Verma modules and preprojective algebras

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Abstract

We give a geometric construction of the Verma modules of a symmetric Kac-Moody Lie algebra $\mathfrak{g}$ in terms of constructible functions on the varieties of nilpotent finite-dimensional modules of the corresponding preprojective algebra Λ.

1 Introduction

Let $\mathfrak{g}$ be the symmetric Kac-Moody Lie algebra associated to a finite unoriented graph Γ without loop. Let $\mathfrak{n}_-$ denote a maximal nilpotent subalgebra of $\mathfrak{g}$. In [**Lu1**, §12], Lusztig has given a geometric construction of $U(\mathfrak{n}_-)$ in terms of certain Lagrangian varieties. These varieties can be interpreted as module varieties for the preprojective algebra Λ attached to the graph Γ by Gelfand and Ponomarev [**GP**]. In Lusztig's construction, $U(\mathfrak{n}_-)$ gets identified with an algebra $(\mathcal{M}, *)$ of constructible functions on these varieties, where $*$ is a convolution product inspired by Ringel's multiplication for Hall algebras.

Later, Nakajima gave a similar construction of the highest weight irreducible integrable $\mathfrak{g}$-modules $L(\lambda)$ in terms of some new Lagrangian varieties which differ from Lusztig's ones by the introduction of some extra vector spaces W_k for each vertex k of Γ, and by considering only stable points instead of the whole variety [**Na**, §10].

The aim of this paper is to extend Lusztig's original construction and to endow $\mathcal{M}$ with the structure of a Verma module $M(\lambda)$.

To do this we first give a variant of the geometrical construction of the integrable $\mathfrak{g}$-modules $L(\lambda)$, using functions on some natural open subvarieties of Lusztig's varieties instead of functions on Nakajima's varieties (Theorem 1). These varieties have a simple description in terms of the preprojective algebra Λ and of certain injective Λ-modules q_λ.

Having realized the integrable modules $L(\lambda)$ as quotients of $\mathcal{M}$, it is possible, using the comultiplication of $U(\mathfrak{n}_-)$, to construct geometrically the raising operators $E_i^\lambda \in \mathrm{End}(\mathcal{M})$ which make $\mathcal{M}$ into the Verma module $M(\lambda)$ (Theorem 2). Note that we manage in this way to realize Verma modules with arbitrary highest weight (not necessarily dominant).

Finally, we dualize this setting and give a geometric construction of the dual Verma module $M(\lambda)^*$ in terms of the delta functions $\delta_x \in \mathcal{M}^*$ attached to the finite-dimensional nilpotent Λ-modules x (Theorem 3).

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2 Verma modules

2.1 Let $\mathfrak{g}$ be the symmetric Kac-Moody Lie algebra associated with a finite unoriented graph Γ without loop. The set of vertices of the graph is denoted by I. The (generalized) Cartan matrix of $\mathfrak{g}$ is $A = (a_{ij})_{i,j\in I}$, where $a_{ii} = 2$ and, for $i \neq j$, $-a_{ij}$ is the number of edges between i and j.

2.2 Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{h}$ is a Cartan subalgebra and $(\mathfrak{n}, \mathfrak{n}_-)$ a pair of opposite maximal nilpotent subalgebras. Let $\mathfrak{b} = \mathfrak{n}\oplus\mathfrak{h}$. The Chevalley generators of $\mathfrak{n}$ (*resp.* $\mathfrak{n}_-$) are denoted by e_i $(i \in I)$ (*resp.* f_i) and we set $h_i = [e_i, f_i]$.

2.3 Let α_i denote the simple root of $\mathfrak{g}$ associated with $i \in I$. Let $(- \,;\, -)$ be a symmetric bilinear form on $\mathfrak{h}^*$ such that $(\alpha_i \,;\, \alpha_j) = a_{ij}$. The lattice of integral weights in $\mathfrak{h}^*$ is denoted by P, and the sublattice spanned by the simple roots is denoted by Q. We put

$$P_+ = \{\lambda \in P \mid (\lambda \,;\, \alpha_i) \geqslant 0,\ (i \in I)\}, \qquad Q_+ = Q \cap P_+.$$

2.4 Let $\lambda \in P$ and let $M(\lambda)$ be the Verma module with highest weight λ. This is the induced $\mathfrak{g}$-module defined by $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}\, u_\lambda$, where u_λ is a basis of the one-dimensional representation of $\mathfrak{b}$ given by

$$h\, u_\lambda = \lambda(h)\, u_\lambda, \quad n\, u_\lambda = 0, \qquad (h \in \mathfrak{h},\ n \in \mathfrak{n}).$$

As a P-graded vector space $M(\lambda) \cong U(\mathfrak{n}_-)$ (up to a degree shift by λ). $M(\lambda)$ has a unique simple quotient denoted by $L(\lambda)$, which is integrable if and only if $\lambda \in P_+$. In this case, the kernel of the $\mathfrak{g}$-homomorphism $M(\lambda) \to L(\lambda)$ is the $\mathfrak{g}$-module $I(\lambda)$ generated by the vectors

$$f_i^{(\lambda\,;\,\alpha_i)+1} \otimes u_\lambda, \qquad (i \in I).$$

3 Constructible functions

3.1 Let X be an algebraic variety over $\mathbb{C}$ endowed with its Zariski topology. A map f from X to a vector space V is said to be constructible if its image $f(X)$ is finite, and for each $v \in f(X)$ the preimage $f^{-1}(v)$ is a constructible subset of X.

3.2 By $\chi(A)$ we denote the Euler characteristic of a constructible subset A of X. For a constructible map $f : X \to V$ one defines

$$\int_{x\in X} f(x) = \sum_{v\in V} \chi(f^{-1}(v))\, v \in V.$$

More generally, for a constructible subset A of X we write

$$\int_{x\in A} f(x) = \sum_{v\in V} \chi(f^{-1}(v) \cap A)\, v.$$

4 Preprojective algebras

4.1 Let Λ be the preprojective algebra associated to the graph Γ (see for example [**Ri**, **GLS**]). This is an associative $\mathbb{C}$-algebra, which is finite-dimensional if and only if Γ is a graph of type A, D, E. Let s_i denote the simple one-dimensional Λ-module associated with $i \in I$, and let p_i be its projective cover and q_i its injective hull. Again, p_i and q_i are finite-dimensional if and only if Γ is a graph of type A, D, E.

4.2 A finite-dimensional Λ-module x is nilpotent if and only if it has a composition series with all factors of the form s_i $(i \in I)$. We will identify the dimension vector of x with an element $\beta \in Q_+$ by setting $\mathbf{dim}(s_i) = \alpha_i$.

4.3 Let q be an injective Λ-module of the form

$$q = \bigoplus_{i\in I} q_i^{\oplus a_i}$$

for some nonnegative integers a_i $(i \in I)$.

Lemma 1 *Let x be a finite-dimensional Λ-module isomorphic to a submodule of q. If $f_1 : x \to q$ and $f_2 : x \to q$ are two monomorphisms, then there exists an automorphism $g : q \to q$ such that $f_2 = gf_1$.*

Proof — Indeed, q is the injective hull of its socle $b = \bigoplus_{i\in I} s_i^{\oplus a_i}$. Let c_j $(j = 1, 2)$ be a complement of $f_j(\mathrm{socle}(x))$ in b. Then $c_1 \cong c_2$ and the maps

$$h_j := f_j \oplus \mathrm{id} : \quad x \oplus c_j \to q, \qquad (j = 1, 2)$$

are injective hulls. The result then follows from the unicity of the injective hull. □

Hence, up to isomorphism, there is a unique way to embed x into q.

4.4 Let $\mathcal{M}$ be the algebra of constructible functions on the varieties of finite-dimensional nilpotent Λ-modules defined by Lusztig [**Lu2**] to give a geometric realization of $U(\mathfrak{n}_-)$. We recall its definition.

For $\beta = \sum_{i\in I} b_i\alpha_i \in Q_+$, let Λ_β denote the variety of nilpotent Λ-modules with dimension vector β. Recall that Λ_β is endowed with an action of the algebraic group $G_\beta = \prod_{i\in I} GL_{b_i}(\mathbb{C})$, so that two points of Λ_β are isomorphic as Λ-modules if and only if they belong to the same G_β-orbit. Let $\widetilde{\mathcal{M}}_\beta$ denote the vector space of constructible functions from Λ_β to $\mathbb{C}$ which are constant on G_β-orbits. Let

$$\widetilde{\mathcal{M}} = \bigoplus_{\beta\in Q_+} \widetilde{\mathcal{M}}_\beta.$$

One defines a multiplication $*$ on $\widetilde{\mathcal{M}}$ as follows. For $f \in \widetilde{\mathcal{M}}_\beta$, $g \in \widetilde{\mathcal{M}}_\gamma$ and $x \in \Lambda_{\beta+\gamma}$, we have

$$(f * g)(x) = \int_U f(x')g(x''), \tag{1}$$

where the integral is over the variety of x-stable subspaces U of x of dimension γ, x'' is the Λ-submodule of x obtained by restriction to U and $x' = x/x''$. In the sequel in order to simplify notation, we will not distinguish between the subspace U and the submodule x'' of x carried by U. Thus we shall rather write

$$(f * g)(x) = \int_{x''} f(x/x'')g(x''), \tag{2}$$

where the integral is over the variety of submodules x'' of x of dimension γ.

For $i \in I$, the variety Λ_{α_i} is reduced to a single point : the simple module s_i. Denote by $\mathbf{1}_i$ the function mapping this point to 1. Let $\mathcal{G}(i, x)$ denote the variety of all submodules y of x such that $x/y \cong s_i$. Then by (2) we have

$$(\mathbf{1}_i * g)(x) = \int_{y \in \mathcal{G}(i,x)} g(y). \tag{3}$$

Let $\mathcal{M}$ denote the subalgebra of $\widetilde{\mathcal{M}}$ generated by the functions $\mathbf{1}_i$ $(i \in I)$. By Lusztig [**Lu2**], $(\mathcal{M}, *)$ is isomorphic to $U(\mathfrak{n}_-)$ by mapping $\mathbf{1}_i$ to the Chevalley generator f_i.

4.5 In the identification of $U(\mathfrak{n}_-)$ with $\mathcal{M}$, formula (3) represents the left multiplication by f_i. In order to endow $\mathcal{M}$ with the structure of a Verma module we need to introduce the following important definition. For $\nu \in P_+$, let

$$q_\nu = \bigoplus_{i \in I} q_i^{\oplus(\nu\,;\,\alpha_i)}.$$

Lusztig has shown [**Lu3**, §2.1] that Nakajima's Lagrangian varieties for the geometric realization of $L(\nu)$ are isomorphic to the Grassmann varieties of Λ-submodules of q_ν with a given dimension vector.

Let x be a finite-dimensional nilpotent Λ-module isomorphic to a submodule of the injective module q_ν. Let us fix an embedding $F : x \to q_\nu$ and identify x with a submodule of q_ν via F.

Definition 1 *For $i \in I$ let $\mathcal{G}(x, \nu, i)$ be the variety of submodules y of q_ν containing x and such that y/x is isomorphic to s_i.*

This is a projective variety which, by 4.3, depends only (up to isomorphism) on i, ν and the isoclass of x.

5 Geometric realization of integrable irreducible $\mathfrak{g}$-modules

5.1 For $\lambda \in P_+$ and $\beta \in Q_+$, let Λ_β^λ denote the variety of nilpotent Λ-modules of dimension vector β which are isomorphic to a submodule of q_λ. Equivalently Λ_β^λ consists of the nilpotent modules of dimension vector β whose socle contains s_i with multiplicity at most $(\lambda\,;\,\alpha_i)$ $(i \in I)$. This variety has been considered by Lusztig [**Lu4**, §1.5]. In particular it is known that Λ_β^λ is an open subset of Λ_β, and that the number of its irreducible components is equal to the dimension of the $(\lambda - \beta)$-weight space of $L(\lambda)$.

5.2 Define $\widetilde{\mathcal{M}}_\beta^\lambda$ to be the vector space of constructible functions on Λ_β^λ which are constant on G_β-orbits. Let $\mathcal{M}_\beta^\lambda$ denote the subspace of $\widetilde{\mathcal{M}}_\beta^\lambda$ obtained by restricting elements of $\mathcal{M}_\beta$ to Λ_β^λ.

Put $\widetilde{\mathcal{M}}^\lambda = \bigoplus_\beta \widetilde{\mathcal{M}}^\lambda_\beta$ and $\mathcal{M}^\lambda = \bigoplus_\beta \mathcal{M}^\lambda_\beta$. For $i \in I$ define endomorphisms E_i, F_i, H_i of $\widetilde{\mathcal{M}}^\lambda$ as follows:

$$(E_i f)(x) = \int_{y \in \mathcal{G}(x,\lambda,i)} f(y), \qquad (f \in \widetilde{\mathcal{M}}^\lambda_\beta,\ x \in \Lambda^\lambda_{\beta-\alpha_i}), \tag{4}$$

$$(F_i f)(x) = \int_{y \in \mathcal{G}(i,x)} f(y), \qquad (f \in \widetilde{\mathcal{M}}^\lambda_\beta,\ x \in \Lambda^\lambda_{\beta+\alpha_i}), \tag{5}$$

$$(H_i f)(x) = (\lambda - \beta; \alpha_i)\, f(x), \qquad (f \in \widetilde{\mathcal{M}}^\lambda_\beta,\ x \in \Lambda^\lambda_\beta). \tag{6}$$

Theorem 1 *The endomorphisms E_i, F_i, H_i of $\widetilde{\mathcal{M}}^\lambda$ leave stable the subspace $\mathcal{M}^\lambda$. Denote again by E_i, F_i, H_i the induced endomorphisms of $\mathcal{M}^\lambda$. Then the assignments $e_i \mapsto E_i$, $f_i \mapsto F_i$, $h_i \mapsto H_i$, give a representation of $\mathfrak{g}$ on $\mathcal{M}^\lambda$ isomorphic to the irreducible representation $L(\lambda)$.*

5.3 The proof of Theorem 1 will involve a series of lemmas.

5.3.1 For $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ and $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{N}^r$, define the variety $\mathcal{G}(x, \lambda, (\mathbf{i}, \mathbf{a}))$ of flags of Λ-modules

$$\mathfrak{f} = (x = y_0 \subset y_1 \subset \cdots \subset y_r \subset q_\lambda)$$

with $y_k/y_{k-1} \cong s_{i_k}^{\oplus a_k}$ $(1 \leqslant k \leqslant r)$. As in Definition 1, this is a projective variety depending (up to isomorphism) only on $(\mathbf{i}, \mathbf{a})$, λ and the isoclass of x and not on the choice of a specific embedding of x into q_λ.

Lemma 2 *Let $f \in \widetilde{\mathcal{M}}^\lambda_\beta$ and $x \in \Lambda^\lambda_{\beta - a_1\alpha_{i_1} - \cdots - a_r\alpha_{i_r}}$. Put $E_i^{(a)} = (1/a!)E_i^a$. We have*

$$(E_{i_r}^{(a_r)} \cdots E_{i_1}^{(a_1)} f)(x) = \int_{\mathfrak{f} \in \mathcal{G}(x,\lambda,(\mathbf{i},\mathbf{a}))} f(y_r).$$

The proof is standard and will be omitted.

5.3.2 By [**Lu1**, 12.11] the endomorphisms F_i satisfy the Serre relations

$$\sum_{p=0}^{1-a_{ij}} (-1)^p\, F_j^{(p)}\, F_i\, F_j^{(1-a_{ij}-p)} = 0$$

for every $i \neq j$. A similar argument shows that

Lemma 3 *The endomorphisms E_i satisfy the Serre relations*

$$\sum_{p=0}^{1-a_{ij}} (-1)^p\, E_j^{(p)}\, E_i\, E_j^{(1-a_{ij}-p)} = 0$$

for every $i \neq j$.

Proof — Let $f \in \widetilde{\mathcal{M}}^\lambda_\beta$ and $x \in \Lambda^\lambda_{\beta-\alpha_i-(1-a_{ij})\alpha_j}$. By Lemma 2,

$$(E_j^{(p)}\, E_i\, E_j^{(1-a_{ij}-p)} f)(x) = \int_{\mathfrak{f}} f(y_3)$$

the integral being taken on the variety of flags

$$\mathfrak{f} = (x \subset y_1 \subset y_2 \subset y_3 \subset q_\lambda)$$

with $y_1/x \cong s_j^{\oplus 1-a_{ij}-p}$, $y_2/y_1 \cong s_i$ and $y_3/y_2 \cong s_j^{\oplus p}$. This integral can be rewritten as

$$\int_{y_3} f(y_3)\, \chi(\mathcal{F}[y_3;p])$$

where the integral is now over all submodules y_3 of q_λ of dimension β containing x and $\mathcal{F}[y_3;p]$ is the variety of flags $\mathfrak{f}$ as above with fixed last step y_3. Now, by moding out the submodule x at each step of the flag, we are reduced to the same situation as in [**Lu1**, 12.11], and the same argument allows to show that

$$\sum_{p=0}^{1-a_{ij}} \chi(\mathcal{F}[y_3;p]) = 0,$$

which proves the Lemma. □

5.3.3 Let $x \in \Lambda_\beta^\lambda$. Let $\varepsilon_i(x)$ denote the multiplicity of s_i in the head of x. Let $\varphi_i(x)$ denote the multiplicity of s_i in the socle of q_λ/x.

Lemma 4 *Let $i, j \in I$ (not necessarily distinct). Let y be a submodule of q_λ containing x and such that $y/x \cong s_j$. Then*

$$\varphi_i(y) - \varepsilon_i(y) = \varphi_i(x) - \varepsilon_i(x) - a_{ij}.$$

Proof — We have short exact sequences

$$0 \to x \to q_\lambda \to q_\lambda/x \to 0, \tag{7}$$
$$0 \to y \to q_\lambda \to q_\lambda/y \to 0, \tag{8}$$
$$0 \to x \to y \to s_j \to 0, \tag{9}$$
$$0 \to s_j \to q_\lambda/x \to q_\lambda/y \to 0. \tag{10}$$

Clearly, $\varepsilon_i(x) = |\mathrm{Hom}_\Lambda(x, s_i)|$, the dimension of $\mathrm{Hom}_\Lambda(x, s_i)$. Similarly $\varepsilon_i(y) = |\mathrm{Hom}_\Lambda(y, s_i)|$, $\varphi_i(x) = |\mathrm{Hom}_\Lambda(s_i, q_\lambda/x)|$, $\varphi_i(y) = |\mathrm{Hom}_\Lambda(s_i, q_\lambda/y)|$. Hence we have to show that

$$|\mathrm{Hom}_\Lambda(x, s_i)| - |\mathrm{Hom}_\Lambda(y, s_i)| = |\mathrm{Hom}_\Lambda(s_i, q_\lambda/x)| - |\mathrm{Hom}_\Lambda(s_i, q_\lambda/y)| - a_{ij}. \tag{11}$$

In our proof, we will use a property of preprojective algebras proved in [**CB**, §1], namely, for any finite-dimensional Λ-modules m and n there holds

$$|\mathrm{Ext}^1_\Lambda(m, n)| = |\mathrm{Ext}^1_\Lambda(n, m)|. \tag{12}$$

(a) If $i = j$ then $a_{ij} = 2$, $|\mathrm{Hom}_\Lambda(s_j, s_i)| = 1$ and $|\mathrm{Ext}^1_\Lambda(s_j, s_i)| = 0$ since Γ has no loops. Applying $\mathrm{Hom}_\Lambda(-, s_i)$ to (9) we get the exact sequence

$$0 \to \mathrm{Hom}_\Lambda(s_j, s_i) \to \mathrm{Hom}_\Lambda(y, s_i) \to \mathrm{Hom}_\Lambda(x, s_i) \to 0,$$

hence

$$|\mathrm{Hom}_\Lambda(x, s_i)| - |\mathrm{Hom}_\Lambda(y, s_i)| = -1.$$

Similarly applying $\mathrm{Hom}_\Lambda(s_i, -)$ to (10) we get an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(s_i, s_j) \to \mathrm{Hom}_\Lambda(s_i, q_\lambda/x) \to \mathrm{Hom}_\Lambda(s_i, q_\lambda/y) \to 0,$$

hence

$$|\mathrm{Hom}_\Lambda(s_i, q_\lambda/x)| - |\mathrm{Hom}_\Lambda(s_i, q_\lambda/y)| = 1,$$

and (11) follows.

(b) If $i \neq j$, we have $|\mathrm{Hom}_\Lambda(s_i, s_j)| = 0$ and $|\mathrm{Ext}^1_\Lambda(s_i, s_j)| = |\mathrm{Ext}^1_\Lambda(s_j, s_i)| = -a_{ij}$. Applying $\mathrm{Hom}_\Lambda(s_i, -)$ to (9) we get an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(s_i, x) \to \mathrm{Hom}_\Lambda(s_i, y) \to 0,$$

hence

$$|\mathrm{Hom}_\Lambda(s_i, x)| - |\mathrm{Hom}_\Lambda(s_i, y)| = 0. \tag{13}$$

Moreover, by [**Bo**, §1.1], $|\mathrm{Ext}^2_\Lambda(s_i, s_j)| = 0$ because there are no relations from i to j in the defining relations of Λ. (Note that the proof of this result in [**Bo**] only requires that $I \subseteq J^2$ (here we use the notation of [**Bo**]). One does not need the additional assumption $J^n \subseteq I$ for some n. Compare also the discussion in [**BK**].)

Since q_λ is injective $|\mathrm{Ext}^1_\Lambda(s_i, q_\lambda)| = 0$, thus applying $\mathrm{Hom}_\Lambda(s_i, -)$ to (7) we get an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(s_i, x) \to \mathrm{Hom}_\Lambda(s_i, q_\lambda) \to \mathrm{Hom}_\Lambda(s_i, q_\lambda/x) \to \mathrm{Ext}^1_\Lambda(s_i, x) \to 0,$$

hence

$$|\mathrm{Hom}_\Lambda(s_i, x)| - |\mathrm{Hom}_\Lambda(s_i, q_\lambda)| + |\mathrm{Hom}_\Lambda(s_i, q_\lambda/x)| - |\mathrm{Ext}^1_\Lambda(s_i, x)| = 0. \tag{14}$$

Similarly, applying $\mathrm{Hom}_\Lambda(s_i, -)$ to (8) we get

$$|\mathrm{Hom}_\Lambda(s_i, y)| - |\mathrm{Hom}_\Lambda(s_i, q_\lambda)| + |\mathrm{Hom}_\Lambda(s_i, q_\lambda/y)| - |\mathrm{Ext}^1_\Lambda(s_i, y)| = 0. \tag{15}$$

Subtracting (14) from (15) and taking into account (12) and (13) we obtain

$$|\mathrm{Ext}^1_\Lambda(x, s_i)| - |\mathrm{Ext}^1_\Lambda(y, s_i)| = |\mathrm{Hom}_\Lambda(s_i, q_\lambda/x)| - |\mathrm{Hom}_\Lambda(s_i, q_\lambda/y)|. \tag{16}$$

Now applying $\mathrm{Hom}_\Lambda(-, s_i)$ to (9) we get the long exact sequence

$$0 \to \mathrm{Hom}_\Lambda(y, s_i) \to \mathrm{Hom}_\Lambda(x, s_i) \to \mathrm{Ext}^1_\Lambda(s_j, s_i) \to \mathrm{Ext}^1_\Lambda(y, s_i) \to \mathrm{Ext}^1_\Lambda(x, s_i) \to 0,$$

hence

$$|\mathrm{Hom}_\Lambda(y, s_i)| - |\mathrm{Hom}_\Lambda(x, s_i)| - a_{ij} - |\mathrm{Ext}^1_\Lambda(y, s_i)| + |\mathrm{Ext}^1_\Lambda(x, s_i)| = 0,$$

thus, taking into account (16), we have proved (11). □

Lemma 5 *With the same notation we have*

$$\varphi_i(x) - \varepsilon_i(x) = (\lambda - \beta; \alpha_i).$$

Proof — We use an induction on the height of β. If $\beta = 0$ then x is the zero module and $\varepsilon_i(x) = 0$. On the other hand $q_\lambda/x = q_\lambda$ and $\varphi_i(x) = (\lambda; \alpha_i)$ by definition of q_λ. Now assume that the lemma holds for $x \in \Lambda^\lambda_\beta$ and let $y \in \Lambda^\lambda_{\beta+\alpha_j}$ be a submodule of q_λ containing x. Using Lemma 4 we get that

$$\varphi_i(y) - \varepsilon_i(y) = (\lambda - \beta; \alpha_i) - a_{ij} = (\lambda - \beta - \alpha_j; \alpha_i),$$

as required, and the lemma follows. □

Lemma 6 *Let $f \in \widetilde{\mathcal{M}}^\lambda_\beta$. We have*

$$(E_iF_j - F_jE_i)(f) = \delta_{ij}(\lambda - \beta; \alpha_i)f.$$

Proof — Let $x \in \Lambda^\lambda_{\beta-\alpha_i+\alpha_j}$. By definition of E_i and F_j we have

$$(E_iF_jf)(x) = \int_{\mathfrak{p}\in\mathfrak{P}} f(y)$$

where $\mathfrak{P}$ denotes the variety of pairs $\mathfrak{p} = (u, y)$ of submodules of q_λ with $x \subset u$, $y \subset u$, $u/x \cong s_i$ and $u/y \cong s_j$. Similarly,

$$(F_jE_if)(x) = \int_{\mathfrak{q}\in\mathfrak{Q}} f(y)$$

where $\mathfrak{Q}$ denotes the variety of pairs $\mathfrak{q} = (v, y)$ of submodules of q_λ with $v \subset x$, $v \subset y$, $x/v \cong s_j$ and $y/v \cong s_i$.

Consider a submodule y such that there exists in $\mathfrak{P}$ (*resp.* in $\mathfrak{Q}$) at least one pair of the form (u, y) (*resp.* (v, y)). Clearly, the subspaces carrying the submodules x and y have the same dimension d and their intersection has dimension at least $d - 1$. If this intersection has dimension exactly $d - 1$ then there is a unique pair (u, y) (*resp.* (v, y)), namely $(x + y, y)$ (*resp.* $(x \cap y, y)$). This means that

$$\int_{\mathfrak{p}\in\mathfrak{P};\ y\neq x} f(y) = \int_{\mathfrak{q}\in\mathfrak{Q};\ y\neq x} f(y).$$

In particular, since when $i \neq j$ we cannot have $y = x$, it follows that

$$(E_iF_j - F_jE_i)(f) = 0, \qquad (i \neq j).$$

On the other hand if $i = j$ we have

$$((E_iF_i - F_iE_i)(f))(x) = f(x)(\chi(\mathfrak{P}') - \chi(\mathfrak{Q}'))$$

where $\mathfrak{P}'$ is the variety of submodules u of q_λ containing x such that $u/x \cong s_i$, and $\mathfrak{Q}'$ is the variety of submodules v of x such that $x/v \cong s_i$. Clearly we have $\chi(\mathfrak{Q}') = \varepsilon_i(x)$ and $\chi(\mathfrak{P}') = \varphi_i(x)$. The result then follows from Lemma 5. □

5.3.4 The following relations for the endomorphisms E_i, F_i, H_i of $\widetilde{\mathcal{M}}^\lambda$ are easily checked

$$[H_i, H_j] = 0, \quad [H_i, E_j] = a_{ij}E_j, \quad [H_i, F_j] = -a_{ij}F_j.$$

The verification is left to the reader. Hence, using Lemmas 3 and 6, we have proved that the assignments $e_i \mapsto E_i$, $f_i \mapsto F_i$, $h_i \mapsto H_i$, give a representation of $\mathfrak{g}$ on $\widetilde{\mathcal{M}}^\lambda$.

Lemma 7 *The endomorphisms E_i, F_i, H_i leave stable the subspace $\mathcal{M}^\lambda$.*

Proof — It is obvious for H_i, and it follows from the definition of $\mathcal{M}^\lambda$ for F_i. It remains to prove that if $f \in \mathcal{M}^\lambda_\beta$ then $E_i f \in \mathcal{M}^\lambda_{\beta-\alpha_i}$. We shall use induction on the height of β. We can assume that f is of the form $F_j g$ for some $g \in \mathcal{M}^\lambda_{\beta-\alpha_j}$. By induction we can also assume that $E_i g \in \mathcal{M}^\lambda_{\beta-\alpha_i-\alpha_j}$. We have

$$E_i f = E_i F_j g = F_j E_i g + \delta_{ij}(\lambda - \beta + \alpha_j ; \alpha_i) g,$$

and the right-hand side clearly belongs to $\mathcal{M}^\lambda_{\beta-\alpha_i}$. □

Lemma 8 *The representation of $\mathfrak{g}$ carried by $\mathcal{M}^\lambda$ is isomorphic to $L(\lambda)$.*

Proof — For all $f \in \mathcal{M}_\beta$ and all $x \in \Lambda^\lambda_{\beta+(a_i+1)\alpha_i}$ we have $f * \mathbf{1}_i^{*(a_i+1)}(x) = 0$. Indeed, by definition of Λ^λ the socle of x contains s_i with multiplicity at most a_i. Therefore the left ideal of $\mathcal{M}$ generated by the functions $\mathbf{1}_i^{*(a_i+1)}$ is mapped to zero by the linear map $\mathcal{M} \to \mathcal{M}^\lambda$ sending a function f on Λ_β to its restriction to Λ^λ_β. It follows that for all β the dimension of $\mathcal{M}^\lambda_\beta$ is at most the dimension of the $(\lambda - \beta)$-weight space of $L(\lambda)$.

On the other hand, the function $\mathbf{1}_0$ mapping the zero Λ-module to 1 is a highest weight vector of $\mathcal{M}^\lambda$ of weight λ. Hence $\mathbf{1}_0 \in \mathcal{M}^\lambda$ generates a quotient of the Verma module $M(\lambda)$, and since $L(\lambda)$ is the smallest quotient of $M(\lambda)$ we must have $\mathcal{M}^\lambda = L(\lambda)$. □

This finishes the proof of Theorem 1.

6 Geometric realization of Verma modules

6.1 Let $\beta \in Q_+$ and $x \in \Lambda_{\beta-\alpha_i}$. Let $q = \bigoplus_{i\in I} q_i^{\oplus a_i}$ be the injective hull of x. For every $\nu \in P_+$ such that $(\nu;\alpha_i) \geqslant a_i$ the injective module q_ν contains a submodule isomorphic to x. Hence, for such a weight ν and for any $f \in \mathcal{M}_\beta$, the integral

$$\int_{y\in\mathcal{G}(x,\nu,i)} f(y)$$

is well-defined.

Proposition 1 *Let $\lambda \in P$ and choose $\nu \in P_+$ such that $(\nu;\alpha_i) \geqslant a_i$ for all $i \in I$. The number*

$$\int_{y\in\mathcal{G}(x,\nu,i)} f(y) \;-\; (\nu - \lambda\,;\,\alpha_i)\, f(x \oplus s_i) \tag{17}$$

does not depend on the choice of ν. Denote this number by $(E_i^\lambda f)(x)$. Then, the function

$$E_i^\lambda f : x \mapsto (E_i^\lambda f)(x)$$

belongs to $\mathcal{M}_{\beta-\alpha_i}$.

Denote by E_i^λ the endomorphism of $\mathcal{M}$ mapping $f \in \mathcal{M}_\beta$ to $E_i^\lambda f$. Notice that Formula (5), which is nothing but (3), also defines an endomorphism of $\mathcal{M}$ independent of λ which we again denote by F_i. Finally Formula (6) makes sense for any λ, not necessarily dominant, and any $f \in \mathcal{M}_\beta$. This gives an endomorphism of $\mathcal{M}$ that we shall denote by H_i^λ.

Theorem 2 *The assignments $e_i \mapsto E_i^\lambda$, $f_i \mapsto F_i$, $h_i \mapsto H_i^\lambda$, give a representation of $\mathfrak{g}$ on $\mathcal{M}$ isomorphic to the Verma module $M(\lambda)$.*

The rest of this section is devoted to the proofs of Proposition 1 and Theorem 2.

6.2 Denote by e_i^λ the endomorphism of the Verma module $M(\lambda)$ implementing the action of the Chevalley generator e_i. Let $\mathcal{E}_i^\lambda$ denote the endomorphism of $U(\mathfrak{n}_-)$ obtained by transporting e_i^λ via the natural identification $M(\lambda) \cong U(\mathfrak{n}_-)$. Let Δ be the comultiplication of $U(\mathfrak{n}_-)$.

Lemma 9 *For $\lambda, \mu \in P$ and $u \in U(\mathfrak{n}_-)$ we have*

$$\Delta(\mathcal{E}_i^{\lambda+\mu} u) = (\mathcal{E}_i^\lambda \otimes 1 + 1 \otimes \mathcal{E}_i^\mu)\Delta u.$$

Proof — By linearity it is enough to prove this for u of the form $u = f_{i_1} \cdots f_{i_r}$. A simple calculation in $U(\mathfrak{g})$ shows that

$$e_i f_{i_1} \cdots f_{i_r} = f_{i_1} \cdots f_{i_r} e_i + \sum_{k=1}^{r} \delta_{i i_k} f_{i_1} \cdots f_{i_{k-1}} h_i f_{i_{k+1}} \cdots f_{i_r}$$

$$= f_{i_1} \cdots f_{i_r} e_i + \sum_{k=1}^{r} \delta_{i i_k} \left(f_{i_1} \cdots f_{i_{k-1}} f_{i_{k+1}} \cdots f_{i_r} h_i - \left(\sum_{s=k+1}^{r} a_{i i_s} \right) f_{i_1} \cdots f_{i_{k-1}} f_{i_{k+1}} \cdots f_{i_r} \right).$$

It follows that, for $\nu \in P$,

$$\mathcal{E}_i^\nu (f_{i_1} \cdots f_{i_r}) = \sum_{k=1}^{r} \delta_{i i_k} \left((\nu; \alpha_i) - \sum_{s=k+1}^{r} a_{i i_s} \right) f_{i_1} \cdots f_{i_{k-1}} f_{i_{k+1}} \cdots f_{i_r}.$$

Now, using that Δ is the algebra homomorphism defined by $\Delta(f_i) = f_i \otimes 1 + 1 \otimes f_i$, one can finish the proof of the lemma. Details are omitted. □

6.3 We endow $U(\mathfrak{n}_-)$ with the Q_+-grading given by $\deg(f_i) = \alpha_i$. Let u be a homogeneous element of $U(\mathfrak{n}_-)$. Write $\Delta u = u \otimes 1 + u^{(i)} \otimes f_i + A$, where A is a sum of homogeneous terms of the form $u' \otimes u''$ with $\deg(u'') \neq \alpha_i$. This defines $u^{(i)}$ unambiguously.

Lemma 10 *For $\lambda, \mu \in P$ we have*

$$\mathcal{E}_i^{\lambda+\mu} u = \mathcal{E}_i^\lambda u + (\mu; \alpha_i)\, u^{(i)}.$$

Proof — We calculate in two ways the unique term of the form $E \otimes 1$ in $\Delta(\mathcal{E}_i^{\lambda+\mu} u)$. On the one hand, we have obviously $E \otimes 1 = \mathcal{E}_i^{\lambda+\mu} u \otimes 1$. On the other hand, using Lemma 9, we have

$$E \otimes 1 = \mathcal{E}_i^\lambda u \otimes 1 + (1 \otimes \mathcal{E}_i^\mu)(u^{(i)} \otimes f_i) = \mathcal{E}_i^\lambda u \otimes 1 + (\mu; \alpha_i)\, u^{(i)} \otimes 1.$$

Therefore,

$$E = \mathcal{E}_i^{\lambda+\mu} u = \mathcal{E}_i^\lambda u + (\mu; \alpha_i)\, u^{(i)}.$$

□

6.4 Now let us return to the geometric realization $\mathcal{M}$ of $U(\mathfrak{n}_-)$. Let E_i^λ denote the endomorphism of $\mathcal{M}$ obtained by transporting e_i^λ via the identification $M(\lambda) \cong \mathcal{M}$.

Lemma 11 *Let $\lambda \in P_+$, $f \in \mathcal{M}_\beta$ and $x \in \Lambda^\lambda_{\beta-\alpha_i}$. Then*

$$(E_i^\lambda f)(x) = \int_{y \in \mathcal{G}(x,\lambda,i)} f(y).$$

Proof — Let $r_\lambda : \mathcal{M} \to \mathcal{M}^\lambda$ be the linear map sending $f \in \mathcal{M}_\beta$ to its restriction to Λ^λ_β. By Theorem 1, this is a homomorphism of $U(\mathfrak{n}_-)$-modules mapping the highest weight vector of $\mathcal{M} \cong M(\lambda)$ to the highest weight vector of $\mathcal{M}^\lambda \cong L(\lambda)$. It follows that r_λ is in fact a homomorphism of $U(\mathfrak{g})$-modules, hence the restriction of $E_i^\lambda f$ to $\Lambda^\lambda_{\beta-\alpha_i}$ is given by Formula (4) of Section 5. □

Let again $\lambda \in P$ be arbitrary, and pick $f \in \mathcal{M}_\beta$. It follows from Lemma 10 that for any $\mu \in P$

$$E_i^{\lambda+\mu} f - (\mu; \alpha_i)\, f^{(i)} = E_i^\lambda f.$$

Let $x \in \Lambda_{\beta-\alpha_i}$. Choose $\nu = \lambda + \mu$ sufficiently dominant so that x is isomorphic to a submodule of q_ν. Then by Lemma 11, we have

$$(E_i^\nu f)(x) = \int_{y \in \mathcal{G}(x,\nu,i)} f(y).$$

On the other hand, by the geometric description of Δ given in [**GLS**, §6.1], if we write

$$\Delta f = f \otimes 1 + f^{(i)} \otimes \mathbf{1}_i + A$$

where A is a sum of homogeneous terms of the form $f' \otimes f''$ with $\deg(f'') \neq \alpha_i$, we have that $f^{(i)}$ is the function on $\Lambda_{\beta-\alpha_i}$ given by $f^{(i)}(x) = f(x \oplus s_i)$. Hence we obtain that for $x \in \Lambda_{\beta-\alpha_i}$

$$(E_i^\lambda f)(x) = \int_{y \in \mathcal{G}(x,\nu,i)} f(y) \; - \; (\nu - \lambda; \alpha_i) f(x \oplus s_i).$$

This proves both Proposition 1 and Theorem 2. □

6.5 Let $\lambda \in P_+$. We note the following consequence of Lemma 11.

Proposition 2 *Let $\lambda \in P_+$. The linear map $r_\lambda : \mathcal{M} \to \mathcal{M}^\lambda$ sending $f \in \mathcal{M}_\beta$ to its restriction to Λ^λ_β is the geometric realization of the homomorphism of $\mathfrak{g}$-modules $M(\lambda) \to L(\lambda)$.* □

7 Dual Verma modules

7.1 Let S be the anti-automorphism of $U(\mathfrak{g})$ defined by

$$S(e_i) = f_i, \quad S(f_i) = e_i, \quad S(h_i) = h_i, \qquad (i \in I).$$

Recall that, given a left $U(\mathfrak{g})$-module M, the dual module M^* is defined by

$$(u\,\varphi)(m) = \varphi(S(u)\,m), \qquad (u \in U(\mathfrak{g}),\ m \in M,\ \varphi \in M^*).$$

This is also a left module. If M is an infinite-dimensional module with finite-dimensional weight spaces M_ν, we take for M^* the graded dual $M^* = \bigoplus_{\nu \in P} M^*_\nu$.

For $\lambda \in P$ we have $L(\lambda)^* \cong L(\lambda)$, hence the quotient map $M(\lambda) \to L(\lambda)$ gives by duality an embedding $L(\lambda) \to M(\lambda)^*$ of $U(\mathfrak{g})$-modules.

7.2 Let $\mathcal{M}^* = \bigoplus_{\beta\in Q_+} \mathcal{M}^*_\beta$ denote the vector space graded dual of $\mathcal{M}$. For $x \in \Lambda_\beta$, we denote by δ_x the delta function given by

$$\delta_x(f) = f(x), \qquad (f \in \mathcal{M}_\beta).$$

Note that the map $\delta : x \mapsto \delta_x$ is a constructible map from Λ_β to $\mathcal{M}^*_\beta$. Indeed the preimage of δ_x is the intersection of the constructible subsets

$$\mathcal{M}_{(i_1,\ldots,i_r)} = \{y \in \Lambda_\beta \mid (\mathbf{1}_{i_1} * \cdots * \mathbf{1}_{i_r})(y) = (\mathbf{1}_{i_1} * \cdots * \mathbf{1}_{i_r})(x)\}, \quad (\alpha_{i_1} + \cdots + \alpha_{i_r} = \beta).$$

7.3 We can now dualize the results of Sections 5 and 6 as follows. For $\lambda \in P$ and $x \in \Lambda_\beta$ put

$$(E_i^*)(\delta_x) = \int_{y\in\mathcal{G}(i,x)} \delta_y, \tag{18}$$

$$(F_i^{\lambda *})(\delta_x) = \int_{y\in\mathcal{G}(x,\nu,i)} \delta_y - (\nu-\lambda\,;\,\alpha_i)\,\delta_{x\oplus s_i}, \tag{19}$$

$$(H_i^{\lambda *})(\delta_x) = (\lambda-\beta;\alpha_i)\,\delta_x, \tag{20}$$

where in (19) the weight $\nu \in P_+$ is such that x is isomorphic to a submodule of q_ν. The following theorem then follows immediately from Theorems 1 and 2.

Theorem 3 (i) *The formulas above define endomorphisms $E_i^*, F_i^{\lambda *}, H_i^{\lambda *}$ of $\mathcal{M}^*$, and the assignments $e_i \mapsto E_i^*$, $f_i \mapsto F_i^{\lambda *}$, $h_i \mapsto H_i^{\lambda *}$, give a representation of $\mathfrak{g}$ on $\mathcal{M}^*$ isomorphic to the dual Verma module $M(\lambda)^*$.*

(ii) *If $\lambda \in P_+$, the subspace $\mathcal{M}^{\lambda *}$ of $\mathcal{M}^*$ spanned by the delta functions δ_x of the finite-dimensional nilpotent submodules x of q_λ carries the irreducible submodule $L(\lambda)$. For such a module x, Formula (19) simplifies as follows*

$$(F_i^{\lambda *})(\delta_x) = \int_{y\in\mathcal{G}(x,\lambda,i)} \delta_y\,.$$

□

Example 1 Let $\mathfrak{g}$ be of type A_2. Take $\lambda = \varpi_1 + \varpi_2$, where ϖ_i is the fundamental weight corresponding to $i \in I$. Thus $L(\lambda)$ is isomorphic to the 8-dimensional adjoint representation of $\mathfrak{g} = \mathfrak{sl}_3$.

A Λ-module x consists of a pair of linear maps $x_{21} : V_1 \to V_2$ and $x_{12} : V_2 \to V_1$ such that $x_{12}x_{21} = x_{21}x_{12} = 0$. The injective Λ-module $q = q_\lambda$ has the following form :

$$q = \begin{pmatrix} u_1 \longrightarrow u_2 \\ v_1 \longleftarrow v_2 \end{pmatrix}$$

This diagram means that (u_1, v_1) is a basis of V_1, that (u_2, v_2) is a basis of V_2, and that

$$q_{21}(u_1) = u_2, \quad q_{21}(v_1) = 0, \quad q_{12}(v_2) = v_1, \quad q_{12}(u_2) = 0.$$

Using the same type of notation, we can exhibit the following submodules of q :

$$x_1 = \big(v_1\big)\,, \quad x_2 = \big(u_2\big)\,, \quad x_3 = \big(v_1 \qquad u_2\big)\,, \quad x_4 = \big(u_1 \longrightarrow u_2\big)\,, \quad x_5 = \big(v_1 \longleftarrow v_2\big)\,,$$

$$x_6 = \begin{pmatrix} u_1 \longrightarrow u_2 \\ v_1 \end{pmatrix}, \qquad x_7 = \begin{pmatrix} & u_2 \\ v_1 \longleftarrow & v_2 \end{pmatrix}.$$

This is not an exhaustive list. For example, $x_4' = \big((u_1 + v_1) \longrightarrow u_2\big)$ is another submodule, isomorphic to x_4. Denoting by **0** the zero submodule, we see that $\delta_{\mathbf{0}}$ is the highest weight vector of $L(\lambda) \subset M(\lambda)^*$. Next, writing for simplicity δ_i instead of δ_{x_i} and F_i instead of F_i^λ, Theorem 3 (ii) gives the following formulas for the action of the F_i's on $L(\lambda)$.

$$F_1\delta_{\mathbf{0}} = \delta_1, \quad F_2\delta_{\mathbf{0}} = \delta_2, \quad F_1\delta_2 = \delta_3 + \delta_4, \quad F_2\delta_1 = \delta_3 + \delta_5,$$

$$F_1\delta_3 = F_1\delta_4 = \delta_6, \quad F_2\delta_3 = F_2\delta_5 = \delta_7, \quad F_2\delta_3 = F_1\delta_6 = \delta_q, \quad F_1\delta_q = F_2\delta_q = 0.$$

Now consider the Λ-module $x = s_1 \oplus s_1$. Since x is not isomorphic to a submodule of q_λ, the vector δ_x does not belong to $L(\lambda)$. Let us calculate $F_i\delta_x$ $(i = 1, 2)$ by means of Formula (19). We can take $\nu = 2\varpi_1$. The injective Λ-module q_ν has the following form :

$$q_\nu = \begin{pmatrix} w_1 \longleftarrow w_2 \\ v_1 \longleftarrow v_2 \end{pmatrix}$$

It is easy to see that the variety $\mathcal{G}(x, \nu, 2)$ is isomorphic to a projective line $\mathbb{P}_1$, and that all points on this line are isomorphic to

$$y = \begin{pmatrix} w_1 & \\ v_1 \longleftarrow & v_2 \end{pmatrix}$$

as Λ-modules. Hence,

$$F_2\delta_x = \chi(\mathbb{P}_1)\, \delta_y - (\nu - \lambda; \alpha_2)\, \delta_{x\oplus s_2} = 2\, \delta_y + \delta_{s_1\oplus s_1\oplus s_2}.$$

On the other hand, $\mathcal{G}(x, \nu, 1) = \emptyset$, so that

$$F_1\delta_x = -(\nu - \lambda; \alpha_1)\, \delta_{x\oplus s_1} = -\delta_{s_1\oplus s_1\oplus s_1}.$$

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