Winter term 2025/26

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Exercises for **Algebraic Topology I**
Sheet 1

Exercise 1 (*7 points*)**.** Let $0 < k < n$. Show that the inclusion $\mathbb{RP}^k \hookrightarrow \mathbb{RP}^n$ does not admit a retraction.

Exercise 2 (*18 points*)**.**

1. Let X, Y be well-pointed spaces such that X is p-connected and Y is q-connected for $p, q \geq 1$. Show that the inclusions induce an isomorphism
$$(i_{1*}, i_{2*}) \colon \pi_k(X) \oplus \pi_k(Y) \xrightarrow{\cong} \pi_k(X \vee Y)$$
for $1 \leq k \leq p + q$, with inverse induced by the collapse maps
$$c_1 \coloneqq (\mathrm{id}_X, 0) \colon X \vee Y \to X \qquad c_2 \coloneqq (0, \mathrm{id}_Y) \colon X \vee Y \to Y.$$
2. Show that $(c_{1*}, c_{2*}) \colon \pi_3(S^2 \vee S^2) \to \pi_3(S^2) \oplus \pi_3(S^2)$ is split surjective, but not injective.
3. Show that the kernel of the split surjection from the previous subtask is infinite cyclic, and describe a generator of this kernel. In particular, this shows $\pi_3(S^2 \vee S^2) \cong \mathbb{Z}^3$.

Exercise 3 (*15 points*)**.** Show that $\pi_2(S^1 \vee S^2)$ is a free module of rank 1 over $\mathbb{Z}[\pi_1] \cong \mathbb{Z}[t, t^{-1}]$.

∗ **Exercise 4** (*5 bonus points*)**.** Show that $\pi_3(S^1 \vee S^2)$ is not finitely generated as a $\mathbb{Z}[\pi_1]$-module.

Please hand in your solutions by **October 27, 2pm** via eCampus.