RIGID MODULES OVER PREPROJECTIVE ALGEBRAS

CHRISTOF GEISS, BERNARD LECLERC, AND JAN SCHRÖER

Abstract. Let Λ be a preprojective algebra of simply laced Dynkin type Δ. We study maximal rigid Λ-modules, their endomorphism algebras and a mutation operation on these modules. This leads to a representation-theoretic construction of the cluster algebra structure on the ring $\mathbb{C}[N]$ of polynomial functions on a maximal unipotent subgroup N of a complex Lie group of type Δ. As an application we obtain that all cluster monomials of $\mathbb{C}[N]$ belong to the dual semicanonical basis.

Contents

1. Introduction

1.1. Preprojective algebras were introduced by Gelfand and Ponomarev in 1979 [19], and since then played an important role in representation theory and Lie theory. For example, let $U(\mathfrak{n})$ be the enveloping algebra of a maximal nilpotent subalgebra $\mathfrak{n}$ of a simple Lie algebra $\mathfrak{g}$ of type $\mathbb{A}, \mathbb{D}, \mathbb{E}$, and let Λ denote the preprojective algebra associated to the Dynkin diagram of $\mathfrak{g}$. In [30], [32], Lusztig has given a geometric construction of $U(\mathfrak{n})$ as an algebra $(\mathcal{M}, *)$ of constructible functions on varieties of finite-dimensional Λ-modules, where $*$ is a convolution product inspired by Ringel's multiplication for Hall algebras [37]. This yields a new basis $\mathcal{S}$ of $U(\mathfrak{n})$ given by the irreducible components of these varieties of modules, called the *semicanonical basis*.

1.2. Cluster algebras were introduced by Fomin and Zelevinsky [13] to provide, among other things, an algebraic and combinatorial framework for the study of canonical bases and of total positivity. Of particular interest is the algebra $\mathbb{C}[N]$ of polynomial functions on a unipotent group N with Lie algebra $\mathfrak{n}$. One can identify $\mathbb{C}[N]$ with the graded dual $U(\mathfrak{n})^*$ and thus think of the dual $\mathcal{S}^*$ of $\mathcal{S}$ as a basis of $\mathbb{C}[N]$. We call $\mathcal{S}^*$ the *dual semicanonical*

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basis. It follows from [4] that $\mathbb{C}[N]$ has a natural (upper) cluster algebra structure. This consists of a distinguished family of regular functions on N, grouped into subsets called clusters. It is generated inductively from an initial cluster defined in a combinatorial way. In type $\mathbb{A}_n$ ($n \leq 4$), there are finitely many clusters and their elements, the cluster variables, can all be described explicitly. Moreover the set of all cluster monomials (that is, monomials in the cluster variables supported on a single cluster) coincides with $\mathcal{S}^*$. In general, though, there are infinitely many clusters and cluster variables in $\mathbb{C}[N]$, and very little is known about them.

1.3. To a finite-dimensional Λ-module M one can attach the linear form $\delta_M \in \mathcal{M}^*$ which maps a constructible function $f \in \mathcal{M}$ to its evaluation $f(M)$ at M. Under the isomorphism $\mathcal{M}^* \cong U(\mathfrak{n})^* \cong \mathbb{C}[N]$, δ_M gets identified to a regular function $\varphi_M \in \mathbb{C}[N]$. Thus, to study special elements of $\mathbb{C}[N]$, like cluster monomials, we may try to lift them to $\mathrm{mod}(\Lambda)$ via the map $M \mapsto \varphi_M$. For example, by construction, the element of $\mathcal{S}^*$ attached to an irreducible component Z of a variety of Λ-modules is equal to φ_M, where M is a "generic module" in Z.

1.4. A Λ-module M is called *rigid* provided $\mathrm{Ext}^1_\Lambda(M, M) = 0$. We characterize rigid Λ-modules as the modules having an open orbit in the corresponding module variety. In particular, the closure of such an orbit is an irreducible component.

In the first part of this paper, we show that the endomorphism algebras of rigid modules have astonishing properties which we believe are interesting in themselves. In particular, maximal rigid modules are examples of maximal 1-orthogonal modules, which play a role in the higher dimensional Auslander-Reiten theory recently developed by Iyama [25], [26]. This yields a direct link between preprojective algebras and classical tilting theory.

In the second part, we use these results to show that the operation of *mutation* involved in the cluster algebra structure of $\mathbb{C}[N]$ can be entirely understood in terms of maximal rigid Λ-modules. More precisely we define a mutation operation on maximal rigid modules, and, taking into account the results of [15], [16], [17], we prove that this gives a lifting of the cluster structure to the category $\mathrm{mod}(\Lambda)$. In particular, this implies that all cluster monomials of $\mathbb{C}[N]$ belong to $\mathcal{S}^*$. This theorem establishes for the first time a bridge between Lusztig's geometric construction of canonical bases and Fomin and Zelevinsky's approach to this topic via cluster algebras.

1.5. Our way of understanding the cluster algebra $\mathbb{C}[N]$ via the category $\mathrm{mod}(\Lambda)$ is very similar to the approach of [6], [7], [8], [10], [11], [28] which study cluster algebras attached to quivers via some new *cluster categories*. There are however two main differences.

First, the theory of cluster categories relies on the well developed representation theory of hereditary algebras, and covers only a special class of cluster algebras called *acyclic*. With the exception of Dynkin types $\mathbb{A}_n$ with $n \leq 4$, the cluster algebras $\mathbb{C}[N]$ are not believed to belong to this class. This indicates that hereditary algebras cannot be used in this context. In contrast, we use the preprojective algebras which have infinite global dimension and whose representation theory is much less developed.

Secondly, cluster categories are not abelian categories but only triangulated categories, defined as orbit categories of derived categories of representations of quivers. In our approach, we just use the concrete abelian category $\mathrm{mod}(\Lambda)$.

2. Main results

2.1. Let A be a finite-dimensional algebra over an algebraically closed field K. By $\mathrm{mod}(A)$ we denote the category of finite-dimensional left A-modules. If not mentioned otherwise, modules are assumed to be left modules. In this article we only consider finite-dimensional modules. We often do not distinguish between a module and its isomorphism class. For an A-module M let $\mathrm{add}(M)$ be the full subcategory of $\mathrm{mod}(A)$ formed by all modules isomorphic to direct summands of finite direct sums of copies of M. The opposite algebra of A is denoted by A^{op}. Let

$$\mathrm{D} = \mathrm{Hom}_K(-, K)\colon \mathrm{mod}(A) \to \mathrm{mod}(A^{\mathrm{op}})$$

be the usual duality functor.

If $f\colon U \to V$ and $g\colon V \to W$ are morphisms, then the composition is denoted by $gf\colon U \to W$. With this convention concerning compositions of homomorphisms we get that $\mathrm{Hom}_A(M, N)$ is a left $\mathrm{End}_A(N)$-module and a right $\mathrm{End}_A(M)$-module.

For natural numbers $a \leq b$ let $[a, b] = \{i \in \mathbb{N} \mid a \leq i \leq b\}$.

2.2. An A-module T is called *rigid* if $\mathrm{Ext}^1_A(T, T) = 0$. A rigid module T is *maximal* if any indecomposable A-module T' such that $T \oplus T'$ is rigid, is isomorphic to a direct summand of T (in other words, $T' \in \mathrm{add}(T)$).

2.3. Throughout the article let $Q = (Q_0, Q_1, s, t)$ be a Dynkin quiver of simply laced type

$$\Delta \in \{\mathbb{A}_n\ (n \geq 2),\ \mathbb{D}_n\ (n \geq 4),\ \mathbb{E}_n\ (n = 6, 7, 8)\}.$$

Thus Q is given by a simply laced Dynkin diagram together with an arbitrary orientation on the edges. Here Q_0 and Q_1 denote the set of vertices and arrows of Q, respectively. For an arrow $\alpha\colon i \to j$ in Q let $s(\alpha) = i$ and $t(\alpha) = j$ be its starting and terminal vertex, respectively. By n we always denote the number of vertices of Q. Note that we exclude the trivial case $\mathbb{A}_1$.

Let $\overline{Q}$ be the *double quiver* of Q, which is obtained from Q by adding an arrow $\alpha^*\colon j \to i$ whenever there is an arrow $\alpha\colon i \to j$ in Q. The *preprojective algebra* associated to Q is defined as

$$\Lambda = \Lambda_Q = K\overline{Q}/(c)$$

where (c) is the ideal generated by the element

$$c = \sum_{\alpha \in Q_1} (\alpha^*\alpha - \alpha\alpha^*),$$

and $K\overline{Q}$ is the path algebra associated to $\overline{Q}$, see [38]. Since Q is a Dynkin quiver it follows that Λ is a finite-dimensional selfinjective algebra. One can easily show that Λ does not depend on the orientation of Q. More precisely, if Q and Q' are Dynkin quivers of the same Dynkin type Δ, then Λ_Q and $\Lambda_{Q'}$ are isomorphic algebras.

Let r be the set of positive roots of Q, or equivalently, let r be the number of isomorphism classes of indecomposable representations of Q, compare [14]. Here are all possible values for r:

Q	$\mathbb{A}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
r	$\frac{n(n+1)}{2}$	$n^2 - n$	36	63	120

For a module M let $\Sigma(M)$ be the number of isomorphism classes of indecomposable direct summands of M. The following theorem is proved in [18]:

Theorem 2.1. *For any rigid Λ-module T we have $\Sigma(T) \leq r$.*

We call a rigid Λ-module T *complete* if $\Sigma(T) = r$. It follows from the definitions and from Theorem 2.1 that any complete rigid module is also maximal rigid. In Theorem 2.2 we will show the converse, namely that $\Sigma(T) = r$ for any maximal rigid module T .

2.4. Let A be a finite-dimensional algebra. Following Iyama we call an additive full subcategory $\mathcal{T}$ of $\mathrm{mod}(A)$ *maximal 1-orthogonal* if for every A-module M the following are equivalent:

- $M \in \mathcal{T}$;
- $\mathrm{Ext}^1_A(M, T) = 0$ for all $T \in \mathcal{T}$;
- $\mathrm{Ext}^1_A(T, M) = 0$ for all $T \in \mathcal{T}$.

An A-module T is called *maximal 1-orthogonal* if $\mathrm{add}(T)$ is maximal 1-orthogonal.

An A-module C is a *generator* (resp. *cogenerator*) of $\mathrm{mod}(A)$ if for every A-module M there exists some $m \geq 1$ and an epimorphism $C^m \to M$ (resp. a monomorphism $M \to C^m$). One calls C a *generator-cogenerator* if it is both a generator and a cogenerator. It follows that C is a generator (resp. cogenerator) if and only if all indecomposable projective (resp. injective) A-modules occur as direct summands of C, up to isomorphism.

It follows from the definitions that any maximal 1-orthogonal A-module T is a generator-cogenerator of $\mathrm{mod}(A)$. It also follows that T is rigid. In general, maximal 1-orthogonal modules need not exist.

The *global dimension* $\mathrm{gl.dim}(A)$ of an algebra A is the supremum over all projective dimensions $\mathrm{proj.dim}(M)$ of all A-modules M in $\mathrm{mod}(A)$. The *representation dimension* of A is defined as

$$\mathrm{rep.dim}(A) = \inf\{\mathrm{gl.dim}(\mathrm{End}_A(C)) \mid C \text{ a generator-cogenerator of } \mathrm{mod}(A)\}.$$

Auslander proved that A is of finite representation type if and only if $\mathrm{rep.dim}(A) \leq 2$, see [1, p.559]. Iyama [24] showed that $\mathrm{rep.dim}(A)$ is always finite. Next, let

$$0 \to A \to I_0 \to I_1 \to I_2 \to \cdots$$

be a minimal injective resolution of A. Then the *dominant dimension* of A is defined as

$$\mathrm{dom.dim}(A) = \inf\{i \geq 0 \mid I_i \text{ is non-projective}\}.$$

Let M be a finite-dimensional A-module. Thus $M = M_1^{n_1} \oplus \cdots \oplus M_t^{n_t}$ where the M_i are pairwise non-isomorphic indecomposable modules and $n_i = [M : M_i] \geq 1$ is the multiplicity of M_i in M. The endomorphism algebra $\mathrm{End}_A(M)$ is Morita equivalent to an algebra $K\Gamma_M/I$, where Γ_M is some uniquely determined finite quiver and I is some admissible ideal in the path algebra $K\Gamma_M$. (An ideal I in a path algebra $K\Gamma$ is called *admissible* if I is generated by a set of elements of the form $\sum_{i=1}^m \lambda_i p_i$, where the p_i are paths of length at least two in Γ and the λ_i are in K, and if $K\Gamma/I$ is finite-dimensional.)

The module M is called *basic* if $n_i = 1$ for all i. In fact we have $\mathrm{End}_A(M_1 \oplus \cdots \oplus M_t) \cong K\Gamma_M/I$. Thus, if M' is the basic module $M_1 \oplus \cdots \oplus M_t$, then the categories $\mathrm{mod}(\mathrm{End}_A(M))$ and $\mathrm{mod}(\mathrm{End}_A(M'))$ are equivalent, and we can restrict to the study of $\mathrm{mod}(\mathrm{End}_A(M'))$.

We call Γ_M the *quiver* of $\mathrm{End}_A(M)$. The vertices $1, \ldots, t$ of Γ_M correspond to the modules M_i. By S_{M_i} or just S_i we denote the simple $\mathrm{End}_A(M)$-module corresponding to i. The indecomposable projective $\mathrm{End}_A(M)$-module P_i with top S_i is just $\mathrm{Hom}_A(M_i, M)$.

Our first main result shows that endomorphism algebras of maximal rigid modules over preprojective algebras have surprisingly nice properties:

Theorem 2.2. *Let Λ be a preprojective algebra of Dynkin type Δ. For a Λ-module T the following are equivalent:*

- *T is maximal rigid;*
- *T is complete rigid;*
- *T is maximal 1-orthogonal.*

If T satisfies one of the above equivalent conditions, then the following hold:

- $\mathrm{gl.dim}(\mathrm{End}_\Lambda(T)) = 3$;
- $\mathrm{dom.dim}(\mathrm{End}_\Lambda(T)) = 3$;

- *The quiver Γ_T of $\mathrm{End}_\Lambda(T)$ has no sinks, no sources, no loops and no 2-cycles.*

Corollary 2.3. $\mathrm{rep.dim}(\Lambda) \leq 3$.

The following is a consequence of Theorem 2.2, see Proposition 4.4 for a proof:

Corollary 2.4. *For a maximal rigid Λ-module T the functor*

$$\mathrm{Hom}_\Lambda(-,T)\colon\ \mathrm{mod}(\Lambda) \to \mathrm{mod}(\mathrm{End}_\Lambda(T))$$

is fully faithful, and its image is the category of $\mathrm{End}_\Lambda(T)$-modules of projective dimension at most one.

The proof of Theorem 2.2 uses the following variation of a result by Iyama (see Theorem 4.3 below). The definition of a tilting module can be found in Section 3.3.

Proposition 2.5. *Let T_1 and T_2 be maximal rigid Λ-modules. Then $T = \mathrm{Hom}_\Lambda(T_2, T_1)$ is a tilting module over $\mathrm{End}_\Lambda(T_1)$, and we have*

$$\mathrm{End}_{\mathrm{End}_\Lambda(T_1)}(T) \cong \mathrm{End}_\Lambda(T_2)^{\mathrm{op}}.$$

In particular, $\mathrm{End}_\Lambda(T_1)$ and $\mathrm{End}_\Lambda(T_2)$ are derived equivalent.

2.5. If $\widetilde{B} = (b_{ij})$ is any $r \times (r-n)$-matrix, then the *principal part* B of $\widetilde{B}$ is obtained from $\widetilde{B}$ by deleting the last n rows. The following definition is due to Fomin and Zelevinsky [13]: Given some $k \in [1, r-n]$ define a new $r \times (r-n)$-matrix $\mu_k(\widetilde{B}) = (b'_{ij})$ by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise,} \end{cases}$$

where $i \in [1, r]$ and $j \in [1, r-n]$. One calls $\mu_k(\widetilde{B})$ a *mutation* of $\widetilde{B}$. If $\widetilde{B}$ is an integer matrix whose principal part is skew-symmetric, then it is easy to check that $\mu_k(\widetilde{B})$ is also an integer matrix with skew-symmetric principal part.

2.6. Let $T = T_1 \oplus \cdots \oplus T_r$ be a basic complete rigid Λ-module with T_i indecomposable for all i. Without loss of generality assume that $T_{r-n+1}, \dots, T_r$ are projective. Let $B(T) = (t_{ij})_{1 \leq i,j \leq r}$ be the $r \times r$-matrix defined by

$$t_{ij} = (\text{number of arrows } j \to i \text{ in } \Gamma_T) - (\text{number of arrows } i \to j \text{ in } \Gamma_T).$$

Since the quiver Γ_T does not have 2-cycles, at least one of the two summands in the definition of t_{ij} is zero. Define $B(T)^\circ = (t_{ij})$ to be the $r \times (r-n)$-matrix obtained from $B(T)$ by deleting the last n columns.

For $k \in [1, r-n]$ there is a short exact sequence

$$0 \to T_k \xrightarrow{f} \bigoplus_{t_{ik} > 0} T_i^{t_{ik}} \to T_k^* \to 0$$

where f is a minimal left $\mathrm{add}(T/T_k)$-approximation of T_k (i.e. the map $\mathrm{Hom}_\Lambda(f, T)$ is surjective, and every morphism g with $gf = f$ is an isomorphism, see Section 3.1 for a review of basic results on approximations.) Set

$$\mu_{T_k}(T) = T_k^* \oplus T/T_k.$$

We show that $\mu_{T_k}(T)$ is again a basic complete rigid module (Proposition 5.6). In particular, T_k^* is indecomposable. We call $\mu_{T_k}(T)$ the *mutation of T in direction T_k*.

Our second main result shows that the quivers of the endomorphism algebras $\mathrm{End}_\Lambda(T)$ and $\mathrm{End}_\Lambda(\mu_{T_k}(T))$ are related via Fomin and Zelevinsky's mutation rule:

Theorem 2.6. *Let Λ be a preprojective algebra of Dynkin type Δ. For a basic complete rigid Λ-module T as above and $k \in [1, r-n]$ we have*

$$B(\mu_{T_k}(T))^\circ = \mu_k(B(T)^\circ).$$

2.7. By $\Gamma_Q = (\Gamma_0, \Gamma_1, s, t, \tau)$ we denote the Auslander-Reiten quiver of Q, where τ is the Auslander-Reiten translation.

Recall that the vertices of Γ_Q correspond to the isomorphism classes of indecomposable KQ-modules. For an indecomposable KQ-module M let $[M]$ be its corresponding vertex. A vertex of Γ_Q is called *projective* provided it corresponds to an indecomposable projective KQ-module. For every indecomposable non-projective KQ-module X there exists an Auslander-Reiten sequence

$$0 \to \tau(X) \to \bigoplus_{i=1}^{n(X)} X_i \to X \to 0.$$

Here $\tau(X)$ and the X_i are pairwise non-isomorphic indecomposable KQ-modules, which are uniquely dermined by X. We also write $\tau([X]) = [\tau(X)]$. The arrows of Γ_Q are defined as follows: Whenever there is an irreducible homomorphism $X \to Y$ between indecomposable KQ-modules X and Y, then there is an arrow from $[X]$ to $[Y]$. The Auslander-Reiten sequence above yields arrows $[\tau(X)] \to [X_i]$ and $[X_i] \to [X]$, and all arrows of Γ_Q are obtained in this way from Auslander-Reiten sequences.

In general, there can be more than one arrow between two vertices of an Auslander-Reiten quiver of an algebra. But since we just work with path algebras of Dynkin quivers, this does not occur in our situation. For details on Auslander-Reiten sequences we refer to [3] and [36].

It is well known (Gabriel's Theorem) that Γ_Q has exactly r vertices, say $1, \dots, r$, and n of these are projective. Without loss of generality assume that the projective vertices are labelled by $r-n+1, \dots, r$. We define a new quiver Γ_Q^* which is obtained from Γ_Q by adding an arrow $x \to \tau(x)$ for each non-projective vertex x of Γ_Q. For the proof of Theorem 2.2 we need the following result [16, Theorem 1]:

Theorem 2.7. *There exists a basic complete rigid Λ-module T_Q such that the quiver of the endomorphism algebra $\mathrm{End}_\Lambda(T_Q)$ is Γ_Q^*.*

In comparison to [16] we changed our notation slightly: Here we denote the module $\mathrm{I}_{Q^{op}}$ which we constructed in [16] by T_Q. Also we write Γ_Q^* instead of $\check{A}_Q$.

Since Λ does not depend on the orientation of Q, we get that for every Dynkin quiver Q' of type Δ there is a basic complete rigid Λ-module $T_{Q'}$ such that the quiver of its endomorphism algebra is $\Gamma_{Q'}^*$.

2.8. Note that in [4] only the cluster algebra structure on $\mathbb{C}[G/N]$ is defined explicitly. But one can easily modify it to get the one on $\mathbb{C}[N]$. The cluster algebra structure on $\mathbb{C}[N]$ is defined as follows, see [16] for details: To any reduced word $\mathbf{i}$ of the longest element w_0 of the Weyl group, one can associate an initial seed $(\tilde{\mathbf{x}}', \widetilde{B}(\mathbf{i})')$ consisting of a list $\tilde{\mathbf{x}}' = (\Delta(1, \mathbf{i})', \dots, \Delta(r, \mathbf{i})')$ of r distinguished elements of $\mathbb{C}[N]$ together with an $r \times (r-n)$-matrix $\widetilde{B}(\mathbf{i})'$. (Here we use the notation from [16].) This seed is described combinatorially in terms of $\mathbf{i}$. In particular, the elements of $\tilde{\mathbf{x}}'$ are certain explicit *generalized minors* attached to certain subwords of $\mathbf{i}$. The other seeds are then produced inductively from $(\tilde{\mathbf{x}}', \widetilde{B}(\mathbf{i})')$ by a process of seed mutation introduced by Fomin and Zelevinsky:

Assume $((f_1, \dots, f_r), \widetilde{B})$ is a seed which was obtained by iterated seed mutation from our initial seed. Thus the f_i are certain elements in $\mathbb{C}[N]$ and $\widetilde{B} = (b_{ij})$ is a certain

$r \times (r-n)$-matrix with integer entries. For $1 \le k \le r-n$ define another $r \times (r-n)$-matrix $\mu_k(\widetilde{B}) = (b'_{ij})$ as in Section 2.5 above, and let

$$f'_k = \frac{\prod_{b_{ik}>0} f_i^{b_{ik}} + \prod_{b_{ik}<0} f_i^{-b_{ik}}}{f_k}.$$

The choice of the initial seed ensures that f'_k is again an element in $\mathbb{C}[N]$, this follows from the results in [4]. Let $\mu_k(f_1, \dots, f_r)$ be the r-tuple obtained from $(f_1, \dots, f_r)$ by replacing the entry f_k by f'_k. Then $(\mu_k(f_1, \dots, f_r), \mu_k(\widetilde{B}))$ is the seed obtained from $((f_1, \dots, f_r), \widetilde{B})$ by mutation in direction k. Thus, starting with our initial seed, an inductive combinatorial procedure gives all the other seeds. Each seed has $r-n$ neighbouring seeds obtained by just one mutation.

Suppose that $\mathbf{i}$ is *adapted* to the quiver Q, in the sense of [30]. Then it is shown in [16] that the matrix $\widetilde{B}(\mathbf{i})'$ coincides with the matrix $B(T_Q)^\circ$, where T_Q is the basic complete rigid module of Theorem 2.7. Moreover, if we write $T_Q = T_1 \oplus \cdots \oplus T_r$ and $x_i = \varphi_{T_i}$ $(1 \le i \le r)$ (see 1.3 above), then $\tilde{\mathbf{x}}'$ coincides with $(x_1, \dots, x_r)$. In other words, the rigid module T_Q can be regarded as a lift to $\mathrm{mod}(\Lambda)$ of the initial seed $(\tilde{\mathbf{x}}', \widetilde{B}(\mathbf{i})')$.

Let $\mathcal{T}_\Lambda$ be the graph with vertices the isomorphism classes of basic maximal rigid Λ-modules and with edges given by mutations. Let $\mathcal{T}_\Lambda^\circ$ denote the connected component of $\mathcal{T}_\Lambda$ containing T_Q. To a vertex $R = R_1 \oplus \cdots \oplus R_r$ we attach the r-tuple of regular functions $\mathbf{x}(R) = (\varphi_{R_1}, \dots, \varphi_{R_r})$.

For a rigid Λ-module M the closure of the orbit $\mathcal{O}_M$ in the corresponding module variety is an irreducible component, see Section 3.5 (in particular Corollary 3.15). Therefore M is a generic point of the irreducible component $\overline{\mathcal{O}_M}$, see Section 9.2. This implies that $\mathbf{x}(R)$ is a collection of elements of the dual semicanonical basis $\mathcal{S}^*$. Define $B(R)^\circ$ as in Section 2.6.

Let $\mathcal{G}$ be the *exchange graph* of the cluster algebra $\mathbb{C}[N]$, that is, the graph with vertices the seeds and edges given by seed mutation.

Using Theorem 2.6 and the multiplication formula for functions φ_M of [17], we can then deduce the following theorem.

Theorem 2.8. *With the notation above the following hold:*

(1) *For each vertex R of $\mathcal{T}_\Lambda^\circ$, $\mathbf{x}(R)$ is a cluster of $\mathbb{C}[N]$;*
(2) *The map $R \mapsto (\mathbf{x}(R), B(R)^\circ)$ induces an isomorphism of graphs from $\mathcal{T}_\Lambda^\circ$ to $\mathcal{G}$;*
(3) *The cluster monomials belong to the dual semicanonical basis $\mathcal{S}^*$.*

Note that this proves in particular that the cluster monomials are linearly independent, which is not obvious from their definition. On the other hand, with the exception of Dynkin types $\mathbb{A}_n$ with $n \le 4$, it is known that there exist irreducible components of varieties of Λ-modules without rigid modules. Therefore, it also follows from Theorem 2.8 that when $\mathbb{C}[N]$ is a cluster algebra of infinite type, the cluster monomials form a proper subset of $\mathcal{S}^*$ and do not span $\mathbb{C}[N]$.

It is also worth noting that there is no known algorithm to calculate the semicanonical basis (in [31], a similar basis of constructible functions for the group algebra of a Weyl group is considered to be "probably uncomputable"). Therefore it is remarkable that, by Theorem 2.8, a large family of elements of $\mathcal{S}^*$ can be obtained by a combinatorial algorithm, namely by repeated applications of the exchange formula for cluster mutation.

We conjecture that the graph $\mathcal{T}_\Lambda$ is connected, so that Theorem 2.8 should hold with $\mathcal{T}_\Lambda$ instead of $\mathcal{T}_\Lambda^\circ$.

2.9. The paper is organized as follows:

In Section 3 we recall mostly known results from the representation theory of algebras. Section 4 discusses maximal 1-orthogonal modules, which were recently introduced and studied by Iyama [25], [26]. We repeat some of Iyama's results, and then study the functor $\mathrm{Hom}_A(-, T)$ associated to a maximal 1-orthogonal module T. In Section 5 we introduce a mutation operation on basic rigid Λ-modules and study the corresponding exchange sequences.

The endomorphism algebras $\mathrm{End}_\Lambda(T)$ of maximal rigid Λ-modules are studied in detail in Section 6, which contains a proof of Theorem 2.2.

Next, in Section 7 we use the results from Section 6 to prove Theorem 2.6. Some examples are given in Section 8.

In Section 9 we recall some backround results on cluster algebras and semicanonical bases, and we prove Theorem 2.8.

3. Preliminary results

In this section let A be a finite-dimensional algebra over an algebraically closed field K. For backround material concerning the representation theory of finite-dimensional algebras we refer to [3] and [36].

3.1. **Approximations of modules.** We recall some well known results from the theory of approximations of modules. Let M be an A-module. A homomorphism $f\colon X \to M'$ in $\mathrm{mod}(A)$ is a *left* $\mathrm{add}(M)$*-approximation* of X if $M' \in \mathrm{add}(M)$ and the induced map

$$\mathrm{Hom}_A(f, M)\colon \mathrm{Hom}_A(M', M) \to \mathrm{Hom}_A(X, M)$$

(which maps a homomorphism $g\colon M' \to M$ to $gf\colon X \to M$) is surjective, i.e. every homomorphism $X \to M$ factors through f.

A morphism $f\colon V \to W$ is called *left minimal* if every morphism $g\colon W \to W$ with $gf = f$ is an isomorphism.

Dually, one defines right $\mathrm{add}(M)$-approximations and right minimal morphisms: A homomorphism $f\colon M' \to X$ is a *right* $\mathrm{add}(M)$*-approximation* of X if $M' \in \mathrm{add}(M)$ and the induced map

$$\mathrm{Hom}_A(M, f)\colon \mathrm{Hom}_A(M, M') \to \mathrm{Hom}_A(M, X)$$

(which maps a homomorphism $g\colon M \to M'$ to $fg\colon M \to X$) is surjective, i.e. every homomorphism $M \to X$ factors through f. A morphism $f\colon V \to W$ is called *right minimal* if every morphism $g\colon V \to V$ with $fg = f$ is an isomorphism.

The following results are well known:

Lemma 3.1. *Let $f\colon X \to M$ be a homomorphism of A-modules. Then there exists a decomposition $M = M_1 \oplus M_2$ with $\mathrm{Im}(f) \subseteq M_1$ and $f_1\colon X \to M_1$ is left minimal, where $f_1 = \pi_1 f$ with $\pi_1\colon M \to M_1$ the canonical projection.*

Lemma 3.2. *Let*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

be a non-split short exact sequence of A-modules. Then the following hold:

- *If Z is indecomposable, then f is left minimal;*
- *If X is indecomposable, then g is right minimal.*

Lemma 3.3. *If $f_i\colon X \to M_i$ $(i = 1, 2)$ are minimal left* $\mathrm{add}(M)$*-approximations, then $M_1 \cong M_2$ and* $\mathrm{Coker}(f_1) \cong \mathrm{Coker}(f_2)$.

Lemma 3.4. *If $f_i\colon X \to M_i$ $(i = 1, 2)$ are left* $\mathrm{add}(M)$*-approximations with $M_1 \cong M_2$, then* $\mathrm{Coker}(f_1) \cong \mathrm{Coker}(f_2)$.

There are obvious duals of Lemmas 3.1, 3.3 and 3.4.

Lemma 3.5. *Let*

$$0 \to X \xrightarrow{f} T' \xrightarrow{g} Y \to 0$$

be a non-split short exact sequence with $T' \in \operatorname{add}(T)$ *for some module* T. *Then the following hold:*

- *If* $\operatorname{Ext}^1_A(Y,T) = 0$, *then* f *is a left* $\operatorname{add}(T)$*-approximation;*
- *If* $\operatorname{Ext}^1_A(T,X) = 0$, *then* g *is a right* $\operatorname{add}(T)$*-approximation.*

Proof. Applying $\operatorname{Hom}_A(-,T)$ we get an exact sequence

$$0 \to \operatorname{Hom}_A(Y,T) \to \operatorname{Hom}_A(T',T) \xrightarrow{\operatorname{Hom}_A(f,T)} \operatorname{Hom}_A(X,T) \to \operatorname{Ext}^1_A(Y,T) = 0.$$

Thus $\operatorname{Hom}_A(f,T)$ is surjective, i.e. f is a left $\operatorname{add}(T)$-approximation. If we apply the functor $\operatorname{Hom}_A(T,-)$ we obtain the corresponding dual result for g. □

3.2. **Endomorphism algebras and their quivers.** Let $M = M_1^{n_1} \oplus \cdots \oplus M_t^{n_t}$ be a finite-dimensional A-module, where the M_i are pairwise non-isomorphic indecomposable modules and $n_i \geq 1$. As before let $S_i = S_{M_i}$ be the simple $\operatorname{End}_A(M)$-module corresponding to M_i, and let $P_i = \operatorname{Hom}_A(M_i, M)$ be the indecomposable projective $\operatorname{End}_A(M)$-module with top S_i. For $1 \leq i,j \leq t$ the following numbers are equal:

- The number of arrows $i \to j$ in the quiver Γ_M of $\operatorname{End}_A(M)$;
- $\dim \operatorname{Ext}^1_{\operatorname{End}_A(M)}(S_i,S_j)$;
- The dimension of the space of irreducible maps $M_i \to M_j$ in the category $\operatorname{add}(M)$;
- The dimension of the space of irreducible maps $P_j \to P_i$ in the category $\operatorname{add}(P_1 \oplus \cdots \oplus P_t)$ of projective $\operatorname{End}_A(M)$-modules.

Furthermore, let $f\colon M_i \to M'$ (resp. $g\colon M'' \to M_i$) be a minimal left (resp. right) $\operatorname{add}(M/(M_i^{n_i}))$-approximation of M_i. If $i \neq j$, then we have

$$\dim \operatorname{Ext}^1_{\operatorname{End}_A(M)}(S_i,S_j) = [M' : M_j],$$
$$\dim \operatorname{Ext}^1_{\operatorname{End}_A(M)}(S_j,S_i) = [M'' : M_j].$$

The above facts are important and will be often used, in particular in Sections 6 and 7.

3.3. **Tilting modules and derived equivalences.** An A-module T is a *tilting module* if the following three conditions hold:

(1) $\operatorname{proj.dim}(T) \leq 1$;
(2) $\operatorname{Ext}^1_A(T,T) = 0$;
(3) There exists a short exact sequence

$$0 \to A \to T' \to T'' \to 0$$

with $T', T'' \in \operatorname{add}(T)$.

By $D^b(A)$ we denote the derived category of bounded complexes of A-modules. Let B be another finite-dimensional K-algebra. The algebras A and B are *derived equivalent* if the categories $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories, see for example [21, Section 0].

Theorem 3.6 ([21, Section 1.7])**.** *If* T *is a tilting module over* A*, then* A *and* $\operatorname{End}_A(T)^{\mathrm{op}}$ *are derived equivalent.*

The following theorem is also well known:

Theorem 3.7 ([21, Section 1.4])**.** *If* A *and* B *are derived equivalent, then* $\operatorname{gl.dim}(A) < \infty$ *if and only if* $\operatorname{gl.dim}(B) < \infty$.

For details on tilting theory and derived categories we refer to [20], [21].

3.4. **Loops and 2-cycles.** For the rest of this section we assume that $A = K\Gamma/I$, where Γ is a finite quiver and I is an admissible ideal. We denote the simple A-module corresponding to a vertex l of Γ by S_l. The path of length 0 at l is denoted by e_l. Thus $e_pK\Gamma e_q$ is the vector space with basis the set of paths in Γ which start at q and end at p.

The following result is proved in [23]:

Theorem 3.8 (Igusa)**.** *If* $\mathrm{gl.dim}(A) < \infty$*, then the quiver of* A *has no loops.*

Let $[A, A]$ be the *commutator subgroup* of A. This is the subspace of A generated by all commutators $[a, b] = ab - ba$ with $a, b \in A$. The following result can be found in [29, Satz 5]:

Theorem 3.9 (Lenzing)**.** *If* $\mathrm{gl.dim}(A) < \infty$*, then every nilpotent element* $a \in A$ *lies in* $[A, A]$*.*

As mentioned in [23], Theorem 3.8 is already contained implicitly in Theorem 3.9. The next lemma follows directly from [5], see also [9]:

Lemma 3.10. *Let* w *be a path of length two in* $e_pK\Gamma e_q$*. If* $w - c \in I$ *for some* $c \in e_pK\Gamma e_q$ *with* $w \neq c$*, then* $\mathrm{Ext}^2_A(S_q, S_p) \neq 0$*.*

From Theorem 3.9 we can deduce the following result:

Proposition 3.11. *Assume that* $\mathrm{gl.dim}(A) < \infty$ *and that the quiver of* A *has a 2-cycle. Then* $\mathrm{Ext}^2_A(S, S) \neq 0$ *for some simple* A*-module* S*.*

Proof. For some vertices $i \neq j$ in Γ there are arrows $a\colon j \to i$ and $b\colon i \to j$ in Γ. We have $ab \in e_iK\Gamma e_i$. We claim that $\mathrm{Ext}^2_A(S_i, S_i) \neq 0$ or $\mathrm{Ext}^2_A(S_j, S_j) \neq 0$:

We know that ab is nilpotent in A. Thus by Theorem 3.9 we have $ab \in [A, A]$. One easily checks that every commutator is a linear combination of commutators of the form $[v, w]$ where v and w are paths in Γ. Thus

$$ab - \sum_{l=1}^{m} \lambda_l [v_l, w_l] \in I$$

for some appropriate paths v_l and w_l in Γ and some scalars $\lambda_l \in K$. Without loss of generality we can assume the following:

- $v_1 = a$ and $w_1 = b$;
- $\{v_s, w_s\} \neq \{v_t, w_t\}$ for all $s \neq t$;
- $\{v_lw_l, w_lv_l\} \cap \{ab, ba\} = \emptyset$ for all $2 \leq l \leq m$.

Note that we do not assume that the scalars λ_l are all non-zero. Set

$$C = ab - \sum_{l=1}^{m} \lambda_l [v_l, w_l].$$

We get

$$e_iCe_i = ab - \lambda_1 ab - \sum_{l=2}^{m} \lambda_l e_i [v_l, w_l] e_i \in I.$$

If $\lambda_1 \neq 1$, then Lemma 3.10 applies and we get $\mathrm{Ext}^2_A(S_i, S_i) \neq 0$. If $\lambda_1 \neq 0$, then

$$e_jCe_j = \lambda_1 ba - \sum_{l=2}^{m} \lambda_l e_j [v_l, w_l] e_j \in I.$$

Again we can apply Lemma 3.10 to this situation and get $\mathrm{Ext}^2_A(S_j, S_j) \neq 0$. This finishes the proof. □

3.5. **Preprojective algebras.** We recall some results on preprojective algebras. There is a symmetric bilinear form $(-,-) : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ associated to Q defined by

$$(d,e) = 2\sum_{i \in Q_0} d_i e_i - \sum_{\alpha \in Q_1} (d_{s(\alpha)} e_{t(\alpha)} + e_{s(\alpha)} d_{t(\alpha)}).$$

The dimension vector of a module M is denoted by $\underline{\dim}(M)$. The following lemma is due to Crawley-Boevey [12, Lemma 1].

Lemma 3.12. *For any Λ-modules X and Y we have*

$$\dim \operatorname{Ext}^1_\Lambda(X,Y) = \dim \operatorname{Hom}_\Lambda(X,Y) + \dim \operatorname{Hom}_\Lambda(Y,X) - (\underline{\dim}(X), \underline{\dim}(Y)).$$

Corollary 3.13. $\dim \operatorname{Ext}^1_\Lambda(X,X)$ *is even, and* $\dim \operatorname{Ext}^1_\Lambda(X,Y) = \dim \operatorname{Ext}^1_\Lambda(Y,X)$.

Let $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$. By

$$\operatorname{rep}(Q,\beta) = \prod_{\alpha \in Q_1} \operatorname{Hom}_K(K^{\beta_{s(\alpha)}}, K^{\beta_{t(\alpha)}})$$

we denote the affine space of representations of Q with dimension vector β. Furthermore, let Λ_β be the affine variety of elements

$$(f_\alpha, f_{\alpha^*})_{\alpha \in Q_1} \in \prod_{\alpha \in Q_1} \left(\operatorname{Hom}_K(K^{\beta_{s(\alpha)}}, K^{\beta_{t(\alpha)}}) \times \operatorname{Hom}_K(K^{\beta_{t(\alpha)}}, K^{\beta_{s(\alpha)}}) \right)$$

such that for all $i \in Q_0$ we have

$$\sum_{\alpha \in Q_1 : s(\alpha) = i} f_{\alpha^*} f_\alpha = \sum_{\alpha \in Q_1 : t(\alpha) = i} f_\alpha f_{\alpha^*}.$$

Note that $\operatorname{rep}(Q,\beta)$ can be considered as a subvariety of Λ_β. Since Q is a Dynkin quiver, the variety Λ_β coincides with Lusztig's nilpotent variety associated to Q. One can interpret Λ_β as the variety of Λ-modules with dimension vector β. The group

$$\mathrm{G}_\beta = \prod_{i=1}^n \mathrm{GL}_{\beta_i}(K)$$

acts by conjugation on $\operatorname{rep}(Q,\beta)$ and Λ_β:

For $g = (g_1, \ldots, g_n) \in \mathrm{G}_\beta$ and $x = (f_\alpha, f_{\alpha^*})_{\alpha \in Q_1} \in \Lambda_\beta$ define

$$g \cdot x = (g_{t(\alpha)} f_\alpha g_{s(\alpha)}^{-1}, g_{s(\alpha)} f_{\alpha^*} g_{t(\alpha)}^{-1})_{\alpha \in Q_1}.$$

The action on $\operatorname{rep}(Q,\beta)$ is obtained via restricting the action on Λ_β to $\operatorname{rep}(Q,\beta)$.

The isomorphism classes of representations in $\operatorname{rep}(Q,\beta)$ and Λ-modules in Λ_β, respectively, correspond to the orbits of these actions. The G_β-orbit of some $M \in \operatorname{rep}(Q,\beta)$ or $M \in \Lambda_\beta$ is denoted by $\mathcal{O}_M$. Since Q is a Dynkin quiver, there are only finitely many orbits in $\operatorname{rep}(Q,\beta)$. Let $\pi_\beta \colon \Lambda_\beta \to \operatorname{rep}(Q,\beta)$ be the canonical projection morphism. Lusztig shows that $\mathcal{O} \mapsto \overline{\pi_\beta^{-1}(\mathcal{O})}$ defines a one-to-one correspondence between the G_β-orbits in $\operatorname{rep}(Q,\beta)$ and the set $\operatorname{Irr}(\Lambda_\beta)$ of irreducible components of Λ_β. He also proved that all irreducible components of Λ_β have dimension

$$\sum_{\alpha \in Q_1} \beta_{s(\alpha)} \beta_{t(\alpha)},$$

see [30, Section 12]. For a G_β-orbit $\mathcal{O}$ in Λ_β let $\operatorname{codim} \mathcal{O} = \dim \Lambda_\beta - \dim \mathcal{O}$ be its codimension.

Lemma 3.14. *For any Λ-module M we have* $\dim \operatorname{Ext}^1_\Lambda(M,M) = 2 \operatorname{codim} \mathcal{O}_M$.

Proof. Set $\beta = \underline{\dim}(M)$. By Lemma 3.12 we have $\dim \operatorname{Ext}^1_\Lambda(M,M) = 2\dim \operatorname{End}_\Lambda(M) - (\beta,\beta)$. Furthermore, $\dim \mathcal{O}_M = \dim \mathrm{G}_\beta - \dim \operatorname{End}_\Lambda(M)$. Thus

$$\operatorname{codim} \mathcal{O}_M = \dim \Lambda_\beta - \dim \mathcal{O}_M = \sum_{\alpha \in Q_1} \beta_{s(\alpha)}\beta_{t(\alpha)} - \sum_{i=1}^n \beta_i^2 + \dim \operatorname{End}_\Lambda(M).$$

Combining these equations yields the result. □

Corollary 3.15. *For a Λ-module M with dimension vector β the following are equivalent:*

- *The closure $\overline{\mathcal{O}_M}$ of $\mathcal{O}_M$ is an irreducible component of Λ_β;*
- *The orbit $\mathcal{O}_M$ is open in Λ_β;*
- $\operatorname{Ext}^1_\Lambda(M,M) = 0$.

4. Maximal 1-orthogonal modules

In this section let A be a finite-dimensional algebra over an algebraically closed field K.

4.1. **Iyama's results.**

We recall some of Iyama's recent results on maximal 1-orthogonal modules [25], [26].

Theorem 4.1 ([26, Theorem 5.1 (3)]). *Let T be a rigid A-module and assume that T is a generator-cogenerator. If $\operatorname{gl.dim}(\operatorname{End}_A(T)) \leq 3$, then T is maximal 1-orthogonal.*

Theorem 4.2 ([26, Theorem 0.2]). *If T is a maximal 1-orthogonal A-module, then*

$$\operatorname{gl.dim}(\operatorname{End}_A(T)) \leq 3 \text{ and } \operatorname{dom.dim}(\operatorname{End}_A(T)) \geq 3.$$

Theorem 4.3 ([26, Theorem 5.3.2]). *Let T_1 and T_2 be maximal 1-orthogonal A-modules. Then $T = \operatorname{Hom}_A(T_2, T_1)$ is a tilting module over $\operatorname{End}_A(T_1)$, and we have*

$$\operatorname{End}_{\operatorname{End}_A(T_1)}(T) \cong \operatorname{End}_A(T_2)^{\mathrm{op}}.$$

In particular, $\operatorname{End}_A(T_1)$ and $\operatorname{End}_A(T_2)$ are derived equivalent, and $\Sigma(T_1) = \Sigma(T_2)$.

4.2. **A Hom-functor.**

Proposition 4.4. *Let T be a maximal 1-orthogonal A-module. Then the contravariant functor*

$$F_T = \operatorname{Hom}_A(-,T)\colon \operatorname{mod}(A) \to \operatorname{mod}(\operatorname{End}_A(T))$$

yields an anti-equivalence of categories

$$\operatorname{mod}(A) \to \mathcal{P}(\operatorname{End}_A(T))$$

where $\mathcal{P}(\operatorname{End}_A(T)) \subset \operatorname{mod}(\operatorname{End}_A(T))$ denotes the full subcategory of all $\operatorname{End}_A(T)$-modules of projective dimension at most one.

Proof. Set $E = \operatorname{End}_A(T)$. Let $X \in \operatorname{mod}(A)$, and let $f\colon X \to T'$ be a minimal left $\operatorname{add}(T)$-approximation of X. Since T is a cogenerator of $\operatorname{mod}(A)$, we know that f is a monomorphism. Thus there is a short exact sequence

$$0 \to X \xrightarrow{f} T' \to T'' \to 0.$$

Applying $\operatorname{Hom}_A(-,T)$ to this sequence yields an exact sequence

$$0 \to \operatorname{Hom}_A(T'',T) \to \operatorname{Hom}_A(T',T) \xrightarrow{\operatorname{Hom}_A(f,T)} \operatorname{Hom}_A(X,T)$$
$$\to \operatorname{Ext}^1_A(T'',T) \to \operatorname{Ext}^1_A(T',T).$$

Since T is rigid and $T' \in \mathrm{add}(T)$, we get $\mathrm{Ext}^1_A(T',T) = 0$. The map f is an $\mathrm{add}(T)$-approximation, thus $\mathrm{Hom}_A(f,T)$ is surjective. This implies $\mathrm{Ext}^1_A(T'',T) = 0$. But T is maximal 1-orthogonal, which yields $T'' \in \mathrm{add}(T)$. Thus the exact sequence

$$0 \to \mathrm{Hom}_A(T'',T) \to \mathrm{Hom}_A(T',T) \xrightarrow{\mathrm{Hom}_A(f,T)} \mathrm{Hom}_A(X,T) \to 0$$

is a projective resolution of $F_T(X) = \mathrm{Hom}_A(X,T)$, and we conclude

$$\mathrm{proj.\,dim}_E(F_T(X)) \leq 1.$$

Now the dual of [2, Lemma 1.3(b)] yields that F_T induces a fully faithful functor $\mathrm{mod}(A) \to \mathcal{P}(E)$.

Finally, we show that every object $Y \in \mathcal{P}(E)$ lies in the image of F_T. There exists a projective resolution of the form

$$0 \to \mathrm{Hom}_A(T'',T) \xrightarrow{G} \mathrm{Hom}_A(T',T) \to Y \to 0$$

with $T', T'' \in \mathrm{add}(T)$. Since $\mathrm{Hom}_A(T',T)$ and $\mathrm{Hom}_A(T'',T)$ are projective, there exists some $g \in \mathrm{Hom}_A(T',T'')$ such that $G = \mathrm{Hom}_A(g,T)$. We claim that g must be surjective. Let us assume that g is not surjective. Then there exists a non-zero homomorphism $h \in \mathrm{Hom}_A(T'',T)$ such that $hg = 0$. Here we use that T is a cogenerator of $\mathrm{mod}(A)$. This implies $G(h) = 0$, which is a contradiction to G being injective.

Thus there is a short exact sequence

$$0 \to \mathrm{Ker}(g) \to T' \xrightarrow{g} T'' \to 0.$$

Applying $\mathrm{Hom}_A(-,T)$ to this sequence yields an exact sequence

$$0 \to \mathrm{Hom}_A(T'',T) \xrightarrow{G} \mathrm{Hom}_A(T',T) \to \mathrm{Hom}_A(\mathrm{Ker}(g),T) \to 0.$$

Here we used that $\mathrm{Ext}^1_A(T'',T) = 0$. Thus we get $Y \cong \mathrm{Hom}_A(\mathrm{Ker}(g),T)$. This finishes the proof of Proposition 4.4. □

A more general result than the above Proposition 4.4 can be found in [26, Theorem 5.3.4].

Let F_T be as in Proposition 4.4. Then F_T has the following properties:

- If X is an indecomposable A-module, then $F_T(X)$ is indecomposable;
- F_T reflects isomorphism classes, i.e. if $F_T(X) \cong F_T(Y)$ for some A-modules X and Y, then $X \cong Y$.

Note that the functor F_T is not exact.

The following proposition is due to Iyama:

Proposition 4.5. *Let T be a basic maximal 1-orthogonal A-module, and let X be an indecomposable direct summand of T. Then there is at most one indecomposable A-module Y such that $X \ncong Y$ and $Y \oplus T/X$ is maximal 1-orthogonal.*

Proof. Assume there are two non-isomorphic indecomposable A-modules Y_i where $i = 1, 2$ such that $X \ncong Y_i$ and $Y_i \oplus T/X$ is maximal 1-orthogonal. By Theorem 4.3 we know that $\mathrm{Hom}_A(Y_i \oplus T/X, T)$ is a tilting module over $\mathrm{End}_A(T)$.

Thus the almost complete tilting module $\mathrm{Hom}_A(T/X,T)$ over $\mathrm{End}_A(T)$ has three indecomposable complements, namely $\mathrm{Hom}_A(X,T)$, $\mathrm{Hom}_A(Y_1,T)$ and $\mathrm{Hom}_A(Y_2,T)$. Here we use that the functor F_T as defined in Proposition 4.4 preserves indecomposables and reflects isomorphism classes. By [35, Proposition 1.3] an almost complete tilting module has at most two indecomposable complements, a contradiction. □

5. Mutations of rigid modules

5.1. In this section we work with modules over the preprojective algebra Λ.

Lemma 5.1. *Let T and X be rigid Λ-modules. If*

$$0 \to X \xrightarrow{f} T' \xrightarrow{g} Y \to 0$$

is a short exact sequence with f a left $\mathrm{add}(T)$-approximation, then $T \oplus Y$ is rigid.

Proof. First, we prove that $\mathrm{Ext}^1_\Lambda(Y,T) = 0$: We apply $\mathrm{Hom}_\Lambda(-,T)$ and get an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(Y,T) \to \mathrm{Hom}_\Lambda(T',T) \xrightarrow{\mathrm{Hom}_\Lambda(f,T)} \mathrm{Hom}_\Lambda(X,T) \to \mathrm{Ext}^1_\Lambda(Y,T) \to \mathrm{Ext}^1_\Lambda(T',T) = 0.$$

Since f is a left $\mathrm{add}(T)$-approximation, we know that $\mathrm{Hom}_\Lambda(f,T)$ is surjective. Thus $\mathrm{Ext}^1_\Lambda(Y,T) = 0$. Next, we show that $\mathrm{Ext}^1_\Lambda(Y,Y) = 0$. This is similar to the proof of [6, Lemma 6.7]. We apply $\mathrm{Hom}_\Lambda(X,-)$ and get an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(X,X) \to \mathrm{Hom}_\Lambda(X,T') \xrightarrow{\mathrm{Hom}_\Lambda(X,g)} \mathrm{Hom}_\Lambda(X,Y) \to \mathrm{Ext}^1_\Lambda(X,X) = 0.$$

Thus $\mathrm{Hom}_\Lambda(X,g)$ is surjective, i.e. every morphism $h\colon X \to Y$ factors through $g\colon T' \to Y$. We have $\mathrm{Ext}^1_\Lambda(Y,T) = 0$, and by Lemma 3.12 we get $\mathrm{Ext}^1_\Lambda(T,Y) = 0$.

Applying $\mathrm{Hom}_\Lambda(-,Y)$ yields an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(Y,Y) \to \mathrm{Hom}_\Lambda(T',Y) \xrightarrow{\mathrm{Hom}_\Lambda(f,Y)} \mathrm{Hom}_\Lambda(X,Y) \to \mathrm{Ext}^1_\Lambda(Y,Y) \to \mathrm{Ext}^1_\Lambda(T',Y) = 0.$$

To show that $\mathrm{Ext}^1_\Lambda(Y,Y) = 0$ it is enough to show that $\mathrm{Hom}_\Lambda(f,Y)$ is surjective, i.e. that every map $h\colon X \to Y$ factors through $f\colon X \to T'$. Since $\mathrm{Hom}_\Lambda(X,g)$ is surjective, there is a morphism $t\colon X \to T'$ such that $gt = h$. Since f is a left $\mathrm{add}(T)$-approximation, there is a morphism $s\colon T' \to T'$ such that $sf = t$. So

$$h = gt = gsf.$$

Thus h factors through f, which implies $\mathrm{Ext}^1_\Lambda(Y,Y) = 0$. □

Corollary 5.2. *Let T and X be rigid Λ-modules. If T is maximal rigid, then there exists a short exact sequence*

$$0 \to X \to T' \to T'' \to 0$$

with $T', T'' \in \mathrm{add}(T)$.

Proof. In the situation of Lemma 5.1, the maximality of T implies $Y \in \mathrm{add}(T)$. □

Note that there exist dual results for Lemma 5.1 and Corollary 5.2, involving right instead of left $\mathrm{add}(T)$-approximations.

Corollary 5.3. *Let T and X be rigid Λ-modules, and set $E = \mathrm{End}_\Lambda(T)$. If T is maximal rigid, then*

$$\mathrm{proj.\,dim}_E(\mathrm{Hom}_\Lambda(X,T)) \le 1.$$

Proof. Apply $\mathrm{Hom}_\Lambda(-,T)$ to the short exact sequence in Corollary 5.2. This yields a projective resolution

$$0 \to \mathrm{Hom}_\Lambda(T'',T) \to \mathrm{Hom}_\Lambda(T',T) \to \mathrm{Hom}_\Lambda(X,T) \to 0.$$

□

Theorem 5.4. *For $i = 1, 2$ let T_i be a maximal rigid Λ-module, let $E_i = \mathrm{End}_\Lambda(T_i)$, and set $T = \mathrm{Hom}_\Lambda(T_2, T_1)$. Then T is a tilting module over E_1 and we have*

$$\mathrm{End}_{E_1}(T) \cong E_2^{\mathrm{op}}.$$

In particular, E_1 and E_2 are derived equivalent, and $\Sigma(T_1) = \Sigma(T_2)$.

Proof. It follows from Corollary 5.2 that there exists a short exact sequence

$$0 \to T_2 \to T_1' \to T_1'' \to 0$$

with $T_1', T_1'' \in \mathrm{add}(T_1)$. Then Corollary 5.3 yields that the projective dimension of $T = \mathrm{Hom}_\Lambda(T_2, T_1)$ regarded as an E_1-module is at most one. This shows that T satisfies condition (1) in the definition of a tilting module.

Now the rest of the proof is identical to Iyama's proof of Theorem 4.3. □

The above theorem is very similar to Iyama's Theorem 4.3. However there are two differences: Theorem 4.3 holds for arbitrary finite-dimensional algebras, whereas we restrict to the class of finite-dimensional preprojective algebras. On the other hand, the modules T_i in Theorem 4.3 are assumed to be maximal 1-orthogonal, which is stronger than our assumption that the T_i are maximal rigid.

In both cases, one has to prove that $\mathrm{proj.dim}(T) \leq 1$. This is the only place in the proof of Theorem 4.3 where the maximal 1-orthogonality of the T_i is needed. For the rest of his proof Iyama just needs that the T_i are maximal rigid. If we restrict now to finite-dimensional preprojective algebras, then one can prove that $\mathrm{proj.dim}(T) \leq 1$ under the weaker assumption that the T_i are maximal rigid.

Later on we then use Theorem 5.4 in order to show that the maximal rigid Λ-modules are in fact maximal 1-orthogonal and also complete rigid.

Corollary 5.5. *For a Λ-module M the following are equivalent:*

- *M is maximal rigid;*
- *M is complete rigid.*

Proof. By Theorem 2.1 every complete rigid module is maximal rigid. For the other direction, let T_1 be a complete rigid Λ-module. Such a module exists by Theorem 2.7. Let T_2 be a maximal rigid module. Theorem 5.4 implies that $\mathrm{End}_\Lambda(T_1)$ and $\mathrm{End}_\Lambda(T_2)$ are derived equivalent. Thus $\Sigma(T_1) = \Sigma(T_2) = r$, in other words, T_2 is also complete rigid. □

Proposition 5.6. *Let $T \oplus X$ be a basic rigid Λ-module such that the following hold:*

- *X is indecomposable;*
- *$\Lambda \in \mathrm{add}(T)$.*

Then there exists a short exact sequence

$$0 \to X \xrightarrow{f} T' \xrightarrow{g} Y \to 0$$

such that the following hold:

- *f is a minimal left $\mathrm{add}(T)$-approximation;*
- *g is a minimal right $\mathrm{add}(T)$-approximation;*
- *$T \oplus Y$ is basic rigid;*
- *Y is indecomposable and $X \ncong Y$.*

Proof. Let $f\colon X \to T'$ be a minimal left $\mathrm{add}(T)$-approximation of X. Since $\Lambda \in \mathrm{add}(T)$, we know that f is a monomorphism. Let Y be the cokernel of f and let $g\colon T' \to Y$ be the projection map. Thus we have a short exact sequence

$$0 \to X \xrightarrow{f} T' \xrightarrow{g} Y \to 0.$$

Since $X \notin \mathrm{add}(T)$, this sequence does not split. Thus $X \ncong Y$, because X is rigid. By Lemma 5.1 we know that $T \oplus Y$ is rigid.

Using Lemma 3.2 and Lemma 3.5 yields that g is a minimal right $\mathrm{add}(T)$-approximation. Thus, if $Y \in \mathrm{add}(T)$, then $T' \cong Y$ and g would be an isomorphism. But this would imply $X = 0$, a contradiction since X is indecomposable. Thus $Y \notin \mathrm{add}(T)$.

Next, we prove that Y is indecomposable. Assume $Y = Y_1 \oplus Y_2$ with Y_1 and Y_2 non-zero. For $i = 1, 2$ let $f_i \colon T_i' \to Y_i$ be a minimal right $\mathrm{add}(T)$-approximation, and let X_i be the kernel. Thus we have short exact sequences

$$0 \to X_i \to T_i' \to Y_i \to 0.$$

The direct sum $T_1' \oplus T_2' \to Y_1 \oplus Y_2$ is then a minimal right $\mathrm{add}(T)$-approximation. Thus by the dual of Lemma 3.3 the kernel $X_1 \oplus X_2$ is isomorphic to X. Thus $X_1 = 0$ or $X_2 = 0$. If $X_1 = 0$, then $0 \to T_1'$ is a direct summand of $f \colon X \to T'$. This is a contradiction to f being minimal. Similarly $X_2 = 0$ also leads to a contradiction. Thus Y must be indecomposable. □

Note that the assumption $\Lambda \in \mathrm{add}(T)$ in Proposition 5.6 can be replaced by the weaker assumption that there exists a monomorphism from X to some object in $\mathrm{add}(T)$.

The proof of Proposition 5.6 is similar to the proofs of Lemma 6.3 - Lemma 6.6 in [6]. For convenience we gave a complete proof. Note that we work with modules over preprojective algebras, whereas [6] deals with cluster categories. However both have the crucial property that for all objects M and N the extension groups $\mathrm{Ext}^1(M, N)$ and $\mathrm{Ext}^1(N, M)$ have the same dimension.

In the situation of the above proposition, we call $\{X, Y\}$ an *exchange pair associated to* T, and we write

$$\mu_X(T \oplus X) = T \oplus Y.$$

We say that $T \oplus Y$ is the mutation of $T \oplus X$ in direction X. The short exact sequence

$$0 \to X \xrightarrow{f} T' \xrightarrow{g} Y \to 0$$

is the *exchange sequence* starting in X and ending in Y.

For example, if $T = \Lambda$ and $\{X, Y\}$ is an exchange pair associated to T, then

$$\mu_X(T \oplus X) = T \oplus \Omega^{-1}(X)$$

where Ω is the syzygy functor.

Exchange sequences appear in tilting theory, compare for example [22, Theorem 1.1], [34, Section 3], [35, Proposition 1.3] and [39, Theorem 2.1]. A special case of an exchange sequence in the context of tilting theory can be found already in [2, Lemma 1.6].

Proposition 5.7. *Let X and Y be indecomposable rigid Λ-modules with* $\dim \mathrm{Ext}^1_\Lambda(Y, X) = 1$*, and let*

$$0 \to X \xrightarrow{f} M \xrightarrow{g} Y \to 0$$

be a non-split short exact sequence. Then $M \oplus X$ and $M \oplus Y$ are rigid and $X, Y \notin \mathrm{add}(M)$. If we assume additionally that $T \oplus X$ and $T \oplus Y$ are basic maximal rigid Λ-modules for some T, then f is a minimal left $\mathrm{add}(T)$*-approximation and g is a minimal right* $\mathrm{add}(T)$*-approximation.*

Before we prove Proposition 5.7 let us state a corollary:

Corollary 5.8. *Let $\{X, Y\}$ be an exchange pair associated to some basic rigid module T such that $T \oplus X$ and $T \oplus Y$ are maximal rigid, and assume* $\dim \mathrm{Ext}^1_\Lambda(Y, X) = 1$*. Then*

$$\mu_Y(\mu_X(T \oplus X)) = T \oplus X.$$

Proof. Let
$$0 \to X \to T' \to Y \to 0$$
be the short exact sequence from Proposition 5.6. Thus $\mu_X(T \oplus X) = T \oplus Y$. Since $\dim \operatorname{Ext}^1_\Lambda(Y,X) = \dim \operatorname{Ext}^1_\Lambda(X,Y) = 1$ and since $T \oplus X$ and $T \oplus Y$ are maximal rigid, Proposition 5.7 yields a non-split short exact sequence
$$0 \to Y \xrightarrow{h} M \to X \to 0$$
with h a minimal left $\operatorname{add}(T)$-approximation. Thus $\mu_Y(T \oplus Y) = T \oplus X$. □

5.2. **Proof of Proposition 5.7.** Let X and Y be indecomposable rigid Λ-modules with $\dim \operatorname{Ext}^1_\Lambda(Y,X) = 1$, and let

$$(1) \qquad 0 \to X \xrightarrow{f} M \xrightarrow{g} Y \to 0$$

be a non-split short exact sequence.

Lemma 5.9. $\operatorname{Ext}^1_\Lambda(M,M) = 0$.

Proof. By Lemma 3.12 we have
$$\dim \operatorname{Ext}^1_\Lambda(X \oplus Y, X \oplus Y) = 2 \dim \operatorname{Hom}_\Lambda(X \oplus Y, X \oplus Y) - (\underline{\dim}(X \oplus Y), \underline{\dim}(X \oplus Y)),$$
$$\dim \operatorname{Ext}^1_\Lambda(M,M) = 2 \dim \operatorname{Hom}_\Lambda(M,M) - (\underline{\dim}(M), \underline{\dim}(M)).$$
Then our assumptions on X and Y yield
$$2 = \dim \operatorname{Ext}^1_\Lambda(X \oplus Y, X \oplus Y) = 2 \dim \operatorname{Hom}_\Lambda(X \oplus Y, X \oplus Y) - (\underline{\dim}(M), \underline{\dim}(M)).$$
Since Sequence (1) does not split, we get $M <_{\deg} X \oplus Y$, where $\leq_{\deg}$ is the usual degeneration order, see for example [33]. Thus $\dim \operatorname{Hom}_\Lambda(M,M) < \dim \operatorname{Hom}_\Lambda(X \oplus Y, X \oplus Y)$, which implies $\operatorname{Ext}^1_\Lambda(M,M) = 0$. □

Lemma 5.10. $X, Y \notin \operatorname{add}(M)$.

Proof. Assume $X \in \operatorname{add}(M)$. Since X is indecomposable, $M \cong X \oplus M'$ for some M', and we get a short exact sequence
$$0 \to X \to X \oplus M' \to Y \to 0.$$
By [33, Proposition 3.4] we get $M' \leq_{\deg} Y$. Since $\operatorname{Ext}^1_\Lambda(Y,Y) = 0$ this implies $M' = Y$. Thus the above sequence splits, a contradiction. Dually, one shows that $Y \notin \operatorname{add}(M)$. □

Lemma 5.11. $\operatorname{Ext}^1_\Lambda(M, X \oplus Y) = 0$.

Proof. Apply $\operatorname{Hom}_\Lambda(-,X)$ to Sequence (1). This yields an exact sequence
$$0 \to \operatorname{Hom}_\Lambda(Y,X) \to \operatorname{Hom}_\Lambda(M,X) \xrightarrow{\operatorname{Hom}_\Lambda(f,X)} \operatorname{Hom}_\Lambda(X,X) \xrightarrow{\delta} \operatorname{Ext}^1_\Lambda(Y,X)$$
$$\to \operatorname{Ext}^1_\Lambda(M,X) \to \operatorname{Ext}^1_\Lambda(X,X) = 0.$$
Suppose that $\operatorname{Hom}_\Lambda(f,X)$ is surjective. Then the identity morphism $X \to X$ factors through $f\colon X \to M$. Thus $X \in \operatorname{add}(M)$, a contradiction to the previous lemma. So the morphism δ has to be non-zero. Since $\dim \operatorname{Ext}^1_\Lambda(Y,X) = 1$ this implies that δ is surjective, thus $\operatorname{Ext}^1_\Lambda(M,X) = 0$. Dually, one proves that $\operatorname{Ext}^1_\Lambda(M,Y) = 0$. □

Now assume additionally that $T \oplus X$ and $T \oplus Y$ are basic maximal rigid for some T.

Lemma 5.12. $\operatorname{Ext}^1_\Lambda(M,T) = 0$.

Proof. Applying $\operatorname{Hom}_\Lambda(-,T)$ to Sequence (1) yields an exact sequence
$$0 = \operatorname{Ext}^1_\Lambda(Y,T) \to \operatorname{Ext}^1_\Lambda(M,T) \to \operatorname{Ext}^1_\Lambda(X,T) = 0.$$
Thus $\operatorname{Ext}^1_\Lambda(M,T) = 0$. □

Lemma 5.13. $M \in \mathrm{add}(T)$.

Proof. We know already that X and Y cannot be direct summands of M, and what we proved so far yields that $T \oplus X \oplus M$ is rigid. Since $T \oplus X$ is a maximal rigid module, we get $M \in \mathrm{add}(T)$. □

Lemma 3.2 and Lemma 3.5 imply that f is a minimal left $\mathrm{add}(T)$-approximation. Dually, g is a minimal right $\mathrm{add}(T)$-approximation. This finishes the proof of Proposition 5.7.

6. Endomorphism algebras of maximal rigid modules

In this section we work only with basic rigid Λ-modules. However, all our results on their endomorphism algebras are Morita invariant, thus they hold for endomorphism algebras of arbitrary rigid Λ-modules.

6.1. Global dimension and quiver shapes.

Lemma 6.1. *Let $\{X, Y\}$ be an exchange pair associated to a basic rigid Λ-module T. Then the following are equivalent:*

- *The quiver of* $\mathrm{End}_\Lambda(T \oplus X)$ *has no loop at X;*
- *Every non-isomorphism $X \to X$ factors through* $\mathrm{add}(T)$*;*
- $\dim \mathrm{Ext}^1_\Lambda(Y, X) = 1$.

Proof. The equivalence of the first two statements is easy to show, we leave this to the reader. Let

$$0 \to X \xrightarrow{f} T' \to Y \to 0$$

be the exchange sequence starting in X. Applying $\mathrm{Hom}_\Lambda(-, X)$ yields an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(Y, X) \to \mathrm{Hom}_\Lambda(T', X) \xrightarrow{\mathrm{Hom}_\Lambda(f,X)} \mathrm{Hom}_\Lambda(X, X) \to \mathrm{Ext}^1_\Lambda(Y, X) \to 0.$$

Since f is an $\mathrm{add}(T)$-approximation, every non-isomorphism $X \to X$ factors through $\mathrm{add}(T)$ if and only if it factors through f. Clearly, this is equivalent to the cokernel $\mathrm{Ext}^1_\Lambda(Y, X)$ of $\mathrm{Hom}_\Lambda(f, X)$ being 1-dimensional. Here we use that K is algebraically closed, which implies $\mathrm{Hom}_\Lambda(X, X)/\mathrm{rad}_\Lambda(X, X) \cong K$. □

Proposition 6.2. *Let T be a basic maximal rigid Λ-module. If the quiver of* $\mathrm{End}_\Lambda(T)$ *has no loops, then*

$$\mathrm{gl.dim}(\mathrm{End}_\Lambda(T)) = 3.$$

Proof. Set $E = \mathrm{End}_\Lambda(T)$. By assumption, the quiver of E has no loops. Thus $\mathrm{Ext}^1_E(S, S) = 0$ for all simple E-modules S. Let

$$T = T_1 \oplus \cdots \oplus T_r$$

with T_i indecomposable for all i. As before, denote the simple E-module corresponding to T_i by S_{T_i}.

Assume that $X = T_i$ is non-projective. We claim that $\mathrm{proj.dim}_E(S_X) = 3$. Let $\{X, Y\}$ be the exchange pair associated to T/X. Note that $\Lambda \in \mathrm{add}(T/X)$. By Lemma 6.1 we have $\dim \mathrm{Ext}^1_\Lambda(Y, X) = 1$. Let

$$0 \to X \xrightarrow{f} T' \to Y \to 0$$

and

$$0 \to Y \to T'' \to X \to 0$$

be the corresponding non-split short exact sequences. Applying $\mathrm{Hom}_\Lambda(-,T)$ to both sequences yields exact sequences

$$0 \to \mathrm{Hom}_\Lambda(Y,T) \to \mathrm{Hom}_\Lambda(T',T) \xrightarrow{\mathrm{Hom}_\Lambda(f,T)} \mathrm{Hom}_\Lambda(X,T)$$
$$\to \mathrm{Ext}^1_\Lambda(Y,T) \cong \mathrm{Ext}^1_\Lambda(Y,X) \to \mathrm{Ext}^1_\Lambda(T',T) = 0$$

and

$$0 \to \mathrm{Hom}_\Lambda(X,T) \to \mathrm{Hom}_\Lambda(T'',T) \to \mathrm{Hom}_\Lambda(Y,T) \to \mathrm{Ext}^1_\Lambda(X,T) = 0.$$

Since $\dim \mathrm{Ext}^1_\Lambda(Y,X) = 1$ we know that the cokernel of $\mathrm{Hom}_\Lambda(f,T)$ is 1-dimensional. Thus, since $\mathrm{Hom}_\Lambda(X,T)$ is the indecomposable projective E-module with top S_X, the cokernel must be isomorphic to S_X. Combining the two sequences above yields an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(X,T) \to \mathrm{Hom}_\Lambda(T'',T) \to \mathrm{Hom}_\Lambda(T',T) \to \mathrm{Hom}_\Lambda(X,T) \to S_X \to 0.$$

This is a projective resolution of S_X. Thus $\mathrm{proj.\,dim}_E(S_X) \leq 3$.

By Proposition 5.6 we have $X \notin \mathrm{add}(T'')$. Thus $\mathrm{Hom}_E(\mathrm{Hom}_\Lambda(T'',T), S_X) = 0$ and $\mathrm{Ext}^3_E(S_X,S_X) \cong \mathrm{Hom}_E(\mathrm{Hom}_\Lambda(X,T), S_X)$ is one-dimensional, in particular it is non-zero. Thus $\mathrm{proj.\,dim}_E(S_X) = 3$.

Next, assume that $P = T_i$ is projective. We claim that $\mathrm{proj.\,dim}_E(S_P) \leq 2$. Set $X = P/S$ where S is the (simple) socle of P. First, we prove that X is rigid: Applying $\mathrm{Hom}_\Lambda(-,X)$ to the short exact sequence

$$0 \to S \to P \xrightarrow{\pi} X \to 0$$

yields an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(X,X) \to \mathrm{Hom}_\Lambda(P,X) \to \mathrm{Hom}_\Lambda(S,X) \to \mathrm{Ext}^1_\Lambda(X,X) \to \mathrm{Ext}^1_\Lambda(P,X) = 0.$$

The quiver of the preprojective algebra Λ does not contain any loops. Thus the socle of X does not contain S as a composition factor. This implies $\mathrm{Hom}_\Lambda(S,X) = 0$, and thus $\mathrm{Ext}^1_\Lambda(X,X) = 0$. Let $f\colon X \to T'$ be a minimal left $\mathrm{add}(T/P)$-approximation. Clearly, f is injective. We get a short exact sequence

$$0 \to X \xrightarrow{f} T' \to Y \to 0.$$

Now Lemma 5.1 yields that $Y \oplus T/P$ is rigid. Since P is projective, there is only one indecomposable module C such that $C \oplus T/P$ is maximal rigid, namely $C = P$. Here we use the assumption that T is maximal rigid. This implies $Y \in \mathrm{add}(T)$. The projection $\pi\colon P \to X$ yields an exact sequence

$$P \xrightarrow{h} T' \to Y \to 0$$

where $h = f\pi$. Applying $\mathrm{Hom}_\Lambda(-,T)$ to this sequence gives an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(Y,T) \to \mathrm{Hom}_\Lambda(T',T) \xrightarrow{\mathrm{Hom}_\Lambda(h,T)} \mathrm{Hom}_\Lambda(P,T) \to Z \to 0.$$

We have $\mathrm{Hom}_\Lambda(P,T) = \mathrm{Hom}_\Lambda(P,T/P) \oplus \mathrm{Hom}_\Lambda(P,P)$. For each morphism $g\colon P \to T/P$ there exists some morphism $g'\colon X \to T/P$ such that $g = g'\pi$. Since f is a left $\mathrm{add}(T/P)$-approximation of X, there is a morphism $g''\colon T' \to T/P$ such that $g' = g''f$. Thus we get a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{g} & T/P \\ {\scriptstyle\pi}\big\downarrow & \overset{g'}{\nearrow} & \big\uparrow{\scriptstyle g''} \\ X & \xrightarrow[f]{} & T'. \end{array}$$

This implies $g = g'\pi = g''f\pi = g''h$. Thus g factors through h.

Since there is no loop at S_P, all non-isomorphisms $P \to P$ factor through T/P. Thus by the argument above they factor through h. Thus the cokernel Z of $\mathrm{Hom}_\Lambda(h,T)$ must be 1-dimensional, which implies $Z \cong S_P$. Since $Y \in \mathrm{add}(T)$, we know that $\mathrm{Hom}_\Lambda(Y,T)$ is projective. Thus the above is a projective resolution of S_P, so $\mathrm{proj.dim}_E(S_P) \leq 2$.

This finishes the proof of Proposition 6.2. □

Corollary 6.3. $\mathrm{rep.dim}(\Lambda) \leq 3$.

Proof. Clearly, the module T_Q from Section 2.7 is a generator-cogenerator of $\mathrm{mod}(\Lambda)$. The quiver of T_Q has no loops, thus by Proposition 6.2 we know that $\mathrm{gl.dim}(\mathrm{End}_\Lambda(T_Q)) = 3$. This implies $\mathrm{rep.dim}(\Lambda) \leq 3$. □

The statements in the following theorem are presented in the order in which we prove them.

Theorem 6.4. *Let T be a basic maximal rigid Λ-module, and set $E = \mathrm{End}_\Lambda(T)$. Then the following hold:*

1. *The quiver of E has no loops;*
2. $\mathrm{gl.dim}(E) = 3$*;*
3. *T is maximal 1-orthogonal;*
4. $\mathrm{dom.dim}(E) = 3$*;*
5. *The quiver of E has no sinks and no sources;*
6. *For all simple E-modules S we have $\mathrm{Ext}^1_E(S,S) = 0$ and $\mathrm{Ext}^2_E(S,S) = 0$;*
7. *The quiver of E has no 2-cycles.*

Proof. By Theorem 5.4 we know that $\mathrm{End}_\Lambda(T_Q)$ and $\mathrm{End}_\Lambda(T)$ are derived equivalent, where T_Q is the complete rigid module mentioned in Section 2.7. Since the quiver of $\mathrm{End}_\Lambda(T_Q)$ has no loops, Proposition 6.2 implies that $\mathrm{gl.dim}(\mathrm{End}_\Lambda(T_Q)) = 3 < \infty$. This implies $\mathrm{gl.dim}(\mathrm{End}_\Lambda(T)) < \infty$. Thus by Theorem 3.8 the quiver of $\mathrm{End}_\Lambda(T)$ has no loops. Then again Proposition 6.2 yields $\mathrm{gl.dim}(\mathrm{End}_\Lambda(T)) = 3$. Thus T is maximal 1-orthogonal by Theorem 4.1, and by Theorem 4.2 we get $\mathrm{dom.dim}(\mathrm{End}_\Lambda(T)) = 3$. (It follows from the definitions that for an algebra A and some $n \geq 1$, $\mathrm{gl.dim}(A) = n$ implies $\mathrm{dom.dim}(A) \leq n$.) This proves parts (1)-(4) of the theorem.

For any indecomposable direct summand M of T there are non-zero homomorphisms $M \to P_i$ and $P_j \to M$ for some indecomposable projective Λ-modules P_i and P_j which are not isomorphic to M. Thus the vertex S_M in the quiver of E is neither a sink nor a source. So (5) is proved.

Since the quiver of E has no loops, we have $\mathrm{Ext}^1_E(S,S) = 0$ for all simple E-modules S. Let X be a non-projective direct summand of T. In the proof of Proposition 6.2, we constructed a projective resolution

$$0 \to \mathrm{Hom}_\Lambda(X,T) \to \mathrm{Hom}_\Lambda(T'',T) \to \mathrm{Hom}_\Lambda(T',T) \to \mathrm{Hom}_\Lambda(X,T) \to S_X \to 0,$$

and we also know that $X \notin \mathrm{add}(T'')$. Thus applying $\mathrm{Hom}_E(-,S_X)$ to this resolution yields $\mathrm{Ext}^2_E(S_X,S_X) = 0$. Next, assume P is an indecomposable projective direct summand of T. As in the proof of Proposition 6.2 we have a projective resolution

$$0 \to \mathrm{Hom}_\Lambda(Y,T) \to \mathrm{Hom}_\Lambda(T',T) \xrightarrow{\mathrm{Hom}_\Lambda(h,T)} \mathrm{Hom}_\Lambda(P,T) \to S_P \to 0$$

where $P \notin \mathrm{add}(T')$. Since the module T' projects onto Y, we conclude that $P \notin \mathrm{add}(Y)$. Applying $\mathrm{Hom}_E(-,S_P)$ to the above resolution of S_P yields $\mathrm{Ext}^2_E(S_P,S_P) = 0$. This finishes the proof of (6).

We proved that $\mathrm{Ext}^2_E(S,S) = 0$ for all simple E-modules S. We also know that $\mathrm{gl.dim}(E) = 3 < \infty$. Then it follows from Proposition 3.11 that the quiver of E cannot have 2-cycles. Thus (7) holds. This finishes the proof. □

Corollary 6.5. *Let $T = T_1 \oplus \cdots \oplus T_r$ be a basic maximal rigid Λ-module with T_i indecomposable for all i. For a non-projective $X = T_i$ let*

$$0 \to X \to T' \to Y \to 0$$

be the corresponding exchange sequence starting in X. Then the following hold:

- *We have* $\dim \operatorname{Ext}^1_\Lambda(Y, X) = \dim \operatorname{Ext}^1_\Lambda(X, Y) = 1$, *and the exchange sequence ending in X is of the form*

$$0 \to Y \to T'' \to X \to 0$$

for some $T'' \in \operatorname{add}(T/X)$;
- *The simple* $\operatorname{End}_\Lambda(T)$*-module S_X has a minimal projective resolution of the form*

$$0 \to \operatorname{Hom}_\Lambda(X, T) \to \operatorname{Hom}_\Lambda(T'', T) \to \operatorname{Hom}_\Lambda(T', T) \to \operatorname{Hom}_\Lambda(X, T) \to S_X \to 0;$$

- *We have* $\operatorname{add}(T') \cap \operatorname{add}(T'') = 0$.

Proof. By Theorem 6.4 the quiver of $\operatorname{End}_\Lambda(T)$ has no loops. Now Lemma 6.1 yields that $\dim \operatorname{Ext}^1_\Lambda(Y, X) = 1$, and by Lemma 3.12 we get $\dim \operatorname{Ext}^1_\Lambda(X, Y) = 1$. Corollary 5.8 implies that the exchange sequence ending in X starts at Y. The minimal projective resolution of S_X is obtained from the proof of Proposition 6.2. By Theorem 6.4 there are no 2-cycles in the quiver of $\operatorname{End}_\Lambda(T)$. This implies $\operatorname{add}(T') \cap \operatorname{add}(T'') = 0$. □

6.2. **Ext-group symmetries.**

Proposition 6.6. *Let T be a basic maximal rigid Λ-module, and let X be a non-projective indecomposable direct summand of T. Set $E = \operatorname{End}_\Lambda(T)$. Then for any simple E-module S we have*

$$\dim \operatorname{Ext}^{3-i}_E(S_X, S) = \dim \operatorname{Ext}^i_E(S, S_X)$$

where $0 \leq i \leq 3$.

Proof. We have $S = S_Z$ for some indecomposable direct summand Z of T. Since S_X and S_Z are simple, we get

$$\dim \operatorname{Hom}_E(S_X, S_Z) = \dim \operatorname{Hom}_E(S_Z, S_X) = \begin{cases} 1 & \text{if } S_Z \cong S_X, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$(2) \qquad 0 \to \operatorname{Hom}_\Lambda(X, T) \to \operatorname{Hom}_\Lambda(T'', T) \to \operatorname{Hom}_\Lambda(T', T) \to \operatorname{Hom}_\Lambda(X, T) \to S_X \to 0$$

be the minimal projective resolution of S_X as constructed in the proof of Proposition 6.2. Dually we get a minimal injective resolution

(3)

$$0 \to S_X \to \mathrm{D}\operatorname{Hom}_\Lambda(T, X) \to \mathrm{D}\operatorname{Hom}_\Lambda(T, T'') \to \mathrm{D}\operatorname{Hom}_\Lambda(T, T') \to \mathrm{D}\operatorname{Hom}_\Lambda(T, X) \to 0$$

of S_X. We apply $\operatorname{Hom}_E(-, S_Z)$ to (2) and $\operatorname{Hom}_E(S_Z, -)$ to (3) and get

$$\dim \operatorname{Ext}^3_E(S_X, S_Z) = \dim \operatorname{Ext}^3_E(S_Z, S_X) = \begin{cases} 1 & \text{if } S_Z \cong S_X, \\ 0 & \text{otherwise.} \end{cases}$$

Here we use that $X \notin \operatorname{add}(T' \oplus T'')$. Similarly, if $S_Z \cong S_X$, then $\operatorname{Ext}^1_E(S_X, S_Z) = 0 = \operatorname{Ext}^2_E(S_X, S_Z)$ and $\operatorname{Ext}^1_E(S_Z, S_X) = 0 = \operatorname{Ext}^2_E(S_Z, S_X)$. Thus the proposition is true for $S_Z \cong S_X$.

From now on assume $S_X \ncong S_Z$. The projective resolution (2) yields a complex

$$0 \to \operatorname{Hom}_E(\operatorname{Hom}_\Lambda(T', T), S_Z) \xrightarrow{\theta} \operatorname{Hom}_E(\operatorname{Hom}_\Lambda(T'', T), S_Z) \to 0$$

whose homology groups are the extension groups $\mathrm{Ext}^i_E(S_X, S_Z)$. By Corollary 6.5 we have $\mathrm{add}(T') \cap \mathrm{add}(T'') = 0$, which implies that either $\mathrm{Hom}_E(\mathrm{Hom}_\Lambda(T', T), S_Z) = 0$ or $\mathrm{Hom}_E(\mathrm{Hom}_\Lambda(T'', T), S_Z) = 0$. Hence $\theta = 0$. Therefore we have $\dim \mathrm{Ext}^2_E(S_X, S_Z) = [T'' : Z]$. The discussion in Section 3.2 yields

$$\dim \mathrm{Ext}^1_E(S_X, S_Z) = [T' : Z] = \text{number of arrows } S_X \to S_Z \text{ in } \Gamma_T,$$
$$\dim \mathrm{Ext}^1_E(S_Z, S_X) = [T'' : Z] = \text{number of arrows } S_Z \to S_X \text{ in } \Gamma_T.$$

(As before Γ_T denotes the quiver of $E = \mathrm{End}_\Lambda(T)$.)

This implies $\dim \mathrm{Ext}^2_E(S_X, S_Z) = \dim \mathrm{Ext}^1_E(S_Z, S_X)$. Using the injective resolution (3) we get $\dim \mathrm{Ext}^2_E(S_Z, S_X) = \dim \mathrm{Ext}^1_E(S_X, S_Z)$. This finishes the proof. □

6.3. **The graph of basic maximal rigid modules.**

Proposition 6.7. *Let T be a basic maximal rigid Λ-module, and let X be an indecomposable direct summand of T. If X is non-projective, then up to isomorphism there exists exactly one indecomposable Λ-module Y such that $X \ncong Y$ and $Y \oplus T/X$ is maximal rigid.*

Proof. Recall that every maximal rigid Λ-module is maximal 1-orthogonal. Proposition 5.6 yields an exchange sequence

$$0 \to X \to T' \to Y \to 0$$

such that Y satisfies the required properties. Proposition 4.5 implies that Y is uniquely determined up to isomorphism. □

Let T be a maximal rigid Λ-module. Let $\mathcal{T}_{\mathrm{End}_\Lambda(T)}$ be the graph of basic tilting modules over $\mathrm{End}_\Lambda(T)$. By definition the vertices of this graph are the isomorphism classes of basic tilting modules over $\mathrm{End}_\Lambda(T)$. Two such basic tilting modules M_1 and M_2 are connected with an edge if and only if $M_1 = M \oplus M_1'$ and $M_2 = M \oplus M_2'$ for some M and some indecomposable modules M_1' and M_2' with $M_1' \ncong M_2'$.

Similarly, let $\mathcal{T}_\Lambda$ be the graph with set of vertices the isomorphism classes of basic maximal rigid Λ-modules, and an edge between vertices T_1 and T_2 if and only if $T_1 = T \oplus T_1'$ and $T_2 = T \oplus T_2'$ for some T and some indecomposable modules T_1' and T_2' with $T_1' \ncong T_2'$.

Lemma 6.8. *Let $T = T_1 \oplus \cdots \oplus T_r$ be a basic maximal rigid Λ-module with T_i indecomposable for all i, and assume that $T_{r-n+1}, \dots, T_r$ are projective. There are exactly n indecomposable projective-injective $\mathrm{End}_\Lambda(T)$-modules up to isomorphism, namely $\mathrm{Hom}_\Lambda(T_{r-n+1}, T), \dots, \mathrm{Hom}_\Lambda(T_r, T)$. These are direct summands of any tilting module over $\mathrm{End}_\Lambda(T)$.*

Proof. For $1 \le i \le r-n$ we know that $\mathrm{Hom}_\Lambda(\mu_{T_i}(T), T)$ is a tilting module over $\mathrm{End}_\Lambda(T)$, which does not contain $\mathrm{Hom}_\Lambda(T_i, T)$ as a direct summand. Thus $\mathrm{Hom}_\Lambda(T_i, T)$ cannot be projective-injective, otherwise it would occur as a direct summand of any tilting module over $\mathrm{End}_\Lambda(T)$.

Let $\nu_\Lambda = \mathrm{D}\,\mathrm{Hom}_\Lambda(-, \Lambda)$ be the Nakayama automorphism of $\mathrm{mod}(\Lambda)$. It is well known that ν_Λ maps projective modules to injective ones. But Λ is a selfinjective algebra. This implies that for $r-n+1 \le i \le r$ we get

$$\mathrm{Hom}_\Lambda(T_i, T) \cong \mathrm{D}\,\mathrm{Hom}_\Lambda(T, \nu_\Lambda(T_i)) = \mathrm{D}\,\mathrm{Hom}_\Lambda(T, T_j)$$

for some j. In particular, $\mathrm{Hom}_\Lambda(T_i, T)$ is a projective-injective $\mathrm{End}_\Lambda(T)$-module. □

Proposition 6.9. *Let T be a basic maximal rigid Λ-module. The functor*

$$F_T\colon \mathrm{mod}(\Lambda) \to \mathrm{mod}(\mathrm{End}_\Lambda(T))$$

induces an embedding of graphs

$$\psi_T\colon \mathcal{T}_\Lambda \to \mathcal{T}_{\mathrm{End}_\Lambda(T)}$$

whose image is a union of connected components of $\mathcal{T}_{\mathrm{End}_\Lambda(T)}$. *Each vertex of* $\mathcal{T}_\Lambda$ *(and therefore each vertex of the image of* ψ_T*) has exactly* $r-n$ *neighbours.*

Proof. By Theorem 4.3 we know that every vertex of $\mathcal{T}_\Lambda$ gets mapped to a vertex of $\mathcal{T}_{\mathrm{End}_\Lambda(T)}$. Proposition 4.4 implies that F_T reflects isomorphism classes, therefore ψ_T is injective on vertices. Proposition 5.6 and Proposition 6.7 and its proof yield that ψ_T is injective on edges as well. It also follows that every vertex of $\mathcal{T}_\Lambda$ has exactly $r-n$ neighbours. Thus every vertex in the image of ψ_T has at least $r-n$ neighbours. But by [35, Proposition 1.3] there are at most two complements for an almost complete tilting module. Combining this with Lemma 6.8 implies that every vertex in the image of ψ_T has exactly $r-n$ neighbours. Thus the image of ψ_T is a union of connected components. □

Conjecture 6.10. *The graphs* $\mathcal{T}_\Lambda$ *and* $\mathcal{T}_{\mathrm{End}_\Lambda(T)}$ *are connected.*

To prove Conjecture 6.10 it is enough to show that $\mathcal{T}_{\mathrm{End}_\Lambda(T)}$ is connected.

7. From mutation of modules to mutation of matrices

In this section we prove Theorem 2.6.

7.1. Let $B=(b_{ij})$ be an $l\times m$-matrix with real entries such that $l\geq m$, and let $k\in[1,m]$. Following Fomin and Zelevinsky, the *mutation* of B in direction k is an $l\times m$-matrix

$$\mu_k(B)=(b'_{ij})$$

defined by

$$b'_{ij}=\begin{cases}-b_{ij} & \text{if } i=k \text{ or } j=k,\\ b_{ij}+\dfrac{|b_{ik}|b_{kj}+b_{ik}|b_{kj}|}{2} & \text{otherwise,}\end{cases}$$

where $i\in[1,l]$ and $j\in[1,m]$. If $l=m$ and B is skew-symmetric, then it is easy to check that $\mu_k(B)$ is also a skew-symmetric matrix.

For an $m\times m$-matrix B and some $k\in[1,m]$ we define an $m\times m$-matrix $S=S(B,k)=(s_{ij})$ by

$$s_{ij}=\begin{cases}-\delta_{ij}+\dfrac{|b_{ij}|-b_{ij}}{2} & \text{if } i=k,\\ \delta_{ij} & \text{otherwise.}\end{cases}$$

By S^t we denote the transpose of the matrix $S=S(B,k)$. If B is skew-symmetric, then one easily checks that

$$S^2=1$$

is the identity matrix. The following lemma is a special case of [4, (3.2)]. For convenience we include a proof.

Lemma 7.1. *Let* B *be a skew-symmetric* $m\times m$*-matrix, and let* $S=S(B,k)$ *for some* $k\in[1,m]$*. Then we have*

$$\mu_k(B)=S^tBS.$$

Proof. Let b'_{ij} be the ijth entry of S^tBS. Thus

$$b'_{ij}=\sum_{p=1}^m\sum_{q=1}^m s_{pi}b_{pq}s_{qj}.$$

Since B is skew-symmetric we have $b_{ll} = 0$ for all l. From the definition of S we get

$$b'_{ij} = \begin{cases} s_{kk}b_{kj}s_{jj} & \text{if } i = k, \\ s_{ii}b_{ik}s_{kk} & \text{if } j = k, \\ s_{ii}b_{ij}s_{jj} + s_{ii}b_{ik}s_{kj} + s_{ki}b_{kj}s_{jj} & \text{otherwise.} \end{cases}$$

Now an easy calculation yields $\mu_k(B) = S^t BS$. □

7.2. Let A be a finite-dimensional algebra, and let $P_1, \dots, P_m$ be a complete set of representatives of isomorphism classes of indecomposable projective A-modules. By S_i we denote the top of P_i. Thus $S_1, \dots, S_m$ is a complete set of representatives of isomorphism classes of simple A-modules. The *Cartan matrix* C of A is by definition $C = (c_{ij})_{1 \le i,j \le m}$ where

$$c_{ij} = \dim \operatorname{Hom}_A(P_i, P_j).$$

Now assume that the global dimension of A is finite. Then the *Ringel form* of A is a bilinear form $\langle -,- \rangle \colon \mathbb{Z}^m \times \mathbb{Z}^m \to \mathbb{Z}$ defined by

$$\langle \underline{\dim}(M), \underline{\dim}(N) \rangle = \sum_{i \ge 0} (-1)^i \dim \operatorname{Ext}^i_A(M, N)$$

where $M, N \in \operatorname{mod}(A)$. Here we use the convention $\operatorname{Ext}^0_A(M,N) = \operatorname{Hom}_A(M,N)$. We often just write $\langle M, N \rangle$ instead of $\langle \underline{\dim}(M), \underline{\dim}(N) \rangle$. The matrix of the Ringel form of A is then

$$R = (\langle S_i, S_j \rangle)_{1 \le i,j \le m}.$$

The following lemma can be found in [36, Section 2.4]:

Lemma 7.2. *If* $\operatorname{gl.dim}(A) < \infty$, *then the Cartan matrix* C *of* A *is invertible, and we have*

$$R = C^{-t},$$

where C^{-t} *denotes the transpose of the inverse of* C.

7.3. Let $T = T_1 \oplus \cdots \oplus T_r$ be a basic complete rigid Λ-module with T_i indecomposable for all i. Without loss of generality we assume that $T_{r-n+1}, \dots, T_r$ are projective. For $1 \le i \le r$ let S_i be the simple $\operatorname{End}_\Lambda(T)$-module corresponding to T_i. The matrix

$$C_T = (c_{ij})_{1 \le i,j \le r}$$

where

$$c_{ij} = \dim \operatorname{Hom}_{\operatorname{End}_\Lambda(T)}(\operatorname{Hom}_\Lambda(T_i, T), \operatorname{Hom}_\Lambda(T_j, T)) = \dim \operatorname{Hom}_\Lambda(T_j, T_i)$$

is the Cartan matrix of the algebra $\operatorname{End}_\Lambda(T)$.

By Theorem 6.4 we know that $\operatorname{gl.dim}(\operatorname{End}_\Lambda(T)) = 3$. Thus by Lemma 7.2

$$R_T = (r_{ij})_{1 \le i,j \le r} = C_T^{-t}$$

is the matrix of the Ringel form of $\operatorname{End}_\Lambda(T)$, where

$$r_{ij} = \langle S_i, S_j \rangle = \sum_{i=0}^{3} (-1)^i \dim \operatorname{Ext}^i_{\operatorname{End}_\Lambda(T)}(S_i, S_j).$$

Lemma 7.3. *Assume that* $i \le r - n$ *or* $j \le r - n$. *Then the following hold:*

- $r_{ij} = \dim \operatorname{Ext}^1_{\operatorname{End}_\Lambda(T)}(S_j, S_i) - \dim \operatorname{Ext}^1_{\operatorname{End}_\Lambda(T)}(S_i, S_j)$;
- $r_{ij} = -r_{ji}$;
- $r_{ij} = \begin{cases} \textit{number of arrows } j \to i \textit{ in } \Gamma_T & \textit{if } r_{ij} > 0, \\ -(\textit{number of arrows } i \to j \textit{ in } \Gamma_T) & \textit{if } r_{ij} < 0, \\ 0 & \textit{otherwise.} \end{cases}$

Proof. The first two statements follow from Proposition 6.6. Since by Theorem 6.4 there are no 2-cycles in the quiver of $\mathrm{End}_\Lambda(T)$, at least one of the two summands in the equation

$$r_{ij} = \dim \mathrm{Ext}^1_{\mathrm{End}_\Lambda(T)}(S_j, S_i) - \dim \mathrm{Ext}^1_{\mathrm{End}_\Lambda(T)}(S_i, S_j)$$

has to be 0. Now the third statement follows from the remarks in Section 3.2. □

Recall that $B(T) = (t_{ij})_{1 \leq i,j \leq r}$ is the $r \times r$-matrix defined by

$$t_{ij} = (\text{number of arrows } j \to i \text{ in } \Gamma_T) - (\text{number of arrows } i \to j \text{ in } \Gamma_T).$$

Let $B(T)^\circ = (t_{ij})$ and $R_T^\circ = (r_{ij})$ be the $r \times (r-n)$-matrices obtained from $B(T)$ and R_T, respectively, by deleting the last n columns. As a consequence of Lemma 7.3 we get the following:

Corollary 7.4. $R_T^\circ = B(T)^\circ$.

Note that the dimension vector of the indecomposable projective $\mathrm{End}_\Lambda(T)$-module $\mathrm{Hom}_\Lambda(T_i, T)$ is the ith column of the matrix C_T.

For $1 \leq k \leq r-n$ let

$$0 \to T_k \to T' \to T_k^* \to 0 \tag{4}$$

and

$$0 \to T_k^* \to T'' \to T_k \to 0 \tag{5}$$

be exchange sequences associated to the direct summand T_k of T. Keeping in mind the remarks in Section 3.2, it follows from Lemma 7.3 that

$$T' = \bigoplus_{r_{ik}>0} T_i^{r_{ik}} \quad \text{and} \quad T'' = \bigoplus_{r_{ik}<0} T_i^{-r_{ik}}.$$

Set

$$T^* = \mu_{T_k}(T) = T_k^* \oplus T/T_k$$

and $S = S(R_T, k)$.

Proposition 7.5. *With the above notation we have*

$$C_{T^*} = S C_T S^t.$$

Proof. Set $C_T' = S C_T S^t = (c_{ij}')_{1 \leq i,j \leq r}$. Thus we have

$$c_{ij}' = \sum_{p=1}^r \sum_{q=1}^r s_{ip} c_{pq} s_{jq}$$

where

$$s_{ip} = \begin{cases} -\delta_{ip} + \dfrac{|r_{ip}| - r_{ip}}{2} & \text{if } i = k, \\ \delta_{ip} & \text{otherwise} \end{cases}$$

and

$$s_{jq} = \begin{cases} -\delta_{qj} + \dfrac{|r_{qj}| + r_{qj}}{2} & \text{if } j = k, \\ \delta_{qj} & \text{otherwise.} \end{cases}$$

It follows that the transformation $C_T \mapsto C_T'$ only changes the kth row and the kth column of C_T. We denote the ijth entry of C_{T^*} by c_{ij}^*. Clearly, $c_{ij}^* = c_{ij} = c_{ij}'$ provided $i \neq k$ and $j \neq k$. Assume $i \neq k$ and $j = k$. We get

$$c_{ik}' = \sum_{q=1}^r c_{iq} s_{kq} = -c_{ik} + \sum_{r_{qk}>0} r_{qk} c_{iq}.$$

Applying $\mathrm{Hom}_\Lambda(-, T_i)$ to Sequence (4) yields

$$\begin{aligned} c_{ik}^* &= \dim \mathrm{Hom}_\Lambda(T_k^*, T_i) = -\dim \mathrm{Hom}_\Lambda(T_k, T_i) + \sum_{r_{qk}>0} r_{qk} \dim \mathrm{Hom}_\Lambda(T_q, T_i) \\ &= -c_{ik} + \sum_{r_{qk}>0} r_{qk} c_{iq}. \end{aligned}$$

Thus $c'_{ik} = c^*_{ik}$. Similarly, if $i = k$ and $j \neq k$, then

$$c'_{kj} = \sum_{p=1}^{r} s_{kp} c_{pj} = -c_{kj} + \sum_{r_{kp}<0} |r_{kp}| c_{pj}.$$

Applying $\mathrm{Hom}_\Lambda(T_j, -)$ to Sequence (4) yields

$$\begin{aligned} c_{kj}^* &= \dim \mathrm{Hom}_\Lambda(T_j, T_k^*) = -\dim \mathrm{Hom}_\Lambda(T_j, T_k) + \sum_{r_{pk}>0} r_{pk} \dim \mathrm{Hom}_\Lambda(T_j, T_p) \\ &= -c_{kj} + \sum_{r_{pk}>0} r_{pk} c_{pj} = -c_{kj} + \sum_{r_{kp}<0} |r_{kp}| c_{pj}. \end{aligned}$$

Thus $c'_{kj} = c^*_{kj}$. Finally, let $i = j = k$. Thus

$$c'_{kk} = \sum_{p=1}^{r} \sum_{q=1}^{r} s_{kp} c_{pq} s_{kq} = c_{kk} - \sum_{r_{kp}<0} |r_{kp}| c_{pk} - \sum_{r_{qk}>0} r_{qk} c_{kq} + \sum_{r_{qk}>0} \sum_{r_{kp}<0} r_{qk} |r_{kp}| c_{pq}.$$

We apply $\mathrm{Hom}_\Lambda(-, T_k^*)$ to Sequence (4) and get

$$c_{kk}^* = \dim \mathrm{Hom}_\Lambda(T_k^*, T_k^*) = -\dim \mathrm{Hom}_\Lambda(T_k, T_k^*) + \sum_{r_{qk}>0} r_{qk} \dim \mathrm{Hom}_\Lambda(T_q, T_k^*).$$

By applying $\mathrm{Hom}_\Lambda(T_k, -)$ and $\mathrm{Hom}_\Lambda(T_q, -)$ to the same sequence, we can compute the values $\dim \mathrm{Hom}_\Lambda(T_k, T_k^*)$ and $\dim \mathrm{Hom}_\Lambda(T_q, T_k^*)$ in the above equation. We get

$$\begin{aligned} c_{kk}^* &= c_{kk} - \sum_{r_{pk}>0} r_{pk} c_{pk} + \left(\sum_{r_{qk}>0} r_{qk} \left(-c_{kq} + \sum_{r_{pk}>0} r_{pk} c_{pq} \right) \right) \\ &= c_{kk} - \sum_{r_{pk}>0} r_{pk} c_{pk} - \sum_{r_{qk}>0} r_{qk} c_{kq} + \sum_{r_{qk}>0} \sum_{r_{pk}>0} r_{qk} r_{pk} c_{pq} \\ &= c_{kk} - \sum_{r_{kp}<0} |r_{kp}| c_{pk} - \sum_{r_{qk}>0} r_{qk} c_{kq} + \sum_{r_{qk}>0} \sum_{r_{kp}<0} r_{qk} |r_{kp}| c_{pq}. \end{aligned}$$

This proves that $c'_{kk} = c^*_{kk}$. □

Note that in the proof of Proposition 7.5 we only made use of Sequence (4). There is an alternative proof using Sequence (5).

Corollary 7.6. *$R_{T^*} = S^t R_T S$.*

Proof. From Proposition 7.5 and and Lemma 7.2 we get

$$C_{T^*}^{-1} = (S^t)^{-1} C_T^{-1} S^{-1} = S^t C_T^{-1} S$$

and therefore

$$R_{T^*} = C_{T^*}^{-t} = S^t C_T^{-t} S = S^t R_T S.$$

□

Corollary 7.7. *$R^\circ_{T^*} = \mu_k(R^\circ_T)$.*

Proof. For an $r \times r$-matrix $B = (b_{ij})_{1 \le i,j \le r}$ define a matrix $B^\vee = (b_{ij}^\vee)_{1 \le i,j \le r}$ by

$$b_{ij}^\vee = \begin{cases} 0 & \text{if } r-n+1 \le i,j \le r, \\ b_{ij} & \text{otherwise.} \end{cases}$$

It follows from Lemma 7.3 that the matrix $R_T^\vee$ is skew-symmetric. One easily checks that $S(R_T^\vee, k) = S(R_T, k) = S$. By Corollary 7.6 we have $R_{T^*} = S^t R_T S$.

It follows from the definition that the matrix S is of the form

$$S = \begin{pmatrix} S_1 & S_2 \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

where $\mathbf{0}$ is the 0-matrix of size $n \times (r-n)$, and $\mathbf{I}$ is the identity matrix of size $n \times n$.

We partition R_T and $R_T^\vee$ into blocks of the corresponding sizes and obtain

$$R_T = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix} \text{ and } R_T^\vee = \begin{pmatrix} R_1 & R_2 \\ R_3 & 0 \end{pmatrix}.$$

This implies

$$R_{T^*} = S^t R_T S = \begin{pmatrix} S_1^t R_1 S_1 & S_1^t R_1 S_2 + S_1^t R_2 \\ S_2^t R_1 S_1 + R_3 S_1 & S_2^t R_1 S_2 + S_2^t R_2 + R_3 S_2 + R_4 \end{pmatrix}.$$

Since $R_T^\vee$ is skew-symmetric and $S = S(R_T^\vee, k)$, Lemma 7.1 implies

$$\mu_k(R_T^\vee) = S^t R_T^\vee S = \begin{pmatrix} S_1^t R_1 S_1 & S_1^t R_1 S_2 + S_1^t R_2 \\ S_2^t R_1 S_1 + R_3 S_1 & S_2^t R_1 S_2 + S_2^t R_2 + R_3 S_2 \end{pmatrix}.$$

It follows from the definitions that $\mu_k(R_T^\circ)$ is obtained from $\mu_k(R_T^\vee)$ by deleting the last n columns. This yields

$$\mu_k(R_T^\circ) = \begin{pmatrix} S_1^t R_1 S_1 \\ S_2^t R_1 S_1 + R_3 S_1 \end{pmatrix} = R_{T^*}^\circ.$$

□

Now we combine Corollary 7.4 and Corollary 7.7 and obtain the following theorem:

Theorem 7.8. $B(\mu_{T_k}(T))^\circ = \mu_k(B(T)^\circ)$.

8. Examples

In this section we want to illustrate some of our results with examples. We often describe modules by just indicating the multiplicities of composition factors in the socle series. For example for the preprojective algebra Λ of type $\mathbb{A}_3$ we write

$$M = \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix}.$$

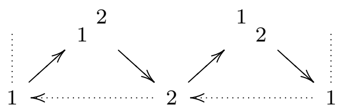
This means that M has a socle isomorphic to the simple labelled by 2, the next layer of the socle series is isomorphic to $1 \oplus 3$, and finally the third layer is isomorphic to 2 again.

The examples discussed are for preprojective algebras of type $\mathbb{A}_2$ and $\mathbb{A}_3$. These are easy to deal with since they are representation finite algebras. The only other finite type case is $\mathbb{A}_4$ (recall that we excluded $\mathbb{A}_1$), and the tame cases are $\mathbb{A}_5$ and $\mathbb{D}_4$. Beyond that, all preprojective algebras are of wild representation type.

8.1. **Case $\mathbb{A}_2$.** Let Λ be the preprojective algebra of type $\mathbb{A}_2$. There are exactly four indecomposable Λ-modules up to isomorphism, namely

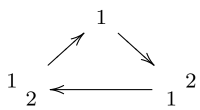
$$T_1 = 1, \qquad T_2 = 2, \qquad T_3 = {}^{1}{}_{2}, \qquad T_4 = {}_{1}{}^{2}.$$

The modules T_3 and T_4 are the indecomposable projective Λ-modules. The Auslander-Reiten quiver of Λ looks as follows:

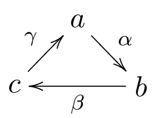


In the above picture one has to identify the two vertical dotted lines. The dotted arrows describe the Auslander-Reiten translation.

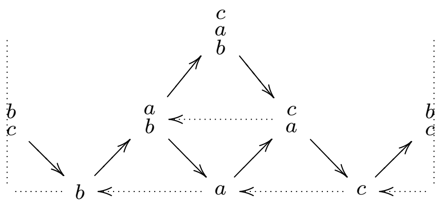
The module $T = T_1 \oplus T_3 \oplus T_4$ is complete rigid, and the quiver of the endomorphism algebra $\mathrm{End}_\Lambda(T)$ looks as follows:



We want to illustrate Proposition 4.4: The algebra $\mathrm{End}_\Lambda(T)$ is isomorphic to the path algebra of the quiver



with zero relations $\beta\alpha$ and $\gamma\beta$. There are exactly 7 indecomposable $\mathrm{End}_\Lambda(T)$-modules, all of which are serial. The Auslander-Reiten quiver of $\mathrm{End}_\Lambda(T)$ looks as follows:



Again, the dotted arrows describe the Auslander-Reiten translation, and the two dotted vertical lines have to be identified.

One can easily check that there are exactly four indecomposable $\mathrm{End}_\Lambda(T)$-modules of projective dimension at most one, namely the three indecomposable projective modules and the simple module corresponding to the vertex c. One easily checks that

$$F_T(T_1) = \begin{smallmatrix} a \\ b \end{smallmatrix}, \qquad F_T(T_2) = c, \qquad F_T(T_3) = \begin{smallmatrix} c \\ a \\ b \end{smallmatrix}, \qquad F_T(T_4) = \begin{smallmatrix} b \\ c \end{smallmatrix},$$

where F_T is the functor defined in Proposition 4.4.

For T we get

$$C_T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad R_T = C_T^{-t} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \quad S = S(R_T, 1) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Besides T there exists only one more basic complete rigid Λ-module, namely $T^* = T_2 \oplus T_3 \oplus T_4$. The endomorphism algebras $\mathrm{End}_\Lambda(T)$ and $\mathrm{End}_\Lambda(T^*)$ are isomorphic. The two corresponding exchange sequences are

$$0 \to 2 \to {}^{1}{}_{2} \to 1 \to 0$$

and

$$0 \to 1 \to \begin{smallmatrix}&2\\1&\end{smallmatrix} \to 2 \to 0.$$

For T^* we get

$$C_{T^*} = \begin{pmatrix}1&0&1\\1&1&1\\0&1&1\end{pmatrix} \text{ and } R_{T^*} = C_{T^*}^{-t} = \begin{pmatrix}0&-1&1\\1&1&-1\\-1&0&1\end{pmatrix}.$$

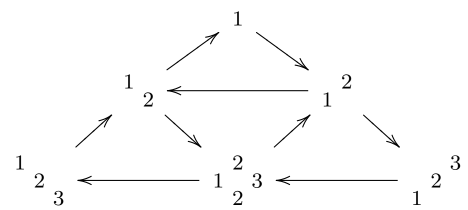
Now one can check that $SC_TS^t = C_{T^*}$ and $S^tR_TS = R_{T^*}$.

8.2. **Case $\mathbb{A}_3$.** Let Λ be the preprojective algebra of type $\mathbb{A}_3$. There are exactly 12 indecomposable Λ-modules up to isomorphism, and all of these are rigid.

Define $T = T_1 \oplus \cdots \oplus T_6$ where

$$T_1 = 1, \quad T_2 = \begin{smallmatrix}1&\\&2\end{smallmatrix}, \quad T_3 = \begin{smallmatrix}&2\\1&\end{smallmatrix}, \quad T_4 = \begin{smallmatrix}1&&\\&2&\\&&3\end{smallmatrix}, \quad T_5 = \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix}, \quad T_6 = \begin{smallmatrix}&&3\\&2&\\1&&\end{smallmatrix}.$$

By computing the dimension of $\mathrm{End}_\Lambda(T)$ one can easily check that T is complete rigid. The quiver Γ_T of the endomorphism algebra $\mathrm{End}_\Lambda(T)$ looks as follows:



Set $T^* = \mu_{T_2}(T)$. It turns out that T^* is obtained from T by replacing the direct summand T_2 by the module

$$T_2^* = \begin{smallmatrix}&2&\\1&&3\end{smallmatrix}.$$

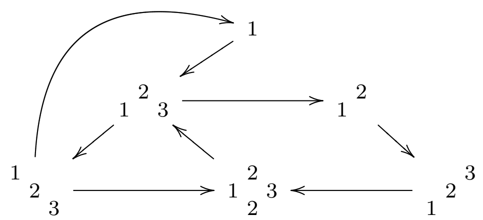
The two exchange sequences are

$$0 \to \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \to \begin{smallmatrix}&2\\1&\end{smallmatrix} \oplus \begin{smallmatrix}1&&\\&2&\\&&3\end{smallmatrix} \to \begin{smallmatrix}1&\\&2\end{smallmatrix} \to 0$$

and

$$0 \to \begin{smallmatrix}1&\\&2\end{smallmatrix} \to 1 \oplus \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix} \to \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \to 0.$$

The quiver Γ_{T^*} of $\mathrm{End}_\Lambda(T^*)$ looks as follows:



For T an easy calculation yields

$$C_T = \begin{pmatrix}1&1&0&1&0&0\\0&1&1&1&1&0\\1&1&1&1&1&0\\0&0&0&1&1&1\\0&1&1&1&2&1\\1&1&1&1&1&1\end{pmatrix} \text{ and } R_T = C_T^{-t} = \begin{pmatrix}0&1&-1&0&0&0\\-1&0&1&1&-1&0\\1&-1&0&0&1&-1\\0&-1&0&1&0&0\\0&1&-1&-1&1&0\\0&0&1&0&-1&1\end{pmatrix},$$

and for T^* we get

$$C_{T^*} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and } R_{T^*} = C_{T^*}^{-t} = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix}.$$

Furthermore, we have

$$S = S(R_T, 2) = \begin{pmatrix} 1 & & & & & \\ 1 & -1 & & & 1 & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}.$$

In the above matrix we only displayed the non-zero entries of S. Now one can check that $SC_TS^t = C_{T^*}$ and $S^tR_TS = R_{T^*}$.

For the case $\mathbb{A}_3$ there are exactly 14 basic complete rigid Λ-modules up to isomorphism.

8.3. **Case** $\mathbb{A}_4$**.** Let Λ be the preprojective algebra of type $\mathbb{A}_4$. There are exactly 40 indecomposable Λ-modules up to isomorphism, and all of these are rigid. The number of isomorphism classes of basic complete rigid Λ-modules is 672. For more details we refer to [15].

9. Relation with cluster algebras

From now on let $K = \mathbb{C}$ be the field of complex numbers.

9.1. We are going to outline the construction of the map $\varphi\colon M \mapsto \varphi_M$ introduced in 1.3. As before let Λ_β be the affine variety of Λ-modules with dimension vector $\beta \in \mathbb{N}^n$. Let $\mathcal{M} = \oplus_{\beta\in\mathbb{N}^n}\mathcal{M}_\beta$ be the algebra of G_β-invariant constructible functions introduced by Lusztig as a geometric model for $U(\mathfrak{n})$. Here $\mathcal{M}_\beta$ is the vector space of functions from Λ_β to $\mathbb{C}$ spanned by certain functions $d_{\mathbf{i}}$ defined as follows. For $M \in \Lambda_\beta$, let $\Phi_{\mathbf{i},M}$ be the variety of composition series of M of type $\mathbf{i} = (i_1, \dots, i_k)$, that is of flags of submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_k = 0$$

with M_{j-1}/M_j isomorphic to the simple module S_{i_j} for $j = 1, \dots, k$. Then

$$d_{\mathbf{i}}(M) := \chi(\Phi_{\mathbf{i},M}), \tag{6}$$

where χ denotes the Euler characteristic.

Let $\mathcal{M}^* = \oplus_{\beta\in\mathbb{N}^n}\mathcal{M}_\beta^*$ be the graded dual of $\mathcal{M}$, and let $\delta_M \in \mathcal{M}^*$ be the linear form which maps a constructible function $f \in \mathcal{M}$ to its evaluation $f(M)$ at M. It is completely determined by the numbers $\delta_M(d_{\mathbf{i}}) = \chi(\Phi_{\mathbf{i},M})$ where $\mathbf{i}$ varies over the possible composition types. Under the isomorphism $\mathcal{M}^* \cong U(\mathfrak{n})^* \cong \mathbb{C}[N]$, δ_M gets identified with a regular function $\varphi_M \in \mathbb{C}[N]$.

To describe φ_M explicitly, we introduce the one-parameter subgroups

$$x_i(t) = \exp(te_i), \quad (t \in \mathbb{C},\ i \in Q_0),$$

of N associated with the Chevalley generators e_i of $\mathfrak{n}$.

Lemma 9.1. *For any sequence* $\mathbf{i} = (i_1, \dots, i_k)$ *of elements of* Q_0*, we have*

$$\varphi_M(x_{i_1}(t_1)\cdots x_{i_k}(t_k)) = \sum_{\mathbf{a}=(a_1,\dots,a_k)\in\mathbb{N}^k} \chi(\Phi_{\mathbf{i}^{\mathbf{a}},M}) \frac{t_1^{a_1}}{a_1!} \cdots \frac{t_k^{a_k}}{a_k!}\ , \qquad (t_1, \dots, t_k \in \mathbb{C}),$$

where we use the short-hand notation $\mathbf{i}^{\mathbf{a}} := (\underbrace{i_1, \dots, i_1}_{a_1}, \dots, \underbrace{i_k, \dots, i_k}_{a_k})$.

The proof follows easily from the classical description of the duality between $U(\mathfrak{n})$ and $\mathbb{C}[N]$. Note that if $\mathbf{i}$ is a reduced word for the longest element of the Weyl group of $\mathfrak{g}$ then the set $\{x_{i_1}(t_1)\cdots x_{i_k}(t_k) \mid t_1, \dots, t_k \in \mathbb{C}\}$ is dense in N, so the above formula completely determines the polynomial function φ_M.

It is shown in [15, Lemma 7.3] that the functions φ_M are multiplicative, in the sense that

$$\varphi_M \cdot \varphi_N = \varphi_{M\oplus N}, \qquad (M, N \in \mathrm{mod}(\Lambda)). \tag{7}$$

9.2. Let Z be an irreducible component of the variety Λ_β. The map φ being constructible, there exists a dense open subset O_Z of Z such that for all $M, N \in O_Z$ we have $\varphi_M = \varphi_N$. A point $M \in O_Z$ is called a *generic point* of Z. Put $\rho_Z = \varphi_M$ for a generic point M of Z. Then the collection $\{\rho_Z\}$ where Z runs over all irreducible components of all varieties Λ_β is dual to the semicanonical basis of $U(\mathfrak{n})$. We call it the *dual semicanonical basis* of $\mathbb{C}[N]$ and denote it by $\mathcal{S}^*$.

If $M \in \Lambda_\beta$ is a rigid Λ-module in the irreducible component Z, its G_β-orbit is open so M is generic and $\varphi_M = \rho_Z$ belongs to $\mathcal{S}^*$.

9.3. Recall the setting of 2.8. From the complete rigid Λ-module $T_Q = T_1 \oplus \cdots \oplus T_r$ of Theorem 2.7 we get the initial seed $(\mathbf{x}(T_Q), B(T_Q)^\circ)$ where

$$\mathbf{x}(T_Q) = (x_1(T_Q), \dots, x_r(T_Q)) := (\varphi_{T_1}, \dots, \varphi_{T_r}).$$

By [16], it coincides with one of the initial seeds of [4] for the upper cluster algebra structure on $\mathbb{C}[N]$. Here we assume as before that $T_{r-n+1}, \dots, T_r$ are the n indecomposable projective Λ-modules, so that $x_1(T_Q), \dots, x_{r-n}(T_Q)$ are the exchangeable cluster variables of $\mathbf{x}(T_Q)$ and the remaining ones generate the ring of coefficients.

Recall the definition of the mutation of a seed $(\mathbf{x}, \widetilde{B})$ in direction $k \in [1, r-n]$. This is the new seed $(\mathbf{x}', \widetilde{B}')$, where $\widetilde{B}' = \mu_k(\widetilde{B})$ is given by 2.5, and $\mathbf{x}'$ is obtained from $\mathbf{x} = (x_1, \dots, x_r)$ by replacing x_k by

$$x'_k = \frac{\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}}{x_k}. \tag{8}$$

Here the exponents b_{ik} are the entries of the matrix $\widetilde{B}$.

The mutation class $\mathcal{C}$ of the seed $(\mathbf{x}(T_Q), B(T_Q)^\circ)$ is defined to be the set of all seeds $(\mathbf{x}, \widetilde{B})$ which can be obtained from $(\mathbf{x}(T_Q), B(T_Q)^\circ)$ by a sequence of mutations.

9.4. We will need the following result of [17].

Theorem 9.2. *Let* M *and* N *be* Λ*-modules such that* $\dim \mathrm{Ext}^1_\Lambda(M, N) = 1$*, and let*

$$0 \to M \to X \to N \to 0 \quad \textit{and} \quad 0 \to N \to Y \to M \to 0$$

be non-split short exact sequences. Then $\varphi_M \cdot \varphi_N = \varphi_X + \varphi_Y$.

9.5. Now everything is ready for the proof of Theorem 2.8. Let $R = R_1 \oplus \cdots \oplus R_r$ be a vertex of $\mathcal{T}_\Lambda^\circ$. Thus R is obtained from T_Q by means of a finite number of mutations, say ℓ. We want to prove that $(\mathbf{x}(R), B(R)^\circ)$ is a seed in $\mathcal{C}$. We argue by induction on ℓ. Put $\mathbf{x}(R) = (\varphi_{R_1}, \dots, \varphi_{R_r})$. If $\ell = 0$ then $R = T_Q$ and we know already that $(\mathbf{x}(R), B(R)^\circ)$ is an initial seed of $\mathcal{C}$. Otherwise we have $R = \mu_{S_k}(S)$ for some vertex S of $\mathcal{T}_\Lambda^\circ$ and some k, and by induction we can assume that $(\mathbf{x}(S), B(S)^\circ)$ is a seed in $\mathcal{C}$. By Corollary 6.5, we know that $\dim \mathrm{Ext}_\Lambda^1(S_k, R_k) = 1$, so we can apply Theorem 9.2. Let

$$0 \to R_k \to X \to S_k \to 0 \quad \text{and} \quad 0 \to S_k \to Y \to R_k \to 0$$

be non-split short exact sequences. Then $\varphi_{R_k} \cdot \varphi_{S_k} = \varphi_X + \varphi_Y$. By Corollary 7.4 and the remarks which follow it, we have

$$X = \bigoplus_{b_{ik}<0} S_i^{-b_{ik}}, \qquad Y = \bigoplus_{b_{ik}>0} S_i^{b_{ik}},$$

where the b_{ik} here are the entries of $B(S)^\circ$. Using (7), it follows that

$$\varphi_X = \prod_{b_{ik}<0} \varphi_{S_i}^{-b_{ik}}, \qquad \varphi_Y = \prod_{b_{ik}>0} \varphi_{S_i}^{b_{ik}}.$$

Hence, comparing with (8) and taking into account Theorem 7.8, we see that $(\mathbf{x}(R), B(R)^\circ)$ is obtained from $(\mathbf{x}(S), B(S)^\circ)$ by a seed mutation in direction k. This shows that the map $R \mapsto (\mathbf{x}(R), B(R)^\circ)$ gives a covering of graphs from $\mathcal{T}_\Lambda^\circ$ to the exchange graph $\mathcal{G}$ of $\mathbb{C}[N]$.

Now if R and R' are such that $\mathbf{x}(R) = \mathbf{x}(R')$, then $\varphi_R = \varphi_{R'}$. In particular, R and R' have the same dimension vector β. Since R and R' are both generic $\varphi_R = \varphi_{R'}$ belongs to $\mathcal{S}^*$ and R and R' have to be in the same irreducible component of Λ_β. Therefore R and R' are isomorphic. Hence the covering of graphs is in fact an isomorphism. (Two seeds which can be obtained from each other by reordering of the entries of the clusters and a corresponding reordering of the columns and rows of the exchange matrices are considered to be identical.)

Finally, using (7), we get that all cluster monomials are of the form φ_M for a rigid module M (not necessarily basic or maximal), hence by 9.2 they belong to $\mathcal{S}^*$.

This finishes the proof of Theorem 2.8.

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Christof Geiss
Instituto de Matemáticas, UNAM
Ciudad Universitaria
04510 Mexico D.F.
Mexico
E-mail address: christof@math.unam.mx

Bernard Leclerc
Laboratoire LMNO
Université de Caen
F-14032 Caen Cedex
France
E-mail address: leclerc@math.unicaen.fr

Jan Schröer
Mathematisches Institut
Universität Bonn
Beringstr. 1
D-53115 Bonn
Germany
E-mail address: schroer@math.uni-bonn.de