

# EXTENSION-ORTHOGONAL COMPONENTS OF PREPROJECTIVE VARIETIES

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ABSTRACT. Let  $Q$  be a Dynkin quiver, and let  $\Lambda$  be the corresponding preprojective algebra. Let  $\mathcal{C} = \{C_i \mid i \in I\}$  be a set of pairwise different indecomposable irreducible components of varieties of  $\Lambda$ -modules such that generically there are no extensions between  $C_i$  and  $C_j$  for all  $i, j$ . We show that the number of elements in  $\mathcal{C}$  is at most the number of positive roots of  $Q$ . Furthermore, we give a module theoretic interpretation of Leclerc's counterexample to a conjecture of Berenstein and Zelevinsky.

## 1. INTRODUCTION

Let  $k$  be an algebraically closed field. For a finitely generated  $k$ -algebra  $A$  let  $\text{mod}_A(\mathbf{d})$  be the affine variety of (left)  $A$ -modules with dimension vector  $\mathbf{d}$ . Throughout, we only consider finite-dimensional modules.

For irreducible components  $C_1 \subseteq \text{mod}_A(\mathbf{d}_1)$  and  $C_2 \subseteq \text{mod}_A(\mathbf{d}_2)$  define

$$\text{ext}_A^1(C_1, C_2) = \min\{\dim \text{Ext}_A^1(M_1, M_2) \mid (M_1, M_2) \in C_1 \times C_2\}.$$

An irreducible component  $C \subseteq \text{mod}_A(\mathbf{d})$  is *indecomposable* if it contains a dense subset of indecomposable  $A$ -modules. A general theory of irreducible components and their decomposition into indecomposable irreducible components was developed in [5]. Our aim is to apply this to the preprojective varieties.

If not mentioned otherwise, we always assume that  $Q$  is a Dynkin quiver of type  $\mathbb{A}_n$ ,  $\mathbb{D}_n$ ,  $\mathbb{E}_6$ ,  $\mathbb{E}_7$  or  $\mathbb{E}_8$ . By  $R^+$  we denote the set of positive roots of  $Q$ , and by  $\Lambda$  we denote the preprojective algebra associated to  $Q$ , see [20]. Let  $n$  be the number of vertices of  $Q$ , and let  $\Lambda(\mathbf{d}) = \text{mod}_\Lambda(\mathbf{d})$ ,  $\mathbf{d} \in \mathbb{N}^n$ , be the variety of  $\Lambda$ -modules with dimension vector  $\mathbf{d}$ . The varieties  $\Lambda(\mathbf{d})$  are called *preprojective varieties*. Since we consider only preprojective algebras of Dynkin type, the preprojective varieties coincide with the nilpotent varieties defined by Lusztig. We refer to [16, Section 12] for basic properties. Our main result is the following:

**Theorem 1.1.** *If  $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$  is a set of pairwise different indecomposable irreducible components such that  $\text{ext}_\Lambda^1(C_i, C_j) = 0$  for all  $i, j \in I$ , then  $|I| \leq |R^+|$ .*

The upper bound given in the theorem seems to be optimal. For example, if  $Q$  is of type  $\mathbb{A}_n$ , then it is easy to construct sets  $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$  of pairwise different indecomposable irreducible components such that  $\text{ext}_\Lambda^1(C_i, C_j) = 0$  for all  $i, j \in I$  and  $|I| = |R^+|$ , see the end of Section 4.

As a consequence of the above theorem we get the following result on  $\Lambda$ -modules without self-extensions:

**Corollary 1.2.** *Let  $M$  be a  $\Lambda$ -module with  $\text{Ext}_\Lambda^1(M, M) = 0$ . Then the number of pairwise non-isomorphic indecomposable direct summands of  $M$  is at most  $|R^+|$ .*

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Let  $U_v^-$  be the negative part of the quantized enveloping algebra of the Lie algebra corresponding to  $Q$ . We regard  $U_v^-$  as a  $\mathbb{Q}(v)$ -algebra. Let  $\mathcal{B}$  be the canonical basis and  $\mathcal{B}^*$  the dual canonical basis of  $U_v^-$ , see [2], [14], [16] or [18] for definitions. By [12, Section 5], [15] the elements of  $\mathcal{B}$  (and thus of  $\mathcal{B}^*$ ) correspond to the irreducible components of the preprojective varieties  $\Lambda(\mathbf{d})$ ,  $\mathbf{d} \in \mathbb{N}^n$ . Let  $b^*(C)$  be the dual canonical basis vector corresponding to an irreducible component  $C$ . We denote the structure constants of  $U_v^-$  with respect to the basis  $\mathcal{B}^*$  by  $\lambda_{C,D}^E$ , i.e.

$$b^*(C)b^*(D) = \sum_E \lambda_{C,D}^E b^*(E).$$

Following the terminology in [2], two dual canonical basis vectors  $b^*(C)$  and  $b^*(D)$  are called *multiplicative* if

$$b^*(C)b^*(D) = \lambda b^*(E)$$

for some irreducible component  $E$  and some  $0 \neq \lambda \in \mathbb{Q}(v)$ . One calls  $b^*(C)$  and  $b^*(D)$  *quasi-commutative* if

$$b^*(C)b^*(D) = \lambda b^*(D)b^*(C)$$

for some  $0 \neq \lambda \in \mathbb{Q}(v)$ . The following conjecture was stated in [2, Section 1]:

**Conjecture 1.3** (Berenstein, Zelevinsky). *Two dual canonical basis vectors are multiplicative if and only if they are quasi-commutative.*

One direction of this conjecture was proved by Reineke [18, Corollary 4.5]. The other direction turned out to be wrong. Namely, Leclerc [13] constructed examples of quasi-commutative elements in  $\mathcal{B}^*$  which are not multiplicative. Using preprojective algebras, we give a module theoretic interpretation of one of his examples.

Marsh and Reineke [17] conjectured that the multiplicative behaviour of dual canonical basis vectors should be related to sets of irreducible components with  $\text{Ext}^1$  vanishing generically between them. This was the principal motivation for our work.

The paper is organized as follows: In Section 2 we review the main results from [5]. In Section 3 we recall known results for the case that  $\Lambda$  is an algebra of finite or tame representation type. The proof of Theorem 1.1 and its corollary can be found in Section 4. Finally, Section 5 is devoted to the interpretation of Leclerc's example.

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## 2. VARIETIES OF MODULES - DEFINITIONS AND KNOWN RESULTS

In this section, we work with arbitrary finite quivers.

2.1. Let  $Q = (Q_0, Q_1)$  be a finite quiver, where  $Q_0$  denotes the set of vertices and  $Q_1$  the set of arrows of  $Q$ . Assume that  $|Q_0| = n$ . For an arrow  $\alpha$  let  $s\alpha$  be its starting vertex and  $e\alpha$  its end vertex. An element  $\mathbf{d} = (d_i)_{i \in Q_0} \in \mathbb{N}^n$  is called a *dimension vector* for  $Q$ . A *representation* of  $Q$  with dimension vector  $\mathbf{d}$  is a matrix tuple  $M = (M_\alpha)_{\alpha \in Q_1}$  with  $M_\alpha \in \text{M}_{d_{e\alpha} \times d_{s\alpha}}(k)$ . A *path* of length  $l \geq 1$  in  $Q$  is a sequence  $p = \alpha_1 \cdots \alpha_l$  of arrows in  $Q_1$  such that  $s\alpha_i = e\alpha_{i+1}$  for  $1 \leq i \leq l-1$ . Define  $sp = s\alpha_l$  and  $ep = e\alpha_1$ . For a representation  $M$  and a path  $p = \alpha_1 \cdots \alpha_l$  define  $M_p = M_{\alpha_1} \cdots M_{\alpha_l}$  which is a matrix in  $\text{M}_{d_{ep} \times d_{sp}}(k)$ . A *relation* for  $Q$  is a  $k$ -linear combination  $\sum_{i=1}^t \lambda_i p_i$  of paths  $p_i$  of length at least two such that  $sp_i = sp_j$  and  $ep_i = ep_j$  for all  $1 \leq i, j \leq t$ . A representation  $M$  satisfies such a relation if  $\sum_{i=1}^t \lambda_i M_{p_i} = 0$ . Given a set  $\rho$  of relations for  $Q$  let  $\text{rep}_{(Q,\rho)}(\mathbf{d})$  be the affine variety of representations of  $Q$  with dimension vector  $\mathbf{d}$  which satisfy all relations in  $\rho$ .

2.2. One can interpret this construction in a module theoretic way. Namely, let  $kQ$  be the path algebra of  $Q$ , and let  $A = kQ/(\rho)$ , where  $(\rho)$  is the ideal generated by the elements in  $\rho$ . Then  $\text{mod}_A(\mathbf{d}) = \text{rep}_{(Q,\rho)}(\mathbf{d})$  is the affine *variety of  $A$ -modules* with dimension vector  $\mathbf{d}$ . If  $A = kQ/(\rho)$  is finite-dimensional, then  $A$  is a basic algebra. In this case, the vertices of  $Q$  correspond to the isomorphism classes of simple  $A$ -modules. The entry  $d_i$ ,  $i \in Q_0$ , of  $\mathbf{d}$  is the multiplicity of the simple module corresponding to  $i$  in a composition series of any  $M \in \text{mod}_A(\mathbf{d})$ . The group  $\text{GL}(\mathbf{d}) = \prod_{i \in Q_0} \text{GL}_{d_i}(k)$  acts on  $\text{mod}_A(\mathbf{d})$  by conjugation, i.e.

$$g \cdot M = (g_{e\alpha} M_{\alpha} g_{s\alpha}^{-1})_{\alpha \in Q_1}.$$

The orbit of  $M$  under this action is denoted by  $\mathcal{O}(M)$ . There is a 1-1 correspondence between the set of orbits in  $\text{mod}_A(\mathbf{d})$  and the set of isomorphism classes of  $A$ -modules with dimension vector  $\mathbf{d}$ . For further details on varieties of modules, in particular on the close relation between representations of bounded quivers and modules over finite-dimensional algebras, we refer to [3].

2.3. Given irreducible components  $C_i \subseteq \text{mod}_A(\mathbf{d}_i)$ ,  $1 \leq i \leq t$ , we consider all  $A$ -modules with dimension vector  $\mathbf{d} = \mathbf{d}_1 + \cdots + \mathbf{d}_t$ , which are isomorphic to  $M_1 \oplus \cdots \oplus M_t$  where  $M_i \in C_i$  for all  $i$ . By

$$C_1 \oplus \cdots \oplus C_t$$

we denote the corresponding subset of  $\text{mod}_A(\mathbf{d})$ . This is the image of the map

$$\begin{aligned} \text{GL}(\mathbf{d}) \times C_1 \times \cdots \times C_t &\longrightarrow \text{mod}_A(\mathbf{d}) \\ (g, M_1, \dots, M_t) &\mapsto g \cdot \left( \bigoplus_{i=1}^t M_i \right). \end{aligned}$$

We call  $C_1 \oplus \cdots \oplus C_t$  the *direct sum* of the components  $C_i$ . It follows that the closure  $\overline{C_1 \oplus \cdots \oplus C_t}$  is irreducible. For an irreducible component  $C$  define  $C^n = \bigoplus_{i=1}^n C$ . We call  $C$  *indecomposable* if  $C$  contains a dense subset of indecomposable  $A$ -modules. The following result from [5] is an analogue of the Krull-Remak-Schmidt Theorem.

**Theorem 2.1.** *If  $C \subseteq \text{mod}_A(\mathbf{d})$  is an irreducible component, then*

$$C = \overline{C_1 \oplus \cdots \oplus C_t}$$

*for some indecomposable irreducible components  $C_i \subseteq \text{mod}_A(\mathbf{d}_i)$ ,  $1 \leq i \leq t$ . The components  $C_1, \dots, C_t$  are uniquely determined by this, up to reordering. The above direct sum is called the canonical decomposition of  $C$ .*

However, the closure of a direct sum of irreducible components is not in general an irreducible component. The next result is also proved in [5].

**Theorem 2.2.** *If  $C_i \subseteq \text{mod}_A(\mathbf{d}_i)$ ,  $1 \leq i \leq t$ , are irreducible components, and  $\mathbf{d} = \mathbf{d}_1 + \cdots + \mathbf{d}_t$ , then  $\overline{C_1 \oplus \cdots \oplus C_t}$  is an irreducible component of  $\text{mod}_A(\mathbf{d})$  if and only if  $\text{ext}_A^1(C_i, C_j) = 0$  for all  $i \neq j$ .*

Instead of taking direct sums of the modules in two irreducible components, one can take extensions. Let  $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$  be dimension vectors, let  $G = \text{GL}(\mathbf{d}_1) \times \text{GL}(\mathbf{d}_2)$ , and let  $S$  be a  $G$ -stable subset of  $\text{mod}_A(\mathbf{d}_1) \times \text{mod}_A(\mathbf{d}_2)$ . We denote by  $\mathcal{E}(S)$  the  $\text{GL}(\mathbf{d})$ -stable subset of  $\text{mod}_A(\mathbf{d})$  corresponding to all modules  $M$  which belong to a short exact sequence

$$0 \longrightarrow M_2 \longrightarrow M \longrightarrow M_1 \longrightarrow 0$$

with  $(M_1, M_2) \in S$ , see [5] for more details.

For an irreducible component  $C \subseteq \text{mod}_A(\mathbf{d})$  let

$$\mu_g(C) = \dim C - \max\{\dim \mathcal{O}(M) \mid M \in C\}$$

be the *generic number of parameters* of  $C$ . Thus  $\mu_g(C) = 0$  if and only if  $C$  contains a dense orbit  $\mathcal{O}(M)$ . For example, if  $P$  is a projective  $A$ -module, then  $\text{Ext}_A^1(P, P) = 0$ . This implies that the closure of the orbit  $\mathcal{O}(P)$  is an irreducible component, and we get  $\mu_g(\overline{\mathcal{O}(P)}) = 0$ . Also, if  $C = \overline{C_1 \oplus \cdots \oplus C_t}$ , then

$$\mu_g(C) = \sum_{i=1}^t \mu_g(C_i).$$

### 3. THE FINITE AND TAME CASES

Let  $A$  be a finite-dimensional  $k$ -algebra. Then  $A$  is called *representation finite* if there are only finitely many isomorphism classes of indecomposable  $A$ -modules. The algebra  $A$  is *tame* if  $A$  is not representation finite, and if for all dimension vectors  $\mathbf{d}$  the indecomposable  $A$ -modules in  $\text{mod}_A(\mathbf{d})$  can be parametrized by a finite number of affine lines. Otherwise  $A$  is called *wild*. For precise definitions we refer to [4, Section 5.3]. The preprojective algebras of Dynkin type are selfinjective. We refer to [9] for the theory of representation finite selfinjective algebras. The general theory of arbitrary representation finite algebras is explained in [8]. Introductions to Auslander-Reiten theory and the representation theory of finite-dimensional algebras can be found in [1] and [19].

**Proposition 3.1.** *Let  $Q$  be a Dynkin quiver, and let  $\Lambda$  be the associated preprojective algebra. Then the following hold:*

- $\Lambda$  is representation finite if and only if  $Q$  is of type  $\mathbb{A}_i$ ,  $i \leq 4$ ;
- $\Lambda$  is tame if and only if  $Q$  is of type  $\mathbb{A}_5$  or  $\mathbb{D}_4$ .

The above proposition is well known to the experts. Let us sketch a proof: There always exists a simply connected Galois covering  $F : \tilde{\Lambda} \rightarrow \Lambda$  of  $\Lambda$ . In the cases  $\mathbb{A}_i$ ,  $i \leq 4$ , one can construct all indecomposable  $\tilde{\Lambda}$ -modules via the knitting procedure of preprojective components. One gets that  $\tilde{\Lambda}$  is locally representation finite, thus  $\Lambda$  is representation finite in these cases. For  $\mathbb{A}_5$  and  $\mathbb{D}_4$  the algebra  $\tilde{\Lambda}$  is the repetitive algebra of a tubular algebra. It follows that the push down functor  $\text{mod}(\tilde{\Lambda}) \rightarrow \text{mod}(\Lambda)$  is dense, and that  $\Lambda$  is tame in these two cases. We refer to [6, Section 6] and [11] for more details. In all other cases, one can show that the algebra  $\tilde{\Lambda}$  contains wild full convex subalgebras, thus also  $\Lambda$  is wild. For the basics of covering theory we refer to [7] and [10].

If  $\Lambda$  is representation finite, and if  $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$  is a maximal set of pairwise different indecomposable irreducible components such that  $\text{ext}_\Lambda^1(C_i, C_j) = 0$  for all  $i, j$ , then  $|I| = |R^+|$ . This was proved by Marsh and Reineke, compare also [2].

For a tame algebra  $A$  one has  $\mu_g(C) \leq 1$  for any indecomposable irreducible component  $C \subseteq \text{mod}_A(\mathbf{d})$ . For  $\Lambda$  tame a complete classification of the indecomposable irreducible components, and a necessary and sufficient condition for  $\text{ext}_\Lambda^1(C, D) = 0$  for any two irreducible components  $C$  and  $D$  was obtained in [11].

If  $\Lambda$  is of wild representation type, one should expect irreducible components  $C$  with  $\text{ext}_\Lambda^1(C, C) \neq 0$ . Thus, maybe one should study sets  $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$  of irreducible components with the weaker condition  $\text{ext}_\Lambda^1(C_i, C_j) = 0$  for all  $i \neq j$ . However, we do not know how to generalize Theorem 1.1 to this case.

For the two tame cases, the following was proved in [11]:

**Theorem 3.2.** *Assume that  $Q$  is of type  $\mathbb{A}_5$  or  $\mathbb{D}_4$ . Then the following hold:*

- (1) *For any irreducible component  $C \subseteq \Lambda(\mathbf{d})$  we have  $\text{ext}_\Lambda^1(C, C) = 0$ ;*
- (2) *If  $C \subseteq \Lambda(\mathbf{d})$  is an indecomposable irreducible component, then we have  $\mu_g(C) = 0$  or  $\mu_g(C) = 1$ . For suitable  $\mathbf{d}$  there exists an indecomposable irreducible component  $C \subseteq \Lambda(\mathbf{d})$  with  $\mu_g(C) = 1$ ;*

- (3) Let  $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$  be a maximal set of pairwise different indecomposable irreducible components such that  $\text{ext}_\Lambda^1(C_i, C_j) = 0$  for all  $i, j$ . Then there is at most one  $C_i$  with  $\mu_g(C_i) = 1$ . In this case, we have  $|I| = |R^+| - 1$ , and we get  $|I| = |R^+|$ , otherwise.

This leads us to the following conjecture for arbitrary Dynkin quivers of type  $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7$  or  $\mathbb{E}_8$ :

**Conjecture 3.3.** *If  $\{C_i \subseteq \Lambda(\mathbf{d}_i) \mid i \in I\}$  is a maximal set of pairwise different indecomposable irreducible components such that  $\text{ext}_\Lambda^1(C_i, C_j) = 0$  for all  $i, j$ , then*

$$|I| = |R^+| - \sum_{i \in I} \mu_g(C_i).$$

#### 4. PROOF OF THEOREM 1.1

As before let  $Q$  be a Dynkin quiver, and let

$$R^+ = \{\mathbf{a}_i \mid 1 \leq i \leq N\}$$

be the set of positive roots of  $Q$ . From now on  $N = |R^+|$  will always be the number of positive roots of  $Q$ .

By Gabriel's Theorem there is a 1-1 correspondence between  $R^+$  and the set of isomorphism classes of indecomposable  $kQ$ -modules. This correspondence associates to a root  $\mathbf{a}_i$  the isomorphism class  $[M(\mathbf{a}_i)]$  of an indecomposable  $kQ$ -module  $M(\mathbf{a}_i)$  with dimension vector  $\mathbf{a}_i$ . By the Theorem of Krull-Remak-Schmidt each  $kQ$ -module is isomorphic to a unique (up to reordering) direct sum of the indecomposable modules  $M(\mathbf{a}_i)$ . For  $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$  set

$$M_\alpha = \bigoplus_{i=1}^N M(\mathbf{a}_i)^{\alpha_i} \text{ and } C_\alpha = \overline{\pi_{\mathbf{d}}^{-1}(\mathcal{O}(M_\alpha))}$$

where  $\mathbf{d}$  is the dimension vector of  $M_\alpha$  and

$$\pi_{\mathbf{d}} : \Lambda(\mathbf{d}) \longrightarrow \text{mod}_{kQ}(\mathbf{d})$$

is the canonical projection map. Let  $\text{Irr}(\Lambda)$  be the set of irreducible components of the varieties  $\Lambda(\mathbf{d})$ ,  $\mathbf{d} \in \mathbb{N}^n$ . The three maps

$$\begin{aligned} \mathbb{N}^N &\longrightarrow \{\mathcal{O}(M) \mid M \in \text{mod}_{kQ}(\mathbf{d}), \mathbf{d} \in \mathbb{N}^n\} \longrightarrow \text{Irr}(\Lambda) \longrightarrow \mathcal{B}^* \\ \alpha &= (\alpha_1, \dots, \alpha_N) \mapsto \mathcal{O}(M_\alpha) \mapsto C_\alpha \mapsto b^*(C_\alpha) \end{aligned}$$

are all bijective.

Let  $\alpha, \beta \in \mathbb{N}^N$ . By Theorem 2.2 the closure  $\overline{C_\alpha \oplus C_\beta}$  is an irreducible component if and only if  $\text{ext}_\Lambda^1(C_\alpha, C_\beta) = \text{ext}_\Lambda^1(C_\beta, C_\alpha) = 0$ . In this case, we have  $\overline{C_\alpha \oplus C_\beta} = C_{\alpha+\beta}$ .

Let  $\delta_j = (\delta_{1j}, \dots, \delta_{Nj})$ ,  $1 \leq j \leq N+1$ , be non-zero pairwise different elements in  $\mathbb{N}^N$  such that  $C_{\delta_j}$  is an indecomposable irreducible component for all  $j$ . To get a contradiction, we assume that  $\text{ext}_\Lambda^1(C_{\delta_i}, C_{\delta_j}) = 0$  for all  $1 \leq i, j \leq N+1$ . For  $\mathbf{m} = (m_1, \dots, m_{N+1}) \in \mathbb{N}^{N+1}$  define

$$C(\mathbf{m}) = C_{\Delta \mathbf{m}} \text{ where } \Delta = (\delta_{ij}) \in \mathbb{N}^{N \times (N+1)}.$$

Thus  $\delta_j$  is the  $j$ th column of the matrix  $\Delta$ . By our assumption we get

$$C(\mathbf{m}) = \overline{C_{\delta_1}^{m_1} \oplus \dots \oplus C_{\delta_{N+1}}^{m_{N+1}}}.$$

Since the  $C_{\delta_j}$  are indecomposable, the above is the canonical decomposition of the irreducible component  $C(\mathbf{m})$ .

Now there exist some elements  $\mathbf{m} = (m_1, \dots, m_{N+1}) \neq \mathbf{1} = (l_1, \dots, l_{N+1})$  in  $\mathbb{N}^{N+1}$  such that

$$\Delta \mathbf{1} = \Delta \mathbf{m} \in \mathbb{N}^N$$

This implies  $C(\mathbf{m}) = C(\mathbf{1})$ . Thus, we get a contradiction to the unicity of the canonical decomposition of irreducible components, see Theorem 2.1.

In fact, let  $0 \neq \mathbf{z} \in \mathbb{Z}^{N+1}$  with  $\Delta \mathbf{z} = 0$ . Clearly, such a  $\mathbf{z}$  always exists. Since all entries of  $\Delta$  are non-negative,  $\mathbf{z}$  has at least one negative entry. Let

$$l = -\min\{z_i \mid 1 \leq i \leq N+1\},$$

then trivially  $\mathbf{1} = (l, l, \dots, l)$  and  $\mathbf{m} = \mathbf{1} + \mathbf{z}$  are as required. This finishes the proof of Theorem 1.1.

Corollary 1.2 follows immediately from the fact that an orbit  $\mathcal{O}(N) \subseteq \text{mod}_A(\mathbf{d})$  of an  $A$ -module  $N$  is open provided  $\text{Ext}_A^1(N, N) = 0$ . Clearly,  $\mathcal{O}(N)$  is open if and only if the closure  $\overline{\mathcal{O}(N)}$  is an irreducible component. Then we use Theorems 1.1 and 2.2.

Finally, assume that  $Q$  is of type  $\mathbb{A}_n$ , and let  $S_1, \dots, S_n$  denote the (isomorphism classes of) simple  $\Lambda$ -modules. Set

$$H = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq n, 1 \leq j \leq n+1-i\},$$

and notice that  $|H| = |R^+|$ . For  $(i, j) \in H$  there exists a unique (up to isomorphism)  $\Lambda$ -module  $L_{(i,j)}$  with  $\text{top}(L_{(i,j)}) \cong S_i$  and  $\text{soc}(L_{(i,j)}) \cong S_j$  that admits  $S_1$  as a composition factor. It is easy to check that  $\text{Ext}_\Lambda^1(L_{(i,j)}, L_{(p,q)}) = 0$  for all  $(i, j), (p, q) \in H$ . Thus

$$\mathcal{C} = \{\overline{\mathcal{O}(L_{(i,j)})} \mid (i, j) \in H\}$$

is a set of pairwise different indecomposable irreducible components with  $\text{ext}_\Lambda^1(C, D) = 0$  for all  $C, D \in \mathcal{C}$ , and we have  $|\mathcal{C}| = |R^+|$ .

## 5. INTERPRETATION OF LECLERC'S EXAMPLE

In the following, we use the notation introduced at the beginning of Section 4. Reineke proved in [18, Lemma 4.6] that the multiplicativity of  $b^*(C_\alpha)$  and  $b^*(C_\beta)$  implies that

$$b^*(C_\alpha)b^*(C_\beta) = v^m b^*(C_{\alpha+\beta})$$

for some  $m \in \mathbb{Z}$ . He also showed that  $\lambda_{C,D}^E \neq 0$  if and only if  $\lambda_{D,C}^E \neq 0$ . This follows from [18, Proposition 4.4]. Thus one direction of Conjecture 1.3 holds, namely if two dual canonical basis vectors are multiplicative, then they are quasi-commutative. The following related problem should be of interest:

**Problem 5.1.** *Describe the elements  $\alpha, \beta \in \mathbb{N}^N$  such that*

$$C_{\alpha+\beta} = \overline{C_\alpha \oplus C_\beta}.$$

As mentioned in the introduction, Leclerc recently constructed in [13] counterexamples for the other direction of the Berenstein-Zelevinsky Conjecture. We give a module theoretic interpretation of one of his examples:

Let  $Q$  be the quiver of type  $\mathbb{A}_5$  with arrows  $a_i : i+1 \rightarrow i$ ,  $1 \leq i \leq 4$ . Thus  $\Lambda$  is given by the quiver

$$\begin{array}{cccccc} & \bar{a}_1 & & \bar{a}_2 & & \bar{a}_3 & & \bar{a}_4 & & \\ 1 & & 2 & & 3 & & 4 & & 5 & \\ & a_1 & & a_2 & & a_3 & & a_4 & & \end{array}$$

and the following set of relations

$$\{a_1 \bar{a}_1, \bar{a}_1 a_1 - a_2 \bar{a}_2, \bar{a}_2 a_2 - a_3 \bar{a}_3, \bar{a}_3 a_3 - a_4 \bar{a}_4, \bar{a}_4 a_4\}.$$

Now  $R^+$  contains exactly 15 elements, namely for each  $1 \leq i \leq j \leq 5$  there is a positive root  $[i, j] = (d_l)_{1 \leq l \leq 5}$  with  $d_l = 1$  for  $i \leq l \leq j$ , and  $d_l = 0$ , otherwise. We identify  $\text{NR}^+$  with  $\mathbb{N}^{15}$  by fixing a linear ordering on  $R^+$ , namely let

$$\begin{aligned} [1, 1] &< [1, 2] < [1, 3] < [1, 4] < [1, 5] < [2, 2] < [2, 3] < [2, 4] < [2, 5] \\ &< [3, 3] < [3, 4] < [3, 5] < [4, 4] < [4, 5] < [5, 5]. \end{aligned}$$

Define

$$\begin{aligned} \alpha &= [1, 2] + [2, 4] + [3, 3] + [4, 5], \\ \beta &= [1, 2] + [1, 4] + [2, 3] + [2, 5] + [3, 4] + [4, 5]. \end{aligned}$$

Thus, regarded as elements in  $\mathbb{N}^{15}$  we have

$$\begin{aligned} \alpha &= (0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0), \\ \beta &= (0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0). \end{aligned}$$

In [13] Leclerc showed that

$$b^*(C_\alpha)^2 = v^{-2}(b^*(C_{\alpha+\alpha}) + b^*(C_\beta)).$$

This is obviously a counterexample to the Berenstein-Zelevinsky Conjecture. Now define

$$\begin{aligned} \beta_1 &= [1, 2] + [2, 3] + [3, 4] + [4, 5], \\ \beta_2 &= [1, 4] + [2, 5]. \end{aligned}$$

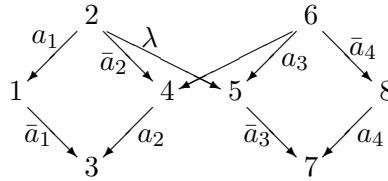
Thus we have  $\beta = \beta_1 + \beta_2$ .

**Proposition 5.2.** *Let  $\alpha, \beta, \beta_1, \beta_2$  be as above. Then the following hold:*

- (1) *The irreducible components  $C_\alpha, C_{\beta_1}$  and  $C_{\beta_2}$  are indecomposable with  $\mu_g(C_\alpha) = 1$  and  $\mu_g(C_{\beta_1}) = \mu_g(C_{\beta_2}) = 0$ ;*
- (2) *We have  $C_{\alpha+\alpha} = \overline{C_\alpha \oplus C_\alpha}$  and  $C_\beta = \overline{C_{\beta_1} \oplus C_{\beta_2}}$ . Thus*

$$b^*(C_\alpha)^2 = v^{-2}(b^*(\overline{C_\alpha \oplus C_\alpha}) + b^*(\overline{C_{\beta_1} \oplus C_{\beta_2}})).$$

*Proof.* For  $\lambda \in k \setminus \{0, 1\}$  let  $M_\lambda$  be the 8-dimensional  $\Lambda$ -module where the arrows of  $\Lambda$  operate on a basis  $\{1, \dots, 8\}$  as in the following picture:



Thus, for example  $a_1 \cdot 2 = 1$ ,  $\bar{a}_2 \cdot 2 = 4 + \lambda 5$ ,  $a_3 \cdot 6 = 4 + 5$ ,  $\bar{a}_1 \cdot 1 = 3$ , etc. Note that  $M_\lambda$  lies in  $C_\alpha$ .

The modules  $M_\lambda$  are indecomposable and  $\dim \text{End}_\Lambda(M_\lambda) = 3$ . It is known that the preprojective varieties  $\Lambda(\mathbf{d})$  are equidimensional of dimension

$$\sum_{\alpha \in Q_1} d_{s\alpha} d_{e\alpha},$$

see for example [16, Section 12].

Thus each irreducible component of  $\Lambda(1, 2, 2, 2, 1)$  has dimension  $2 + 4 + 4 + 2 = 12$ . The group  $\text{GL}(1, 2, 2, 2, 1)$  acts as described in Section 2 on  $\Lambda(1, 2, 2, 2, 1)$  and has dimension

14. Thus we get  $\dim \mathcal{O}(M_\lambda) = 14 - 3 = 11$ . One checks easily that  $M_\lambda$  and  $M_\mu$  are isomorphic if and only if  $\lambda = \mu$ . This implies

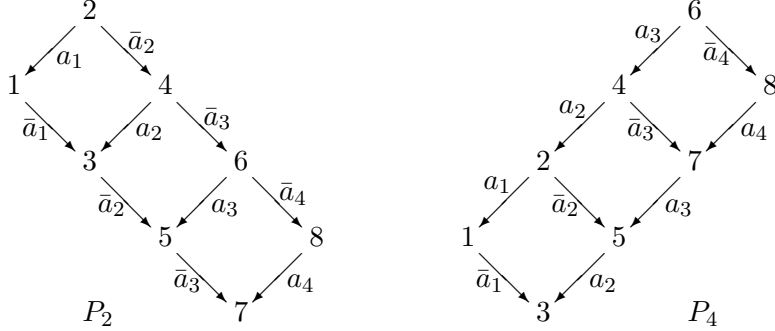
$$\dim \overline{\{\mathcal{O}(M_\lambda) \mid \lambda \in k \setminus \{0, 1\}\}} = 11 + 1 = 12.$$

We get

$$C_\alpha = \overline{\{\mathcal{O}(M_\lambda) \mid \lambda \in k \setminus \{0, 1\}\}}.$$

Thus  $C_\alpha$  is an indecomposable irreducible component with  $\mu_g(C_\alpha) = 1$ .

Next, let  $P_2$  and  $P_4$  be the indecomposable projective  $\Lambda$ -modules corresponding to the vertices 2 and 4, respectively. These modules are both 8-dimensional and with the same convention as above we may describe them as follows:



We have  $\text{Ext}_\Lambda^1(P_i, P_j) = 0$  for all  $i, j \in \{2, 4\}$ . This follows directly from the projectivity of both modules. From this and the above pictures we get

$$\begin{aligned} C_{\beta_1} &= \overline{\mathcal{O}(P_2)}, \\ C_{\beta_2} &= \overline{\mathcal{O}(P_4)}, \\ C_\beta &= \overline{C_{\beta_1} \oplus C_{\beta_2}}. \end{aligned}$$

In particular,  $C_{\beta_1}$  and  $C_{\beta_2}$  are indecomposable irreducible components with  $\mu_g(C_{\beta_1}) = \mu_g(C_{\beta_2}) = 0$ . This finishes the proof.  $\square$

For irreducible components  $C \subseteq \Lambda(\mathbf{d})$  and  $D \subseteq \Lambda(\mathbf{e})$  define

$$\mathcal{V}(C, D) = \bigcap_{U \subseteq C, V \subseteq D} \left\{ E \subseteq \Lambda(\mathbf{d} + \mathbf{e}) \text{ irred. comp.} \mid E \subseteq \overline{\mathcal{E}(U \times V)} \right\},$$

where  $U$  (resp.  $V$ ) runs through all non-empty  $\text{GL}(\mathbf{d})$ -stable (resp.  $\text{GL}(\mathbf{e})$ -stable) open subsets of  $C$  (resp.  $D$ ), see Section 2 for the definition of  $\mathcal{E}(U \times V)$ .

Using the previous proposition, and some well-known results on the representation theory of the algebra  $\Lambda$ , see [11] and [19], one can show that

$$\mathcal{V}(C_\alpha, C_\alpha) = \{C_{\alpha+\alpha}, C_\beta\}.$$

Note that  $\text{ext}_\Lambda^1(C_\alpha, C_\alpha) = 0$ , since  $\text{Ext}_\Lambda^1(M_\lambda, M_\mu) = 0$  for all  $\lambda \neq \mu$ . But one can show that  $\dim \text{Ext}_\Lambda^1(M_\lambda, M_\lambda) = 2$ , see [11, Section 6]. For any  $M_\lambda$  there is a short exact sequence

$$0 \longrightarrow M_\lambda \longrightarrow M_\lambda(2) \longrightarrow M_\lambda \longrightarrow 0,$$

where  $M_\lambda(2)$  is the module of quasi-length two in the same Auslander-Reiten component as  $M_\lambda$  (it is known that  $M_\lambda$  lies in a homogeneous tube). Additionally to this ‘natural’ self-extension, there exists a short exact sequence

$$0 \longrightarrow M_\lambda \longrightarrow P_2 \oplus P_4 \longrightarrow M_\lambda \longrightarrow 0.$$

Motivated by our above analysis, one might conjecture the following:

- Conjecture 5.3.** (1) If  $E \in \mathcal{V}(C, D) \cup \mathcal{V}(D, C)$ , then  $\lambda_{C,D}^E \neq 0$ ;  
 (2) If irreducible components  $C$  and  $D$  contain non-empty stable open subsets  $U \subseteq C$  and  $V \subseteq D$  such that  $\text{Ext}_{\Lambda}^1(M, N) = 0$  for all  $M \in U, N \in V$ , then  $b^*(C)$  and  $b^*(D)$  are multiplicative.

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