

IRREDUCIBLE COMPONENTS OF VARIETIES OF MODULES

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ABSTRACT. We prove some basic results about irreducible components of varieties of modules for an arbitrary finitely generated associative algebra. Our work generalizes results of Kac and Schofield on representations of quivers, but our methods are quite different, being based on deformation theory.

1. INTRODUCTION

Let k be an algebraically closed field. When studying the representation theory of a finitely generated k -algebra A (associative, with unit), one would often like to classify the finite-dimensional A -modules. In general, however, this is a hopeless—or meaningless—task. For example, the 1-dimensional modules for a f.g. commutative algebra correspond to points of the corresponding affine variety, and what should it mean to classify the points? A more basic problem in that case is to compute the irreducible components of the variety. This suggests to study, for each $d \geq 1$, the irreducible components of the ‘module variety’ $\text{mod}_A^d(k)$ of d -dimensional A -modules.

More precisely, $\text{mod}_A^d(k)$ is the set of A -module structures on k^d , or equivalently the set of k -algebra homomorphisms from A to $M_d(k)$. Now a homomorphism is determined by its value on a set of generators of A , and this gives an embedding of $\text{mod}_A^d(k)$ as a closed subvariety of a suitable product of copies of $M_d(k)$. The group $\text{GL}_d(k)$ acts by conjugation on $\text{mod}_A^d(k)$, and the orbits correspond to isomorphism classes of d -dimensional A -modules. By abuse of language, if C is a $\text{GL}_d(k)$ -stable subset of $\text{mod}_A^d(k)$, we say that a d -dimensional A -module M is ‘in’ C if the corresponding orbit is contained in C .

In addition to computing the irreducible components C of $\text{mod}_A^d(k)$, one would like to be able to say something about the properties of a general module contained in C . That is, the properties which hold for all modules in some dense open subset of C . For example, is the general module in C indecomposable? Can one say anything about the possible submodules of a general module in C ? What is the general dimension of the space of homomorphisms or extensions between modules in two irreducible components? In fact the dimensions of these spaces are known to be upper semicontinuous

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functions (see Lemma 4.3), so given irreducible components C_1 in $\text{mod}_A^{d_1}(k)$ and C_2 in $\text{mod}_A^{d_2}(k)$ this amounts to determining the numbers

$$\begin{aligned} \text{hom}_A(C_1, C_2) &= \min\{\dim \text{Hom}_A(M_1, M_2)\}, \text{ and} \\ \text{ext}_A^1(C_1, C_2) &= \min\{\dim \text{Ext}_A^1(M_1, M_2)\} \end{aligned}$$

where the modules M_i range over all modules in C_i .

The problem we consider is motivated by the fact that many natural varieties occur as, or are related to, module varieties. This includes the variety of complexes [5], Lusztig's nilpotent variety [13], §12, and Kleinian singularities and their deformations [4]. The case of Lusztig's nilpotent variety is particularly interesting, since by work of Kashiwara and Saito [12], its irreducible components are in 1-1 correspondence with elements of the crystal basis of a quantum group. Because these varieties arise as module varieties, one can hope to use decomposition properties of modules, and homological algebra techniques, to study the irreducible components. It is this theory that we initiate here.

The examples mentioned actually require a seemingly more general setup. Fixing orthogonal idempotents $e_1, \dots, e_n \in A$ with $e_1 + \dots + e_n = 1$, and a dimension vector $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$, one can consider the set $\text{mod}_A^{\mathbf{d}}(k)$ of k -algebra homomorphisms $A \rightarrow M_{d_1+\dots+d_n}(k)$, sending each e_i to the matrix whose diagonal $d_i \times d_i$ block is the identity, and with all other blocks zero. The natural group which acts is the product of the general linear groups of size d_i . Taking $n = 1$ one recovers $\text{mod}_A^d(k)$. Taking $A = kQ$, the path algebra of a quiver Q , and letting the e_i be the 'trivial paths', one obtains the space $\text{Rep}(Q, \mathbf{d})$ of representations of Q . All of our results below hold in this more general setup, either by adapting the proofs, or by using Bongartz's observation [1] that these varieties are related by a fibre bundle construction.

In fact $\text{Rep}(Q, \mathbf{d})$ is a model for some of our work. There is no problem describing the irreducible components in this case, as $\text{Rep}(Q, \mathbf{d})$ is a vector space, hence irreducible. But the problem of determining when the general element is an indecomposable representation, and of computing the general dimension of homomorphism and extension spaces is nontrivial. A nice theory has, however, been created by Kac [11] and by Schofield [19]. Our more general setup differs from this one, however, in an important way: it is possible to relate certain Ext spaces to tangent spaces in $\text{Rep}(Q, \mathbf{d})$, but in $\text{mod}_A^d(k)$ this is no longer the case. This is because Voigt's Lemma [6] really involves a certain scheme mod_A^d which need not be reduced, not even generically reduced, so its tangent spaces are not the same as for $\text{mod}_A^d(k)$.

Given subsets $C_i \subseteq \text{mod}_A^{d_i}(k)$ which are $\text{GL}_{d_i}(k)$ -stable ($1 \leq i \leq t$), we consider all modules of dimension $d = d_1 + \dots + d_t$ which are of the form $M_1 \oplus \dots \oplus M_t$ with the M_i in C_i , and we denote by $C_1 \oplus \dots \oplus C_t$ the corresponding $\text{GL}_d(k)$ -stable subset of $\text{mod}_A^d(k)$. This is the image of the

map

$$(1) \quad \mathrm{GL}_d(k) \times C_1 \times \cdots \times C_t \longrightarrow \mathrm{mod}_A^d(k)$$

sending a tuple (g, x_1, \dots, x_t) to the conjugation by g of the module structure in $\mathrm{mod}_A^d(k)$ which has the x_i as diagonal blocks. It follows that if the C_i are irreducible locally closed subsets, then the closure $\overline{C_1 \oplus \cdots \oplus C_t}$ is irreducible.

Our first result is an analogue of the Krull-Remak-Schmidt Theorem, and indeed follows from it quite easily. Most of the theorem is already known to the experts, but the only reference seems to be the preprint [15], which is not published elsewhere. For convenience we give a full proof.

Theorem 1.1. *If C is an irreducible component in $\mathrm{mod}_A^d(k)$, then*

$$(2) \quad C = \overline{C_1 \oplus \cdots \oplus C_t}$$

for some irreducible components C_i of module varieties $\mathrm{mod}_A^{d_i}(k)$, with the property that the general module in each C_i is indecomposable. Moreover C_1, \dots, C_t are uniquely determined by this, up to reordering.

One cannot use direct sums quite as freely as this suggests, however, as the closure of a direct sum of irreducible components is not in general an irreducible component. Our main result, proved using deformation theory, is the determination of when this happens.

Theorem 1.2. *If $C_i \subseteq \mathrm{mod}_A^{d_i}(k)$ are irreducible components ($1 \leq i \leq t$), and $d = d_1 + \cdots + d_t$, then $\overline{C_1 \oplus \cdots \oplus C_t}$ is an irreducible component of $\mathrm{mod}_A^d(k)$ if and only if $\mathrm{ext}_A^1(C_i, C_j) = 0$ for all $i \neq j$.*

These two theorems give an analogue of Kac's 'canonical decomposition' for representations of quivers [11], Proposition 3. Thus we call (2) the *canonical decomposition* of C . It reduces the problem of computing all irreducible components of module varieties to the case of components whose general element is indecomposable. By studying the fibres of the map (1), it is easy to see that

$$\dim C = \sum_{i=1}^t \dim C_i + \sum_{i \neq j} (d_i d_j - \mathrm{hom}_A(C_i, C_j)),$$

and also, if D is any irreducible component of a module variety, then

$$\mathrm{hom}_A(C, D) = \sum_{i=1}^t \mathrm{hom}_A(C_i, D),$$

and similarly for ext_A^1 , or with C and D exchanged.

Instead of taking direct sums of the modules in two irreducible components, one can take extensions. Let $d = d_1 + d_2$, let

$$G = \mathrm{GL}_{d_1}(k) \times \mathrm{GL}_{d_2}(k),$$

and let S be a G -stable subset of $\text{mod}_A^{d_1}(k) \times \text{mod}_A^{d_2}(k)$. Again, by abuse of language we say that a pair of modules (M_1, M_2) is ‘in’ S if the product of the corresponding orbits is a subset of S . We denote by $\mathcal{E}(S)$ the $\text{GL}_d(k)$ -stable subset of $\text{mod}_A^d(k)$ corresponding to all modules M which belong to a short exact sequence

$$(3) \quad 0 \longrightarrow M_2 \longrightarrow M \longrightarrow M_1 \longrightarrow 0$$

with (M_1, M_2) in S . We have the following result.

Theorem 1.3. *Let S be a G -stable subset of $\text{mod}_A^{d_1}(k) \times \text{mod}_A^{d_2}(k)$.*

- (i) *If S is a closed subset, then $\mathcal{E}(S)$ is a closed subset of $\text{mod}_A^d(k)$.*
- (ii) *If S is an irreducible locally closed subset, and $\text{Ext}_A^1(M_1, M_2)$ has constant dimension for all (M_1, M_2) in S , then the closure of $\mathcal{E}(S)$ in $\text{mod}_A^d(k)$ is irreducible.*
- (iii) *If S is an open subset and $\text{Ext}_A^1(M_2, M_1) = 0$ for all (M_1, M_2) in S , then $\mathcal{E}(S)$ is an open subset of $\text{mod}_A^d(k)$.*

This has the following application to irreducible components.

Corollary 1.4. *If C_1 and C_2 are irreducible components of $\text{mod}_A^{d_1}(k)$ and $\text{mod}_A^{d_2}(k)$ respectively, and if $\text{ext}_A^1(C_2, C_1) = 0$, then $\mathcal{E}(C_1 \times C_2)$ contains an irreducible component of $\text{mod}_A^d(k)$. It may be realized as $\overline{\mathcal{E}(S)}$, where S is the G -stable subset of $C_1 \times C_2$ corresponding to the pairs of modules (M_1, M_2) with $\dim \text{Ext}_A^1(M_2, M_1) = 0$ and $\dim \text{Ext}_A^1(M_1, M_2) = \text{ext}_A^1(C_1, C_2)$.*

In particular, if $\text{mod}_A^d(k)$ is known to be irreducible then, under the hypotheses of the theorem, associated to every d -dimensional A -module M there must be a short exact sequence (3) with M_i in C_i . Our results thus generalize the implications (i) \Leftrightarrow (ii) \Leftarrow (iii) of [19], Theorem 3.3. The remaining implication is clearly false in general, even for 2-dimensional modules for the algebra of dual numbers.

One source of irreducible components is provided by modules M without self-extensions, i.e. $\text{Ext}_A^1(M, M) = 0$. This condition ensures that the corresponding orbit $\mathcal{O}(M)$ is open in $\text{mod}_A^d(k)$, see [6], and so, since it is irreducible, its closure $\overline{\mathcal{O}(M)}$ is an irreducible component.

It is perhaps worthwhile to record the following consequence of Theorem 1.3(iii). If M_1, \dots, M_t are A -modules, we denote by $\mathcal{F}(M_1, \dots, M_t)$ the set of A -modules N which have an increasing sequence $0 = N_0 \subseteq N_1 \subseteq \dots \subseteq N_s = N$ of submodules such that each quotient N_j/N_{j-1} is isomorphic to one of the M_i . We denote by $\mathcal{F}_d(M_1, \dots, M_t)$ the $\text{GL}_d(k)$ -stable subset of $\text{mod}_A^d(k)$ corresponding to the d -dimensional modules in $\mathcal{F}(M_1, \dots, M_t)$.

Corollary 1.5. *If M_1, \dots, M_t are modules with $\text{Ext}_A^1(M_i, M_j) = 0$ for all $i \leq j$, then $\mathcal{F}_d(M_1, \dots, M_t)$ is open in $\text{mod}_A^d(k)$ for all d .*

In this paper, unless otherwise stated, ‘modules’ are right modules. Although we often write maps on the left hand side, we compose them as if

they were on the right. Thus the composition of a map θ followed by a map ϕ is denoted $\theta\phi$. By a variety we mean a locally closed subset of affine or projective space over k . We prove the results in numerical order, except that Theorem 1.2 is proved last. At the end we give various examples.

2. PROOF OF THEOREM 1.1

If C is an irreducible component of $\text{mod}_A^d(k)$, we denote by C^{ind} the subset of C given by the indecomposable modules. This is a constructible subset of $\text{mod}_A^d(k)$, and clearly every indecomposable d -dimensional module is in C^{ind} for some irreducible component C .

Every d -dimensional module M is isomorphic to a direct sum of indecomposables, and if the summands have dimensions d_1, \dots, d_t then M is in a subset

$$S = C_1^{\text{ind}} \oplus \dots \oplus C_t^{\text{ind}}$$

for some irreducible components $C_i \subseteq \text{mod}_A^{d_i}(k)$. Since the sets S which arise this way are constructible in $\text{mod}_A^d(k)$, and their union is the whole space, if C is an irreducible component of $\text{mod}_A^d(k)$, at least one of the sets S must contain a dense open subset of C . We concentrate on this set.

As observed in the introduction, $\overline{C_1 \oplus \dots \oplus C_t}$ is an irreducible closed subset of $\text{mod}_A^d(k)$. Since it contains C , which is an irreducible component, it follows that

$$C = \overline{C_1 \oplus \dots \oplus C_t}.$$

Now suppose that

$$(g, x_1, \dots, x_t) \in \text{GL}_d(k) \times C_1 \times \dots \times C_t$$

is an element which is sent under the map (1) to an element of S . By the Krull-Remak-Schmidt Theorem, any two decompositions of a module into a direct sum of indecomposables have the same number of terms. It follows that $x_i \in C_i^{\text{ind}}$ for all i . Thus the inverse image of S is $\text{GL}_d(k) \times C_1^{\text{ind}} \times \dots \times C_t^{\text{ind}}$. Since $C_1 \oplus \dots \oplus C_t$ contains a dense open subset of C , $\text{GL}_d(k) \times C_1^{\text{ind}} \times \dots \times C_t^{\text{ind}}$ contains a dense open subset of $\text{GL}_d(k) \times C_1 \times \dots \times C_t$. It follows that each C_i^{ind} contains a dense open subset of C_i . This completes the proof of existence.

Now uniqueness. Suppose that C is an irreducible component with

$$C = \overline{C_1 \oplus \dots \oplus C_t} = \overline{D_1 \oplus \dots \oplus D_s}$$

for irreducible components C_i and D_j with the general element indecomposable.

We denote by C_i^0 the set of elements of C_i which are not contained in any other irreducible component of $\text{mod}_A^{d_i}(k)$. Clearly C_i^0 is a dense open subset of C_i . Now the general element of C_i is indecomposable, so $C_i^{0,\text{ind}} = C_i^0 \cap C_i^{\text{ind}}$ contains a dense open subset of C_i . Thus

$$\text{GL}_d(k) \times C_1^{0,\text{ind}} \times \dots \times C_t^{0,\text{ind}}$$

contains a dense open subset of

$$\mathrm{GL}_d(k) \times C_1 \times \cdots \times C_t.$$

Now if $X \rightarrow Y$ is a dominant morphism between irreducible varieties then the image of a dense open subset of X must contain a dense open subset of Y . It follows that $C_1^{0,\mathrm{ind}} \oplus \cdots \oplus C_t^{0,\mathrm{ind}}$ contains a dense open subset of C . Repeating for the D_j , and using the fact that two dense open subsets of C must intersect, we obtain an element in

$$(C_1^{0,\mathrm{ind}} \oplus \cdots \oplus C_t^{0,\mathrm{ind}}) \cap (D_1^{0,\mathrm{ind}} \oplus \cdots \oplus D_s^{0,\mathrm{ind}}).$$

Thus there is an isomorphism of modules

$$M_1 \oplus \cdots \oplus M_t \cong N_1 \oplus \cdots \oplus N_s$$

with the M_i in $C_i^{0,\mathrm{ind}}$ and the N_j in $D_j^{0,\mathrm{ind}}$. By the Krull-Remak-Schmidt Theorem, we have $s = t$, and after permuting the N_j we have $M_i \cong N_i$ for all i . It follows that $C_i = D_i$.

3. PROOF OF THEOREM 1.3(i)

Let S be a closed subset of $\mathrm{mod}_A^{d_1}(k) \times \mathrm{mod}_A^{d_2}(k)$, stable for the action of $G = \mathrm{GL}_{d_1}(k) \times \mathrm{GL}_{d_2}(k)$, and let $d = d_1 + d_2$. By $V^{\mathrm{ses}}(d_1, d_2)$ we denote the locally closed subset of pairs (θ, ϕ) in $\mathrm{Hom}_k(k^{d_2}, k^d) \times \mathrm{Hom}_k(k^d, k^{d_1})$ such that

$$0 \longrightarrow k^{d_2} \xrightarrow{\theta} k^d \xrightarrow{\phi} k^{d_1} \longrightarrow 0$$

is a short exact sequence.

Let V_A^{ses} be the set of elements $(m, (\theta, \phi))$ in $\mathrm{mod}_A^d(k) \times V^{\mathrm{ses}}(d_1, d_2)$ such that the image of θ is a submodule of k^d , with the module structure given by m . Recall that m is an algebra homomorphism $A \rightarrow M_d(k)$. Identifying elements of $M_d(k)$ with endomorphisms of k^d , the condition is that $\theta m(a)\phi = 0$ for all a in A . Thus V_A^{ses} is a closed subset of $\mathrm{mod}_A^d(k) \times V^{\mathrm{ses}}(d_1, d_2)$.

Let

$$\mu : V_A^{\mathrm{ses}} \longrightarrow \mathrm{mod}_A^{d_1}(k) \times \mathrm{mod}_A^{d_2}(k)$$

be the map sending an element $(m, (\theta, \phi))$ to the pair (m_1, m_2) where m_1 and m_2 are the unique module structures on k^{d_1} and k^{d_2} for which θ and ϕ are module homomorphisms, where k^d is considered as an A -module using the structure m .

It is not obvious from this, but it can be checked, that μ is a morphism of varieties. One uses the covering of $V^{\mathrm{ses}}(d_1, d_2)$ by affine open subsets consisting of the pairs (θ, ϕ) where the image of θ is complementary to a given subspace of k^d . See [17] for a similar situation.

Now $\mu^{-1}(S)$ is a closed subset of V_A^{ses} , and hence a closed subset of $\mathrm{mod}_A^d(k) \times V^{\mathrm{ses}}(d_1, d_2)$.

By $\mathrm{Gr}(d_2 \hookrightarrow d)$ we denote the Grassmannian of d_2 -dimensional subspaces of k^d . Let

$$\pi : V^{\mathrm{ses}}(d_1, d_2) \longrightarrow \mathrm{Gr}(d_2 \hookrightarrow d)$$

be the map which sends a pair (θ, ϕ) to the image of θ . Clearly π is a morphism of varieties, and it is a principal G -bundle. Since S is G -stable we know that $\mu^{-1}(S)$ is a union of fibres of the morphism

$$(1, \pi) : \text{mod}_A^d(k) \times V^{\text{ses}}(d_1, d_2) \longrightarrow \text{mod}_A^d(k) \times \text{Gr}(d_2 \hookrightarrow d).$$

Now the principal bundle property implies that the image $(1, \pi)(\mu^{-1}(S))$ must be closed in $\text{mod}_A^d(k) \times \text{Gr}(d_2 \hookrightarrow d)$.

Let

$$\pi_1 : \text{mod}_A^d(k) \times \text{Gr}(d_2 \hookrightarrow d) \longrightarrow \text{mod}_A^d(k)$$

be the first projection. Since $\text{Gr}(d_2 \hookrightarrow d)$ is a projective variety, the image of $(1, \pi)(\mu^{-1}(S))$ under π_1 is closed. Now this image is clearly $\mathcal{E}(S)$, and the result is proved.

4. DERIVATIONS

Recall that if M is an A - A -bimodule, then a *derivation* $d : A \rightarrow M$ is a linear map with $d(ab) = ad(b) + d(a)b$ for all $a, b \in A$. We denote by $\text{Der}(A, M)$ the vector space of all derivations $A \rightarrow M$. A derivation d is *inner* if there is some $m \in M$ with $d(a) = am - ma$ for all $a \in A$. We denote the subspace of inner derivations by $\text{Der}^0(A, M)$.

Lemma 4.1. *Let a_1, \dots, a_N be k -algebra generators of A . If M is an A - A -bimodule, then $\text{Der}(A, M)$ is isomorphic to the subspace of M^N consisting of the tuples (d_1, \dots, d_N) with the property that*

$$(4) \quad \sum_{i_1, \dots, i_r} \lambda_{i_1, \dots, i_r} \left(\sum_{j=1}^r a_{i_1} \dots a_{i_{j-1}} d_{i_j} a_{i_{j+1}} \dots a_{i_r} \right) = 0$$

for all noncommutative polynomials

$$f(x_1, \dots, x_N) = \sum_{i_1, \dots, i_r} \lambda_{i_1, \dots, i_r} x_{i_1} \dots x_{i_r}$$

over k , with the property that $f(a_1, \dots, a_N) = 0$.

Proof. A derivation d is determined by its values on the generators a_i , and if $d(a_i) = d_i$, then

$$(5) \quad d(a_{i_1} \dots a_{i_r}) = \sum_{j=1}^r a_{i_1} \dots a_{i_{j-1}} d_{i_j} a_{i_{j+1}} \dots a_{i_r}.$$

Thus if $f(a_1, \dots, a_N) = 0$, then $d(f(a_1, \dots, a_N)) = 0$, which implies (4). Conversely, given d_1, \dots, d_N , one can define a function $d : A \rightarrow M$ using (5). The condition (4) ensures that this is well-defined. The construction of d ensures that it is a derivation. \square

Recall that if X is a variety, then a function $f : X \rightarrow \mathbb{Z}$ is *upper semi-continuous* if $\{x \in X \mid f(x) \leq n\}$ is open in X for all $n \in \mathbb{Z}$. If V is a vector space, then a *cone* in V is a subset which contains 0 and is closed under multiplication by elements of the field k . The following result is well-known.

Lemma 4.2. *Let X be a variety, V is a vector space and F a closed subset of $X \times V$. If, for all $x \in X$ the set $F_x = \{v \in V \mid (x, v) \in F\}$ is a subspace of V , or more generally a cone in V , then the function $X \rightarrow \mathbb{Z}$ sending x to $\dim F_x$ is upper semicontinuous.*

If M_1 and M_2 are A -modules, then $\mathrm{Hom}_k(M_1, M_2)$ is naturally an A - A -bimodule, and it is well-known that

$$(6) \quad \mathrm{Ext}_A^1(M_1, M_2) \cong \mathrm{Der}(A, \mathrm{Hom}_k(M_1, M_2)) / \mathrm{Der}^0(A, \mathrm{Hom}_k(M_1, M_2)).$$

From the construction we also have

$$\mathrm{Der}^0(A, \mathrm{Hom}_k(M_1, M_2)) \cong \mathrm{Hom}_k(M_1, M_2) / \mathrm{Hom}_A(M_1, M_2),$$

and hence

$$(7) \quad \dim \mathrm{Ext}_A^1(M_1, M_2) = \dim \mathrm{Der}(A, \mathrm{Hom}_k(M_1, M_2)) + \\ + \dim \mathrm{Hom}_A(M_1, M_2) - \dim M_1 \dim M_2.$$

Lemma 4.3. *The functions*

$$\mathrm{mod}_A^{d_1}(k) \times \mathrm{mod}_A^{d_2}(k) \longrightarrow \mathbb{Z}$$

sending a pair (m_1, m_2) to the dimensions of the spaces $\mathrm{Hom}_A(M_1, M_2)$, $\mathrm{Der}(A, \mathrm{Hom}_k(M_1, M_2))$ and $\mathrm{Ext}_A^1(M_1, M_2)$ are upper semicontinuous (where M_i is the module corresponding to m_i).

For Hom spaces this is well-known. For Ext^1 it is known, but we could not find a convenient reference which applies for all finitely generated k -algebras A , and the usual method, using projective resolutions, does not immediately adapt.

Proof. For $\mathrm{Hom}_A(M_1, M_2)$ one applies Lemma 4.2 with the set F_1 of

$$((m_1, m_2), \theta) \in (\mathrm{mod}_A^{d_1}(k) \times \mathrm{mod}_A^{d_2}(k)) \times \mathrm{Hom}_k(k^{d_1}, k^{d_2})$$

for which θ is an A -module homomorphism, for the module structures given by m_1 and m_2 .

For $\mathrm{Der}(A, \mathrm{Hom}_k(M_1, M_2))$ one uses the set F_2 of

$$((m_1, m_2), (d_1, \dots, d_N)) \in (\mathrm{mod}_A^{d_1}(k) \times \mathrm{mod}_A^{d_2}(k)) \times \mathrm{Hom}_k(k^{d_1}, k^{d_2})^N$$

for which (d_1, \dots, d_N) satisfies the condition (4) using the module structures m_1 and m_2 to turn $\mathrm{Hom}_k(k^{d_1}, k^{d_2})$ into an A - A -bimodule. This is clearly a closed subset.

Finally, a sum of upper semicontinuous functions is upper semicontinuous, so the assertion for $\mathrm{Ext}_A^1(M_1, M_2)$ follows from equation (7). \square

Lemma 4.4. *If X is an irreducible locally closed subset of $\mathrm{mod}_A^{d_1}(k) \times \mathrm{mod}_A^{d_2}(k)$ on which the dimension $\mathrm{Ext}_A^1(M_1, M_2)$ is constant, then so are the dimensions of $\mathrm{Hom}_A(M_1, M_2)$ and $\mathrm{Der}(A, \mathrm{Hom}_k(M_1, M_2))$.*

Proof. The subsets of X on which the dimensions of $\mathrm{Hom}_A(M_1, M_2)$ and $\mathrm{Der}(A, \mathrm{Hom}_k(M_1, M_2))$ take their minimum values are nonempty open subsets of X , and so they must meet. Away from this intersection $\mathrm{Ext}_A^1(M_1, M_2)$ must have non-minimal dimension. \square

5. PROOF OF THEOREM 1.3(ii)

Suppose that S is an irreducible locally closed G -stable subset of $X = \mathrm{mod}_A^{d_1}(k) \times \mathrm{mod}_A^{d_2}(k)$, and suppose that $\mathrm{Ext}_A^1(M_1, M_2)$ has dimension e for all (M_1, M_2) in S . Let $d = d_1 + d_2$.

Let Z be the closed subset of $\mathrm{mod}_A^d(k)$ consisting of the homomorphisms $A \rightarrow M_d(k)$ which take an upper triangular block form for the decomposition $d = d_1 + d_2$ with the lower $d_2 \times d_1$ block zero. A map $m : A \rightarrow M_d(k)$ taking this block form is a homomorphism if and only if the diagonal blocks are homomorphisms $m_i : A \rightarrow M_{d_i}(k)$ and the upper triangular block $A \rightarrow M_{d_1 \times d_2}(k)$ is a derivation for the A - A -bimodule structure on $M_{d_1 \times d_2}(k) \cong \mathrm{Hom}_k(k^{d_1}, k^{d_2})$ given by the module structures m_1 and m_2 . Thus Z is isomorphic to the closed subset F_2 used in the proof of Lemma 4.3.

By Lemma 4.4, the space $\mathrm{Der}(A, \mathrm{Hom}_k(M_1, M_2))$ has constant dimension on S . It follows that the projection $\pi : (S \times \mathrm{Hom}_k(k^{d_1}, k^{d_2})^N) \cap F_2 \rightarrow S$ is a sub-bundle of the trivial bundle $S \times \mathrm{Hom}_k(k^{d_1}, k^{d_2})^N \rightarrow S$. In particular it is a vector bundle in its own right. Thus, since S is irreducible, so is $\pi^{-1}(S)$.

Now the composition $\pi^{-1}(S) \hookrightarrow F_2 \cong Z \hookrightarrow \mathrm{mod}_A^d(k)$ and the conjugation action of $\mathrm{GL}_d(k)$ combine to give a map

$$\mathrm{GL}_d(k) \times \pi^{-1}(S) \longrightarrow \mathrm{mod}_A^d(k).$$

Now the left hand side is irreducible, so its image, which is $\mathcal{E}(S)$, has irreducible closure.

6. DEFORMATIONS

Let $k[[T]]$ be the power series algebra in one variable over k . Given a homomorphism $m : A \rightarrow M_d(k)$, a *deformation* of m is a homomorphism $\mathbf{m} : A \rightarrow M_d(k[[T]])$ whose composition with the natural homomorphism $M_d(k[[T]]) \rightarrow M_d(k)$ is equal to m . Equivalently, for each $a \in A$, the (i, j) component of $m(a)$ is the constant term of the (i, j) component of $\mathbf{m}(a)$.

The next lemma follows the philosophy of Gerstenhaber's deformation theory, see for example [7] or [8]. Let d_1 and d_2 be positive integers and set $d = d_1 + d_2$.

Lemma 6.1. *Let $m : A \rightarrow M_d(k)$ be a homomorphism which takes the upper triangular block form*

$$m(a) = \begin{pmatrix} m_{11}(a) & m_{12}(a) \\ 0 & m_{22}(a) \end{pmatrix},$$

where $m_{ij}(a)$ is a matrix of size $d_i \times d_j$, and let M_i be the A -module of dimension d_i corresponding to m_{ii} . Suppose $\mathrm{Ext}_A^1(M_2, M_1) = 0$. Then for

any deformation of m there exists an element \mathbf{g} with block form

$$\mathbf{g} = \begin{pmatrix} 1 & 0 \\ \mathbf{g}_{21} & 1 \end{pmatrix} \in \mathrm{GL}_d(k[[T]]),$$

where \mathbf{g}_{21} is a $d_2 \times d_1$ matrix with entries in $Tk[[T]]$, such that the conjugate of the deformation by \mathbf{g} is a deformation $\mathbf{m} : A \rightarrow \mathrm{M}_d(k[[T]])$, where \mathbf{m} has the upper triangular block form

$$\mathbf{m}(a) = \begin{pmatrix} \mathbf{m}_{11}(a) & \mathbf{m}_{12}(a) \\ 0 & \mathbf{m}_{22}(a) \end{pmatrix}.$$

Proof. We show first that if \mathbf{m} is a deformation of m and for some $n \geq 1$ the lower triangular block of $\mathbf{m}(a)$ has coefficients in $T^n k[[T]]$ for all $a \in A$, then there is a matrix

$$(8) \quad \mathbf{g}_n = \begin{pmatrix} 1 & 0 \\ T^n \theta_n & 1 \end{pmatrix} \in \mathrm{GL}_d(k[[T]])$$

where $\theta_n \in \mathrm{M}_{d_2 \times d_1}(k)$, such that the lower triangular block of $\mathbf{g}_n \mathbf{m}(a) \mathbf{g}_n^{-1}$ has coefficients in $T^{n+1} k[[T]]$ for all $a \in A$.

Write

$$\mathbf{m}(a) = \sum_{r=0}^{\infty} c^r(a) T^r$$

for some maps $c^r : A \rightarrow \mathrm{M}_d(k)$, and write each c^r in block form

$$c^r(a) = \begin{pmatrix} c_{11}^r(a) & c_{12}^r(a) \\ c_{21}^r(a) & c_{22}^r(a) \end{pmatrix}.$$

Thus

$$\mathbf{m}(a) = \begin{pmatrix} \mathbf{m}_{11}(a) & \mathbf{m}_{12}(a) \\ \mathbf{m}_{21}(a) & \mathbf{m}_{22}(a) \end{pmatrix}$$

where $\mathbf{m}_{ij}(a) = \sum_{r=0}^{\infty} c_{ij}^r(a) T^r$.

By assumption the constant term satisfies $c_{11}^0 = m_{11}$, $c_{22}^0 = m_{22}$, $c_{12}^0 = m_{12}$ and $c_{21}^0 = 0$, and we are assuming that $c_{21}^r = 0$ for $0 \leq r \leq n-1$. The fact that \mathbf{m} is an algebra homomorphism implies that

$$c^r(ab) = \sum_{\substack{s+t=r \\ s,t \geq 0}} c^s(a) c^t(b).$$

for all $a, b \in A$. Compare for example [8], p.24. Now since $c_{21}^r = 0$ for all $0 \leq r \leq n-1$, we have

$$c_{21}^n(ab) = c_{22}^0(a) c_{21}^n(b) + c_{21}^n(a) c_{11}^0(b) = m_{22}(a) c_{21}^n(b) + c_{21}^n(a) m_{11}(b).$$

Identifying $\mathrm{M}_{d_2 \times d_1}(k)$ with $\mathrm{Hom}_k(M_2, M_1)$, and considering this as an A - A -bimodule using the actions of A on M_1 and M_2 , it follows that c_{21}^n defines a derivation $A \rightarrow \mathrm{Hom}_k(M_2, M_1)$. Now by (6) and our assumption that $\mathrm{Ext}_A^1(M_2, M_1) = 0$, every derivation is inner. Therefore, there is a matrix

$\theta_n \in M_{d_2 \times d_1}(k)$ with $c_{21}^n(a) = m_{22}(a)\theta_n - \theta_n m_{11}(a)$ for all $a \in A$. Let \mathbf{g}_n be the matrix given by (8). Then

$$\mathbf{g}_n^{-1} = \begin{pmatrix} 1 & 0 \\ -T^n \theta_n & 1 \end{pmatrix}$$

and so the lower triangular $d_2 \times d_1$ block of $\mathbf{g}_n \mathbf{m}(a) \mathbf{g}_n^{-1}$ is equal to

$$\mathbf{m}_{21}(a) + T^n \theta_n \mathbf{m}_{11}(a) - \mathbf{m}_{22}(a) T^n \theta_n - T^{2n} \theta_n \mathbf{m}_{12}(a) \theta_n.$$

This has entries in $T^{n+1}k[[T]]$, so \mathbf{g}_n has the required property.

We now prove the lemma. Let \mathbf{m}^0 be a deformation of m . For each $a \in A$, the lower triangular block of $\mathbf{m}^0(a)$ has entries in $Tk[[T]]$, so by the construction above there is a matrix \mathbf{g}_1 with the property that if \mathbf{m}^1 is defined by $\mathbf{m}^1(a) = \mathbf{g}_1 \mathbf{m}^0(a) \mathbf{g}_1^{-1}$ then the lower triangular block of $\mathbf{m}^1(a)$ has entries in $T^2k[[T]]$. Thus there is a \mathbf{g}_2 such that the lower triangular block of $\mathbf{m}^2(a) = \mathbf{g}_2 \mathbf{m}^1(a) \mathbf{g}_2^{-1}$ has entries in $T^3k[[T]]$. Repeating in this way gives deformations $\mathbf{m}^1, \mathbf{m}^2, \dots$ of m and matrices $\mathbf{g}_1, \mathbf{g}_2, \dots$ of the form (8). Define

$$\mathbf{g} = \begin{pmatrix} 1 & 0 \\ \sum_{n=1}^{\infty} T^n \theta_n & 1 \end{pmatrix} \in \mathrm{GL}_d(k[[T]]).$$

and define \mathbf{m} by $\mathbf{m}(a) = \mathbf{g} \mathbf{m}^0(a) \mathbf{g}^{-1}$.

We have

$$\mathbf{g} = \mathbf{g}_n \dots \mathbf{g}_1 + O(T^{n+1})$$

where the notation $O(T^{n+1})$ means we ignore powers T^{n+1} and above. Thus

$$\mathbf{m}(a) = \mathbf{g}_n \dots \mathbf{g}_1 \mathbf{m}^0(a) \mathbf{g}_1^{-1} \dots \mathbf{g}_n^{-1} + O(T^{n+1}) = \mathbf{m}^n(a) + O(T^{n+1}).$$

Thus, for all n , the lower triangular block of $\mathbf{m}(a)$ has entries in $T^{n+1}k[[T]]$. Thus it is zero. \square

In another language, a *deformation* of a d -dimensional A -module M is a $k[[T]]$ - A -bimodule \mathbf{M} which is free of rank d over $k[[T]]$ and with

$$k \otimes_{k[[T]]} \mathbf{M} \cong M.$$

The lemma then says that if M belongs to an exact sequence

$$0 \longrightarrow M_2 \longrightarrow M \longrightarrow M_1 \longrightarrow 0$$

with $\mathrm{Ext}_A^1(M_2, M_1) = 0$, then any deformation \mathbf{M} of M belongs to an exact sequence

$$0 \longrightarrow \mathbf{M}_2 \longrightarrow \mathbf{M} \longrightarrow \mathbf{M}_1 \longrightarrow 0$$

with \mathbf{M}_i a deformation of M_i .

7. A VALUATIVE CRITERION

We use the following variation on the valuative criterion for flatness [9], §11.8, to transfer results about deformations to module varieties. Here $k[[T]]$ is the power series algebra in one variable, and $\pi^* : \text{Spec}(k) \rightarrow \text{Spec}(k[[T]])$ is the unique morphism of schemes over k .

Lemma 7.1. *Let $f : X \rightarrow Y$ be a morphism of schemes over k , and assume that Y is of finite type and quasiprojective over k . Suppose that for every commutative square of morphisms of schemes over k ,*

$$\begin{array}{ccc} \text{Spec}(k) & \xrightarrow{p} & X \\ \pi^* \downarrow & & \downarrow f \\ \text{Spec}(k[[T]]) & \xrightarrow{h} & Y \end{array}$$

there exists a morphism

$$h' : \text{Spec}(k[[T]]) \longrightarrow X$$

such that $\pi^ h' = p$ and $h' f = h$, that is, the two resulting triangles commute. Then the image of f is open in Y .*

Proof. Recall that the points of a scheme are in 1-1 correspondence with irreducible closed subsets, with the point z corresponding to the closure $\overline{\{z\}}$. Since Y has finite type over k , any nonempty constructible subset of Y must contain a closed point.

Since $\text{Im}(f)$ is a constructible subset of Y , to show it is open it suffices to show that $\text{Im}(f)$ is closed under generization, that is, if $z \in Y$, $y \in \text{Im}(f)$ and $y \in \overline{\{z\}}$, then $z \in \text{Im}(f)$. By the remark above, it is sufficient to prove this in the case when y is a closed point. For in general, since $y \in \text{Im}(f)$, we know that $\text{Im}(f) \cap \overline{\{y\}}$ is nonempty, so since it is constructible it contains a closed point y' . Then $y' \in \overline{\{y\}} \subseteq \overline{\{z\}}$, and hence $z \in \text{Im}(f)$.

Thus suppose that $z \in Y$, let $Z = \overline{\{z\}}$, let y be a closed point in $\text{Im}(f) \cap Z$, and assume for a contradiction that $z \notin \text{Im}(f)$. Since $\text{Im}(f) \cap Z$ does not contain z , this set must be contained in a proper closed subset F of Z . Choose a closed point $t \in Z \setminus F$. Now it is known that it is possible to join any two points of an irreducible variety by a curve, see [14], p.56. Thus there is a morphism $C \rightarrow Z$, where C is a curve, and with the image containing y and t . By passing to the normalization if necessary, we may assume that C is smooth.

Let c be a closed point of C which is sent to y . The complete local ring of C at c is isomorphic to $k[[T]]$, and this gives a morphism h

$$\text{Spec}(k[[T]]) \longrightarrow C \longrightarrow Z \longrightarrow Y$$

sending the closed point of $\text{Spec}(k[[T]])$ to y . We complete this to a commutative square as in the statement of the lemma using the morphism $\text{Spec}(k) \rightarrow X$ corresponding to a closed point $x \in X$ with $f(x) = y$. Thus the hypotheses of the lemma give a morphism h' .

Let ξ be the generic point of $\text{Spec}(k[[T]])$. Then $h(\xi)$ is the generic point of the closure of the image of C . Thus $\overline{\{h(\xi)\}}$ contains t . On the other hand $h(\xi) = f(h'(\xi)) \in \text{Im}(f)$, and $h(\xi) \in Z$, so $h(\xi) \in \text{Im}(f) \cap Z$. Thus $\overline{\{h(\xi)\}} \subseteq F$, which does not contain t . This is a contradiction. \square

The following observation will also be useful. The proof follows from the fact that $\eta^{-1}(U)$ is closed under generization.

Lemma 7.2. *Let $\eta : \text{Spec}(k[[T]]) \rightarrow X$ be a morphism of schemes. If U is an open subset of X which contains the image of the closed point of $\text{Spec}(k[[T]])$, then the image of η is contained in U .*

8. PROOF OF THEOREM 1.3(iii)

Recall that $\text{mod}_A^d(k)$ can be identified with the set of closed points, or the set of k -valued points, of a scheme mod_A^d over k . It is constructed as follows. Let a_1, \dots, a_N be a set of k -algebra generators of A . For $d \geq 1$, let

$$L = k[X_{ijl} \mid 1 \leq i, j \leq d, 1 \leq l \leq N]$$

be the polynomial ring over k in Nd^2 commuting variables. By $X_l = (X_{ijl})_{1 \leq i, j \leq d}$ we denote the $d \times d$ matrix with ij th entry the variable X_{ijl} . Let $k[A, d]$ be the quotient of L by the ideal generated by the entries of $f(X_1, \dots, X_N)$ for all noncommutative polynomials f over k in N variables, with $f(a_1, \dots, a_N) = 0$. Define

$$\text{mod}_A^d = \text{Spec}(k[A, d]).$$

One can show that this definition is (up to GL_d -equivariant isomorphism) independent of the choice of the generators of A , so mod_A^d is called *the scheme of d -dimensional A -modules*. See [1], [6] or [16] for the basic properties of schemes of modules.

For a finitely generated commutative k -algebra R , let $\text{mod}_A^d(R)$ be the set of R -valued points of the scheme mod_A^d . These are by definition the morphisms

$$\text{Spec}(R) \longrightarrow \text{Spec}(k[A, d]) = \text{mod}_A^d$$

of schemes over k . They are in 1-1 correspondence with the k -algebra homomorphisms

$$k[A, d] \longrightarrow R.$$

Such homomorphisms correspond to N -tuples of matrices in $M_d(R)$ which satisfy the same relations as the generators a_i . Thus they correspond to k -algebra homomorphisms

$$\mathbf{m} : A \longrightarrow M_d(R).$$

Thus the R -valued points of mod_A^d correspond to R - A -bimodule structures on R^d .

By GL_d we denote the general linear affine group scheme. It acts on mod_A^d by conjugation, $(\mathbf{g} \cdot \mathbf{m})(a) = \mathbf{g}\mathbf{m}(a)\mathbf{g}^{-1}$ for $\mathbf{g} \in \text{GL}_d(R)$ and $\mathbf{m} \in \text{mod}_A^d(R)$.

The orbits correspond to isomorphism classes of R - A -bimodules which are free of rank d over R .

Now let $d = d_1 + d_2$, let S be a G -stable open subset of $\text{mod}_A^{d_1}(k) \times \text{mod}_A^{d_2}(k)$, and assume that $\text{Ext}_A^1(M_2, M_1) = 0$ whenever (M_1, M_2) is in S .

Let Z be the closed subscheme of mod_A^d whose R -valued points are the homomorphisms $\mathbf{m} : A \rightarrow M_d(R)$ which take an upper triangular block form with respect to the decomposition $d = d_1 + d_2$. There is a morphism of schemes over k ,

$$\Delta : Z \longrightarrow \text{mod}_A^{d_1} \times \text{mod}_A^{d_2}$$

which on R -valued points sends an upper triangular homomorphism $\mathbf{m} : A \rightarrow M_d(R)$ to the pair $(\mathbf{m}_1, \mathbf{m}_2)$ consisting of the two diagonal blocks of \mathbf{m} . Now S can be considered as an open subscheme of $\text{mod}_A^{d_1} \times \text{mod}_A^{d_2}$, and so there is a corresponding open subscheme $\Delta^{-1}(S)$ of Z .

Let

$$f : \text{GL}_d \times \Delta^{-1}(S) \longrightarrow \text{mod}_A^d$$

be the morphism obtained from the inclusion of $\Delta^{-1}(S)$ in Z , followed by the conjugation action of GL_d . We show that the hypotheses of Lemma 7.1 are satisfied by the morphism f . Thus suppose we are given morphisms p and h forming a commutative square as in the lemma.

The morphism h is a $k[[T]]$ -valued point of mod_A^d , so it corresponds to a homomorphism $\mathbf{m} : A \rightarrow M_d(k[[T]])$.

The morphism p corresponds to a pair (g_0, m) where $g_0 \in \text{GL}_d(k)$ and m is a homomorphism $A \rightarrow M_d(k)$ which has upper triangular block form, and such that the diagonal blocks satisfy $(m_1, m_2) \in S$.

The commutativity of the square implies that \mathbf{m} is a deformation of $g_0 \cdot m$, so that $g_0^{-1} \cdot \mathbf{m}$ is a deformation of m . Applying Lemma 6.1 to this deformation gives an element $\mathbf{g} \in \text{GL}_d(k[[T]])$ such that $\mathbf{m}' = (\mathbf{g}g_0^{-1}) \cdot \mathbf{m}$ is a homomorphism $A \rightarrow M_d(k[[T]])$ which has upper triangular block form, and is a deformation of m .

Now \mathbf{m}' defines a morphism $\theta : \text{Spec}(k[[T]]) \rightarrow Z$. The closed point of $\text{Spec}(k[[T]])$ is sent to m , and then under Δ this is sent to $(m_1, m_2) \in S$. Thus Lemma 7.2 implies that the composition $\theta\Delta$ has image contained in S , so that θ defines a morphism $\theta' : \text{Spec}(k[[T]]) \rightarrow \Delta^{-1}(S)$.

Now $g_0\mathbf{g}^{-1} \in \text{GL}_d(k[[T]])$ defines a morphism $\text{Spec}(k[[T]]) \rightarrow \text{GL}_d$. Combining it with θ' gives a morphism

$$h' : \text{Spec}(k[[T]]) \longrightarrow \text{GL}_d \times \Delta^{-1}(S).$$

Since $(g_0\mathbf{g}^{-1}) \cdot ((\mathbf{g}g_0^{-1}) \cdot \mathbf{m}) = \mathbf{m}$, we have $h'f = h$. We also have $\pi^*h' = p$. Thus the hypotheses of Lemma 7.1 are satisfied, so the image of f , which is $\mathcal{E}(S)$, is open.

9. PROOF OF COROLLARY 1.4

Let C_1 and C_2 be irreducible components of $\text{mod}_A^{d_1}(k)$ and $\text{mod}_A^{d_2}(k)$ respectively, and suppose that $\text{ext}_A^1(C_2, C_1) = 0$. Let $d = d_1 + d_2$.

Let S be the subset of $C_1 \times C_2$ corresponding to the modules (M_1, M_2) with the property that $\dim \text{Ext}_A^1(M_2, M_1) = 0$ and $\dim \text{Ext}_A^1(M_1, M_2) = \text{ext}_A^1(C_1, C_2)$. This is an irreducible locally closed subset of $\text{mod}_A^{d_1}(k) \times \text{mod}_A^{d_2}(k)$, so $\overline{\mathcal{E}(S)}$ is irreducible by Theorem 1.3(ii).

Let C_i^0 be the set of elements of C_i which are contained in no other irreducible component of $\text{mod}_A^{d_i}(k)$. This is an open subset of $\text{mod}_A^{d_i}(k)$. Let $T = S \cap (C_1^0 \times C_2^0)$. This is a nonempty open subset of $\text{mod}_A^{d_1}(k) \times \text{mod}_A^{d_2}(k)$, so by Theorem 1.3(i), $\mathcal{E}(T)$ is an open subset of $\text{mod}_A^d(k)$, clearly nonempty. Thus $\overline{\mathcal{E}(S)}$ contains a nonempty open subset of $\text{mod}_A^d(k)$, so it must be an irreducible component.

Finally, $\mathcal{E}(C_1 \times C_2)$ contains $\overline{\mathcal{E}(S)}$ by Theorem 1.3(i).

10. PROOF OF COROLLARY 1.5

Using the fact that $\text{Ext}_A^1(M_i, M_j) = 0$ for all $i \leq j$, an induction shows that any module N in $\mathcal{F}(M_1, \dots, M_t)$ has a filtration

$$0 = N_0 \subseteq N_1 \subseteq \dots \subseteq N_t = N$$

with each N_j/N_{j-1} isomorphic to a direct sum of copies of M_j . Assume by induction on t that $\mathcal{F}_d(M_1, \dots, M_{t-1})$ is open for all d . Let $\dim M_t = r$, and for $i \geq 0$ let \mathcal{O}_i be the orbit in $\text{mod}_A^{ri}(k)$ corresponding to the direct sum of i copies of M_t . Then

$$\mathcal{F}_d(M_1, \dots, M_t) = \bigcup_{i \geq 0} \mathcal{E}(\mathcal{O}_i \times \mathcal{F}_{d-ri}(M_1, \dots, M_{t-1}))$$

which is open by Theorem 1.3(iii), since $\text{Ext}_A^1(N, M_t) = 0$ for all N in $\mathcal{F}(M_1, \dots, M_{t-1})$.

11. PROOF OF THEOREM 1.2

Let C_1, \dots, C_t be irreducible components of varieties of A -modules. Assuming that

$$C = \overline{C_1 \oplus \dots \oplus C_t}$$

is an irreducible component of $\text{mod}_A^d(k)$, we prove that $\text{ext}_A^1(C_i, C_j) = 0$ for all $i \neq j$.

We say that a d -dimensional A -module M is a *minimal degeneration* if the corresponding orbit $\mathcal{O}(M)$ in $\text{mod}_A^d(k)$ is not contained in the closure of any other orbit. Note that $\dim \mathcal{O}(M) = d^2 - \dim \text{End}_A(M)$, so if an orbit $\mathcal{O}(M)$ is contained in the closure of another orbit $\mathcal{O}(N)$, then $\dim \text{End}_A(M) > \dim \text{End}_A(N)$.

Recall that the function $\text{mod}_A^d(k) \rightarrow \mathbb{Z}$, sending a module structure to the dimension of the endomorphism algebra of the corresponding module, is upper semicontinuous. Thus if C is an irreducible component of a variety of modules, then the set C^{\min} where this function takes its minimal value (on C) is a dense open subset of C . If C^0 denotes the set of elements of

C which are not contained in any other irreducible component of $\text{mod}_A^d(k)$, then $C^{0,\min} = C^0 \cap C^{\min}$ is also a dense open subset of C .

Clearly any module in $C^{0,\min}$ is a minimal degeneration, for if $\mathcal{O}(M)$ is contained in the closure of another orbit $\mathcal{O}(N)$ then the fact that M is in C^0 implies that N is in C , and then $\dim \text{End}_A(M) > \dim \text{End}_A(N)$.

Now assume that $\text{ext}_A^1(C_i, C_j) \neq 0$ for some $i \neq j$. Thus $\text{Ext}_A^1(X, Y) \neq 0$ for all modules X in C_i and Y in C_j . Taking a non-split short exact sequence

$$0 \longrightarrow Y \longrightarrow E \longrightarrow X \longrightarrow 0$$

we have $E \not\cong X \oplus Y$ by [18], §2.3, Lemma 1, so that $E \oplus Z \not\cong X \oplus Y \oplus Z$ for any module Z , and $\mathcal{O}(X \oplus Y \oplus Z) \subset \overline{\mathcal{O}(E \oplus Z)}$ by [3]. It follows that no element of $C_1 \oplus \dots \oplus C_t$ can be a minimal degeneration.

But this set must meet $C^{0,\min}$. Contradiction.

We now prove the converse direction. By induction it suffices to prove the case when $t = 2$, that is, if C_1 and C_2 are irreducible components of $\text{mod}_A^{d_1}(k)$ and $\text{mod}_A^{d_2}(k)$, and if $\text{ext}_A^1(C_1, C_2) = \text{ext}_A^1(C_2, C_1) = 0$, then $\overline{C_1 \oplus C_2}$ is an irreducible component of $\text{mod}_A^d(k)$, where $d = d_1 + d_2$.

Let S be the subset appearing in the statement of Corollary 1.4. If (M_1, M_2) is in S , then $\text{Ext}_A^1(M_1, M_2) = 0$, so that $\mathcal{E}(S) \subseteq C_1 \oplus C_2$. Now the closure of $\mathcal{E}(S)$ contains an irreducible component of $\text{mod}_A^d(k)$ by Corollary 1.4. Thus, since $\overline{C_1 \oplus C_2}$ is irreducible, it must be this irreducible component.

12. REMARKS AND EXAMPLES

12.1. There are only very few algebras A such that all irreducible components of $\text{mod}_A^d(k)$ are known for all d . It should be important to construct more explicit examples. If A is of finite representation type, i.e. there are only finitely many isomorphism classes of indecomposable A -modules, then one can use Auslander-Reiten theory and [21] to compute all irreducible components of varieties of A -modules. One of the known examples of infinite representation type can be found in [20].

12.2. Theorem 1.2 can be proved using tangent space methods, rather than deformation theory, when the scheme mod_A^d is reduced, or at least generically reduced. It should be an important task to find methods that determine when this happens. If A is hereditary, i.e. the global dimension of A is one, then mod_A^d is reduced. One of the few other examples where this is known to be the case is the variety of complexes, studied by De Concini and Strickland [5]. Namely, let Q be the quiver

$$m \longrightarrow \dots \longrightarrow 2 \longrightarrow 1 \longrightarrow 0$$

and let $\Lambda = kQ/I$, where I is the ideal generated by all paths of length 2. Let $\mathbf{d} = (n_m, \dots, n_1, n_0)$ be a dimension vector, and define $k_i = \min(n_{i-1}, n_i)$.

Then $\text{mod}_\Lambda^{\mathbf{d}}$, the scheme of Λ -modules of dimension vector \mathbf{d} , can be identified with $\text{Spec}(A/\mathcal{E}(k_1, \dots, k_m))$, as in the introduction to [5], and Theorem 1.7 of that paper says this scheme is reduced.

12.3. As mentioned in the introduction, our main results imply that the classification of irreducible components of $\text{mod}_A^{\mathbf{d}}(k)$ can be reduced to the study of irreducible components that contain dense subsets of indecomposable modules. So we suggest the following definition. The *component quiver* $\mathcal{C}(A)$ of A has as vertices the set of irreducible components C of varieties of A -modules such that C contains a dense subset of indecomposable A -modules. We draw an arrow $C_1 \rightarrow C_2$ if $\text{ext}_A^1(C_2, C_1) = 0$.

The *component graph* $\mathcal{C}_\Gamma(A)$ is the graph with the same vertices as $\mathcal{C}(A)$ and an edge connecting C_1 and C_2 if $\text{ext}_A^1(C_1, C_2) = \text{ext}_A^1(C_2, C_1) = 0$. One can easily construct examples where $\mathcal{C}_\Gamma(A)$ contains finite and infinite connected components.

12.4. We give an example to show that $\mathcal{E}(C_1 \times C_2)$ in Corollary 1.4 need not be a union of irreducible components. Let Q be the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$$

and let $A = kQ/I$ where I is generated by the paths $\alpha\beta$ and $\beta\gamma$. The variety $\text{mod}_A^{\mathbf{d}}(k)$ breaks into connected components, one for each decomposition $\mathbf{d} = d_1 + d_2 + d_3 + d_4$, with the component corresponding to the representations of Q of dimension vector (d_1, d_2, d_3, d_4) which satisfy the relations $\alpha\beta = 0$ and $\beta\gamma = 0$.

For the variety $\text{mod}_A^1(k)$ there are four connected components, each of which is actually a point. Let C_1 be the connected component for the dimension vector $(1, 0, 0, 0)$.

The variety $\text{mod}_A^3(k)$ has a connected component corresponding to the representations of dimension vector $(0, 1, 1, 1)$. It consists of two irreducible components, one consisting of the representations with $\beta = 0$, the other corresponding to the representations with $\gamma = 0$. Let C_2 be the second of these.

Then $\mathcal{E}(C_1 \times C_2)$ consists of the representations of dimension vector $(1, 1, 1, 1)$ in which $\gamma = 0$. It is the union of two irreducible closed subsets. The subset of representations with $\alpha = 0$ is an irreducible component of $\text{mod}_A^4(k)$. The subset of representations with $\beta = 0$ is not an irreducible component, since it is a proper subset of the set of representations with $\beta = 0$ still, but with both α and γ arbitrary.

12.5. Let Q be the quiver with two vertices 1 and 2 and arrows $\alpha : 1 \rightarrow 1$, $\beta : 1 \rightarrow 1$ and $\gamma : 1 \rightarrow 2$. Define $A = kQ/I$ where the ideal I is generated by $\alpha\gamma$ and $\beta\gamma$. Then $\text{mod}_A^1(k)$ has two connected components, C_1 and C_2 , corresponding to the representations of dimension vector $(1, 0)$ and $(0, 1)$. They are both irreducible, with $\dim(C_1) = 2$ and $\dim(C_2) = 0$,

and $\text{ext}_A^1(C_2, C_1) = 0$. One can show that $\mathcal{E}(C_1 \times C_2)$ is the union of two irreducible components of $\text{mod}_A^2(k)$, of dimensions 3 and 4.

12.6. The condition that $\text{ext}_A^1(C_2, C_1) = 0$ is not necessary in Corollary 1.4. Let Q be the quiver with two vertices 1 and 2 and arrows $\alpha : 1 \rightarrow 2$ and $\beta : 2 \rightarrow 1$. Let $A = kQ/I$ where I is generated by $\alpha\beta$ and $\beta\alpha$. Then $\text{mod}_A^1(k)$ has two connected components, C_1 and C_2 , corresponding to the representations of dimension vector $(1, 0)$ and $(0, 1)$. They both consist of one point. One easily checks that $\text{ext}_A^1(C_1, C_2)$ and $\text{ext}_A^1(C_2, C_1)$ are both non-zero. But $\mathcal{E}(C_1 \times C_2)$ and $\mathcal{E}(C_2 \times C_1)$ are both irreducible components of $\text{mod}_A^2(k)$ of dimension 3.

12.7. We would like to add an application to tilted algebras that we worked out with C. Geiß after the paper was submitted.

Let $A = kQ$ be the path algebra of a quiver Q . Recall that the irreducible components of $\text{mod}_A^d(k)$ are indexed by the dimension vectors α for Q with total dimension d . The numbers $\text{hom}_A(\alpha, \beta)$ and $\text{ext}_A^1(\alpha, \beta)$ may be computed using the theory of general representations of quivers [19], and they are related by the formula $\text{hom}_A(\alpha, \beta) - \text{ext}_A^1(\alpha, \beta) = \langle \alpha, \beta \rangle$ where $\langle -, - \rangle$ is the Ringel form for Q .

Let T_A be a tilting module, let t be its dimension vector, and let $B = \text{End}_A(T)^{op}$ be the corresponding tilted algebra, see [10]. (The opposite algebra is needed, with our conventions, to ensure that T is naturally a B - A -bimodule.) We show that the irreducible components of $\text{mod}_B^d(k)$ are in 1-1 correspondence with the pairs of dimension vectors (α, β) for Q with $\langle t, \alpha - \beta \rangle = d$, $\text{ext}_A^1(t, \alpha) = 0$, $\text{hom}_A(t, \beta) = 0$ and $\text{hom}_A(\beta, \alpha) = 0$.

The Brenner-Butler Theorem [10, Theorem 2.1] gives equivalences $F = \text{Hom}_A(T, -) : \mathcal{T} \rightarrow \mathcal{Y}$ and $F' = \text{Ext}_A^1(T, -) : \mathcal{F} \rightarrow \mathcal{X}$ involving the module classes

$$\begin{aligned} \mathcal{F} &= \{M \text{ an } A\text{-module} \mid \text{Hom}_A(T, M) = 0\}, \\ \mathcal{T} &= \{M \text{ an } A\text{-module} \mid \text{Ext}_A^1(T, M) = 0\}, \\ \mathcal{X} &= \{N \text{ a } B\text{-module} \mid N \otimes_B T = 0\} = \{N \mid \text{Hom}_B(N, DT) = 0\}, \\ \mathcal{Y} &= \{N \text{ a } B\text{-module} \mid \text{Tor}_1^B(N, T) = 0\} = \{N \mid \text{Ext}_B^1(N, DT) = 0\}. \end{aligned}$$

By upper semicontinuity, \mathcal{F} and \mathcal{T} define open subsets $\mathcal{F}(d)$ and $\mathcal{T}(d)$ of $\text{mod}_A^d(k)$, and \mathcal{X} and \mathcal{Y} define open subsets $\mathcal{X}(d)$ and $\mathcal{Y}(d)$ of $\text{mod}_B^d(k)$.

The irreducible components of $\mathcal{T}(d)$ are the ones for $\text{mod}_A^d(k)$ containing a module in \mathcal{T} , and since T corresponds to an open orbit, they are the ones with $\text{ext}_A^1(t, \alpha) = 0$. By a result of Bongartz [2, Theorem 3], the modules $F(M)$ with M in such an irreducible component form an irreducible component of $\mathcal{Y}(d_1)$, where $d_1 = \langle t, \alpha \rangle$. Varying α , this construction gives all irreducible components of $\mathcal{Y}(d_1)$.

Dually, the dimension vectors β with $\text{hom}_A(t, \beta) = 0$ index the irreducible components of $\mathcal{X}(d_2)$, where $d_2 = -\langle t, \beta \rangle$. (This requires the dual of Bongartz's Theorem, but this follows automatically on exchanging the roles of A and B , and using the dual of T .)

Now any B -module is a direct sum of modules in \mathcal{X} and \mathcal{Y} by [10, Theorem 6.3], so any irreducible component of $\text{mod}_B^d(k)$ is of the form $\overline{C_1 \oplus C_2}$ where C_1 is the closure in $\text{mod}_B^{d_1}(k)$ of an irreducible component of $\mathcal{Y}(d_1)$, say corresponding to α , and C_2 is the closure in $\text{mod}_B^{d_2}(k)$ of an irreducible component of $\mathcal{X}(d_2)$, say corresponding to β . Moreover, the summands of the B -module in \mathcal{X} and \mathcal{Y} are unique up to isomorphism, and it follows that different choices of C_1 and C_2 lead to different irreducible components.

Now the condition for $\overline{C_1 \oplus C_2}$ to be an irreducible component is that ext vanishes generically in both directions. In one direction this is automatic, for $\text{Ext}_B^1(\mathcal{Y}, \mathcal{X}) = 0$ by [18, 4.1.6(d)]. In the other direction we have

$$\text{Ext}_B^1(F'(M), F(N)) \cong \text{Hom}_A(M, N)$$

for $M \in \mathcal{F}$ and $N \in \mathcal{T}$. (This is obvious from the interpretation of tilting theory as an equivalence of derived categories.) The condition for this to vanish generically is that $\text{hom}_A(\beta, \alpha) = 0$.

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