RADICAL EMBEDDINGS AND REPRESENTATION DIMENSION

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Abstract. Given a representation-finite algebra B and a subalgebra A of B such that the Jacobson radicals of A and B coincide, we prove that the representation dimension of A is at most three. By a result of Igusa and Todorov, this implies that the finitistic dimension of A is finite.

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1. Introduction and main result

In representation theory there are various homological invariants which measure the deviation of an algebra or its module category from a nice situation which is well understood. Among the first and most important of such invariants is the global dimension of an algebra which roughly speaking describes how far the algebra is away from being semisimple, where all modules are projective. A highlight in exploiting this invariant is the Auslander-Buchsbaum-Serre Theorem, proved in the 1950's, which states that an algebraic variety over an algebraically closed field is smooth if and only if its coordinate ring has finite global dimension, and in this case the global dimension is equal to the dimension of the variety.

Taking the point of view that finite-dimensional modules for an algebra A are to be studied via their endomorphism rings, leads to the concept of representation dimension. This homological invariant was introduced by Auslander around 1970 in [1]. It describes the minimal global dimension of such endomorphism rings for modules which are roughly speaking not too small. To be precise

$$\operatorname{repdim}(A) = \min\{\operatorname{gldim}(\operatorname{End}_A(Z))\}$$

taking the minimum over all A-modules Z which are generator-cogenerators for A. (A generator-cogenerator is a module which has all indecomposable projective and all indecomposable injective modules as direct summands.) It is shown in [13] that $\operatorname{repdim}(A) < \infty$ for all algebras A. Auslander proved in [1] that $\operatorname{repdim}(A) \leq 2$ if and only if A is representation finite (that is, there are only finitely many non-isomorphic indecomposable A-modules).

Mathematics Subject Classification (2000): 16E05, 16E10, 16G20.
Keywords: Representation dimension; Radical embeddings; Finitistic Dimension Conjecture; Special biserial algebras; Quasi-hereditary algebras.

Apart from this, there were only few examples where the precise value of the representation dimension was known.
Our main result is the following.

Theorem 1.1. *Suppose A is an algebra such that there is a radical embedding $f : A \to B$ with B a representation finite algebra, then* $\operatorname{repdim}(A) \leq 3$.

A radical embedding $f : A \to B$ is an algebra monomorphism which maps the Jacobson radical of A onto the Jacobson radical of B. This can be applied to a large class of algebras, we will give details later.

Algebras with small representation dimension are of particular interest due to a result by Igusa and Todorov. They proved that any algebra A with representation dimension ≤ 3 satisfies $\operatorname{findim}(A) < \infty$, i.e. the (second) finitistic dimension conjecture holds for A (Corollary 0.8 in [11]).

In case of infinite global dimension usually many modules have finite projective dimension. In this situation one would like to know whether or not these finite projective dimensions are bounded. For a ring A and an A-module M, let $\operatorname{projdim}(M)$ denote its projective dimension, that is the minimal length of a projective resolution. The little finitistic dimension of A is defined to be

$$\operatorname{findim}(A) = \sup\{\operatorname{projdim}(M) : \operatorname{projdim}(M) < \infty\}$$

the supremum taken over all finitely generated A-modules. Similarly the big finitistic dimension $\operatorname{Findim}(A)$ is defined, allowing arbitrary A-modules.

If A is commutative noetherian then $\operatorname{Findim}(A)$ is equal to the Krull dimension of A ([4], [10]), and if in addition A is local then $\operatorname{findim}(A)$ is equal to the depth of A; in particular the little and the big finitistic dimensions coincide if and only if A is Cohen-Macaulay. The finitistic dimensions are far less understood for non-commutative rings.

We assume that A is a finite-dimensional associative algebra over a field k. In [3] Bass formulated two so-called finitistic dimension conjectures. The first one asserts that $\operatorname{findim}(A) = \operatorname{Findim}(A)$. A counterexample was given in [20]; and [15] shows that the difference can be arbitrary large.

The second finitistic dimension conjecture, which is open in general, states that $\operatorname{findim}(A) < \infty$. It was proved to be true only for few classes of algebras: monomial algebras [8], algebras where the cube of the radical is zero [9], and a few more special cases [2], [16], [19]. If the finitistic dimension conjecture holds then some other well-known homological conjectures also hold, this is explained for example in [21].

To check whether the finitistic dimension is finite is very difficult, since (at least with the naive approach) we have to compute the projective dimension of all modules. To prove that the representation dimension of an algebra is at most three, one 'just' has to guess a suitable module, which is a generator-cogenerator and whose endomorphism algebra has global dimension at most three. So the result of Igusa and Todorov gives a new possible strategy to prove the finitistic dimension conjecture for particular classes of algebras.

Up to now there are no examples known where the representation dimension is bigger than three.[1] For further results on the representation dimension of algebras we refer to [1], [17] and [18]. For proving that a particular algebra has representation dimension at most three, the following result of the present paper is often useful:

Proposition 1.2. *Let A be a basic algebra, and let P be an indecomposable projective-injective A-module. Define $B = A/\operatorname{soc}(P)$. If* $\operatorname{repdim}(B) \leq 3$*, then* $\operatorname{repdim}(A) \leq 3$.

We remark that in the above situation every indecomposable A-module, except P itself, is a B-module, by the rejection lemma of Drozd and Kiricenko [7].

An important class of algebras is given by the *special biserial* algebras (for the definition see section 4). Special biserial algebras have tame representation type. Well known examples are given by blocks of group algebras with cyclic or dihedral defect group, and by algebras occurring in the Gelfand-Ponomarev classification of Harish-Chandra modules over the Lorentz group.

In [14] it was proved that all special biserial algebras with at most two simple modules have finite finitistic dimension. The following application of our main result Theorem 1.1 proves the finitistic dimension conjecture for all special biserial algebras:

Corollary 1.3. *If A is a special biserial algebra, then we have* $\operatorname{repdim}(A) \leq 3$ *and* $\operatorname{findim}(A) < \infty$.

The proof of Theorem 1.1 yields an explicit construction of a generator-cogenerator, whose endomorphism algebra has the desired global dimension. Namely, as before, let $f : A \to B$ be a radical embedding with B representation-finite. Let N be the direct sum of a complete set of representatives of the isomorphism classes of indecomposable B-modules. Note that we can consider N also as an A-module. Then define $C_f = A \oplus A^* \oplus N$. We get the following result on the structure of the endomorphism ring (for the definition of quasi-hereditary algebras see section 5):

Theorem 1.4. *If $f : A \to B$ is a radical embedding with B a representation-finite algebra, then* $\operatorname{End}_A(C_f)$ *is a quasi-hereditary algebra of global dimension at most three.*

The paper is organized as follows: In Section 2 we prove Theorem 1.1 and Proposition 1.2. In Section 3 we give a general construction principal

[1] After submission of this paper, at a conference in November 2002 R. Rouquier announced a proof that the exterior algebra of a 3-dimensional vector space has representation dimension 4.

for radical embeddings. This is applied in Section 4 to prove Corollary 1.3. Section 5 contains the proof of Theorem 1.4. Finally, we discuss an example in Section 6.

In this paper, 'modules' are finite-dimensional left modules. Although we often write maps on the left hand side, we compose them as if they were on the right. Thus the composition of a map θ followed by a map ϕ is denoted $\theta\phi$.

2. Proof of Theorem 1.1 and Proposition 1.2

The proof of the following lemma is implicitly contained in [1, Chapter III, §3]. For convenience we repeat it here.

Lemma 2.1. *Let A be an algebra, and let M be a generator-cogenerator of A. Then for $n \geq 3$ the following are equivalent:*

(1) *For all indecomposable A-modules X there exists an exact sequence*
$$0 \to M_{n-2} \to \cdots \to M_1 \to M_0 \to X \to 0$$
with $M_i \in \operatorname{add}(M)$, such that
$$0 \to \operatorname{Hom}_A(M, M_{n-2}) \to \cdots \to \operatorname{Hom}_A(M, M_1) \\ \to \operatorname{Hom}_A(M, M_0) \to \operatorname{Hom}_A(M, X) \to 0$$
is exact;

(2) *For all indecomposable A-modules X there exists an exact sequence*
$$0 \to X \to M_0 \to M_1 \to \cdots \to M_{n-2} \to 0$$
with $M_i \in \operatorname{add}(M)$, such that
$$0 \to \operatorname{Hom}_A(M_{n-2}, M) \to \cdots \to \operatorname{Hom}_A(M_1, M) \\ \to \operatorname{Hom}_A(M_0, M) \to \operatorname{Hom}_A(X, M) \to 0$$
is exact;

(3) $\operatorname{gldim}(\operatorname{End}_A(M)) \leq n$.

Proof. For brevity set $E = \operatorname{End}_A(M)$. Assume that (1) holds. Let T be an E-module, and let
$$\operatorname{Hom}_A(M, M'') \xrightarrow{F} \operatorname{Hom}_A(M, M') \to T \to 0$$
be a projective presentation of T. Then $F = \operatorname{Hom}_A(M, f)$ for some homomorphism $f : M'' \to M'$. Thus we get an exact sequence
$$0 \to \operatorname{Ker}(f) \to M'' \to M'.$$
Using our assumption, we get an exact sequence
$$0 \to M_{n-2} \to \cdots \to M_1 \to M_0 \to \operatorname{Ker}(f) \to 0$$
having the properties described in (1), here we set $X = \operatorname{Ker}(f)$. This yields an exact sequence
$$0 \to M_{n-2} \to \cdots \to M_1 \to M_0 \to M'' \to M'.$$

Applying $\mathrm{Hom}_A(M,-)$ gives an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_A(M,M_{n-2}) \to \cdots \to \mathrm{Hom}_A(M,M_1) \to \mathrm{Hom}_A(M,M_0) \\ \to \mathrm{Hom}_A(M,M'') \to \mathrm{Hom}_A(M,M') \to T \to 0. \end{aligned}$$

Thus $\mathrm{projdim}(T) \leq n$ for all E-modules T. We get $\mathrm{gldim}(E) \leq n$. Thus (3) is true.

Next, assume that (3) holds. For an A-module X, let

$$0 \to X \to I_0 \xrightarrow{h} I_1$$

be an injective presentation. Note that $I_0, I_1 \in \mathrm{add}(M)$. We get an exact sequence

$$0 \to \mathrm{Hom}_A(M,X) \to \mathrm{Hom}_A(M,I_0) \to \mathrm{Hom}_A(M,I_1) \to Y \to 0$$

with $Y = \mathrm{Cok}(\mathrm{Hom}_A(M,h))$. Now $\mathrm{Hom}_A(M,X)$ is the second syzygy module $\Omega^2(Y)$ of Y. Since $\mathrm{gldim}(E) \leq n$, we know that $\mathrm{projdim}(\Omega^2(Y)) \leq n-2$. Thus there exists an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_A(M,M_{n-2}) \to \cdots \to \mathrm{Hom}_A(M,M_1) \\ \to \mathrm{Hom}_A(M,M_0) \to \mathrm{Hom}_A(M,X) \to 0 \end{aligned}$$

with $M_i \in \mathrm{add}(M)$. This yields an exact sequence

$$0 \to M_{n-2} \to \cdots \to M_1 \to M_0 \to X \to 0.$$

Thus (1) follows. The equivalence of the statements (2) and (3) is proved dually. This finishes the proof. □

Let B be an algebra, and let $A \subseteq B$ be a subalgebra of B. We regard any B-module also as an A-module in the obvious way. For an A-module X, define $X^- = \mathrm{Hom}_A(B,X)$. Furthermore, we identify X and $\mathrm{Hom}_A(A,X)$. Let

$$\epsilon_X : X^- \to X$$

be the natural map induced by the inclusion $A \subseteq B$. Note that ϵ_X is an A-module homomorphism. Observe also that X^- is a B-module.

Lemma 2.2. *Let A be a subalgebra of an algebra B, and let X be an A-module. Then*

$$\begin{gathered} \mathrm{Hom}_B(Y,X^-) \to \mathrm{Hom}_A(Y,X), \\ f \mapsto f\epsilon_X \end{gathered}$$

is an isomorphism for all B-modules Y.

Proof. For all B-modules Y we have

$$\mathrm{Hom}_B(Y,\mathrm{Hom}_A(B,X)) \cong \mathrm{Hom}_A(B \otimes_B Y, X) \cong \mathrm{Hom}_A(Y,X),$$

where the isomorphisms are given by

$$f \mapsto (b \otimes y \mapsto f(y)(b)) \mapsto (y \mapsto f(y)(1)).$$

Thus the composition maps f to $f\epsilon_X$. □

Lemma 2.3. *Let A be a subalgebra of an algebra B such that $J_A = J_B$. Then $\mathrm{Cok}(\epsilon_X)$ and $\mathrm{Ker}(\epsilon_X)$ are semisimple A-modules for all A-modules X.*

Proof. The sequence

$$0 \to \mathrm{Hom}_A(B/A, X) \to X^- \xrightarrow{\epsilon_X} X \to \mathrm{Ext}^1_A(B/A, X)$$

is exact. We have $(B/A)J_A = 0$. Thus we get $J_A \,\mathrm{Ext}^1_A(B/A, X) = 0$, which implies that $\mathrm{Ext}^1_A(B/A, X)$ is a semisimple A-module. Thus $\mathrm{Cok}(\epsilon_X)$ is a semisimple A-module, since it is a submodule of $\mathrm{Ext}^1_A(B/A, X)$. Also $\mathrm{Ker}(\epsilon_X)$ is semisimple, since $J_A \,\mathrm{Hom}_A(B/A, X) = 0$. □

Proof of Theorem 1.1. Assume that B is a representation-finite algebra, and let $M_1, \cdots, M_n$ be a complete set of representatives of isomorphism classes of indecomposable B-modules. Without loss of generality we assume that A is a subalgebra of B such that $J_A = J_B$. Define $N = \bigoplus_{i=1}^n M_i$, and let $M = A \oplus A^* \oplus N$.

We claim that $\mathrm{gldim}(\mathrm{End}_A(M)) \leq 3$. To prove this, we use the criterion presented in Lemma 2.1.

If X is an indecomposable injective A-module, then we get a short exact sequence

$$0 \to 0 \to X \to X \to 0.$$

Setting $M_0 = X$ and $M_1 = 0$, we see that this sequence satisfies the conditions in Lemma 2.1(1).

Assume next that X is an indecomposable non-injective A-module. We know by Lemma 2.3 that $\mathrm{Cok}(\epsilon_X)$ is a semisimple A-module. By $\pi : P \to \mathrm{Cok}(\epsilon_X)$ we denote the projective cover of $\mathrm{Cok}(\epsilon_X)$. Since P is a projective A-module, there exists a homomorphism $p : P \to X$ such that the diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & J_A P \arrow[r, "\iota"] & P \arrow[r, "\pi"] \arrow[d, "p"] & \mathrm{Cok}(\epsilon_X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\epsilon_X) \arrow[r] & X^- \arrow[r, "\epsilon_X"] & X \arrow[r] & \mathrm{Cok}(\epsilon_X) \arrow[r] & 0
\end{tikzcd}$$

commutes and has exact rows. Observe that the map

$$\begin{pmatrix} \epsilon_X \\ p \end{pmatrix} : X^- \oplus P \to X$$

is an epimorphism of A-modules. Note also that $X^- \oplus P \in \mathrm{add}(M)$. We take

$$M_0 = X^- \oplus P$$

and will show that this works. It follows from Lemma 2.2 that the map

$$\mathrm{Hom}_A(Y, X^- \oplus P) \to \mathrm{Hom}_A(Y, X)$$

$$(f, g) \mapsto (f\epsilon_X + gp)$$

is surjective for any B-module Y. Since A is projective, it follows that the map

$$\mathrm{Hom}_A(A, X^- \oplus P) \to \mathrm{Hom}_A(A, X)$$

$$(f,g) \mapsto (f\epsilon_X + gp)$$

is surjective as well.

Finally, take an injective A-module I, and some homomorphism $f \in \mathrm{Hom}_A(I, X)$. Let $I \to I/\operatorname{soc}(I)$ be the canonical projection. Since X is not injective, there exists a homomorphism $g : I/\operatorname{soc}(I) \to X$ such that the diagram

$$\begin{tikzcd} I \arrow[r] \arrow[d, "f"'] & I/\operatorname{soc}(I) \arrow[dl, "g"] \\ X & \end{tikzcd}$$

commutes. We have $I = \bigoplus_{i=1}^{t}(e_iA)^*$ for some primitive idempotents e_i in A. Since $A \subseteq B$, the e_i are also idempotents of B. From $J_A = J_B$ we get $I/\operatorname{soc}(I) = \bigoplus_{i=1}^{t}(e_iJ_B)^*$. Thus $I/\operatorname{soc}(I)$ is also a B-module. Thus by Lemma 2.2, g factors through ϵ_X.

Altogether, we proved that for any A-module $Z \in \mathrm{add}(M)$ and any homomorphism $f : Z \to X$ of A-modules with X indecomposable there exists a homomorphism $g : Z \to X^- \oplus P$ of A-modules, such that the diagram

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[d, "f"] \\ X^- \oplus P \arrow[r, "\left(\begin{smallmatrix}\epsilon_X \\ p\end{smallmatrix}\right)"'] & X \end{tikzcd}$$

commutes. Next, we show that the kernel of the map $\left(\begin{smallmatrix}\epsilon_X \\ p\end{smallmatrix}\right) : M_0 \to X$ belongs to $\mathrm{add}(M)$.

Let $\epsilon'_X : X^- \to \mathrm{Im}(\epsilon_X)$ be the epimorphism induced by ϵ_X. There are the obvious inclusion maps $\mathrm{Im}(\epsilon_X) \to X$ and $J_AP \to P$. Clearly, there exists a homomorphism $p' : J_AP \to \mathrm{Im}(\epsilon_X)$ such that the diagram

$$\begin{tikzcd} J_AP \arrow[r, "\iota"] \arrow[d, "p'"] & P \arrow[d, "p"] \\ \mathrm{Im}(\epsilon_X) \arrow[r] & X \end{tikzcd}$$

commutes. Now we construct the pullback of p' and get the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(\epsilon_X) \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & J_AP \arrow[r] \arrow[d, "p'"] & 0 \\ 0 \arrow[r] & \mathrm{Ker}(\epsilon_X) \arrow[r] & X^- \arrow[r, "\epsilon'_X"] & \mathrm{Im}(\epsilon_X) \arrow[r] & 0. \end{tikzcd}$$

Thus

$$Y = \mathrm{Ker}\left(\begin{smallmatrix}\epsilon_X \\ p\end{smallmatrix}\right).$$

We have $P = \bigoplus_{i=1}^{t} Ae_i$ for some primitive idempotents e_i in A. Since $J_A = J_B$, we get $J_AP = \bigoplus_{i=1}^{t} J_Be_i$. Thus J_AP is also a B-module. Thus, by Lemma 2.2, the homomorphism p' factors through ϵ'_X. Thus the short exact sequence

$$0 \to \mathrm{Ker}(\epsilon_X) \to Y \to J_A P \to 0$$

splits, and we get $Y \cong \mathrm{Ker}(\epsilon_X) \oplus J_A P$. By the construction of M this implies $Y \in \mathrm{add}(M)$. Now set $M_1 = Y$.

Thus for each A-module X we constructed a short exact sequence

$$0 \to M_1 \to M_0 \to X \to 0$$

with the properties required in Lemma 2.1. We get $\mathrm{gldim}(\mathrm{End}_A(M)) \leq 3$. Since M is a generator-cogenerator of A, it follows that $\mathrm{repdim}(A) \leq 3$. This finishes the proof. □

Remark. Theorem 1.1 and its proof hold under the weaker assumption that f is a monomorphism such that $f(J_A)$ is a two-sided ideal of B (not necessarily equal to J_B). So far we are not aware of interesting applications of this slightly more general result. Thus we refrain from giving details here.

Proof of Proposition 1.2. Next, we prove Proposition 1.2. We have $B = A/\operatorname{soc}(P)$. Thus there is a surjective algebra homomorphism $f : A \to B$. We can regard any B-module as an A-module with the A-module structure induced by f.

Let N be a generator-cogenerator of B with $\mathrm{gldim}(\mathrm{End}_B(N)) \leq 3$. Define $M = N \oplus P$. Observe that M is a generator-cogenerator of A. We claim that $\mathrm{gldim}(\mathrm{End}_A(M)) \leq 3$. To check this, we use again Lemma 2.1. Let X be any indecomposable A-module. If $X = P$, then we get a short exact sequence

$$0 \to 0 \to X \to X \to 0.$$

Set $M_0 = X$ and $M_1 = 0$. It is easy to verify that this sequence satisfies the conditions required in Lemma 2.1. Next, assume that X is not isomorphic to P. Thus X is an indecomposable B-module. Applying Lemma 2.1 and our assumption $\mathrm{gldim}(\mathrm{End}_B(N)) \leq 3$, we get a short exact sequence

$$0 \to N_1 \to N_0 \to X \to 0$$

of B-modules with $N_0, N_1 \in \mathrm{add}(N)$ and

$$0 \to \mathrm{Hom}_B(N, N_1) \to \mathrm{Hom}_B(N, N_0) \to \mathrm{Hom}_B(N, X) \to 0$$

an exact sequence. Since P is projective, the functor $\mathrm{Hom}_A(P, -)$ is exact. Thus we get an exact sequence

$$0 \to \mathrm{Hom}_A(M, N_1) \to \mathrm{Hom}_A(M, N_0) \to \mathrm{Hom}_A(M, X) \to 0.$$

This enables us to apply Lemma 2.1 again, and we get $\mathrm{gldim}(\mathrm{End}_A(M)) \leq 3$. This finishes the proof. □

3. Construction of radical embeddings

A *quiver* is a quadruple $Q = (Q_0, Q_1, s, e)$, where Q_0 and Q_1 are finite sets and $s, e : Q_1 \to Q_0$ are maps. We call the elements in Q_0 the *vertices* of Q, and the elements in Q_1 the *arrows* of Q. A *path* of length $n \geq 1$ in Q is of the form $\alpha_1\alpha_2 \cdots \alpha_n$ where the α_i are arrows with $s(\alpha_i) = e(\alpha_{i+1})$ for $1 \leq i \leq n-1$. Additionally, there is a path e_i of length zero for each vertex $i \in Q_0$. By kQ we denote the path algebra of Q with basis the set of all paths in Q. The multiplication is given by concatenation of paths.

A *relation* for Q is a linear combination $\sum_{i=1}^{t} \lambda_i r_i$ such that $\lambda_i \in k^*$ and the r_i are paths of the form $\alpha_i p_i \beta_i$ with $\alpha_i, \beta_i \in Q_1$ such that $s(\beta_i) = s(\beta_j)$ and $e(\alpha_i) = e(\alpha_j)$ for all $1 \leq i, j \leq t$.

A *basic algebra* is a finite-dimensional algebra of the form kQ/I, where the ideal I is generated by a set of relations. By a result of Gabriel, any finite-dimensional k-algebra is Morita equivalent to a basic algebra provided we assume that k is algebraically closed.

Now, let $A = kQ/I$ be a basic algebra with $Q = (Q_0, Q_1, s, e)$. Let $l \in Q_0$ be a vertex. Define

$$S(l) = \{\alpha \in Q_1 \mid s(\alpha) = l\}$$

and

$$E(l) = \{\beta \in Q_1 \mid e(\beta) = l\}.$$

Note that the intersection of $S(l)$ and $E(l)$ might be non-empty.

Let $S(l) = S_1 \cup S_2$ and $E(l) = E_1 \cup E_2$ be disjoint unions. We call (S_1, S_2, E_1, E_2) a *splitting datum at l* if the following hold:

1. For $\alpha \in S_i$ and $\beta \in E_j$ we have $\alpha\beta = 0$ whenever $i \neq j$;
2. The ideal I can be generated by a set ρ of relations of the form $\sum_{i=1}^{t} \lambda_i \alpha_i p_i \beta_i$ such that $\{\alpha_i \mid 1 \leq i \leq t\} \cap E_j = \emptyset$ for $j = 1$ or $j = 2$, and $\{\beta_i \mid 1 \leq i \leq t\} \cap S_j = \emptyset$ for $j = 1$ or $j = 2$.

Note that condition (2) in the above definition is automatically satisfied, if we assume that I is a monomial ideal, i.e. if I can be generated by a set of paths in Q. Given a splitting datum $\mathrm{Sp} = (S_1, S_2, E_1, E_2)$ at l, we construct from Q a new quiver

$$Q^{\mathrm{Sp}} = (Q_0^{\mathrm{Sp}}, Q_1^{\mathrm{Sp}}, s^{\mathrm{Sp}}, e^{\mathrm{Sp}})$$

as follows: Let

$$Q_0^{\mathrm{Sp}} = \{l_1, l_2\} \cup Q_0 \setminus \{l\},$$

and set $Q_1^{\mathrm{Sp}} = Q_1$. The maps $s^{\mathrm{Sp}}, e^{\mathrm{Sp}} : Q_1^{\mathrm{Sp}} \to Q_0^{\mathrm{Sp}}$ are

$$s^{\mathrm{Sp}}(\alpha) = \begin{cases} s(\alpha) & \text{if } s(\alpha) \neq l, \\ l_1 & \text{if } \alpha \in S_1, \\ l_2 & \text{if } \alpha \in S_2, \end{cases}$$

and

$$e^{\mathrm{Sp}}(\alpha) = \begin{cases} e(\alpha) & \text{if } e(\alpha) \neq l, \\ l_1 & \text{if } \alpha \in E_1, \\ l_2 & \text{if } \alpha \in E_2. \end{cases}$$

Let ρ be a set of relations for Q which satisfy condition (2) above. Define

$$\rho^{\mathrm{Sp}} = \rho \setminus \{\alpha\beta \mid \alpha \in S_i, \beta \in E_j, i \neq j\}.$$

Then each element in ρ^{Sp} is also a relation for the quiver Q^{Sp}. Let I^{Sp} be the ideal of kQ^{Sp} generated by the relations in ρ^{Sp}. Set

$$A^{\mathrm{Sp}} = kQ^{\mathrm{Sp}}/I^{\mathrm{Sp}}.$$

We get the following result:

Lemma 3.1. *Let $A = kQ/I$ be a basic algebra, and let* Sp *be a splitting datum at some vertex of Q. Then there exists a radical embedding*

$$A \to A^{\mathrm{Sp}}.$$

Proof. Let Sp be a splitting datum at some vertex $l \in Q_0$. We construct a map $f : A \to A^{\mathrm{Sp}}$ as follows: For the arrows $\alpha \in Q_1$ we just define

$$f(\alpha) = \alpha.$$

For a vertex $j \in Q_0$ let

$$f(e_j) = \begin{cases} e_j & \text{if } j \neq l, \\ e_{l_1} + e_{l_2} & \text{if } j = l. \end{cases}$$

It follows directly from the definition of a splitting datum that f can be extended to an algebra homomorphism. It is also clear that f is a monomorphism and satisfies $f(J_A) = J_{A^{\mathrm{Sp}}}$. □

The above lemma is useful for the construction of radical embeddings. In fact, it can be applied to numerous situations. In the next section, we illustrate this for one of the most important classes of tame algebras, the string algebras.

4. Proof of Corollary 1.3

A basic algebra $A = kQ/I$ is called a *special biserial algebra* if the following hold:

(1) Any vertex of Q is the starting point of at most two arrows and also the end point of at most two arrows;
(2) Let β be an arrow in Q_1. Then there is at most one arrow α with $\alpha\beta \notin I$ and at at most one arrow γ with $\beta\gamma \notin I$;
(3) There exists some N such that each path of length at least N lies in I, i.e. A is finite-dimensional.

A *string algebra* is a special biserial algebra kQ/I which satisfies the additional condition that I is generated by paths. For details and further references on string algebras we refer to [5].

Proof of Corollary 1.3. Let $A = kQ/I$ be a string algebra. Define

$$c(A) = |\{l \in Q_0 \mid |S(l)| = 2\}| + |\{l \in Q_0 \mid |E(l)| = 2\}|.$$

If $c(A) = 0$, then Q is a disjoint union of quivers which are of type $\mathbb{A}$ with linear orientation or of type $\widetilde{\mathbb{A}}$ with cyclic orientation. But string algebras with such underlying quivers are representation-finite. In fact, it is easy to check that for a string algebra A all indecomposable A-modules are serial if and only if $c(A) = 0$.

Thus, assume $c(A) \geq 1$. Let $l \in Q_0$ such that $|S(l)| = 2$ or $|E(l)| = 2$. First, we consider the case $|S(l)| = 2$, say $S(l) = \{\alpha_1, \alpha_2\}$. We construct a splitting datum $\mathrm{Sp} = (S_1, S_2, E_1, E_2)$ at l as follows: Let $S_1 = \{\alpha_1\}$, $S_2 = \{\alpha_2\}$, $E_1 = \{\beta \in E(l) \mid \alpha_2\beta = 0\}$ and $E_2 = E(l) \setminus E_1$. It follows directly from the definition of a string algebra, that Sp is a splitting datum. Now A^{Sp} is again a string algebra, and we have

$$c(A^{\mathrm{Sp}}) \leq c(A) - 1.$$

The case $|E(l)| = 2$ is done in the same way. Repeating this construction a finite number of times and applying Lemma 3.1 yields a chain

$$A = A_1 \to A_2 \to \cdots \to A_t = B$$

of radical embeddings, where B is a string algebra with $c(B) = 0$ (cf. [12]). As observed above, B is representation-finite. Thus, for any string algebra A, we get a radical embedding $A \to B$ with B representation-finite. Then Theorem 1.1 yields that $\mathrm{repdim}(A) \leq 3$.

Next, assume that A is a special biserial algebra. Then we get from A to a string algebra B by successively factoring out socles of indecomposable projective-injective modules. Applying Proposition 1.2 after each step, we get $\mathrm{repdim}(A) \leq 3$. Now we use the result in [11] and get $\mathrm{findim}(A) < \infty$ for any special biserial algebra A. This finishes the proof. □

Note that for a string algebra A, the proofs of Theorem 1.1 and Corollary 1.3 yield an explicit construction of a generator-cogenerator M of A such that $\mathrm{gldim}(\mathrm{End}_A(M)) \leq 3$. Namely, take M as the direct sum of a complete set of representatives of isomorphism classes of string modules, which are projective, injective or serial.

5. Proof of Theorem 1.4

Let A be a subalgebra of an algebra B. We have the 'induction' functor

$$T = {}_BB \otimes_A - : \mathrm{mod}(A) \to \mathrm{mod}(B),$$

which is left adjoint to the 'inclusion' functor

$$F = \mathrm{Hom}_B(B, -) : \mathrm{mod}(B) \to \mathrm{mod}(A).$$

Thus for any A-module Y and any B-module X we get an isomorphism

$$\phi_{X,Y} : \mathrm{Hom}_B(TY, X) \to \mathrm{Hom}_A(Y, FX).$$

For the sake of brevity we will just write ϕ instead if $\phi_{X,Y}$. Sometimes we will omit writing F. Let

$$e : FT \to 1_{\mathrm{mod}(B)}$$

be the corresponding counit, so that

$$e_X = \phi^{-1}(1_{FX}) : B \otimes_A \mathrm{Hom}_B(B, X) = T(FX) \to X$$

is just the multiplication map. This is a B-homomorphism. The unit of this adjunction is the natural transformation

$$\delta : 1_{\mathrm{mod}(A)} \to TF,$$

so that for $Y \in \mathrm{mod}(A)$ we have

$$\delta_Y = \phi(1_{TY}) : Y \to F(TY)$$

$$y \mapsto (1 \otimes y)$$

if $F(TY) = \mathrm{Hom}_B(B, B \otimes_A Y)$ is identified with $B \otimes_A Y$. This is an A-module homomorphism. Note also that $\phi^{-1}(g) = T(g)e_X$ for $g : Y \to FX$ an A-homomorphism, and $\phi(f) = \delta_Y F(f)$ for $f : TY \to X$ a B-homomorphism.

Lemma 5.1. *Let A be a subalgebra of an algebra B such that $J_A = J_B$. If X is a B-module, then as a B-module, we have $T(FX) \cong B \otimes_A FX \cong X \oplus S$ where S is a semisimple B-module.*

Proof. Write $Y = FX$.

(1) First consider the exact sequences and the resulting commutative diagram

$$\begin{array}{ccccccccc}
0 & & Y & \overset{\delta_Y}{} & B \otimes_A Y & & B/A \otimes_A Y & & 0 \\
\\
(0 & &) \, \mathrm{top}(Y) & & \mathrm{top}(B \otimes_A Y) & & \mathrm{top}(B/A \otimes_A Y) & & 0
\end{array}$$

of A-homomorphisms, obtained by taking radical quotients (here $\mathrm{top}(M) = M/J_A M$), where the vertical maps are the canonical epimorphisms. Since $\mathrm{top}(Y)$ is the restriction of a B-module, the map $Y \to \mathrm{top}(Y)$ factors through δ_Y, see the dual of Lemma 2.2, that is by adjointness. Hence the lower row is a split short exact sequence.

(2) Let e_X be the counit of the adjunction. Then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(e_X) \arrow[r] \arrow[d, "p"] & B \otimes_A Y \arrow[r, "e_X"] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{top}(\mathrm{Ker}(e_X)) \arrow[r] & \mathrm{top}(B \otimes_A Y) \arrow[r] & \mathrm{top}(X) \arrow[r] & 0
\end{tikzcd}$$

of B-homomorphisms, which has exact rows. By $l(M)$ we denote the length of a module M. We have (using that the lower sequence in (1) is split exact)

$$\begin{aligned}
l(\mathrm{Ker}(e_X)) &= l(B \otimes_A Y) - l(X) = l(B \otimes_A Y) - l(Y) \\
&= l(B/A \otimes_A Y) = l(\mathrm{top}(B \otimes_A Y)) - l(\mathrm{top}(Y)) \\
&= l(\mathrm{top}(B \otimes_A Y)) - l(\mathrm{top}(X)) \leq l(\mathrm{top}(\mathrm{Ker}(e_X))).
\end{aligned}$$

Thus $\mathrm{Ker}(e_X)$ is a semisimple B-module, and p is an isomorphism, and both rows in the diagram in (2) are split short exact sequences of B-modules. $\square$

Proof of Theorem 1.4. Let $f : A \to B$ be a radical embedding with B a representation-finite algebra. We assume without loss of generality that f is an inclusion map, i.e. $A \subseteq B$ with $J_A = J_B$. Let N be the direct sum of a complete set of representatives of the isomorphism classes of indecomposable B-modules, say $N = \bigoplus_i N_i$ with N_i indecomposable. Define $M = A \oplus A^* \oplus FN$. From the proof of Theorem 1.1 we already know that $\mathrm{gldim}(\mathrm{End}_A(M)) \leq 3$. We claim that $\mathrm{End}_A(M)$ is quasi-hereditary.

Let us recall the definition of quasi-hereditary algebras. Let Γ be a finite-dimensional k-algebra, and assume that there is a partial order $\leq$ on a set of labels for the isomorphism classes of simple Γ-modules. Then $(\Gamma, \leq)$ is called a *quasi-hereditary algebra* if for any label λ with corresponding simple module $L(\lambda)$ there exists a Γ-module $\Delta(\lambda)$ (called standard module) such that

(i) there is a surjection $\Delta(\lambda) \to L(\lambda)$ where the kernel has only composition factors $L(\mu)$ with $\mu < \lambda$

(ii) there exists a map $P(\lambda) \to \Delta(\lambda)$, where $P(\lambda)$ is the projective cover of $L(\lambda)$, whose kernel has a filtration by standard modules $\Delta(\mu)$ with $\mu > \lambda$.

We set $\Gamma = \mathrm{End}_A(M)$ and show that Γ is quasi-hereditary.

(1) Recall that the isomorphism classes of simple modules of the endomorphism algebra of a module are indexed by the isomorphism classes of its indecomposable direct summands. Let

$$R = \mathrm{End}_B(B \otimes_A M) = \mathrm{End}_B(TM) = \mathrm{End}_B(TA \oplus TA^* \oplus T(FN)).$$

Lemma 5.1 implies $\mathrm{add}(B \otimes_A M) = \mathrm{add}(N)$. Since B is a representation-finite algebra, it follows from [1, Chapter III, §4] that $\mathrm{gldim}(R) \leq 2$. This implies that R is a quasi-hereditary algebra with respect to some partial order $\leq_R$ so that the labels given by the simple direct summands of N are maximal, see [6].

Define a partial order $\leq$ on the labels for the simple Γ-modules as follows: Let X and Y be non-isomorphic indecomposable direct summands of the A-module M. Set $X < Y$ if and only if one of the following holds:

- $X \cong FN_i$ and $Y \cong FN_j$ for some i, j such that $N_i <_R N_j$;
- X is not isomorphic to a direct summand of FN, and $Y \cong FN_i$ for some i.

Note that this is a partial order: The only indecomposable B-modules, which could become isomorphic as A-modules, are simple ones. Namely, if N_i and N_j are B-modules with $FN_i \cong FN_j$, then $T(FN_i) \cong T(FN_j)$, and if they are not simple, then Lemma 5.1 implies that N_i and N_j are isomorphic. It follows that all simple direct summands of M are maximal with respect to $\leq$.

For any indecomposable direct summand X of M, we have the submodule $U(X)$ of the projective Γ-module $P(X) = \mathrm{Hom}_A(M, X)$, which is defined to be the span of all homomorphisms $M \to X$, which factor through some Y with $Y > X$. The *standard module* associated to X is defined as $\Delta(X) = P(X)/U(X)$. By $L(X)$ we denote the top of $P(X)$. Thus $L(X)$ is simple.

We have to show that for each X, the module $P(X)$ has a filtration by standard modules with $\Delta(X)$ occurring only once, and if $\Delta(Y)$ occurs, then $Y \geq X$.

(2) For X simple we have $\Delta(X) = P(X)$. Assume now that X is not isomorphic to a direct summand of FN. Thus X is a projective or injective A-module (and not simple). In case X is projective, the radical of X is of the form $\bigoplus_i X_i$ where $X_i = FX_i'$ with X_i' an indecomposable B-module for all i. Then we get the exact sequence

$$0 \to \bigoplus_i P(X_i) \to P(X),$$

and the cokernel is one-dimensional, hence is $L(X)$. Since $X_i > X$ it follows that $\Delta(X) = L(X)$.

In the second case, we have a short exact sequence of A-modules

$$0 \to \mathrm{Hom}_A(B/A, X) \to X^- \xrightarrow{\epsilon_X} X \to 0$$

with the kernel a semisimple A-module, write it as $\bigoplus_i S_i$ with S_i simple. Here we use Lemma 2.3. Write also $X^- = \bigoplus_j X_j$ where $X_j = FX_j'$ with X_j' an indecomposable B-module for all j. This gives an exact sequence

$$0 \to \bigoplus_i \Delta(S_i) \to \bigoplus_j P(X_j) \to P(X).$$

We claim that the cokernel at $P(X)$ is simple. Suppose $g : W \to X$ is an A-homomorphism where W is an indecomposable direct summand of M. If W is isomorphic to a direct summand of FN, then g factors through ϵ_X, via adjointness, and clearly it factors if W is projective. Suppose W is indecomposable injective and not isomorphic to a direct summand of FN. If g is not an isomorphism, then it factors through the socle quotient of W. But this is of the form FW' for some B-module W'. Hence g factors through ϵ_X again. It follows that the cokernel is $L(X)$ and is isomorphic to $\Delta(X)$.

(3) So assume now that $X = FX'$ with X' an indecomposable B-module, which is not simple. Let ϕ be the adjoint isomorphism

$$\phi : \mathrm{Hom}_B(TM, X') \cong \mathrm{Hom}_A(M, X) = P(X).$$

Through the ring homomorphism $T : \Gamma \to R$, ϕ^{-1} induces Γ-isomorphisms $P(X) \to P_R(X)$, $U(X) \to U_R(X)$ and $\Delta(X) \to \Delta_R(X)$.

We only have to show that this is compatible with factorizing through some module $Z \in \mathrm{add}(N)$, modulo maps factorizing through a semisimple module.

Suppose that $g : M \to X$ is an A-homomorphism with a factorization $g = \alpha\beta$, where $\alpha : M \to FZ$ and $\beta : FZ \to X$ are A-homomorphisms. Then we have

$$\phi^{-1}(g) = T(\alpha)\phi^{-1}(\beta) : TM \to X'.$$

Hence $\phi^{-1}(g)$ factors through $T(FZ)$. Since Z is a B-module, Lemma 5.1 implies that $T(FZ) \cong Z \oplus S$ as a B-module with S semisimple, and this is what we need.

Conversely, suppose $f : TM \to X'$ is a B-module homomorphism, which factors through a B-module Z, say $f = \alpha\beta$, where $\alpha : TM \to Z$ and $\beta : Z \to X'$. Then $\phi(f) = \phi(\alpha)F(\beta)$, hence it factors through FZ.

Since R is quasi-hereditary with respect to $\leq_R$, each indecomposable projective module $\mathrm{Hom}_B(TM, X')$ has a filtration by standard modules of the right kind. The above shows that the adjoint isomorphism identifies this filtration with a filtration of $\mathrm{Hom}_A(M, X)$ by standard modules for Γ.

It remains to show that $L(X)$ occurs with multiplicity one as a composition factor of $\Delta(X)$. This is clear if X is simple, and we have already seen it in case X is not isomorphic to a direct summand of FN. If X is isomorphic to a direct summand of FN, then this multiplicity is the same as the multiplicity of $L_R(X)$ in $\Delta_R(X)$, hence it is one. $\square$

6. An Example

Let $A = kQ/I$ where Q is the quiver with one vertex x and two loops a, b and $I = (a^2, b^2, (ab)^2, (ba)^2)$. When the field has characteristic 2 this is the socle quotient of the group algebra of the dihedral group of order 8.

Let $\mathrm{Sp} = (S_1, S_2, E_1, E_2)$ where $S_1 = \{a\}$, $S_2 = \{b\}$, $E_1 = \{b\}$ and $E_2 = \{a\}$. Clearly, Sp is a splitting datum at x. Following the general construction in Section 3, we get the quiver Q^{Sp} with two vertices l_1 and l_2 and arrows $a : l_1 \to l_2$ and $b : l_2 \to l_1$. The ideal I^{Sp} is generated by all paths of length 4 in Q^{Sp}. The algebra A^{Sp} is a Nakayama algebra. Every indecomposable A^{Sp}-module is serial, and visibly its restriction to A remains serial.

Hence $M = A \oplus A^* \oplus N$, where N is the direct sum of a complete set of representatives of isomorphism classes of serial string modules over A. We denote these string modules as $M(C)$ for C in $\{1_x, a, b, ab, ba, aba, bab\}$, and we write $k = M(1_x)$ for the simple A-module. (For example, $M(ab)$ has basis $\{v, bv, abv\}$).

Let $\Gamma = \mathrm{End}_A(M)$. We can see directly that $\mathrm{gldim}(\Gamma) = 3$: For each indecomposable direct summand W of M, we write $P(W)$ for the indecomposable projective Γ-module $\mathrm{Hom}_A(M, W)$. Let $L(W)$ be the simple top of $P(W)$.

(1) The radical J_A belongs to $\mathrm{add}(M)$, and the inclusion $J_A \to A$ gives an inclusion of projective Γ-modules

$$0 \to \mathrm{Hom}_A(M, J_A) \to \mathrm{Hom}_A(M, A) = P(A).$$

Clearly, the cokernel is 1-dimensional, hence it is the simple module $L(A)$. This implies $\mathrm{projdim}(L(A)) \leq 1$.

(2) We have an exact sequence

$$0 \to k \to M(aba) \oplus M(bab) \to A^* \to 0.$$

This gives an exact sequence

$$0 \to P(k) \to P(M(aba)) \oplus P(M(bab)) \to P(A^*).$$

of Γ-modules. We claim that the cokernel is 1-dimensional. Consider $\phi : W \to A^*$ where W is an indecomposable direct summand of M. If $W = A$, then ϕ factors. Suppose W is serial. Then one easily calculates dimensions and gets that ϕ factors. If $W = A^*$ and ϕ is not an isomorphism, then ϕ factors through the socle quotient, and this is a direct sum of serials. Hence ϕ factors by what we have already seen. This shows $\mathrm{projdim}(L(A^*)) \leq 2$.

Next, assume that X is serial but not simple. Similarly as above, one can show that $\mathrm{projdim}(L(X)) \leq 2$ in this case. This uses the short exact sequences

$$0 \to M(ba) \to A \to M(aba) \to 0,$$
$$0 \to M(ab) \to M(b) \oplus M(aba) \to M(ba) \to 0,$$
$$0 \to M(b) \to k \oplus M(ba) \to M(a) \to 0$$

with terms in $\mathrm{add}(M)$.

Now consider the projective dimension of $L(k)$. We start with the exact sequence

$$0 \to D \to M(a) \oplus M(b) \to k \to 0$$

of A-modules, where $D = A/J_A^2$. Applying $\mathrm{Hom}_A(M, -)$ gives the exact sequence

$$0 \to \mathrm{Hom}_A(M, D) \to P(M(a)) \oplus P(M(b)) \to P(k),$$

which has a 1-dimensional cokernel, namely $L(k)$. The exact sequence

$$0 \to J_A \to A \oplus k \oplus k \to D \to 0$$

gives rise to the projective resolution

$$0 \to \mathrm{Hom}_A(M, J_A) \to P(A) \oplus P(k) \oplus P(k) \to \mathrm{Hom}_A(M, D) \to 0.$$

Hence $\mathrm{projdim}(L(k)) \leq 3$. Thus, we get $\mathrm{gldim}(\Gamma) \leq 3$, and therefore $\mathrm{repdim}(A) \leq 3$. Since the algebra A has infinite representation type, we get $\mathrm{repdim}(A) \geq 3$ by Auslander's theorem.

The same method works for arbitrary string algebras.

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