G \longrightarrow C$ be an augmented Lie group, and let A be a finite G-CW-complex. Then the map $[A, \epsilon^*]^\alpha\colon [A, \Omega^\sigma\mathbf{U}]^\alpha \longrightarrow [A, \Omega^\sigma\mathbf{U}]^\alpha$ is multiplication by -1.*

Proof. The commutative diagram (B.31) witnesses that the involution of ultra-commutative C-monoids $\Im\colon \mathbf{U} \longrightarrow \mathbf{U}$ is the inverse morphism; the same is thus true for $\Omega^\sigma\Im\colon \Omega^\sigma\mathbf{U} \longrightarrow \Omega^\sigma\mathbf{U}$. In particular, the map $[A, \Omega^\sigma\Im]^\alpha\colon [A, \Omega^\sigma\mathbf{U}]^\alpha \longrightarrow [A, \Omega^\sigma\mathbf{U}]^\alpha$ is multiplication by -1. The commutative diagram (B.32) shows that the composite

$$[A, \mathbf{Gr}]^\alpha \xrightarrow{[A,\beta]^\alpha} [A, \Omega^\sigma\mathbf{U}]^\alpha \xrightarrow{[A,\epsilon^*]^\alpha} [A, \Omega^\sigma\mathbf{U}]^\alpha$$

equals $[A, (\Omega^\sigma\Im)]^\alpha \circ [A, \beta]^\alpha = -[A, \beta]^\alpha$. So $[A, \epsilon^*]^\alpha$ is multiplication by -1 on the image of $[A, \beta]^\alpha$. The homomorphism

$$[A, i]^\alpha \;:\; [A, \mathbf{Gr}]^\alpha \;\longrightarrow\; [A, \mathbf{BUP}]^\alpha$$

is a group completion of abelian monoids by Proposition B.5. In particular, the group $[A, \mathbf{BUP}]^\alpha$ is generated by the image of $[A, \beta]^\alpha$. The C-global equivalence $\gamma\colon \mathbf{BUP} \xrightarrow{\sim} \Omega^\sigma\mathbf{U}$ from (B.27) satisfies $\gamma \circ i = \beta\colon \mathbf{Gr} \longrightarrow \Omega^\sigma\mathbf{U}$, see (B.29). So the group $[A, \Omega^\sigma\mathbf{U}]^\alpha$ is generated by the image of $[A, \beta]^\alpha$. Since $[A, \epsilon^*]^\alpha$ is a group homomorphism and multiplication by -1 on a generating set, it is multiplication by -1 on all elements. $\square$

B.2. **Connective Real-global K-theory.** The connective global K-theory spectrum $\mathbf{ku}$ is defined in [24, Construction 3.6.9], adapting a configuration space model of Segal [30, Section 1] to the global equivariant context. In this subsection we recall the definition, along with the involution ψ by complex conjugation that enhances it to the connective Real-global K-theory spectrum $\mathbf{kr} = (\mathbf{ku}, \psi)$. In (B.44) we recall the eigenspace morphism $\mathrm{eig}\colon \mathbf{U} \longrightarrow \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{kr})$, which we show to be a C-global H-map in Proposition B.46.

Construction B.34 (The connective Real-global K-theory spectrum $\mathbf{kr}$)**.** We let $\mathcal{U}$ be a hermitian inner product space of countable dimension (finite or infinite). We recall the Γ-space $\mathscr{C}(\mathcal{U})$ of 'orthogonal subspaces in $\mathcal{U}$'. For a finite based set A we let $\mathscr{C}(\mathcal{U}, A)$ be the space of tuples $(E_a)_{a\in A\setminus\{0\}}$, indexed by the non-basepoint elements of A, of finite-dimensional, pairwise orthogonal $\mathbb{C}$-subspaces of $\mathcal{U}$. The topology on $\mathscr{C}(\mathcal{U}, A)$ is that of a disjoint union of subspaces of a product of Grassmannians. The basepoint of $\mathscr{C}(\mathcal{U}, A)$ is the tuple where $E_a = \{0\}$ for all $a \in A \setminus \{0\}$. For a based map $\alpha\colon A \longrightarrow B$ the induced map $\mathscr{C}(\mathcal{U}, \alpha)\colon \mathscr{C}(\mathcal{U}, A) \longrightarrow \mathscr{C}(\mathcal{U}, B)$ sends (E_a) to (F_b) where

$$F_b \ = \ \bigoplus\nolimits_{\alpha(a)=b} E_a \ .$$

Every Γ-space can be evaluated on a based space by a coend construction, see for example [24, (4.5.14)]. Categorically speaking, this coend realizes the enriched Kan extension along the inclusion of Γ into the category of based spaces. We write $\mathscr{C}(\mathcal{U}, K) = \mathscr{C}(\mathcal{U})(K)$ for the value of the Γ-space $\mathscr{C}(\mathcal{U})$ on a based space K. Elements of $\mathscr{C}(\mathcal{U}, K)$ can be interpreted as 'labeled configurations': a point is represented by an unordered tuple

$$[E_1, \dots, E_n;\, k_1, \dots, k_n]$$

where $(E_1, \dots, E_n)$ is an n-tuple of finite-dimensional, pairwise orthogonal subspaces of $\mathcal{U}$, and $k_1, \dots, k_n$ are points of K, for some n. The topology is such that, informally speaking, the labels sum up whenever two points collide, and a label disappears whenever a point approaches the basepoint of K.

The value of the orthogonal spectrum $\mathbf{kr}$ on a euclidean inner product space V is

$$\mathbf{kr}(V) \ = \ \mathscr{C}(\mathrm{Sym}(V_\mathbb{C}), S^V) \ ,$$

the value of the Γ-space $\mathscr{C}(\mathrm{Sym}(V_\mathbb{C}))$ on the sphere S^V; the inner product on the symmetric algebra is described in [24, Proposition 6.3.8]. The action of $O(V)$ on V then extends to a unitary action on $\mathrm{Sym}(V_\mathbb{C})$. We let the orthogonal group $O(V)$ act diagonally, via the action on the sphere S^V and the action on the Γ-space $\mathscr{C}(\mathrm{Sym}(V_\mathbb{C}))$. For the structure maps we refer to [24, Construction 6.3.9]. The involution $\psi(V)\colon \mathbf{kr}(V) \longrightarrow \mathbf{kr}(V)$ is induced by complex conjugation on $\mathrm{Sym}(V_\mathbb{C})$, which preserves orthogonality of subspaces.

Construction B.35. We introduce an orthogonal C-spectrum $\mathbf{kr}^{[2]}$ that is Real-globally equivalent to a product of two copies of $\mathbf{kr}$. The value of the orthogonal spectrum $\mathbf{kr}^{[2]}$ on a euclidean inner product space V is the configuration space

$$\mathbf{kr}^{[2]}(V) \ = \ \mathscr{C}(\mathrm{Sym}(V_\mathbb{C}), S^V \vee S^V) \ ,$$

the value of the Γ-space $\mathscr{C}(\mathrm{Sym}(V_\mathbb{C}))$ on the wedge of two copies of S^V. The action of $O(V)$, the structure maps of $\mathbf{kr}^{[2]}$ and the involution are defined in much the same way as for $\mathbf{kr}$. The projection and fold maps

$$p_1,\, p_2,\, \nabla \ : \ S^V \vee S^V \ \longrightarrow \ S^V$$

induce continuous, based and $(C \times O(V))$-equivariant maps of configuration spaces

$$p_1(V),\, p_2(V),\, \nabla(V) \ : \ \mathbf{kr}^{[2]}(V) \ = \ \mathscr{C}(\mathrm{Sym}(V_\mathbb{C}), S^V \vee S^V) \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_\mathbb{C}), S^V) \ = \ \mathbf{kr}(V) \ .$$

For varying inner product spaces V, these maps assemble into morphisms of orthogonal C-spectra

$$p_1,\, p_2,\, \nabla \ : \ \mathbf{kr}^{[2]} \ \longrightarrow \ \mathbf{kr} \ .$$

Proposition B.36. *The morphism*

$$(p_1,p_2)\ :\ \mathbf{kr}^{[2]}\ \longrightarrow\ \mathbf{kr}\times\mathbf{kr}$$

is a Real-global equivalence of orthogonal C-spectra.

Proof. We let $\alpha\colon G\longrightarrow C$ be a continuous homomorphism of compact Lie groups. We call an orthogonal G-representation V *ample* if the complex symmetric algebra $\alpha^\flat(\mathrm{Sym}(V_{\mathbb{C}}))$ is a complete Real (G,α)-universe. If V is ample, then the G-Γ-space $\mathscr{C}(\alpha^\flat(\mathrm{Sym}(V_{\mathbb{C}})),-)$ is special by [24, Theorem 6.3.19 (i)], and it is G-cofibrant by [24, Example 6.3.16]. So the G-Γ-space $\mathscr{C}(\alpha^\flat(\mathrm{Sym}(V_{\mathbb{C}})),S^1\wedge -)$ is very special and cofibrant, and thus takes wedges of finite based G-CW-complexes to products, up to G-weak equivalence, by [24, Theorem B.61 (i)]. In particular, if V is ample and $V^G\neq 0$, then the map

$$\begin{aligned}(p_1(V),p_2(V))\ :\ \alpha^\flat(\mathbf{kr}^{[2]}(V))=\mathscr{C}(\alpha^\flat(\mathrm{Sym}(V_{\mathbb{C}})),S^V\vee S^V)\ \longrightarrow\\ \mathscr{C}(\alpha^\flat(\mathrm{Sym}(V_{\mathbb{C}})),S^V)\times\mathscr{C}(\alpha^\flat(\mathrm{Sym}(V_{\mathbb{C}})),S^V)=\alpha^\flat(\mathbf{kr}(V)\times\mathbf{kr}(V))\end{aligned}$$

is a G-weak equivalence. The ample G-representations with nonzero G-fixed points are cofinal in all orthogonal G-representations, so this proves the claim that the morphism (p_1,p_2) is a Real-global equivalence. □

Construction B.37. We introduce a morphism of orthogonal C-spaces

$$\varepsilon\ :\ \mathbf{Gr}\ \longrightarrow\Omega^\bullet(\mathbf{kr})\ . \tag{B.38}$$

Its value at an inner product space V is the map

$$\varepsilon(V)\ :\ \mathbf{Gr}(V)\ \longrightarrow\ \mathrm{map}_*(S^V,\mathbf{kr}(V))\ =\ \Omega^\bullet(\mathbf{kr})(V)$$

adjoint to the map

$$S^V\wedge(\coprod_{k\geq 0}Gr_k^{\mathbb{C}}(V_{\mathbb{C}}))\ =\ S^V\wedge\mathbf{Gr}(V)\ \longrightarrow\ \mathbf{kr}(V)\ =\ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}),S^V)\ ,\quad [v,L]\ \longmapsto\ [L;v]\ ,$$

the one-element configuration of the vector v labeled by the subspace L of $V_{\mathbb{C}}$.

For every augmented Lie group $\alpha\colon G\longrightarrow C$ and every G-space A, the set $[A,\mathbf{Gr}]^\alpha$ has an abelian monoid structure from the ultra-commutative multiplication of $\mathbf{Gr}$ by direct sum of subspaces. And the set $[A,\Omega^\bullet(\mathbf{kr})]^\alpha=\mathbf{kr}^0_\alpha(A)$ has an abelian group structure by virtue of coming from an orthogonal G-spectrum. It is not entirely obvious, but nevertheless true, that the map induced by $\varepsilon\colon\mathbf{Gr}\longrightarrow\Omega^\bullet(\mathbf{kr})$ on $[A,-]^\alpha$ is additive, see Proposition B.41 below. If the augmentation is trivial, then this additivity is shown in [24, Theorem 6.3.28] by a different method.

Construction B.39. The multiplication of an ultra-commutative C-monoid makes it into an abelian monoid object in the C-global homotopy category, as follows. We let $\rho_1,\rho_2\colon R\boxtimes R\longrightarrow R$ denote the projections to the two factors. By [5, Proposition 3.9], the morphism $(\rho_1,\rho_2)\colon R\boxtimes R\longrightarrow R\times R$ is a C-global equivalence. Product of orthogonal C-spaces descend to product in the C-global homotopy category, so in there we can form the composite

$$m_R\ :\ R\times R\ \xrightarrow[\cong]{(\rho_1,\rho_2)^{-1}}\ R\boxtimes R\xrightarrow{\ \mu\ }\ R\ .$$

For every orthogonal C-spectrum Y, the stable structure provides an abelian group structure on the object $\Omega^\bullet(Y)$ in the C-global homotopy category, as follows., The canonical morphism $\kappa\colon Y\vee Y\longrightarrow Y\times Y$ from the wedge to the product is a C-global equivalence of orthogonal C-spectra, so it induces a C-global equivalence of orthogonal C-spaces

$$\Omega^\bullet(\kappa)\ :\ \Omega^\bullet(Y\vee Y)\ \xrightarrow{\ \sim\ }\ \Omega^\bullet(Y\times Y)\cong\Omega^\bullet(Y)\times\Omega^\bullet(Y)\ .$$

This morphism becomes an isomorphism in the C-global homotopy category. So in the C-global homotopy category, we can form the composite

$$m_Y \ : \ \Omega^\bullet(Y) \times \Omega^\bullet(Y) \ \xrightarrow[\cong]{\Omega^\bullet(\kappa)^{-1}} \ \Omega^\bullet(Y \vee Y) \xrightarrow{\Omega^\bullet(\nabla)} \ \Omega^\bullet(Y) \ .$$

Here $\nabla \colon Y \vee Y \longrightarrow Y$ is the fold morphism. The morphism m_Y makes $\Omega^\bullet(Y)$ into an abelian group object in the C-global homotopy category.

On two occasions we will need to know that the map $[A, f]^\alpha \colon [A, R]^\alpha \longrightarrow [A, \Omega^\bullet(Y)]^\alpha$ induced by a certain morphism of orthogonal C-spaces $f \colon R \longrightarrow \Omega^\bullet(Y)$ is additive. To this end, we now introduce the notion of a 'C-global H-map'.

Definition B.40. Let R and M be abelian monoid objects in the based C-global homotopy category. A *C-global H-map* is a morphism $f \colon R \longrightarrow M$ in the C-global homotopy category that makes the following diagram commute:

$$\begin{tikzcd}
R \times R \arrow[r, "(f\rho_1)\times(f\rho_2)"] \arrow[d, "m_R"'] & M \times M \arrow[d, "m_M"] \\
R \arrow[r, "f"'] & M
\end{tikzcd}$$

Proposition B.41. *The morphism* $\varepsilon \colon \mathbf{Gr} \longrightarrow \Omega^\bullet(\mathbf{kr})$ *is a C-global H-map.*

Proof. We introduce a morphism $\varepsilon^{[2]} \colon \mathbf{Gr} \boxtimes \mathbf{Gr} \longrightarrow \Omega^\bullet(\mathbf{kr}^{[2]})$ of orthogonal C-spaces, a certain variation of the morphism (B.38), but with two (instead of one) Grassmannian parameters. Its value at an inner product space V is adjoint to the map

$$\varepsilon^{[2]}(V) \ : \ S^V \wedge (\mathbf{Gr} \boxtimes \mathbf{Gr})(V) \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V \vee S^V) = \mathbf{kr}^{[2]}(V)$$

defined as follows. Elements of $(\mathbf{Gr} \boxtimes \mathbf{Gr})(V)$ are pairs (L, L') of orthogonal $\mathbb{C}$-subspaces of $V_{\mathbb{C}}$. We can then define the map $\varepsilon^{[2]}(V)$ by

$$\varepsilon^{[2]}(V)(v \wedge (L, L')) \ = \ [L, L'; i_1(v), i_2(v)] \ .$$

Here $i_1, i_2 \colon S^V \longrightarrow S^V \vee S^V$ denote the embeddings of the two wedge summands. In other words, the subspace L is attached to the point v in the first wedge summand, and the subspace L' is attached to the point v in the second wedge summand.

We let $\rho_1, \rho_2 \colon \mathbf{Gr} \boxtimes \mathbf{Gr} \longrightarrow \mathbf{Gr}$ denote the projections to the two factors. We let $\iota \colon \mathbf{kr} \vee \mathbf{kr} \longrightarrow \mathbf{kr}^{[2]}$ be the morphism that on each wedge summands is induced by the respective wedge summand inclusion $S^V \longrightarrow S^V \vee S^V$. The following diagram then commutes in the category of based orthogonal C-spaces, by direct inspection of the definitions:

(B.42)
$$\begin{tikzcd}
\mathbf{Gr} \times \mathbf{Gr} \arrow[r, "\varepsilon \times \varepsilon"] & \Omega^\bullet(\mathbf{kr}) \times \Omega^\bullet(\mathbf{kr}) & \\
\mathbf{Gr} \boxtimes \mathbf{Gr} \arrow[u, "(\rho_1,\rho_2)", "\sim"'] \arrow[r, "\varepsilon^{[2]}"'] \arrow[d, "\mu"'] & \Omega^\bullet(\mathbf{kr}^{[2]}) \arrow[u, "(\Omega^\bullet(p_1),\Omega^\bullet(p_2))", "\sim"'] \arrow[d, "\Omega^\bullet(\nabla)"] & \Omega^\bullet(\mathbf{kr} \vee \mathbf{kr}) \arrow[l, "\sim", "\Omega^\bullet(\iota)"'] \arrow[ul, bend right, "\Omega^\bullet(\kappa)", "\sim"'] \arrow[dl, bend left, "\Omega^\bullet(\nabla)"] \\
\mathbf{Gr} \arrow[r, "\varepsilon"'] & \Omega^\bullet(\mathbf{kr}) &
\end{tikzcd}$$

The morphism $(p_1, p_2) \colon \mathbf{kr}^{[2]} \longrightarrow \mathbf{kr} \times \mathbf{kr}$ is a Real-global equivalence by Proposition B.36. The functor $\Omega^\bullet$ preserves Real-global equivalences and products, so the upper right vertical morphism is a Real-global equivalence of orthogonal C-spaces. Hence the various morphisms decorated with a tilde are Real-global equivalences. After passing to the Real-global homotopy category, we can invert the three Real-global equivalences, and the resulting diagram witnesses that $\epsilon\colon \mathbf{Gr} \longrightarrow \Omega^\bullet(\mathbf{kr})$ is a C-global H-map. □

Construction B.43 (The eigenspace morphism)**.** We recall from [24, (6.3.26)] the *eigenspace morphism*

$$\text{(B.44)} \qquad \mathrm{eig} \ : \ \mathbf{U} \ \longrightarrow \ \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{kr}) \ .$$

Its value at an inner product space V

$$\begin{aligned} \mathrm{eig}(V) \ : \ \mathbf{U}(V) = U(V_{\mathbb{C}}) \ \longrightarrow \ &\mathrm{map}_*(S^V, \mathscr{C}(\mathrm{Sym}((V\oplus\sigma)_{\mathbb{C}}), S^{V\oplus\sigma})) \\ &= \mathrm{map}_*(S^V, \mathbf{kr}(V\oplus\sigma)) = \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{kr})(V) \end{aligned}$$

is defined as follows. For a unitary endomorphism $A \in U(V_{\mathbb{C}})$, we let $\lambda_1,\dots,\lambda_n \in U(1)\setminus\{1\}$ be the eigenvalues different from 1, and we let $E(\lambda_j)$ be the eigenspace of A for the eigenvalue λ_j. We let

$$\text{(B.45)} \qquad \mathfrak{c}^{-1} \ : \ U(1) \ \xrightarrow{\cong} \ S^\sigma \ , \quad \mathfrak{c}^{-1}(\lambda) = i\cdot(\lambda+1)(\lambda-1)^{-1}$$

be the inverse of the Cayley transform (2.15). Then $\mathrm{eig}(V)$ is defined by

$$\mathrm{eig}(V)(A)(v) \ = \ [E(\lambda_1),\dots,E(\lambda_n);\, (v,\mathfrak{c}^{-1}(\lambda_1)),\dots,(v,\mathfrak{c}^{-1}(\lambda_n))] \ .$$

In other words, $\mathrm{eig}(V)(A)(v)$ is the configuration of the points $(v,\mathfrak{c}^{-1}(\lambda_i)) \in S^{V\oplus\sigma}$ labeled by the eigenspace $E(\lambda_i)$ of A, whence the name. Strictly speaking, $E(\lambda_i)$ is a subspace of $V_{\mathbb{C}}$, which we embed into the linear summand of $\mathrm{Sym}((V\oplus\sigma)_{\mathbb{C}})$.

If $A \in \mathbf{U}(V)$ has eigenspaces $E_1,\dots,E_n$ for eigenvalues $\lambda_1,\dots,\lambda_n$, then the eigenspaces of A^ψ are $\psi(E_1),\dots,\psi(E_n)$ for eigenvalues $\bar\lambda_1,\dots,\bar\lambda_n$. The involution on $(\mathrm{sh}^\sigma \mathbf{kr})(V) = \mathbf{kr}(V\oplus\sigma)$ is by complex conjugation on the labeling vector spaces, and by sign on the σ-coordinate that stores the eigenvalues. The inverse Cayley transform $\mathfrak{c}^{-1}\colon U(1) \longrightarrow S^\sigma$ is C-equivariant in the sense that $\mathfrak{c}^{-1}(\bar\lambda) = -\mathfrak{c}^{-1}(\lambda)$. So the map $\mathrm{eig}(V)\colon \mathbf{U}(V) \longrightarrow \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{kr})(V)$ commutes with the involutions. The upshot is that the eigenspace morphism eig is a morphism of orthogonal C-spaces.

Proposition B.46. *The morphism* $\mathrm{eig}\colon \mathbf{U} \longrightarrow \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{kr})$ *is a C-global H-map.*

Proof. The proof is similar to that of Proposition B.41. We define a morphism of orthogonal C-spaces $\mathrm{eig}^{[2]}\colon \mathbf{U}\boxtimes\mathbf{U} \longrightarrow \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{kr}^{[2]})$, a variation of the eigenspace morphism (B.44), but with two (instead of one) unitary parameters. Its value at an inner product space V is the map

$$\begin{aligned} \mathrm{eig}^{[2]}(V) \ : \ (\mathbf{U}\boxtimes\mathbf{U})(V) \ \longrightarrow \ &\mathrm{map}_*(S^V, \mathscr{C}(\mathrm{Sym}((V\oplus\sigma)_{\mathbb{C}}), S^{V\oplus\sigma}\vee S^{V\oplus\sigma})) \\ &= \mathrm{map}_*(S^V, \mathbf{kr}^{[2]}(V\oplus\sigma)) = \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{kr}^{[2]})(V) \end{aligned}$$

defined as follows. Elements of $(\mathbf{U}\boxtimes\mathbf{U})(V)$ are pairs (A,B) of unitary endomorphisms $A, B \in U(V)$ that are *transverse* in the following sense: there exists an orthogonal direct sum decomposition $V = V'\oplus V''$ such that A is the identity on V', and B is the identity on V''. The transversality hypothesis in particular means that A and B commute (but it is stronger than that), so A and B are simultaneously diagonalizable. We let $\lambda_1,\dots,\lambda_n \in U(1)\setminus\{1\}$ be the set eigenvalues of A and B different from 1, we let $E(\lambda_j)$ be the eigenspace of A for the eigenvalue λ_j, and we let $F(\lambda_j)$ be the eigenspace of B for the eigenvalue λ_j. By the transversality hypothesis, all these eigenspaces $E(\lambda_i)$ and $F(\lambda_j)$ are pairwise orthogonal. We can then define the map $\mathrm{eig}^{[2]}(V)$ by

$$\mathrm{eig}^{[2]}(V)(A,B)(v) \ = \ \{E(\lambda_j), i_1(v,\mathfrak{c}^{-1}(\lambda_j)\}_{1\le j\le n} \ \cup \ \{F(\lambda_j), i_2(v,\mathfrak{c}^{-1}(\lambda_j)\}_{1\le j\le n} \ .$$

As before, $\mathfrak{c}^{-1}$ is the inverse Cayley transform (B.45). And $i_1, i_2\colon S^{V\oplus\sigma} \longrightarrow S^{V\oplus\sigma}\vee S^{V\oplus\sigma}$ denote the embeddings of the two wedge summands. In other words, the eigenspace $E(\lambda_j)$ of A is attached to the point $(v,\mathfrak{c}^{-1}(\lambda_j))$ in the first wedge summand, and the eigenspace $F(\lambda_j)$ of B is attached to the point $(v,\mathfrak{c}^{-1}(\lambda_j))$ in the second wedge summand.

The following diagram of orthogonal C-spaces then commutes:

$$\begin{tikzcd}
\mathbf{U} \times \mathbf{U} \arrow[r, "\mathrm{eig} \times \mathrm{eig}"] & \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{kr}) \times \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{kr}) & \\
\mathbf{U} \boxtimes \mathbf{U} \arrow[u, "(\rho_1, \rho_2)", "\sim"'] \arrow[r, "\mathrm{eig}^{[2]}"'] \arrow[d, "\mu"'] & \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{kr}^{[2]}) \arrow[u, "(\Omega^\bullet(\mathrm{sh}^\sigma p_1), \Omega^\bullet(\mathrm{sh}^\sigma p_2))", "\sim"'] \arrow[d, "\Omega^\bullet(\mathrm{sh}^\sigma \nabla)"'] & \Omega^\bullet((\mathrm{sh}^\sigma \mathbf{kr}) \vee (\mathrm{sh}^\sigma \mathbf{kr})) \arrow[l, "\sim", "\Omega^\bullet(\iota)"'] \arrow[ul, bend right, "\Omega^\bullet(\kappa)"', "\sim"] \arrow[dl, bend left, "\Omega^\bullet(\nabla)"] \\
\mathbf{U} \arrow[r, "\mathrm{eig}"'] & \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{kr}) &
\end{tikzcd}$$

The morphism $(p_1, p_2)\colon \mathbf{kr}^{[2]} \longrightarrow \mathbf{kr} \times \mathbf{kr}$ is a Real-global equivalence by Proposition B.36. The functors sh^σ and $\Omega^\bullet$ preserves Real-global equivalences and products, so the upwards middle vertical morphism is a Real-global equivalence of orthogonal C-spaces. Hence the three morphisms decorated with a tilde are Real-global equivalences. After passing to the Real-global homotopy category, we can invert the three Real-global equivalences, and the resulting diagram witnesses that $\mathrm{eig}\colon \mathbf{U} \longrightarrow \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{kr})$ is a C-global H-map. $\square$

We let R be an ultra-commutative C-monoid. The ultra-commutative multiplication induces an ultra-commutative C-ring spectrum structure on the unreduced suspension spectrum $\Sigma^\infty_+ R$. This, in turn, induces a product structure on the equivariant homotopy groups of $\Sigma^\infty_+ R$. The morphism $\varrho\colon \Sigma^\infty_+ R \longrightarrow \mathbb{S}$ arising from the unique morphism of orthogonal spaces $R \longrightarrow *$ is a morphism of ultra-commutative C-ring spectra, and thus induces a morphism of equivariant homotopy rings

$$\varrho_* \ : \ \pi_\star^G(\Sigma^\infty_+ R) \ \longrightarrow \ \pi_\star^G(\mathbb{S})$$

for every augmented Lie group. Because the morphism $R \longrightarrow *$ has a section given by the multiplicative identity elements of R, ϱ_* is surjective. The *augmentation ideal* of $\pi_\star^G(\Sigma^\infty_+ R)$ is the kernel of this homomorphism ϱ_*.

Theorem B.47. *Let R be an ultra-commutative C-monoid, and let Y be an orthogonal C-spectrum. Let $f\colon R \longrightarrow \Omega^\bullet(Y)$ be a C-global H-map, with adjoint $f^\natural\colon \Sigma^\infty R \longrightarrow Y$. Then for every augmented Lie group $\alpha\colon G \longrightarrow C$ the composite*

$$\pi_\star^G(\Sigma^\infty_+ R) \ \xrightarrow{q_*} \ \pi_\star^G(\Sigma^\infty R) \ \xrightarrow{f^\natural_*} \ \pi_\star^G(Y) \ .$$

satisfies the relation

$$(f^\natural \circ q)_*(x \cdot y) \ = \ (f^\natural \circ q)_*(x) \cdot \varrho_*(y) \ + \ \varrho_*(x) \cdot (f^\natural \circ q)_*(y)$$

for all representation-graded homotopy classes $x, y \in \pi_\star^G(\Sigma^\infty_+ R)$. In particular, the map $(f^\natural \circ q)_*$ annihilates the square of the augmentation ideal.

Proof. The hypothesis that f is a C-global H-map means that the following diagram commutes in the based C-global homotopy category:

$$\begin{tikzcd}
R \times R \arrow[r, "f \times f"] & \Omega^\bullet(Y) \times \Omega^\bullet(Y) \arrow[dd, bend left=60, "m_Y"] \\
R \boxtimes R \arrow[u, "(\rho_1, \rho_2)", "\sim"'] \arrow[d, "\mu"'] & \Omega^\bullet(Y \vee Y) \arrow[u, "\Omega^\bullet(\kappa)", "\sim"'] \arrow[d, "\Omega^\bullet(\nabla)"'] \\
R \arrow[r, "f"'] & \Omega^\bullet(Y)
\end{tikzcd}$$

Passing to adjoints for the adjunction $(\Sigma^\infty, \Omega^\bullet)$ yields the commutativity in the C-global stable homotopy category of the right part of the following diagram:

$$\begin{tikzcd}
(\Sigma^\infty_+ R) \times (\Sigma^\infty_+ R) \arrow[r, "q\times q"] & (\Sigma^\infty R) \times (\Sigma^\infty R) \arrow[r, "f^\natural \times f^\natural"] & Y \times Y \arrow[dd, bend left=60, "\oplus"] \\
\Sigma^\infty_+ (R \boxtimes R) \arrow[u, "(\Sigma^\infty_+ \rho_1, \Sigma^\infty_+ \rho_2)"] \arrow[r, "q^{[2]}"'] \arrow[d, "\Sigma^\infty_+ \mu"'] & \Sigma^\infty (R \boxtimes R) \arrow[u, "(\Sigma^\infty \rho_1, \Sigma^\infty \rho_2)"] \arrow[d, "\Sigma^\infty \mu"'] & Y \vee Y \arrow[u, "\kappa", "\sim"'] \arrow[d, "\nabla"'] \\
\Sigma^\infty_+ R \arrow[r, "q"'] & \Sigma^\infty R \arrow[r, "f^\natural"'] & Y
\end{tikzcd} \tag{B.48}$$

The morphism $\oplus = \nabla \circ \kappa^{-1} \colon Y \times Y \longrightarrow Y$ induces the addition on equivariant homotopy groups. For orthogonal G-representations V and W, we define the exterior multiplication

$$\boxtimes \ : \ \pi_V^G(\Sigma^\infty_+ R) \times \pi_W^G(\Sigma^\infty_+ R) \ \longrightarrow \ \pi_{V\oplus W}^G(\Sigma^\infty_+ (R \boxtimes R))$$

as the composite

$$\pi_V^G(\Sigma^\infty_+ R) \times \pi_W^G(\Sigma^\infty_+ R) \ \xrightarrow{\ \cdot\ } \ \pi_{V\oplus W}^G((\Sigma^\infty_+ R) \wedge (\Sigma^\infty_+ R)) \ \cong \ \pi_{V\oplus W}^G(\Sigma^\infty_+ (R \boxtimes R)) \ ,$$

where the isomorphism is induced by the strong symmetric monoidal structure on the unreduced suspension spectrum functor, see [24, (4.1.17)]. Then

$$(\Sigma^\infty_+ \mu)_*(x \boxtimes y) \ = \ x \cdot y$$

for all representation-graded homotopy classes x, y. The morphism $\Sigma^\infty_+ \rho_1 \colon \Sigma^\infty_+ (R \boxtimes R) \longrightarrow \Sigma^\infty_+ R$ factors as the composite

$$\Sigma^\infty_+ (R \boxtimes R) \ \cong (\Sigma^\infty_+ R) \wedge (\Sigma^\infty_+ R) \ \xrightarrow{\mathrm{Id} \wedge \varrho} \ (\Sigma^\infty_+ R) \wedge \mathbb{S} \ \cong \ \Sigma^\infty_+ R \ ,$$

where the final isomorphism is the unit isomorphism of the smash product, and similarly for $\Sigma^\infty_+ \rho_2$. So

$$(\Sigma^\infty_+ \rho_1)_*(x \boxtimes y) \ = \ x \cdot \varrho_*(y) \qquad \text{and} \qquad (\Sigma^\infty_+ \rho_2)_*(x \boxtimes y) \ = \ \varrho_*(x) \cdot y \ . \tag{B.49}$$

The commutativity of (B.48) then provides the desired relation

$$\begin{aligned}
(f^\natural \circ q)_*(x \cdot y) \ &= \ (f^\natural \circ q \circ (\Sigma^\infty_+ \mu))_*(x \boxtimes y) \\
&= \ (f^\natural \circ (\Sigma^\infty \mu) \circ q^{[2]})_*(x \boxtimes y) \\
&= \ (\oplus \circ (f^\natural \times f^\natural) \circ (\Sigma^\infty \rho_1, \Sigma^\infty \rho_2) \circ q^{[2]})_*(x \boxtimes y) \\
&= \ (\oplus \circ ((f^\natural \circ q) \times (f^\natural \circ q)) \circ (\Sigma^\infty_+ \rho_1, \Sigma^\infty_+ \rho_2))_*(x \boxtimes y) \\
&\overset{\text{(B.49)}}{=} \ (\oplus \circ ((f^\natural \circ q) \times (f^\natural \circ q)))_*(x \cdot \varrho_*(y), \varrho_*(x) \cdot y) \\
&= \ (f^\natural \circ q)_*(x \cdot \varrho_*(y)) + (f^\natural \circ q)_*(\varrho_*(x) \cdot y) \\
&= \ (f^\natural \circ q)_*(x) \cdot \varrho_*(y) \ + \ \varrho_*(x) \cdot (f^\natural \circ q)_*(y) \ .
\end{aligned}$$

□

B.3. **Periodic Real-global K-theory.** The global K-theory spectrum **KU** was introduced by Joachim [15, Definition 4.3] as a commutative orthogonal ring spectrum, see also [24, Construction 6.4.9]. Joachim showed in [15, Theorem 4.4] that the genuine G-spectrum underlying the global spectrum **KU** represents G-equivariant complex K-theory. A different proof can be found in [24, Corollary 6.4.23].

Joachim's orthogonal ring spectrum can be enhanced to an orthogonal C-spectrum **KR** by suitably incorporating complex conjugation; to our knowledge, the additional involution was first discussed in the literature by Halladay and Kamel [12, Section 6], who use the notation $KU_{\mathbb{R}}$ for the resulting orthogonal C-ring spectrum. In [12, Proposition 6.2], Hallady and Kamel identify the genuine C-fixed point spectrum of **KR** with **KO**, thereby verifying that $KU_{\mathbb{R}}$ is a model for Atiyah's Real K-theory spectrum.

We recall the definition of the orthogonal C-spectrum $\mathbf{KR}$ in Construction B.51, and we refer to it as the *Real-global K-theory spectrum.* We show in Theorem B.59 that $\mathbf{KR}$ deserves its name: for every augmented Lie group $\alpha\colon G \longrightarrow C$, the orthogonal G-spectrum $\alpha^*(\mathbf{KR})$ represents α-equivariant Real K-theory KR_α. Our treatment proceeds from a self-contained proof in Theorem B.55 that the pre-Euler class of the C-sign representation is nilpotent in $\pi_\star^C(\mathbf{KR})$, which immediately implies that the geometric fixed point homotopy groups $\Phi_*^G(\alpha^*(\mathbf{KR}))$ vanish for all compact Lie groups equipped with a *surjective* augmentation $\alpha\colon G \longrightarrow C$. This implies the Real-global homotopy type of $\mathbf{KR}$ is coinduced (or 'relative C-Borel') from the underlying global spectrum $\mathbf{KU}$. Theorem B.57 shows that (and how) the Real-global space $\mathbf{U}$ is the Real-global infinite loop space underlying $\mathbf{KR} \wedge S^\sigma$. Consequently, the Real-global space $\mathbf{BUP} \sim \Omega^\sigma \mathbf{U}$ 'is' the Real-global infinite loop space underlying $\mathbf{KR}$, see Remark B.58.

Construction B.50 (Clifford algebras)**.** We let V be a euclidean inner product space. We define the complex Clifford algebra $\mathbb{C}\mathrm{l}(V)$ by

$$\mathbb{C}\mathrm{l}(V) \ = \ (TV)_{\mathbb{C}}/(v \otimes v - |v|^2 \cdot 1) \ ,$$

the quotient of the complexified tensor algebra of V by the ideal generated by the elements $v \otimes v - |v|^2 \cdot 1$ for all $v \in V$. We write $[-]\colon V \longrightarrow \mathbb{C}\mathrm{l}(V)$ for the $\mathbb{R}$-linear and injective composite

$$V \xrightarrow{\text{linear summand}} TV \xrightarrow{1\otimes -} (TV)_{\mathbb{C}} \longrightarrow \mathbb{C}\mathrm{l}(V) \ .$$

With this notation the relation $[v]^2 = |v|^2 \cdot 1$ holds in $\mathbb{C}\mathrm{l}(V)$ for all $v \in V$. The Clifford algebra construction is functorial for $\mathbb{R}$-linear isometric embeddings, so in particular $\mathbb{C}\mathrm{l}(V)$ inherits an action of the orthogonal group $O(V)$. The Clifford algebra is $\mathbb{Z}/2$-graded, coming from the grading of the tensor algebra by even and odd tensor powers.

The complex Clifford algebra is in fact a $\mathbb{Z}/2$-graded C^*-algebra. The $*$-involution on $\mathbb{C}\mathrm{l}(V)$ is defined by declaring $[v]^* = [v]$ for all $v \in V$ and extending this to a $\mathbb{C}$-semilinear anti-automorphism. This makes the elements $[v]$ for $v \in S(V)$ into unitary elements of $\mathbb{C}\mathrm{l}(V)$. The norm on $\mathbb{C}\mathrm{l}(V)$ arises from the operator norm on the endomorphism algebra of the exterior algebra of $V_{\mathbb{C}}$, as explained, for example, in [24, Construction 6.4.5].

The complex Clifford algebra $\mathbb{C}\mathrm{l}(V)$ supports three different involutions relevant for our purposes:

- The grading involution $\alpha\colon \mathbb{C}\mathrm{l}(V) \longrightarrow \mathbb{C}\mathrm{l}(V)$; it is $\mathbb{C}$-linear and multiplicative, and satisfies $\alpha[v] = -[v]$. The $+1$ and -1 eigenspaces of α are, respectively, the even and odd summands of $\mathbb{C}\mathrm{l}(V)$.
- The conjugation involution $(-)^*\colon \mathbb{C}\mathrm{l}(V) \longrightarrow \mathbb{C}\mathrm{l}(V)$; it is conjugate-linear and anti-multiplicative, and satisfies $[v]^* = [v]$.
- We define $\psi\colon \mathbb{C}\mathrm{l}(V) \longrightarrow \mathbb{C}\mathrm{l}(V)$ as the unique conjugate-linear and multiplicative map satisfying $\psi[v] = [v]$. The fixed points of this involution are thus the real Clifford algebra associated to the positive definite form, i.e.,

$$\mathbb{C}\mathrm{l}(V)^\psi \ = \ (TV)/(v \otimes v - |v|^2 \cdot 1) \ .$$

The three involutions α, $(-)^*$ and ψ of $\mathbb{C}\mathrm{l}(V)$ commute with each other. The pair $((-)^*, \alpha)$ makes $\mathbb{C}\mathrm{l}(V)$ into a complex, $\mathbb{Z}/2$-graded C^*-algebra, and only these data enters into the definition of $\mathbf{KU}$ as an orthogonal spectrum. The conjugate-linear involution ψ of $\mathbb{C}(V)$ will enter in the definition of the involution of $\mathbf{KR}$ which makes it an orthogonal C-ring spectrum.

Construction B.51 (The Real-global K-theory spectrum)**.** For a euclidean inner product space V, we write $\mathcal{H}_V$ for the Hilbert space completion of the complexified symmetric algebra $\mathrm{Sym}(V_{\mathbb{C}})$, with respect to the inner product specified in [24, Proposition 6.3.8]. Then $\mathcal{K}_V$ denotes the C^*-algebra of compact operators on $\mathcal{H}_V$, concentrated in even grading. The value of the commutative orthogonal ring spectrum $\mathbf{KU}$ at V is defined by

$$\mathbf{KU}(V) \ = \ C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V) \otimes_{\mathbb{C}} \mathcal{K}_V) \ .$$

The unit map of the ring spectrum **KU** uses the 'functional calculus'

$$\text{(B.52)}\qquad \mathrm{fc} \;:\; S^V \longrightarrow C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V)) \ , \quad v \longmapsto (-)[v] \ ,$$

For $v \in V$ the $*$-homomorphism $\mathrm{fc}(v)$ is given on homogeneous elements of s by

$$f[v] \;=\; \mathrm{fc}(v)(f) \;=\; \begin{cases} f(|v|)\cdot 1 & \text{when } f \text{ is even, and} \\ \frac{f(|v|)}{|v|}\cdot [v] & \text{when } f \text{ is odd.} \end{cases}$$

For $v = 0$ the formula for odd functions is to be interpreted as $f[0] = 0$; this is continuous because for $v \neq 0$, the norm of $f(|v|)/|v| \cdot [v]$ is $f(|v|)$, which tends to $f(0) = 0$ if v tends to 0. If the norm of v tends to infinity, then $f(|v|)$ tends to 0, so $\mathrm{fc}(v)$ tends to the constant $*$-homomorphism with value 0, the basepoint of $C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V))$. The unit map

$$\text{(B.53)}\qquad \eta_V \;:\; S^V \longrightarrow C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V) \otimes_{\mathbb{C}} \mathcal{K}_V) \;=\; \mathbf{KU}(V)$$

is then defined as the composite

$$S^V \xrightarrow{\ \mathrm{fc}\ } C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V)) \xrightarrow{(-\otimes p_0)_*} C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V) \otimes_{\mathbb{C}} \mathcal{K}_V) \;=\; \mathbf{KU}(V) \ ,$$

where $p_0 \in \mathcal{K}_V$ is the orthogonal projection onto the constant summand in the symmetric algebra. The multiplicativity of the unit maps follows from the multiplicativity property [24, (6.4.8)] of the functional calculus maps.

Now we explain the involution on $\mathbf{KU}(V)$, see also [12, Definition 6.2]. Complex conjugation is an conjugate-linear isometry $\psi\colon \mathrm{Sym}(V_{\mathbb{C}}) \longrightarrow \mathrm{Sym}(V_{\mathbb{C}})$; so it extends uniquely to a continuous conjugate-linear isometry $\psi\colon \mathcal{H}_V \longrightarrow \mathcal{H}_V$ of the Hilbert space completion. Conjugation by ψ is then a conjugate-linear involutive automorphism

$$(-)^\psi \;:\; \mathcal{K}_V \longrightarrow \mathcal{K}_V$$

of the C^*-algebra of compact operators. For example, if E is a finite-dimensional $\mathbb{C}$-subspace of $\mathrm{Sym}(V_{\mathbb{C}})$, and $p_E \in \mathcal{K}_V$ the associated orthogonal projection, then

$$(p_E)^\psi \;=\; \psi \circ p_E \circ \psi \;=\; p_{\psi(E)} \ .$$

We combine the conjugate-linear and multiplicative involutions $\psi\colon \mathbb{C}\mathrm{l}(V) \longrightarrow \mathbb{C}\mathrm{l}(V)$ and $(-)^\psi\colon \mathcal{K}_V \longrightarrow \mathcal{K}_V$ into one conjugate-linear and multiplicative involution

$$\psi \otimes (-)^\psi \;:\; \mathbb{C}\mathrm{l}(V) \otimes_{\mathbb{C}} \mathcal{K}_V \longrightarrow \mathbb{C}\mathrm{l}(V) \otimes_{\mathbb{C}} \mathcal{K}_V \ .$$

Finally, we define an involution

$$\psi \;:\; \mathbf{KU}(V) \;=\; C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V) \otimes_{\mathbb{C}} \mathcal{K}_V) \longrightarrow C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(V) \otimes_{\mathbb{C}} \mathcal{K}_V) \;=\; \mathbf{KU}(V)$$

by sending a homomorphism $h\colon s \longrightarrow \mathbb{C}\mathrm{l}(V) \otimes_{\mathbb{C}} \mathcal{K}_V$ to the composite

$$s \xrightarrow{(-)^*} s \xrightarrow{\ h\ } \mathbb{C}\mathrm{l}(V) \otimes_{\mathbb{C}} \mathcal{K}_V \xrightarrow{\psi\otimes(-)^\psi} \mathbb{C}\mathrm{l}(V) \otimes_{\mathbb{C}} \mathcal{K}_V \ .$$

The first map is conjugation on s, i.e., pointwise complex conjugation of functions. If h is a $*$-homomorphism, then in particular $h \circ (-)^* = (-)^* \circ h$. So the involution on $\mathbf{KU}(V)$ is also given by postcomposition of $h\colon s \longrightarrow \mathbb{C}\mathrm{l}(V) \otimes_{\mathbb{C}} \mathcal{K}_V$ with the map

$$\mathbb{C}\mathrm{l}(V) \otimes_{\mathbb{C}} \mathcal{K}_V \xrightarrow{(-)^*} \mathbb{C}\mathrm{l}(V) \otimes_{\mathbb{C}} \mathcal{K}_V \xrightarrow{\psi\otimes(-)^\psi} \mathbb{C}\mathrm{l}(V) \otimes_{\mathbb{C}} \mathcal{K}_V \ .$$

This composite is $\mathbb{C}$-linear and anti-multiplicative. We omit the verification that these involutions are compatible with the multiplication and the structure maps of the orthogonal spectrum **KU**. We write $\mathbf{KR} = (\mathbf{KU}, \psi)$ for the *periodic Real-global K-theory spectrum*, i.e., the commutative orthogonal C-ring spectra **KU** with C-action by this involution.

Construction B.54. We recall from [24, Construction 6.4.13] the construction of the morphism of commutative orthogonal C-ring spectra

$$j \ : \ \mathbf{kr} \ \longrightarrow \ \mathbf{KR}$$

from connective to periodic Real-global K-theory. Its value at a euclidean inner product space V is the map

$$j(V) \ : \ \mathbf{kr}(V) \ = \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ \longrightarrow \ C^*_{\mathrm{gr}}(s, \mathrm{Cl}(V) \otimes_{\mathbb{C}} \mathcal{K}_V) \ = \ \mathbf{KR}(V)$$

defined by

$$j(V)[E_1, \dots, E_n;\, v_1, \dots, v_n](f) \ = \ \sum_{i=1}^n f[v_i] \otimes p_{E_i} \ .$$

Here $f[v_i] = \mathrm{fc}(v_i)(f)$ is the functional calculus map (B.52), and p_{E_i} is the orthogonal projection onto the subspace E_i of $\mathrm{Sym}(V_{\mathbb{C}}) \subset \mathcal{H}_V$. Already in [24, Construction 6.4.13] we omitted the detailed verification that these maps indeed form a morphism of orthogonal ring spectra. We shall honor this tradition here, and also refrain from checking that the map $j(V)$ commutes with the complex-conjugation involutions.

Now we proceed to justify that $\mathbf{KR}$ deserves its name by showing that for every augmented Lie group $\alpha\colon G \longrightarrow C$, the orthogonal G-spectrum $\alpha^*(\mathbf{KR})$ represents α-equivariant Real K-theory KR_α. If the augmentation $\alpha\colon G \longrightarrow C$ is trivial, this amounts to the fact that the underlying G-spectrum of the global spectrum $\mathbf{KU}$ represents G-equivariant complex K-theory, see [15, Theorem 4.4] or [24, Corollary 6.4.23]. If $G = C$ augmented by the identity, this amounts to the fact that the genuine C-spectrum underlying $\mathbf{KR}$ is a model for Atiyah's Real K-theory spectrum, see [12, Proposition 6.2].

In the C-spectrum that represents Atiyah's Real K-theory, the third power of the pre-Euler class of the sign representation vanishes. Our proof that the orthogonal C-spectrum $\mathbf{KR}$ models the 'correct' Real-global homotopy type proceeds by proving this property for $\mathbf{KR}$ from scratch. Nilpotence of the pre-Euler class immediately implies that the Real K-theory C-spectrum has trivial C-geometric fixed points, and is thus a Borel C-spectrum.

We write $a = a_\sigma \in \pi^C_{-\sigma}(\mathbb{S})$ for the pre-Euler class (2.6) of the sign representation of the group C, and also for its image in $\pi^C_{-\sigma}(\mathbf{KR})$ under the unit morphism $\mathbb{S} \longrightarrow \mathbf{KR}$.

Theorem B.55. *The relation $a^3 = 0$ holds in $\pi^C_{-3\sigma}(\mathbf{KR})$.*

Proof. For every $n \geq 0$, the class a^n is represented by the composite

$$S^0 \ \xrightarrow{a^n} \ S^{n\sigma} \ \xrightarrow{\eta_{n\sigma}} \ \mathbf{KR}(n\sigma) \ .$$

By definition (B.53) of the unit maps of $\mathbf{KR}$, that composite factors through

$$S^0 \ \xrightarrow{a^n} \ S^{n\sigma} \ \xrightarrow{\mathrm{fc}} \ C^*_{\mathrm{gr}}(s, \mathrm{Cl}(n\sigma)) \ .$$

So it suffices to show that for $n = 3$, the C-fixed point $\mathrm{fc}(0)$ lies in the same path component of

$$\left(C^*_{\mathrm{gr}}(s, \mathrm{Cl}(3\sigma))\right)^C$$

as the basepoint, the zero homomorphism. We will show that this fixed point space is homeomorphic to a circle, and so in particular path connected. Because $a^2 \neq 0$ in $\pi^C_{-2\sigma}(\mathbf{KR})$, this proof method cannot work for $n \leq 2$; and indeed, the interested reader might want to convince themself that $\left(C^*_{\mathrm{gr}}(s, \mathrm{Cl}(n\sigma))\right)^C = \{0, \mathrm{fc}(0)\}$ for $0 \leq n \leq 2$.

For any n, the C-action on the representation $n\sigma$ is by multiplication by -1. Hence the induced involution on the Clifford algebra

$$\mathrm{Cl}(-\mathrm{Id}) \ : \ \mathrm{Cl}(n\sigma) \ \longrightarrow \ \mathrm{Cl}(n\sigma)$$

is a $\mathbb{C}$-algebra automorphism that negates the element $[v]$ for all $v \in S(n\sigma)$. So for the C-representation $n\sigma$, the involution $\mathrm{Cl}(-\mathrm{Id})$ induced by functoriality of $\mathrm{Cl}(-)$ coincides with the grading involution α. The involution on $C^*_{\mathrm{gr}}(s, \mathrm{Cl}(n\sigma))$ relevant for our present purpose is thus given by

- conjugating a graded $*$-homomorphism $f\colon s \longrightarrow \mathbb{C}\mathrm{l}(n\sigma)$ by the complex conjugations $(-)^*\colon s \longrightarrow s$ (pointwise complex conjugation) and $\psi\colon \mathbb{C}\mathrm{l}(n\sigma) \longrightarrow \mathbb{C}\mathrm{l}(n\sigma)$, and
- applying the involution $\mathbb{C}\mathrm{l}(-\mathrm{Id}_{n\sigma}) = \alpha$ coming from functoriality of $\mathbb{C}\mathrm{l}(-)$.

In other words, we need to identify the fixed points of the involution

$$C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(n\sigma)) \ \longrightarrow \ C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(n\sigma)) \ , \quad f \ \longmapsto \ \alpha \circ \psi \circ f \circ (-)^* \ .$$

For every $\mathbb{Z}/2$-graded complex C^*-algebra A, evaluation at the function

$$r \ : \ \mathbb{R} \ \longrightarrow \ \mathbb{C} \ , \quad r(x) \ = \ \frac{2i}{x-i}$$

in the C^*-algebra s is a homeomorphism

$$C^*_{\mathrm{gr}}(s, A) \ \cong \ \{x \in A \colon xx^* = x^*x = -x - x^* \ , \ \alpha(x) = x^*\} \ ,$$

compare [24, (6.4.3)]. For $A = \mathbb{C}\mathrm{l}(n\sigma)$, the relevant involution on the left hand side corresponds to the involution on the right hand side sending $x \in \mathbb{C}\mathrm{l}(n\sigma)$ to

$$\psi(\alpha(x^*)) \ = \ (\psi(\alpha(x)))^* \ .$$

The previous map thus restricts to a homeomorphism

$$\begin{aligned} \left(C^*_{\mathrm{gr}}(s, \mathbb{C}\mathrm{l}(n))\right)^C \ &\cong \ \{x \in \mathbb{C}\mathrm{l}(n)A \colon xx^* = x^*x = -x - x^* \ , \ \alpha(x^*) = x = \psi(x)\} \\ &\cong \ \{x \in C(n) \colon xx^* = x^*x = -x - x^* \ , \ \alpha(x^*) = x\} \ . \end{aligned}$$

The second homeomorphism uses that $\mathbb{C}\mathrm{l}(n)$ is the complexification of the real Clifford algebra

$$C(n) \ = \ T(\mathbb{R}^n)/([v] \otimes [v] - |v|^2 \cdot 1) \ ,$$

with ψ corresponding to the complex conjugation automorphism of $\mathbb{C} \otimes_{\mathbb{R}} C(n)$.

Now we specialize to $n = 3$. We write $e = [1, 0, 0]$, $f = [0, 1, 0]$ and $g = [0, 0, 1]$ for the multiplicative generators of $C(3)$ coming from the standard orthonormal basis of $\mathbb{R}^3$. Then e, f and g are odd, pairwise anticommuting, fixed by the conjugation $(-)^*$, and satisfy

$$e^2 \ = \ f^2 \ = \ g^2 \ = \ 1 \ .$$

We want to find all $x \in C(3)$ that satisfy

$$x \cdot x^* \ = \ x^* \cdot x \ = \ -x - x^* \quad \text{and} \quad \alpha(x^*) \ = \ x^* \ .$$

The involution $x \mapsto \alpha(x^*) = \alpha(x)^*$ fixes the basis elements 1 and efg, and it negates e, f, g, ef, eg, and fg. The condition $\alpha(x^*) = x$ thus forces x to be an $\mathbb{R}$-linear combination of 1 and efg. So $x = a + b \cdot efg$ for some $a, b \in \mathbb{R}$. The relations $x \cdot x^* = x^* \cdot x = -x - x^*$ then amount to

$$(a+1)^2 + b^2 \ = \ 1 \ .$$

The solutions in $\mathbb{R}^2$ to this equation form a circle, as claimed. $\square$

We write $\tilde{E}C = (EC)^\diamond$ for the unreduced suspension of the universal free C-space EC.

Theorem B.56. *Let M be an orthogonal C-spectrum that admits a $\mathbf{KR}$-module structure in the Real-global stable homotopy category.*

(i) *The Real-global spectrum $\tilde{E}C \wedge M$ is trivial.*
(ii) *The orthogonal C-space $\Omega^\bullet(M)$ is coinduced.*

Proof. (i) We let $\alpha\colon G \longrightarrow C$ be any augmented Lie group. If the augmentation α is trivial, then G acts trivially on $\alpha^*(\tilde{E}C)$, which is thus G-equivariantly contractible. In particular, the geometric fixed point homotopy groups $\Phi^G_*(\tilde{E}C \wedge M)$ vanish.

If the augmentation α is surjective, then we need to argue differently. Because $a^3 = 0$, also $\alpha^*(a)^3 = 0$ in $\pi^G_{-3\alpha^*(\sigma)}(\alpha^*(\mathbf{KR}))$. Because $(\alpha^*(\sigma))^G = \{0\}$, we have So $\Phi^G(\alpha^*(a)) = 1$ in the geometric fixed point ring

$\Phi_0^G(\alpha^*(\mathbf{KR}))$. So $1 = \Phi^G(\alpha^*(a))^3 = \Phi^G(\alpha^*(a^3)) = 0$, and so the ring $\Phi_0^G(\alpha^*(\mathbf{KR}))$ is trivial. Since M is a $\mathbf{KR}$-module spectrum, the group $\Phi_k^G(\alpha^*(M))$ admits a module structure over $\Phi_0^G(\alpha^*(\mathbf{KR})) = 0$, and thus $\Phi_k^G(\alpha^*(M)) = 0$ for all $k \in \mathbb{Z}$. Since $\alpha\colon G \longrightarrow C$ is surjective, we have $(\alpha^*(\tilde{E}C))^G = S^0$, and thus

$$\Phi_*^G(\alpha^*(\tilde{E}C \wedge M)) \;\cong\; \Phi_*^G(\alpha^*(M)) \;=\; 0\ .$$

So all geometric fixed point homotopy groups of $\tilde{E}C \wedge M$ vanish, for all augmented Lie groups, and thus $\tilde{E}C \wedge M$ is trivial in the Real-global stable homotopy category.

(ii) Since M is a $\mathbf{KR}$-module, also the Real-global spectrum $\mathrm{map}(EC, M)$ admits a $\mathbf{KR}$-module structure. So the Real-global spectra $\tilde{E}C \wedge M$ and $\tilde{E}C \wedge \mathrm{map}(EC, M)$ are trivial by part (i). Hence the two horizontal morphisms in the following commutative square are Real-global equivalences:

$$\begin{tikzcd}
M \wedge EC_+ \arrow[r, "p\wedge M", "\sim"'] \arrow[d, "p^*\wedge EC_+"'] & M \arrow[d, "p^*"] \\
\mathrm{map}(EC, M) \wedge EC_+ \arrow[r, "\sim", "p^*\wedge p_+"'] & \mathrm{map}(EC, M)
\end{tikzcd}$$

Since the morphism $p^*\colon M \longrightarrow \mathrm{map}(EC, M)$ is global equivalence of underlying non-Real global spectra, the left horizontal morphism is also a Real-global equivalence, by the stable analog of Proposition A.13. Hence the right vertical morphism $p^*\colon M \longrightarrow \mathrm{map}(EC, M)$ is a Real-global equivalence. The functor $\Omega^\bullet$ takes Real-global equivalences of orthogonal C-spectra to Real-global equivalences of orthogonal C-spaces; and it commutes with $\mathrm{map}(EC, -)$. So the morphism

$$\Omega^\bullet(M) \xrightarrow{\Omega^\infty(p^*)} \Omega^\bullet(\mathrm{map}(EC, M)) \cong \mathrm{map}(EC, \Omega^\bullet M)$$

is a Real-global equivalence. Proposition A.15 then shows that $\Omega^\bullet(M)$ is coinduced. □

The following theorem generalizes [24, Theorem 6.4.21] from the global to the Real-global context; it says that $\mathbf{U}$ is a Real-global genuine infinite loop space, with delooping the σ-suspension of $\mathbf{KR}$.

Theorem B.57. *The morphism of based orthogonal C-spaces*

$$\Omega^\bullet(\mathrm{sh}^\sigma j) \circ \mathrm{eig} \;:\; \mathbf{U} \;\longrightarrow\; \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{KR})$$

is a Real-global equivalence.

Proof. The orthogonal C-space $\mathbf{U}$ is coinduced by Theorem B.22. The orthogonal C-space $\Omega^\bullet(\mathrm{sh}^\sigma \mathbf{KR})$ is coinduced by Theorem B.56. The morphism $\Omega^\bullet(\mathrm{sh}^\sigma j) \circ \mathrm{eig}\colon \mathbf{U} \longrightarrow \Omega^\bullet(\mathrm{sh}^\sigma \mathbf{KR})$ is a global equivalence of underlying non-Real orthogonal spaces by [24, Theorem 6.4.21]. So $\Omega^\bullet(\mathrm{sh}^\sigma j) \circ \mathrm{eig}$ is a Real-global equivalence by Proposition A.14. □

Remark B.58. Theorems B.24 and B.57 provide Real-global equivalences

$$\mathbf{BUP} \;\underset{\sim}{\xrightarrow{\gamma}}\; \Omega^\sigma\mathbf{U} \;\underset{\sim}{\xrightarrow{\Omega^\sigma(\Omega^\bullet(\mathrm{sh}^\sigma j)\circ\mathrm{eig})}}\; \Omega^\sigma(\Omega^\bullet(\mathrm{sh}^\sigma \mathbf{KR})) \;\underset{\sim}{\xleftarrow{\xi_{\mathbf{KR}}}}\; \Omega^\bullet(\mathbf{KR})\ .$$

The Real-global equivalences γ and $\xi_{\mathbf{KR}}$ are defined in (B.27) and (4.12), respectively. This shows that the Real-global K-theory spectrum $\mathbf{KR}$ is an genuine C-global delooping of $\mathbf{BUP}$.

A corollary of Theorem B.57 and the fact that $\mathbf{BUP}$ represents Real-global K-theory (Theorem B.12) is that the Real-global K-theory spectrum $\mathbf{KR}$ represents Real-global K-theory, thereby justifying its name. The following theorem generalizes [15, Theorem 4.4] or [24, Corollary 6.4.23] from equivariant K-theory to Real-equivariant K-theory.

The morphism of orthogonal C-spaces $\varepsilon\colon \mathbf{Gr} \longrightarrow \Omega^\bullet(\mathbf{kr})$ was introduced in (B.38). For every augmented Lie group $\alpha\colon G \longrightarrow C$ and every G-space A, the induced map

$$\varepsilon_* \;:\; [A, \mathbf{Gr}]^\alpha \;\longrightarrow\; [A, \Omega^\bullet(\mathbf{kr})]^\alpha \;=\; \mathbf{kr}_\alpha^0(A)$$

is additive because $\varepsilon\colon \mathbf{Gr} \longrightarrow \Omega^\bullet(\mathbf{kr})$ is a C-global H-map by Proposition B.41. The target of this homomorphism is a group; since $i_*\colon [A,\mathbf{Gr}]^\alpha \longrightarrow [A,\mathbf{BUP}]^\alpha$ is a group completion by Proposition B.5, there is a unique group homomorphism

$$\bar{\varepsilon} \ : \ [A,\mathbf{BUP}]^\alpha \ \longrightarrow \ \mathbf{kr}^0_\alpha(A)$$

such that $\bar{\varepsilon}\circ i_* = \varepsilon_*$. We alert the reader that $\bar{\varepsilon}$ is *not* an isomorphism in general. For example $[S^1,\mathbf{BUP}]^{U(1)}_* \cong \widetilde{KU}_{U(1)}(S^1) = 0$, whereas $\widetilde{\mathbf{ku}}^0_{U(1)}(S^1) \cong \pi_1^{U(1)}(\mathbf{ku})$ is nonzero by [24, Remark 6.3.38]. However, the situation improves if we pass from connective to periodic Real-global K-theory.

Theorem B.59. *For every augmented compact Lie group $\alpha\colon G \longrightarrow C$ and every finite G-CW-complex A, the composite*

$$KR_\alpha(A) \ \cong_{(\mathrm{B.10})} \ [A,\mathbf{BUP}]^\alpha \ \xrightarrow{\ \bar{\varepsilon}\ } \ \mathbf{kr}^0_\alpha(A) \ \xrightarrow{\ j_*\ } \ \mathbf{KR}^0_\alpha(A)$$

is an isomorphism of rings.

Proof. In a first step we show that the composite

$$[A,\mathbf{BUP}]^\alpha \ \xrightarrow{\ \bar{\varepsilon}\ } \ \mathbf{kr}^0_\alpha(A) \ \xrightarrow{\ j_*\ } \ \mathbf{KR}^0_\alpha(A)$$

is an isomorphism of abelian groups. The morphism $\varepsilon\colon \mathbf{Gr} \longrightarrow \Omega^\bullet(\mathbf{kr})$ from (B.38), the eigenspace morphism $\mathrm{eig}\colon \mathbf{U} \longrightarrow \Omega^\bullet(\mathrm{sh}^\sigma\,\mathbf{kr})$ from (B.44), and the morphism of ultra-commutative C-monoids $\beta\colon \mathbf{Gr} \longrightarrow \Omega^\sigma\mathbf{U}$ defined in (B.28) are related by the commutative diagram of based orthogonal C-spaces:

$$\begin{tikzcd}
\mathbf{Gr} \arrow[rr, "\beta"] \arrow[d, "\varepsilon"'] & & \Omega^\sigma\mathbf{U} \arrow[d, "\Omega^\sigma(\mathrm{eig})"] \\
\Omega^\bullet(\mathbf{kr}) \arrow[r, "\sim\ (4.12)", "\xi_{\mathbf{kr}}"'] & \Omega^\sigma(\Omega^\bullet(\mathbf{kr}\wedge S^\sigma)) \arrow[r, "\sim", "\Omega^\sigma(\Omega^\bullet(\lambda^\sigma_{\mathbf{kr}}))"'] & \Omega^\sigma(\Omega^\bullet(\mathrm{sh}^\sigma\,\mathbf{kr}))
\end{tikzcd}$$

The diagram commutes by direct inspection of the definitions, and the two lower horizontal morphisms are Real-global equivalences. So the outer part of the following diagram commutes:

$$\begin{tikzcd}
{[A,\mathbf{Gr}]^\alpha} \arrow[d, "i_*"] \arrow[drr, bend left, "\varepsilon_*"] \arrow[dd, bend right=60, "\beta_*"'] & & \\
{[A,\mathbf{BUP}]^\alpha} \arrow[rr, "\bar{\varepsilon}"] \arrow[d, "\gamma_*", "\cong"'] & & {[A,\Omega^\bullet(\mathbf{kr})]^\alpha} \arrow[d, "(\Omega^\sigma(\Omega^\bullet(\lambda^\sigma_{\mathbf{kr}}))\circ\xi_{\mathbf{kr}})_*", "\cong"'] \\
{[A,\Omega^\sigma\mathbf{U}]^\alpha} \arrow[rr, "(\Omega^\sigma(\mathrm{eig}))_*"'] & & {[A,\Omega^\sigma(\Omega^\bullet(\mathrm{sh}^\sigma\,\mathbf{kr}))]^\alpha}
\end{tikzcd}$$

The eigenspace morphism $\mathrm{eig}\colon \mathbf{U} \longrightarrow \Omega^\bullet(\mathrm{sh}^\sigma\,\mathbf{kr})$ is a C-global H-map by Proposition B.46. Hence also the morphism $\Omega^\sigma(\mathrm{eig})\colon \mathbf{U} \longrightarrow \Omega^\sigma(\Omega^\bullet(\mathrm{sh}^\sigma\,\mathbf{KR}))$ is a C-global H-map. So the induced lower horizontal map in the diagram is additive. Since all maps in the lower rectangle are additive and the abelian group $[A,\mathbf{BUP}]^\alpha$ is generated by the image of $i_*\colon [A,\mathbf{Gr}]^\alpha \longrightarrow [A,\mathbf{BUP}]^\alpha$, this means that also the lower rectangle in the diagram commutes. Now we know that the following diagram of group homomorphisms commutes:

$$\begin{tikzcd}
 & \mathbf{kr}^0_\alpha(A) \arrow[r, "j_*"] \arrow[d, equal] & \mathbf{KR}^0_\alpha(A) \arrow[d, equal] \\
{[A,\mathbf{BUP}]^\alpha} \arrow[r, "\bar{\varepsilon}"] \arrow[d, "\gamma_*", "\cong"'] & {[A,\Omega^\bullet(\mathbf{kr})]^\alpha} \arrow[r, "(\Omega^\bullet j)_*"] \arrow[d, "(\Omega^\sigma(\Omega^\bullet(\lambda^\sigma_{\mathbf{kr}}))\circ\xi_{\mathbf{kr}})_*", "\cong"'] & {[A,\Omega^\bullet(\mathbf{KR})]^\alpha} \arrow[d, "(\Omega^\sigma(\Omega^\bullet(\lambda^\sigma_{\mathbf{kr}}))\circ\xi_{\mathbf{KR}})_*", "\cong"'] \\
{[A,\Omega^\sigma\mathbf{U}]^\alpha} \arrow[r, "(\Omega^\sigma(\mathrm{eig}))_*"'] & {[A,\Omega^\sigma(\Omega^\bullet(\mathrm{sh}^\sigma\,\mathbf{kr}))]^\alpha} \arrow[r, "(\Omega^\sigma(\Omega^\bullet(\mathrm{sh}^\sigma\,j)))_*"'] & {[A,\Omega^\sigma(\Omega^\bullet(\mathrm{sh}^\sigma\,\mathbf{KR}))]^\alpha}
\end{tikzcd}$$

The morphism of based Real-global spaces $\Omega^\bullet(\mathrm{sh}^\sigma j)\circ\mathrm{eig}\colon \mathbf{U}\longrightarrow\Omega^\bullet(\mathrm{sh}^\sigma\,\mathbf{KR})$ is a Real-global equivalence by Theorem B.57. Hence also $\Omega^\sigma(\Omega^\bullet(\mathrm{sh}^\sigma j)\circ\mathrm{eig})$ is a Real-global equivalence, and the lower horizontal composite in the last diagram is an isomorphism. Hence also $j_*\circ\bar{\varepsilon}$ is an isomorphism.

The map (B.10) is an isomorphism of abelian groups by Theorem B.12 (iii). So it remains to show that the composite in the statement of the theorem is also multiplicative, and hence an isomorphism of rings. Since $j\colon\mathbf{kr}\longrightarrow\mathbf{KR}$ is a morphism of ultra-commutative C-ring spectra, the induced map $j_*\colon\mathbf{kr}^0_\alpha(A)\longrightarrow\mathbf{KR}^0_\alpha(A)$ of equivariant cohomology theories is multiplicative. We claim that $\bar{\epsilon}\circ(\mathrm{B.10})\colon KR_\alpha(A)\longrightarrow\mathbf{kr}^0_\alpha(A)$ is also multiplicative. If the augmentation α is trivial, this is shown in [24, Theorem 6.3.31 (ii)]. The proof for surjective augmentations is analogous, and we omit it. $\square$

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