AUSLANDER ALGEBRAS AND INITIAL SEEDS FOR CLUSTER ALGEBRAS

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Abstract. Let Q be a Dynkin quiver and Π the corresponding set of positive roots. For the preprojective algebra Λ associated to Q we produce a rigid Λ-module I_Q with $r = |\Pi|$ pairwise non-isomorphic indecomposable direct summands by pushing the injective modules of the Auslander algebra of $\mathrm{k}Q$ to Λ.

If N is a maximal unipotent subgroup of a complex simply connected simple Lie group of type $|Q|$, then the coordinate ring $\mathbb{C}[N]$ is an upper cluster algebra. We show that the elements of the dual semicanonical basis which correspond to the indecomposable direct summands of I_Q coincide with certain generalized minors which form an initial cluster for $\mathbb{C}[N]$, and that the corresponding exchange matrix of this cluster can be read from the Gabriel quiver of $\mathrm{End}_\Lambda(\mathrm{I}_Q)$.

Finally, we exploit the fact that the categories of injective modules over Λ and over its covering $\widetilde{\Lambda}$ are triangulated in order to show several interesting identities in the respective stable module categories.

Introduction

Let k be a field and Q be a Dynkin quiver. So the underlying graph $|Q|$ of Q is a simply laced Dynkin diagram. We produce for the preprojective algebra Λ over k associated to Q a module I_Q by pushing a minimal injective cogenerator over the Auslander algebra Γ_Q of $\mathrm{k}Q$ to Λ-mod. It is easy to see that I_Q decomposes into $r = |\Pi|$ pairwise non-isomorphic direct summands. We show that I_Q is a rigid module module, i.e. $\mathrm{Ext}^1_\Lambda(\mathrm{I}_Q, \mathrm{I}_Q) = 0$. Moreover, the Gabriel quiver $\check{A}_Q$ of $\mathrm{End}_\Lambda(\mathrm{I}_Q)^{\mathrm{op}}$ is obtained from the Auslander-Reiten quiver A_Q of $\mathrm{k}Q$ by inserting an extra arrow $x \to \tau x$ for each non-projective vertex x.

In [19] we have shown that if $M = \oplus_{i=1}^m M_i$ for pairwise non-isomorphic indecomposable Λ-modules M_i, then $\mathrm{Ext}^1_\Lambda(M, M) = 0$ implies $m \leq r$. So our result shows that this maximum is assumed for each Dynkin quiver. By [16, Theorem 2.2] we conclude that I_Q is a maximal 1-orthogonal Λ-module and thus $\mathrm{End}_\Lambda(\mathrm{I}_Q)$ is a higher Auslander algebra in the sense of Iyama [21]. It follows that each rigid module M as above can be completed to a rigid module with r pairwise non-isomorphic indecomposable direct summands. Note, that for the proof of the main result of [16] it is essential that the quiver $\check{A}_Q$ has no loops.

Let now G be a complex simply connected simple Lie group of type $|Q|$ with $N \subset G$ a maximal unipotent subgroup. Choose $\mathbf{i} = (i_1, i_2, \dots, i_r)$ a reduced expression for the longest element w_0 of the Weyl group W of G. It follows from [4] that the coordinate ring $\mathbb{C}[N]$ is an (upper) cluster algebra. Associated to $\mathbf{i}$ one obtains an initial seed $(\Delta(j, \mathbf{i})'_{j=1,2,\dots,r}, \widetilde{B}(\mathbf{i})')$, where the $\Delta(j, \mathbf{i})'$ are certain generalized minors, and the exchange matrix $\widetilde{B}(\mathbf{i})'$ is obtained naturally from the quiver $\check{A}_Q$ described above.

Next, $\mathbf{i}$ provides us with a convenient labelling of the indecomposable direct summands of I_Q, that is, $\mathrm{I}_Q = \oplus_{j=1}^r \mathrm{I}(j, \mathbf{i})$. Clearly the $\mathrm{I}(j, \mathbf{i})$ are rigid. Thus, if we restrict to the

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special case $k = \mathbb{C}$ the Λ-modules $\mathrm{I}(j, \mathbf{i})$ serve as natural labels for elements $\rho_{\mathrm{I}(j,\mathbf{i})}$ of the dual of Lusztig's semicanonical basis. This is a natural basis for $\mathbb{C}[N]$ and we show that $\rho_{\mathrm{I}(j,\mathbf{i})} = \Delta(j, \mathbf{i})'$.

1. Main results

1.1. We say that a quiver Q is a *Dynkin quiver* if its underlying graph $|Q|$ is a Dynkin diagram of type $\mathsf{A}, \mathsf{D}, \mathsf{E}$. For a field k we consider the *path category* $k[Q]$ (or kQ for short). The category kQ-mod of finitely presented k-functors $M\colon kQ \to$ k-mod is equivalent to the category of finitely presented left modules over the corresponding path algebra which we denote by some abuse also by kQ.

For a quiver Q we consider the double $\overline{Q}$ which is obtained from Q by adding a new arrow $i \xleftarrow{a^*} j$ for each arrow $i \xrightarrow{a} j$ in Q. The preprojective algebra Λ is the quotient of the path algebra $k\overline{Q}$ by the ideal generated by the elements

$$\rho_q = \sum_{\substack{a \in Q_1 \\ t(a)=q}} a^* a - \sum_{\substack{a \in Q_1 \\ h(a)=q}} a a^* \qquad \text{for} \quad q \in Q_0,$$

see also 2.1.

In what follows, Q will always be a connected Dynkin quiver. This implies that Λ is a finite-dimensional selfinjective algebra, which depends only on $|Q|$. Like kQ, the algebra Λ can also be considered as a k-category.

We have the universal covering $F\colon \widetilde{\Lambda} \to \Lambda$ where $\widetilde{\Lambda}$ is the path category $k[\mathbb{Z}Q]$ modulo the usual mesh relations. The fundamental group $\mathbb{Z}$ of Λ acts on $\widetilde{\Lambda}$ via the translation τ. Associated to F we have the push-down functor $F_\lambda\colon \widetilde{\Lambda}$-mod $\to \Lambda$-mod, see 2.3.

In $\mathbb{Z}Q$ we find the Auslander-Reiten quiver A_Q of kQ as a full convex subquiver. The Auslander category Γ_Q is the full subcategory of $\widetilde{\Lambda}$ which has the vertices of A_Q as objects. Denote the inclusion of Γ_Q into $\widetilde{\Lambda}$ by J. There is a natural equivalence R_Q from Γ_Q to the category of indecomposable kQ-modules, kQ-ind. We say that an object $x \in \mathrm{Obj}(\Gamma_Q)$ is projective if $\tau x \notin \mathrm{Obj}(\Gamma_Q)$, dually x is injective if $\tau^{-1} x \notin \mathrm{Obj}(\Gamma_Q)$, see 2.4.

Associated to Γ_Q we consider the $\mathbb{N}_0$-graded category $\check{\Gamma}_Q$. It has the same objects as Γ_Q but the morphisms of degree i are given by $\check{\Gamma}_{Q,i}(x, y) = \Gamma_Q(\tau^i x, y)$ if $\tau^i x \in \mathrm{Obj}(\Gamma_Q)$. We equip $\check{\Gamma}_Q$ with the natural composition. The Gabriel quiver $\check{A}_Q$ of $\check{\Gamma}_Q$ is obtained from A_Q by inserting an additional (degree 1) arrow $x \to \tau x$ for each non-projective $x \in \mathrm{Obj}(\Gamma_Q)$, see 3.3.

1.2. **Start modules.** Let us write D for the usual duality $\mathrm{Hom}_k(-, k)$. We denote by $J^\cdot$ the functor which considers a Γ_Q-module (trivially) as a $\widetilde{\Lambda}$-module. Thus if apply the functor $F_\lambda J^\cdot$ to the injective Γ_Q-module $D\Gamma_Q(-, x)$ we obtain a Λ-module. We call

$$\mathrm{I}_Q := \bigoplus_{x \in \mathrm{Obj}(\Gamma_Q)} F_\lambda J^\cdot (D\Gamma_Q(-, x)).$$

the *start module* for Λ associated to Q. Note that $F_\lambda J^\cdot(D\Gamma_Q(-, x))$ is isomorphic to a submodule of $F_\lambda J^\cdot(D\Gamma_Q(-, \tau^{-1} x))$ if $x \in \mathrm{Obj}(\Gamma_Q)$ is not injective, and $F_\lambda J^\cdot(D\Gamma_Q(-, x))$ is an injective Λ-module if $x \in \mathrm{Obj}(\Gamma_Q)$ is injective, see 2.4 and 3.1.

Consider $\mathcal{E} = \oplus_{x,y \in \mathrm{Obj}(\Gamma_Q)} \check{\Gamma}_Q(x, y)$ as a (graded) associative k-algebra with multiplication induced from the composition of morphisms.

Theorem 1. *Let Λ be the preprojective algebra associated to a Dynkin quiver Q. Then $\mathcal{E}$ is isomorphic to* $\mathrm{End}_\Lambda(\mathrm{I}_Q)^{op}$. *In particular, the Gabriel quiver of* $\mathrm{End}_\Lambda(\mathrm{I}_Q)^{op}$ *is identified with $\check{A}_Q$ as described above in 1.1. Moreover* I_Q *is rigid in the sense that* $\mathrm{Ext}^1_\Lambda(\mathrm{I}_Q, \mathrm{I}_Q) = 0$.

The proof of this result is prepared in 3.4, 3.5, 3.6 and finished in 3.7.

1.3. **Reduced expressions.** Let $\pi\colon \operatorname{Obj}(\Gamma_Q) \to Q_0$ denote the map induced by the composition FJ. We call a total ordering $x(1) < x(2) < \ldots < x(r)$ of the objects of Γ_Q *adapted (to Q)* if $\Gamma_Q(x(i), x(j)) = 0$ for $i < j$. It is easy to find such orderings given that the quiver A_Q is directed.

We call a vertex $i \in Q_0$ a *source* in Q if no arrow ends at i. In this case we denote by $s_i(Q)$ the quiver which is obtained from Q by reversing each arrow starting in i.

Let $\mathbf{i} = (i_1, i_2, \ldots, i_r)$ be a reduced expression for the longest element $w_0 \in W$, that is $w_0 = s_{i_1} s_{i_2} \cdots s_{i_r}$ where $r = |\Pi|$. We say that $\mathbf{i}$ is *adapted to* Q if i_1 is a source in Q and i_{k+1} is a source in $s_{i_k} \cdots s_{i_1}(Q)$ if $1 \leq k < r$. This is dual to the original definition in [26].

In this situation $\mathbf{i} := (\pi(x(1)), \ldots, \pi(x(r)))$ is a reduced expression for the longest element w_0 of the Weyl group W associated to $|Q|$, see 1.4 below. In fact, the adapted orderings of $\operatorname{Obj}(\Gamma_Q)$ correspond in this way bijectively to the reduced expressions for w_0 which are adapted to Q, see [3, Theorem 2.5]. We set

$$\mathrm{I}(j, \mathbf{i}) := F_\lambda J^{\cdot} D\Gamma_Q(-, x(j)) \quad \text{for } 1 \leq j \leq r.$$

Thus, $\mathbf{i}$ provides us with a convenient way of labelling the direct summands of I_Q.

1.4. Let now $\mathfrak{g}$ be a complex simple Lie algebra of type $|Q|$ with the usual Serre generators e_i, h_i, f_i for $i \in Q_0$. Thus the h_i form a basis of the abelian subalgebra $\mathfrak{h}$ and the e_i resp. f_i generate maximal nilpotent subalgebras $\mathfrak{n}$ resp. $\mathfrak{n}_-$. The simple roots α_i form a basis of the dual space $\mathfrak{h}^*$ such that $\alpha_i(h_j) = a_{i,j}$ where $(a_{i,j})_{i,j \in Q_0}$ is the Cartan matrix of $|Q|$. The fundamental weights $(\varpi_i)_{i \in Q_0}$ are the basis of $\mathfrak{h}^*$ dual to the basis $(h_i)_{i \in Q_0}$ of $\mathfrak{h}$.

The Weyl group W is the subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ which is generated by the reflections s_i for $i \in Q_0$ such that

$$s_i(\alpha) = \alpha - \alpha(h_i) \cdot \alpha_i \text{ for } \alpha \in \mathfrak{h}^*.$$

This is a finite reflection group.

Let now G be a complex simply connected simple algebraic group with $\mathrm{Lie}(G) = \mathfrak{g}$. It has maximal unipotent subgroups N resp. N_- with $\mathrm{Lie}(N) = \mathfrak{n}$ resp. $\mathrm{Lie}(N_-) = \mathfrak{n}_-$, and the maximal torus H has $\mathrm{Lie}(H) = \mathfrak{h}$. Moreover, we have standard embeddings $\varphi_i\colon \mathrm{SL}_2 \to G$ such that

$$\exp(tf_i) = \varphi_i\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right) \text{ and } \exp(te_i) = \varphi_i\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \text{ for } i \in Q_0.$$

Moreover set

$$\eta_i(t) := \varphi_i\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \in H \text{ for } i \in Q_0 \text{ and } t \in \mathbb{C}^*.$$

Next, recall that $N_G(H)/H$ is canonically isomorphic to the Weyl group W defined above. In fact, it is possible to choose representatives $\bar{w} \in N_G(H)$ for the elements $w \in W$ such that

$$\bar{s_i} = \exp(f_i) \exp(-e_i) \exp(f_i),$$
$$\overline{uv} = \bar{u}\bar{w} \text{ if } l(uv) = l(u) + l(v).$$

We identify the weight lattice $P = \oplus_{i \in Q_0} \mathbb{Z}\varpi_i$ with the group of multiplicative characters of H in such a way that $\eta_i(t)^{\varpi_j} = t^{\delta_{i,j}}$ for $i, j \in Q_0$. If we write $?^\lambda$ for the character of H corresponding to the weight λ it follows that

$$h^{w(\lambda)} = (\bar{w}^{-1} h \bar{w})^\lambda \text{ for } h \in H, \lambda \in P, w \in W.$$

1.5. **Cluster algebras.** The coordinate ring of the affine base space $\mathbb{C}[N_-\backslash G]$ consists of the functions $f \in \mathbb{C}[G]$ which are invariant under N_-, i.e. $f(g) = f(ng)$ for all $g \in G$ and $n \in N_-$. Now $\mathbb{C}[N_-\backslash G]$ is naturally a G-module via $gf(x) = f(xg)$ for $g, x \in G$. It is well-known that each irreducible highest weight G-module $L(\lambda)$ can be realized as a direct summand of $\mathbb{C}[N_-\backslash G]$ by taking

$$L(\lambda) = \{f \in \mathbb{C}[N_-\backslash G] \mid f(hg) = h^\lambda f(g) \text{ for } h \in H, g \in G\}.$$

For each $L(\lambda)$ we choose a highest weight vector u_λ which we normalize by the condition $u_\lambda(1_G) = 1$. Following [5] we define for each fundamental weight ϖ_i generalized minors

$$\Delta_{\varpi_i, w(\varpi_i)} := \bar{w} \cdot u_{\varpi_i} \in L(\varpi_i)$$

for any $w \in W$. In [4] it is shown in particular that the coordinate ring of the double Bruhat cell $G^{e,w_0} = B \cap (B_-\bar{w}_0 B_-)$ has the structure of an (upper) cluster algebra. Here, B and B_- are opposite Borel subgroups of G with $B \supset N$ and $B_- \supset N_-$.

For a reduced expression $\mathbf{i} = (i_1, i_2, \dots, i_r)$ for w_0 which is adapted to Q and $k \in [-n, -1] \cup [1, r]$ we set $v_{>k} := s_{i_r} s_{i_{r-1}} \cdots s_{i_{k+1}}$ if $k \geq 1$ and $v_{>k} := w_0$ if $k \leq -1$. Then, following [4] set

$$\Delta(k, \mathbf{i}) := \Delta_{\varpi_{i_k}, v_{>k}(\varpi_{i_k})}$$

where we take $i_k = -k$ for $k \in [-n, -1]$. The $\Delta(k, \mathbf{i})$ for $k \in [-n, -1] \cup [1, r]$ form an initial cluster for $\mathbb{C}[G^{e,w_0}]$. There is also an easy algorithm to calculate from $\mathbf{i}$ the corresponding exchange matrix. Now set

$$e(\mathbf{i}) := \{i \in [1, r] \mid x(i) \in \mathrm{Obj}(\Gamma_Q) \text{ non-projective}\}.$$

There is a closely related upper cluster algebra structure on the coordinate ring $\mathbb{C}[N]$, whose initial cluster is given by the restrictions to N of the functions $\Delta(k, \mathbf{i})$ with $k \in [-n, -1] \cup e(\mathbf{i})$ (see 4.4). The quiver associated to the corresponding exchange matrix is obtained from $\tilde{A}_Q$ by removing the arrows between injective vertices. See Section 4 for a more detailed discussion.

1.6. **Semicanonical basis.** In [27] Lusztig introduces the semicanonical basis of the enveloping algebra $U(\mathfrak{n})$. Its elements are labelled naturally by the irreducible components of the preprojective varieties $\Lambda_{\mathbf{v}}$ for $\mathbf{v} \in \mathbb{N}_0^{Q_0}$. In [18] we started the study of the dual semicanonical basis which can be regarded as a basis of $\mathbb{C}[N]$. In particular, we found that to (the isoclass of) a rigid Λ-module M there corresponds naturally an element ρ_M of this basis. If we set

$$\theta\colon [1, r] \to [-n, -1] \cup e(\mathbf{i}), j \mapsto \begin{cases} -i & \text{if } \mathrm{R}_Q(x(j)) \cong D\mathrm{k}Q(-, i), \\ k & \text{if } \tau^{-1}x(j) = x(k), \end{cases}$$

we can state our second main result precisely:

Theorem 2. *For $j \in [1, r]$ we have $\Delta(j, \mathbf{i})' := \Delta(\theta(j), \mathbf{i}) = \rho_{\mathrm{I}(j,\mathbf{i})}$.*

The proof of this result is done after some preparation in Section 5.

1.7. **Dualities.** Denote by $?^*$ the involution on $\overline{Q}$ with $a \mapsto a^*$ and $a^* \mapsto a$ for all $a \in Q_1$. This induces an anti-automorphism of Λ which we also denote by $?^*$. Thus we have a self-duality

$$S\colon \Lambda\text{-mod} \longleftrightarrow \Lambda\text{-mod} \text{ with } SM(?) = DM(?^*).$$

Let us define for a reduced expression $\mathbf{i} = (i_1, \dots, i_r)$ for w_0 which is adapted to Q

$$\mathrm{P}(j, \mathbf{i}) := F_\lambda J^{\cdot}\Gamma_Q(x(j), -) \quad \text{and } \mathrm{P}_Q := \bigoplus_{j=1}^{r} \mathrm{P}(j, \mathbf{i}).$$

Then it is easy to see that $\mathrm{P}_Q \cong S\,\mathrm{I}_{Q^{\mathrm{op}}}$. Thus, $\mathrm{Ext}^1_\Lambda(\mathrm{P}_Q, \mathrm{P}_Q) = 0$ and $\mathrm{End}_\Lambda(\mathrm{P}_Q) \cong \mathrm{End}_\Lambda(\mathrm{I}_{Q^{\mathrm{op}}})^{\mathrm{op}}$. Now, $\check{A}_{Q^{\mathrm{op}}}$ is the Gabriel quiver of $\mathrm{End}_\Lambda(\mathrm{I}_{Q^{\mathrm{op}}})^{\mathrm{op}}$, see Theorem 1, and it is not hard to see that $\check{A}_{Q^{\mathrm{op}}}$ may be identified with $\check{A}_Q^{\mathrm{op}}$ (recall that the same happens for Auslander-Reiten quivers: $A_{Q^{\mathrm{op}}} \cong A_Q^{\mathrm{op}}$). So $\mathrm{End}_\Lambda(\mathrm{I}_{Q^{\mathrm{op}}}) \cong \mathrm{End}_\Lambda(\mathrm{P}_Q)^{\mathrm{op}}$ and $\mathrm{End}_\Lambda(\mathrm{I}_Q)^{\mathrm{op}}$ have the same Gabriel quiver $\check{A}_Q$, but they are usually not isomorphic.

On the other hand, S induces the canonical anti-automorphism $?^*$ on Lusztig's algebra of constructible functions $\mathcal{M} \cong U(\mathfrak{n})$ which commutes with the comultiplication, see [27, Section 3.4]. This yields by duality an automorphism ω of the coordinate ring $\mathbb{C}[N]$ which anti-commutes with the comultiplication. The corresponding anti-automorphism of N leaves the one-parameter subgroups $\mathbb{C} \to N, t \mapsto \exp(te_i)$ invariant.

Note that $\mathbf{i}^* := (\mu(i_r), \mu(i_{r-1}), \dots, \mu(i_1))$ is a reduced expression for w_0 which is adapted to Q^{op}, see 2.3 for the definition of μ. We leave it to the reader to verify the following:

$$\rho_{\mathrm{P}(j,\mathbf{i})} = \omega(\rho_{\mathrm{I}(r+1-j,\mathbf{i}^*)}) \quad \text{for } 1 \le j \le r,$$

see also 1.6.

1.8. **Triangulated structures.** In the appendix (Section 7) we point out that the category of injective $\widetilde{\Lambda}$-modules is triangulated. This implies that the category of injective Λ-modules is also triangulated. This fact helps us to explain the unusual symmetries in the stable module categories over both categories by an old result of Freyd. Among other useful formulas we can recover the famous "6-periodicity" for modules over the preprojective algebra.

2. The universal cover of a preprojective algebra

2.1. **Quiver categories.** Let $Q = (Q_0, Q_1, t, h)$ be a quiver with vertices Q_0, arrows Q_1 and $t, h \colon Q_1 \to Q_0$ such that we have $a \colon t(a) \to h(a)$ for each arrow $a \in Q_1$. A *path* in Q is a sequence of arrows $a_n a_{n-1} \cdots a_1$ such that $t(a_{i+1}) = h(a_i)$ for $i = 1, 2, \dots, n-1$.

On $\mathbb{Z}^{Q_0}$ we have the Ringel bilinear form

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i \in Q_0} \mathbf{v}(i)\mathbf{w}(i) - \sum_{a \in Q_1} \mathbf{v}(t(a))\mathbf{w}(h(a)).$$

Let k be a field. Since we need to consider infinite coverings of preprojective algebras we have to consider k-categories rather than k-algebras.

If Q is a quiver we denote by $\mathrm{k}[Q] = \mathrm{k}Q$ the k-category which has Q_0 as objects and with the morphism space $\mathrm{k}Q(p,q)$ having the paths from p to q as a basis. The composition is naturally induced from the concatenation of paths.

2.2. **Conventions.** If $\mathcal{D}$ is a k-category we denote by $\mathcal{D}$-mod the category of finitely presented (covariant) k-functors $\mathcal{D} \to \mathrm{k}$-mod. These functors are also called left modules, see for example [15, Section 2.2] for more details.

Let G be a group of k-automorphisms of $\mathcal{D}$ which acts from the left on $\mathcal{D}$. Then G acts naturally from the right on $\mathcal{D}$-mod. If $g \in G$ and M is a left $\mathcal{D}$-module, then we write $M^g(-) := M(g^{-1}-)$ for the twisted module. For example, if $x \in \mathcal{D}$ we get for the projective module $\mathcal{D}(x,-)$ an isomorphism $\mathcal{D}^g(x,-) \cong \mathcal{D}(gx,-)$.

2.3. **Construction of $\widetilde{\Lambda}$.** Let $Q = (Q_0, Q_1, t, h)$ be a (connected) Dynkin quiver, so the underlying graph $|Q|$ is one of the following:

A_n: $1 — 2 — 3 — \cdots — (n{-}2) — (n{-}1) — n$

D_n: $1 — 2 — 3 — \cdots — (n{-}2)$, with $(n{-}2)$ joined to both $(n{-}1)$ and n; $n \geq 4$

E_n: $1 — 2 — 3 — \cdots — (n{-}2) — (n{-}1)$, with 3 joined to n; $n = 6,7,8$

Define a (possibly trivial) involution μ on the vertices of Q by

$$\mu(q) = \begin{cases} n+1-q & \text{in case } \mathsf{A}_n, \\ 2n-1-q & \text{in case } \mathsf{D}_n \text{ and } (n \text{ odd and } q \geq n-1), \\ 6-q & \text{in case } \mathsf{E}_6 \text{ and } q \leq 5, \\ q & \text{otherwise.} \end{cases}$$

Following [13] let $\mathbb{Z}Q$ be the quiver with vertices $\mathbb{Z}Q_0 = \mathbb{Z} \times Q_0$ and arrows $\mathbb{Z}Q_1 = \mathbb{Z} \times \{Q_1 \cup Q_1^*\}$ where

$$\tilde{t}(i,a) = (i+1, t(a)), \qquad \tilde{t}(i,a^*) = (i, h(a)),$$
$$\tilde{h}(i,\alpha) = (i, h(a)), \qquad \tilde{h}(i,a^*) = (i, t(a)).$$

We think of Q^{op} as a subquiver of $\mathbb{Z}Q$ via the embedding $q \mapsto (0,q)$. The quiver $\mathbb{Z}Q$ admits a translation automorphism τ induced by $\tau(i,q) = (i+1,q)$. Moreover we have a "Nakayama permutation" $\hat{\nu}$ of $\mathbb{Z}Q$. In order to define it we need the following auxiliary function: For any vertex q denote by $l(q)$ the number of arrows pointing towards 1 on the unique walk from 1 to q. Now, $\hat{\nu}$ is defined on the vertices by

$$\hat{\nu}(p,q) = \begin{cases} (p+q-1+l(\mu(q))-l(q), \mu(q)) & \text{in case } \mathsf{A}_n, \\ (p+n-2+l(\mu(q))-l(q), \mu(q)) & \text{in case } \mathsf{D}_n, \\ (p+q+2+l(\mu(q))-l(q), \mu(q)) & \text{in case } \mathsf{E}_6 \text{ and } q \leq 5, \\ (p+5, \mu(q)) & \text{in case } \mathsf{E}_6 \text{ and } q = 6, \\ (p+8, \mu(q)) & \text{in case } \mathsf{E}_7, \\ (p+14, \mu(q)) & \text{in case } \mathsf{E}_8. \end{cases}$$

So, $\hat{\nu}$ is a "translation reflection" stabilizing the "middle line" in case A_n and a translation in the cases D_{2k}, E_7 and E_8. At the beginning of 6.1 we show as an example part of $\mathbb{Z}Q$ for a quiver of type D_5, together with the action of $\hat{\nu}$.

We consider the *mesh ideal* I in the category $\mathrm{k}[\mathbb{Z}Q]$ which is generated by the elements

$$\sum_{\substack{a \in Q_1 \\ t(a)=q}} (i,a^*) \cdot (i,a) - \sum_{\substack{a \in Q_1 \\ h(a)=q}} (i,a) \cdot (i+1,a^*) \quad \text{for all } (i,q) \in \mathbb{Z}Q_0. \tag{2.1}$$

Note that $\widetilde{\Lambda} := \mathrm{k}[\mathbb{Z}Q]/I$ is independent of the orientation of Q (up to relabelling of the vertices), and that I is invariant under the quiver automorphisms τ and $\hat{\nu}$. Thus we will use the same names for the induced automorphisms of $\widetilde{\Lambda}$. Note further that the commuting automorphisms τ and $\hat{\nu}$ of $\widetilde{\Lambda}$ are determined up to isomorphism by their effect on objects.

The action of the group $\mathbb{Z}$ via τ on $\widetilde{\Lambda}$ induces self-equivalences $?^{(i)}$ of $\widetilde{\Lambda}$-mod with $M^{(i)}(-) := M(\tau^{-i}-)$ for $i \in \mathbb{Z}$. Moreover we obtain the covering functor $F\colon \widetilde{\Lambda} \to \Lambda$ which sends (i,q) to q. Associated to F we have the push-down $F_\lambda\colon \widetilde{\Lambda}\text{-mod} \to \Lambda\text{-mod}$ with

$$(F_\lambda M)(q) = \oplus_{i\in\mathbb{Z}} M(i,q)$$

and the obvious effect on morphisms.

2.4. **Auslander category.** We define a function $N\colon Q_0 \to \mathbb{N}_0$ by the property

$$\tau^{N(q)}(0,q) = \hat{\nu}(0,\mu(q)).$$

This is well-defined by the construction of $\hat{\nu}$ since μ is an involution on Q_0, see 2.3. The function N depends on the orientation of Q in case A_n, D_{2k+1} and E_6. In any case we have $N(q) + N(\mu(q)) = h(Q) - 2$, where $h(Q)$ denotes the Coxeter number of $|Q|$.

Define now Γ_Q as the full subcategory of $\widetilde{\Lambda}$ which has the objects of the form $(i,q) = \tau^{i-N(q)}\hat{\nu}(0,\mu(q))$ with $q \in Q_0$ and $0 \le i \le N(q)$. In other words, we take the objects which lie between the two copies of Q^{op} in $\mathbb{Z}Q$ which are obtained via $q \mapsto (0,q)$ resp. $q \mapsto \hat{\nu}(0,q)$. This is the *Auslander category* of $\mathrm{k}Q$. Note that Γ_Q depends on the orientation of Q.

By construction we have a full embedding $\iota = \iota_Q\colon \mathrm{k}Q^{\mathrm{op}} \to \Gamma_Q$ induced by $q \mapsto (0,q)$. Moreover, Γ_Q is canonically equivalent to the category of indecomposable $\mathrm{k}Q$-modules via the functor

$$\mathrm{R}_Q\colon \Gamma_Q \to \mathrm{k}Q\text{-ind},\ x \mapsto \Gamma_Q(\hat{\nu}\iota, x), \tag{2.2}$$

where $\Gamma_Q(\hat{\nu}\iota, x) = \Gamma_Q(-,x)\circ\hat{\nu}\iota\colon \mathrm{k}Q^{\mathrm{op}} \to \mathrm{k}\text{-mod}$ is a contravariant functor which we have to interpret as a left $\mathrm{k}Q$-module. For example, $\mathrm{R}_Q(0,q) \cong D\mathrm{k}Q(-,q)$ and $\mathrm{R}_Q(\hat{\nu}(0,q)) \cong \mathrm{k}Q(q,-)$.

Thus the Gabriel quiver of Γ_Q (which is the full subquiver of $\mathbb{Z}Q$ with the vertices from Γ_Q) is the Auslander-Reiten quiver A_Q of $\mathrm{k}Q$.

Similarly, since we consider left modules, one obtains an equivalence

$$\Gamma_Q\text{-inj} \to \mathrm{k}Q^{\mathrm{op}}\text{-mod} \cong (\mathrm{k}Q\text{-mod})^{\mathrm{op}}, I \mapsto I \circ \hat{\nu}\iota.$$

Here, Γ_Q-inj denotes the category of injective left Γ_Q-modules.

Note that the $q \in Q_0$ parametrize the indecomposable projective-injective Γ_Q-modules, namely we have

$$D\Gamma_Q(-,(0,q)) \cong \Gamma_Q(\hat{\nu}(0,q),-).$$

Recall, that as an Auslander category Γ_Q has dominant dimension at least 2 [2, VI.5]. This means by duality that each (indecomposable) injective Γ_Q-module $D\Gamma_Q(-,x)$ has a projective presentation

$$P_{1,x} \to P_{0,x} \to D\Gamma_Q(-,x) \to 0$$

with $P_{0,x}$ and $P_{1,x}$ also injective.

3. Start modules

3.1. **Adjoint functors.** Let $J\colon \Gamma_Q \to \widetilde{\Lambda}$ be the full embedding of locally bounded categories. Since Γ_Q is convex in $\widetilde{\Lambda}$ we get an exact functor "extension by 0"

$$J^{\cdot}\colon \Gamma_Q\text{-mod} \to \widetilde{\Lambda}\text{-mod with } (J^{\cdot}M)(x) = \begin{cases} M(x) & \text{if } x \in \Gamma_Q, \\ 0 & \text{else.} \end{cases}$$

For a Γ_Q-module N and $x \in \Gamma_Q$ we have natural isomorphisms

$$\mathrm{Hom}_{\Gamma_Q}(N, D\Gamma_Q(-,x)) \cong DN(x) \cong \mathrm{Hom}_{\widetilde{\Lambda}}(J^{\cdot}N, D\widetilde{\Lambda}(-,x)).$$

We conclude that $J^{\cdot}$ has a right adjoint J^{ρ} which is defined by being left exact and

$$J^{\rho}D\widetilde{\Lambda}(-,x) = \begin{cases} D\Gamma_Q(-,x) & \text{if } x \in \mathrm{Obj}(\Gamma_Q), \\ 0 & \text{else.} \end{cases}$$

The adjunction morphism $\iota_M \colon J^{\cdot}J^{\rho}M \to M$ is injective, its co-kernel is co-generated by $\bigoplus_{y\in\mathrm{Obj}(\widetilde{\Lambda})\setminus\mathrm{Obj}(\Gamma_Q)} M(y)$.

3.2. **Lemma.** *Let $x, y \in \mathrm{Obj}(\Gamma_Q)$ and $i \in \mathbb{Z}$. Then*

$$\mathrm{Hom}_{\widetilde{\Lambda}}(J^{\cdot}D\Gamma_Q(-,x), J^{\cdot}D\Gamma_Q^{(i)}(-,y)) \cong \begin{cases} \Gamma_Q(\tau^i y, x) & \text{if } i \geq 0 \text{ and } \tau^i y \in \mathrm{Obj}(\Gamma_Q), \\ 0 & \text{else.} \end{cases}$$

In the first case we have more generally $J^{\rho}(J^{\cdot}D\Gamma_Q^{(i)}(-,y)) \cong D\Gamma_Q(-,\tau^i y)$.

Note, that $J^{\cdot}D\Gamma(-,y)$ is the injective Γ_Q-module whith socle concentrated in y, but seen as $\widetilde{\Lambda}$-module, thus we may apply to it the translation functor $?^{(i)}$, see 2.3.

Proof: The $\widetilde{\Lambda}$-module $M = J^{\cdot}D\Gamma_Q^{(i)}(-,y)$ has simple socle concentrated in the one-dimensional space $M(\tau^i y)$. If $\tau^i y \notin \mathrm{Obj}(\Gamma_Q)$ then this does not belong to the support of $J^{\cdot}D\Gamma_Q(-,x)$. On the other hand, if $i < 0$, it is sufficient to show that there are no maps from an induced projective-injective module to M since Γ_Q has dominant dimension ≥ 2, see 2.4. Now,

$$0 = M(\hat{\nu}(0,q)) = \mathrm{Hom}_{\widetilde{\Lambda}}(\widetilde{\Lambda}(\hat{\nu}(0,q),-), M) = \mathrm{Hom}_{\widetilde{\Lambda}}(J^{\cdot}\Gamma_Q(\hat{\nu}(0,q),-), M)$$

with the first identity holding for $i < 0$.

Thus, let $i \geq 0$ and $\tau^i y \in \Gamma_Q$. In this case we have in $\widetilde{\Lambda}$-mod an injective presentation

$$0 \to M \to D\widetilde{\Lambda}(-,\tau^i x) \to \oplus_j D\widetilde{\Lambda}(-,y_j)^{m(j)}$$

for certain $y_j \in \mathrm{Obj}(\widetilde{\Lambda}) \setminus \mathrm{Obj}(\Gamma_Q)$. Our claim follows now from the construction of J^{ρ}. □

3.3. **A graded category.** We construct the $\mathbb{N}_0$-graded category $\check{\Gamma}_Q$. It has the same objects as Γ_Q, but the homogenous components are $\check{\Gamma}_{Q,i}(x,y) := \Gamma_Q(\tau^i x, y)$ if $\tau^i x \in \mathrm{Obj}(\Gamma_Q)$ and $\check{\Gamma}_{Q,i}(x,-) = 0$ otherwise. The natural composition is given by

$$\check{\Gamma}_{Q,j}(y,z) \otimes \check{\Gamma}_{Q,i}(x,y) \to \check{\Gamma}_{Q,i+j}(x,z), \quad (\psi \otimes \phi) \mapsto \begin{cases} \psi \circ (\tau^j \phi) & \text{if } \tau^{i+j}x \in \mathrm{Obj}(\Gamma_Q), \\ 0, & \text{else.} \end{cases}$$

By construction, each morphism in $\check{\Gamma}_{Q,\geq 1}(x,-)$ factors through $\mathbb{1}_{\tau x} \in \check{\Gamma}_{Q,1}(x,\tau x)$ if $\tau x \in \mathrm{Obj}(\Gamma_Q)$, otherwise $\check{\Gamma}_{Q,\geq 1}(x,-) = 0$. Moreover we have

$$\mathbb{1}_{\tau^n x} \circ \cdots \circ \mathbb{1}_{\tau^2 x} \circ \mathbb{1}_{\tau x} = \mathbb{1}_{\tau^n x} \in \check{\Gamma}_{Q,n}(x,\tau^n x) \text{ if } \tau^n x \in \mathrm{Obj}(\Gamma_Q),$$

where we consider on the left hand side $\mathbb{1}_{\tau^i x} \in \check{\Gamma}_{Q,1}(\tau^{i-1}x,\tau^i x)$. Now, $\check{\Gamma}_Q$ can be described easily by a graded quiver $\check{A}_Q$. It has the same vertices and degree 0 arrows as the Auslander-Reiten quiver A_Q of $\mathrm{k}Q$ (i.e. the quiver of Γ_Q) moreover there is a degree 1 arrow $t_x \colon x \to \tau x$ for each $x \in \mathrm{Obj}(\Gamma_Q)$ with $\tau x \in \mathrm{Obj}(\Gamma_Q)$. The degree 0 relations are the mesh relations for Γ_Q. Moreover, each degree 0 arrow $a \colon x \to y$ with y not projective gives rise to a degree 1 relation

$$t_y a - (\tau a) t_x.$$

This has to be interpreted as a zero-relation if x is projective. A nice way to remember these relations is the following: For each arrow between to vertices which are not both injective there is a (generic) homogeneous length 2 relation in the opposite direction, see also 6.2.

3.4. **Dynkin quivers.** Let us collect some basic facts about the representation theory of a Dynkin quiver Q. Define $\mathbf{i}_q := \underline{\dim}\, D\mathrm{k}Q(-,q)$ and $\mathbf{p}_q := \underline{\dim}\, \mathrm{k}Q(q,-)$ for $q \in Q_0$, the dimension vectors of the indecomposable injective and projective kQ-modules, respectively. We have then for $0 \le i \le N(q)$

$$\Phi^i \mathbf{i}_q = \underline{\dim}(\tau_Q^i D\mathrm{k}Q(-,q)) = \underline{\dim}(\tau_Q^{i-N(q)} \mathrm{k}Q(q,-)) = \Phi^{i-N(q)} \mathbf{p}_{\mu(q)}, \tag{3.1}$$

where Φ denotes the Coxeter transformation and τ_Q the Auslander-Reiten translate in kQ-mod. Next, if $\langle -,- \rangle$ denotes the Ringel bilinear form of kQ we have

- $\langle \underline{\dim}\, M, \underline{\dim}\, N \rangle = \dim \mathrm{Hom}_Q(M,N) - \dim \mathrm{Ext}^1_Q(M,N)$,
- $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \Phi\mathbf{v}, \Phi\mathbf{w} \rangle = -\langle \mathbf{w}, \Phi\mathbf{v} \rangle$,
- $\langle \mathbf{v}, \mathbf{i}_q \rangle = \mathbf{v}(q)$ thus $\langle \mathbf{v}, \sum_{q \in Q_0} \mathbf{i}_q \rangle = |\mathbf{v}|$, where $|\mathbf{v}| := \sum_{q\in Q_0} \mathbf{v}(q)$,

see for example [28, 2.4].

3.4.1. **Lemma.** *Let $\tau^i(0,p) = (i,p)$ and $\tau^{-j}\hat{\nu}(0,q)$ belong to* $\mathrm{Obj}(\Gamma_Q)$*, then*

$$\dim \Gamma_Q(\tau^{-j}\hat{\nu}(0,q), \tau^i(0,p)) = \begin{cases} (\Phi^{i+j}\mathbf{i}_p)(q) & \text{if } i+j \le N(p), \\ (\Phi^{-i-j}\mathbf{p}_q)(p) & \text{if } i+j \le N(\mu(q)), \\ 0 & \text{if } i+j > \min\{N(p), N(\mu(q))\}. \end{cases}$$

Note that the three cases are not exclusive, however they cover obviously all possibilities for $(i,j) \in [0,N(p)] \times [0, N(\mu(q))]$.

Proof: In the first case we have

$$\dim \Gamma_Q(\tau^{-j}\hat{\nu}(0,q), \tau^i(0,p)) = \dim \Gamma_Q(\hat{\nu}(0,q), \tau^{i+j}(0,q)). \tag{3.2}$$

On the other hand, the equivalence R_Q from Γ_Q to the category of indecomposable kQ-modules commutes with translations

$$\mathrm{R}_Q(\tau^i(0,q)) \cong \tau_Q^i D\mathrm{k}Q(-,q) \quad \text{for } 0 \le i \le N(q).$$

Thus (3.2) is equal to

$$\dim \mathrm{Hom}_Q(\mathrm{k}Q(q,-), \tau_Q^{i+j} D\mathrm{k}Q(-,p)) = \dim \mathrm{Hom}_Q(\tau_Q^{-j}\mathrm{k}Q(q,-), \tau_Q^i D\mathrm{k}Q(-,p)).$$

Our claim follows now from (3.1) since for a finite-dimensional kQ-module M we have that

$$\mathrm{Hom}_Q(\mathrm{k}Q(q,-), M) \cong M(q) \cong D\,\mathrm{Hom}_Q(M, D\mathrm{k}Q(-,q)).$$

The second case is treated similarly. Finally we have

$$\Gamma_Q(\tau^{-j}\hat{\nu}(0,q), \tau^i(0,p)) \cong \widetilde{\Lambda}(\tau^{-j}\hat{\nu}(0,q), \tau^i(0,p)) \cong \widetilde{\Lambda}(\hat{\nu}(0,q), \tau^{i+j-N(p)}\hat{\nu}(0,\mu(p))).$$

The last term vanishes obviously for $i+j > N(p)$. A similar argument shows that $\Gamma_Q(\tau^{-j}\hat{\nu}(0,q), \tau^i(0,p)) = 0$ for $i+j > N(\mu(q))$. □

3.4.2. **Lemma.** *If $N(p) - N(q) > j \ge 0$ holds for some $p,q \in Q_0$, then $\langle \Phi^j \mathbf{i}_p, \mathbf{i}_q \rangle = 0$.*

Proof: Since $D\mathrm{k}Q(-,q)$ is injective we have

$$\langle \Phi^j \mathbf{i}_p, \mathbf{i}_q \rangle = \dim \mathrm{Hom}_Q(\tau_Q^j D\mathrm{k}Q(-,p), D\mathrm{k}Q(-,q)) = \dim \widetilde{\Lambda}(\tau^j(0,p), (0,q)).$$

On the other hand for $N(p) - j > N(q)$ there is no path from $\tau^j(0,p) = (N(p)-j, \mu(p))$ to $(0,q)$ in $\mathbb{Z}Q$. □

3.5. **Proposition.** *With the notation of 1.2 and 3.4 we have*

$$(\underline{\dim}\, \mathrm{I}_Q)(\mu(p)) = \sum_{q\in Q_0} \sum_{i=0}^{N(q)} (i+1)(\Phi^i \mathbf{i}_q)(p) \tag{3.3}$$

for $p \in Q_0$*, and*

$$\dim \mathcal{E} = \sum_{q\in Q_0} \sum_{i=0}^{N(q)} \left(\binom{N(q)+2}{2} - \binom{i+1}{2} \right) |\Phi^i \mathbf{i}_q|. \tag{3.4}$$

Proof: For (3.3) we observe first that

$$\underline{\dim}\, F_\lambda J^{\cdot} D\Gamma_Q(-,(j,q))(\mu(p)) = \sum_{i=j}^{N(q)} (\Phi^i \mathbf{i}_q)(p)$$

for $(j,q) \in \mathrm{Obj}(\Gamma_Q)$ by 3.4.1 and the definition of the push-down F_λ. Now (3.3) follows from the definition of I_Q.

For (3.4) we observe first that

$$|\underline{\dim}\, F_\lambda J^{\cdot} \Gamma_Q(\tau^{-j}\hat{\nu}(0,q),-)| = \sum_{k=j}^{N(\mu(q))} |\Phi^{-k} \mathbf{p}_q|$$

for $\tau^{-j}\hat{\nu}(0,q) \in \mathrm{Obj}(\Gamma_Q)$ again by 3.4.1 and the definition of the push-down. Now, by construction of $\mathcal{E}$ we have

$$\begin{aligned} \dim \mathcal{E} &= \sum_{q\in Q_0} \sum_{i=0}^{N(\mu(q))} \sum_{j=0}^{i} |\underline{\dim}\, F_\lambda J^{\cdot} \Gamma_Q(\tau^{-j}\hat{\nu}(0,q),-)| \\ &= \sum_{q\in Q_0} \sum_{i=0}^{N(\mu(q))} \sum_{j=0}^{i} \sum_{k=i}^{N(\mu(q))} |\Phi^{-k} \mathbf{p}_q| \\ &= \sum_{q\in Q_0} \sum_{i=0}^{N(q)} \sum_{j=0}^{i} \sum_{k=i}^{N(q)-j} |\Phi^k \mathbf{i}_q| = \sum_{q\in Q_0} \sum_{i=0}^{N(q)} \sum_{j=i}^{N(q)+1} j |\Phi^i \mathbf{i}_q|. \end{aligned}$$

□

3.6. **Proposition.** *Let* Q *be a Dynkin quiver. Then for* $\mathbf{v} = \sum_{q\in Q_0} \sum_{d=0}^{N(q)} (i+1)\Phi^i \mathbf{i}_q$ *holds*

$$\langle \mathbf{v}, \mathbf{v} \rangle = \sum_{p\in Q_0} \sum_{j=0}^{N(p)} \left(\binom{N(p)+2}{2} - \binom{d+1}{2} \right) |\Phi^d \mathbf{i}_p|.$$

Proof: For convenience let us write $N(p,q) := \{0,1,\dots,N(p)\} \times \{0,1,\dots,N(q)\}$ and $N(p,q,\geq) := \{(i,j) \in N(p,q) \mid i \geq j\}$, similarly $N(p,q,<) := N(p,q) \setminus N(p,q,\geq)$. Now we have

$$\begin{aligned} \langle \mathbf{v}, \mathbf{v} \rangle = & \sum_{p,q\in Q_0} \sum_{(i,j)\in N(p,q)} (i+1)(j+1)\langle \Phi^i \mathbf{i}_p, \Phi^j \mathbf{i}_q \rangle \\ = & \sum_{p,q\in Q_0} \sum_{(i,j)\in N(p,q,\geq)} (i+1)(j+1)\langle \Phi^i \mathbf{i}_p, \Phi^j \mathbf{i}_q \rangle \\ & - \sum_{p,q\in Q_0} \sum_{(i,j)\in N(p,q,<)} (i+1)(j+1)\langle \Phi^j \mathbf{i}_q, \Phi^{i+1} \mathbf{i}_p \rangle \end{aligned}$$

$$= \sum_{p,q\in Q_0} \sum_{(i,j)\in N(p,q,\geq)} (i+1)\langle \Phi^i \mathbf{i}_p, \Phi^j \mathbf{i}_q\rangle$$
$$- \sum_{p,q\in Q_0} \sum_{i=N(q)+1}^{N(p)} (i+1)(N(q)+1)\langle \Phi^i \mathbf{i}_p, \Phi^{N(q)+1}\mathbf{i}_q\rangle$$

here, the second sum vanishes by Lemma 3.4.2, thus from 3.4

$$= \sum_{p,q\in Q_0} \sum_{(i,j)\in N(p,q,\geq)} (i+1)\langle \Phi^{i-j} \mathbf{i}_p, \mathbf{i}_q\rangle$$
$$= \sum_{p,q\in Q_0} \sum_{d=0}^{N(p)} \left(\sum_{k=d}^{\min\{N(p),N(q)+d\}} (k+1)\right) \langle \Phi^d \mathbf{i}_p, \mathbf{i}_q\rangle$$
$$= \sum_{p\in Q_0} \sum_{d=0}^{N(p)} \sum_{k=d}^{N(p)} (k+1)\langle \Phi^d \mathbf{i}_p, \sum_{q\in Q_0} \mathbf{i}_q\rangle$$
$$- \sum_{p,q\in Q_0} \sum_{d=0}^{N(p)} \sum_{k=N(p)+d}^{N(p)} (k+1)\langle \Phi^d \mathbf{i}_p, \mathbf{i}_q\rangle.$$

Here again, the second sum vanishes by Lemma 3.4.2. Finally, $\langle \Phi^d \mathbf{i}_p, \sum_{q\in Q_0} \mathbf{i}_q\rangle = |\Phi^d \mathbf{i}_p|$ as observed at the beginning of 3.4, so our claim follows. □

3.7. **Proof of Theorem 1.** The first claim follows directly from the construction of $\check{\Gamma}_Q$, Lemma 3.2 and the fact that

$$\mathrm{Hom}_\Lambda(F_\lambda N, F_\lambda N) \cong \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\widehat{\Lambda}}(N, N^{(i)}).$$

For the second claim we have to show by [8, Lemma 1] that $\langle \underline{\dim}\, \mathrm{I}_Q, \underline{\dim}\, \mathrm{I}_Q\rangle = \dim \mathcal{E}$. This follows from the dimension formulas (3.3) and (3.4) of Proposition 3.5 together with Proposition 3.6.

4. The cluster algebras $\mathbb{C}[G^{e,w_0}]$ and $\mathbb{C}[N]$

In the next two sections we will use the setup from 1.4 and 1.5. We will need the following result, see for instance [22, Section 4.4.3]

4.1. **Lemma.** *Let $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ be a reduced expression for some element $w^{-1} \in W$, and $L(\lambda)$ an irreducible representation of highest weight λ for G. If u_λ is a highest weight vector for $L(\lambda)$ then we have*

$$\bar{w}u_\lambda = f_{i_m}^{(b_m)} \cdots f_{i_2}^{(b_2)} f_{i_1}^{(b_1)}(u_\lambda) \text{ and } f_{i_m}(\bar{w}u_\lambda) = 0.$$

Here $b_1 := \lambda(h_{i_1})$, $b_k := (s_{i_{k-1}} \cdots s_{i_2} s_{i_1}(\lambda))(h_{i_k})$ for $2 \leq k \leq m$ and $f_{i_k}^{(b_k)} := \frac{1}{b_k!} f_{i_k}^{b_k} \in U(\mathfrak{g})$.

For $v \in L(\lambda)$, we write $f_j^{\max}(v) := (1/m!)f_j^m(v)$ where $m = \max\{p \mid f_j^p(v) \neq 0\}$. With this notation we could restate the equality of the lemma as

$$\bar{w}u_\lambda = f_{i_m}^{\max} \cdots f_{i_2}^{\max} f_{i_1}^{\max}(u_\lambda).$$

4.2. The group G has Bruhat decompositions with respect to B and B_-, namely

$$G = \bigcup_{u \in W} B\bar{u}B = \bigcup_{v \in W} B_-\bar{v}B_-.$$

The intersection of two cells $G^{u,v} = (B\bar{u}B) \cap (B_-\bar{v}B_-)$ is called a *double Bruhat cell.*

In particular taking $u = e$, the unit in W, and $v = w_0$ we obtain $G^{e,w_0} = B \cap (B_-\bar{w}_0B_-)$, the intersection of B with the big cell relative to B_-. By [4, Proposition 2.8], G^{e,w_0} consists of all elements x of B such that $\Delta_{\varpi_i, w_0(\varpi_i)}(x) \neq 0$ for every i. This is a Zariski open subset of B, hence an algebraic variety of dimension $n+r$, where $r = |\Pi|$ is the number of positive roots associated to the Dynkin type of G. Moreover, we see that the algebra of regular functions $\mathbb{C}[G^{e,w_0}]$ is obtained from $\mathbb{C}[B]$ by adjoining formal inverses to the functions $\Delta_{\varpi_i, w_0(\varpi_i)}$.

On the other hand, N can be described as the subvariety of B given by the equations

$$\Delta_{\varpi_i,\varpi_i}(x) = 1, \qquad (1 \leq i \leq n).$$

Hence the algebra $\mathbb{C}[N]$ is the quotient of $\mathbb{C}[B]$ by the ideal generated by the elements $(\Delta_{\varpi_i,\varpi_i} - 1)_{i=1,2,\dots,n}$.

4.3. In [12], Fomin and Zelevinsky have introduced a transcendence basis $F(\mathbf{i})$ of the field of rational functions $\mathbb{C}(G^{e,w_0})$ consisting of certain generalized minors. In [4] Berenstein, Fomin and Zelevinsky have shown that each $F(\mathbf{i})$ can be taken as the initial cluster for a natural cluster algebra structure on the ring $\mathbb{C}[G^{e,w_0}]$. We are now going to recall their construction.

4.3.1. We add n additional letters $i_{-n}, \dots, i_{-1}$ at the beginning of $\mathbf{i}$, where $i_{-j} = -j$, and obtain an $(r+n)$-tuple

$$(i_{-n}, \dots, i_{-1}, i_1, \dots, i_r) = (-n, \dots, -1, i_1, \dots, i_r).$$

For $k \in [-n,-1] \cup [1,r]$ let

$$k^+ = \begin{cases} r+1 & \text{if } |i_l| \neq |i_k| \text{ for all } l > k, \\ \min\{l \mid l > k \text{ and } |i_l| = |i_k|\} & \text{otherwise.} \end{cases}$$

Then k is called $\mathbf{i}$-*exchangeable* if k and k^+ are in $[1,r]$. Let $e(\mathbf{i}) \subset [1,r]$ be the set of $\mathbf{i}$-exchangeable elements. One easily checks that $e(\mathbf{i})$ contains $r-n$ elements. More precisely, the set of indices i_k for $k \in [1,r] - e(\mathbf{i})$ is exactly $[1,n]$.

4.3.2. Next, one defines a quiver $\widetilde{A}_{\mathbf{i}}$ with set of vertices $[-n,-1] \cup [1,r]$. Assume that k and l are vertices such that the following hold:

- $k < l$;
- $\{k,l\} \cap e(\mathbf{i}) \neq \emptyset$.

There is an arrow $k \to l$ in $\widetilde{A}_{\mathbf{i}}$ if and only if $k^+ = l$, and there is an arrow $l \to k$ if and only if $l < k^+ < l^+$ and $a_{|i_k|,|i_l|} = -1$. Here, $(a_{ij})_{1 \leq i,j \leq n}$ denotes the Cartan matrix of the root system of G. By definition these are all the arrows of $\widetilde{A}_{\mathbf{i}}$.

4.3.3. **Remark.** If $\mathbf{i}$ is a reduced expression for w_0 which is adapted to a Dynkin quiver Q, then it is easy to obtain $\widetilde{A}_{\mathbf{i}}$ from the Auslander-Reiten quiver A_Q. See the examples in 6.5 and 6.6.

4.3.4. Now define an $(r+n) \times (r-n)$-matrix

$$\widetilde{B}(\mathbf{i}) = (b_{kl})$$

as follows. The columns of $\widetilde{B}(\mathbf{i})$ are indexed by the elements in $e(\mathbf{i})$, and the rows by $[-n,-1] \cup [1,r]$. Set

$$b_{kl} = \begin{cases} 1 & \text{if there is an arrow } k \to l \text{ in } \widetilde{A}_{\mathbf{i}}, \\ -1 & \text{if there is an arrow } l \to k \text{ in } \widetilde{A}_{\mathbf{i}}, \\ 0 & \text{otherwise.} \end{cases}$$

4.3.5. For $k \in [-n,-1] \cup [1,r]$ one defines a generalized minor $\Delta(k,\mathbf{i})$ as follows. For $k \in [1,r]$ set $v_{>k} = s_{i_r} s_{i_{r-1}} \cdots s_{i_{k+1}}$ and for $k \in [-n,-1]$ put $v_{>k} = w_0$. Then define

$$\Delta(k,\mathbf{i}) = \Delta_{\varpi_{|i_k|},\, v_{>k}(\varpi_{|i_k|})}.$$

Since $s_j(\varpi_i) = \varpi_i$ for $j \neq i$, it is easy to see that if $k \in [1,r]$ is not exchangeable then $\Delta(k,\mathbf{i}) = \Delta_{\varpi_{i_k},\varpi_{i_k}}$. On the other hand for $-i \in [-n,-1]$ we have $\Delta(-i,\mathbf{i}) = \Delta_{\varpi_i,\, w_0(\varpi_i)}$.

It is known [12, Theorem 1.12] that this collection of $n+r$ minors is a transcendence basis of the field $\mathbb{C}(G^{e,w_0})$, for any reduced expression $\mathbf{i}$ of w_0. By 4.2, we see that if we remove from this collection the n minors $\Delta_{\varpi_i,\varpi_i}$ we obtain a transcendence basis of the field $\mathbb{C}(N)$.

4.3.6. Let $\mathcal{F}$ be the field of rational functions over $\mathbb{C}$ in $n+r$ independent variables $\tilde{\mathbf{x}} = (x_{-n}, \dots, x_{-1}, x_1, \dots, x_r)$. Let $\overline{\mathcal{A}}(\mathbf{i})_{\mathbb{C}}$ denote the upper cluster algebra associated to the seed $(\tilde{\mathbf{x}}, \widetilde{B}(\mathbf{i}))$, a subalgebra of $\mathcal{F}$ (see [4, Definition 1.6]). Here the non-exchangeable indices in $[-n,-1] \cup [1,r]$ label the generators of the coefficient group (see [4, §2.2]).

Berenstein, Fomin and Zelevinsky then show that the isomorphism of fields $\varphi_{\mathbf{i}}$ from $\mathcal{F}$ to $\mathbb{C}(G^{e,w_0})$ defined by

$$\varphi_{\mathbf{i}}(x_k) = \Delta(k,\mathbf{i}), \qquad (k \in [-n,-1] \cup [1,r]),$$

restricts to an algebra isomorphism $\overline{\mathcal{A}}(\mathbf{i})_{\mathbb{C}} \to \mathbb{C}[G^{e,w_0}]$, see [4, Theorem 2.10].

Note that by varying the reduced expression $\mathbf{i}$ we obtain a priori several cluster algebra structures on $\mathbb{C}[G^{e,w_0}]$, but according to [4, Remark 2.14] all these structures coincide and give rise to the same cluster variables and clusters. Note also that in type A_n, the upper cluster algebra $\overline{\mathcal{A}}(\mathbf{i})_{\mathbb{C}}$ coincides with the cluster algebra $\mathcal{A}(\mathbf{i})_{\mathbb{C}}$, see [4, Remark 2.18].

4.4. Let $\tilde{\mathbf{x}}'$ be the subset of $\tilde{\mathbf{x}}$ obtained by removing the variables indexed by the n non-exchangeable elements in $[1,r]$. Let $\mathcal{F}'$ be the field of rational functions over $\mathbb{C}$ in the r variables of $\tilde{\mathbf{x}}'$. Finally, let $\widetilde{B}(\mathbf{i})'$ be the matrix obtained from $\widetilde{B}(\mathbf{i})$ by removing the rows labelled by the n non-exchangeable elements in $[1,r]$, and let $\overline{\mathcal{A}}(\mathbf{i})'_{\mathbb{C}}$ denote the upper cluster algebra associated to the seed $(\tilde{\mathbf{x}}', \widetilde{B}(\mathbf{i})')$, a subalgebra of $\mathcal{F}'$. By 4.2, we see that the isomorphism of fields $\varphi'_{\mathbf{i}} : \mathcal{F}' \to \mathbb{C}(N)$ defined by

$$\varphi'_{\mathbf{i}}(x_k) = \Delta(k,\mathbf{i}), \qquad (x_k \in \tilde{\mathbf{x}}'),$$

restricts to an algebra isomorphism $\overline{\mathcal{A}}(\mathbf{i})'_{\mathbb{C}} \to \mathbb{C}[N]$.

Clearly the same remarks as in 4.3.6 apply to the cluster algebra structures of $\mathbb{C}[N]$.

4.5. We shall now describe in representation theoretic terms the restriction to N of the regular function $\Delta(k, \mathbf{i})$. Let $L(\varpi_i)$ be the fundamental irreducible $\mathfrak{g}$-module with highest weight ϖ_i. Fix a highest weight vector u_{ϖ_i}.

It is well-known that $L(\varpi_i)$ can be realized in a canonical way as a subspace of the vector space $\mathbb{C}[N]$ by restricting the summand $L(\varpi_i)$ of $\mathbb{C}[N_-\backslash G]$ to $\mathbb{C}[N]$. Thus, the highest weight vector u_{ϖ_i} becomes identified with the constant regular function $\mathbf{1}$. Using this identification we obtain the following lemma:

4.5.1. **Lemma.** *For $k \in [-n, -1] \cup [1, r]$ we have*

$$\Delta(k, \mathbf{i}) = \begin{cases} f_{i_r}^{\max} f_{i_{r-1}}^{\max} \cdots f_{i_{k+1}}^{\max}(u_{\varpi_{i_k}}) & \text{if } k \in [1, r], \\ f_{i_r}^{\max} f_{i_{r-1}}^{\max} \cdots f_{i_1}^{\max}(u_{\varpi_{|i_k|}}) & \text{if } k \in [-n, -1]. \end{cases}$$

In particular for $-k \in [-n, -1]$, the minor $\Delta(-k, \mathbf{i}) = \Delta_{\varpi_k, w_0(\varpi_k)}$ is a lowest weight vector of $L(\varpi_k)$.

Proof: This follows from [5, p. 150–151] and 4.1 by restricting regular functions on G to the subgroup N, see also [6, p. 113]. □

5. Cluster variables and semicanonical basis

5.1. Let Q be a Dynkin quiver such that $|Q|$ is the diagram of G. In this section we prove that if $\mathbf{i}$ is a reduced expression for w_0 which is adapted to Q, the minors $\Delta(k, \mathbf{i}) \in \mathbb{C}[N]$ coincide with certain dual semicanonical basis vectors coming from the injective modules of the Auslander algebra of $\mathbb{C}Q$. In particular, this shows that the set of minors $\{\Delta(k, \mathbf{i})\}$ depends only on Q, not on the choice of a particular expression $\mathbf{i}$ adapted to Q.

5.2. Recall that by pushing down the injective modules of the Auslander algebra of $\mathbb{C}Q$ we obtain a set of indecomposable rigid modules $\mathrm{I}(j, \mathbf{i})$ over the preprojective algebra Λ, see 1.2 and 1.3. Since these modules are rigid, they have an open orbit in their module variety, and the closure of this orbit is an irreducible component. Therefore the module $\mathrm{I}(j, \mathbf{i})$ can be used to label an element $\rho_{\mathrm{I}(j,\mathbf{i})}$ of the dual semicanonical basis (see [27], [18, Section 7.2]).

5.3. The proof of Theorem 2 will make use of certain results of [17] that we shall now recall. Let I_i denote the injective envelope of the simple Λ-module S_i with dimension vector $\mathbf{e}_i$, thus $I_i = D\Lambda(-, i)$. In [17] the fundamental $\mathfrak{g}$-module $L(\varpi_i)$ was realized in terms of the lattice of submodules of I_i. This goes as follows.

Let $\mathcal{M}$ denote Lusztig's algebra of constructible functions on the varieties of finite-dimensional Λ-modules, and let $\mathcal{M}^*$ be its graded Hopf dual, an algebra isomorphic to $\mathbb{C}[N]$. For a finite-dimensional Λ-module X, let δ_X denote the linear form on $\mathcal{M}$ obtained by evaluation at X. Then in the identification $\mathbb{C}[N] \cong \mathcal{M}^*$, the subspace $L(\varpi_i)$ gets identified to the subspace of $\mathcal{M}^*$ spanned by the linear forms δ_X where X runs over the lattice of submodules of I_i, and one has explicit formulas for the action of the Chevalley generators of $\mathfrak{g}$ on each vector δ_X [17, Theorem 3]. In particular $\delta_{I_i} = \rho_{I_i}$ is a lowest weight vector of $L(\varpi_i)$, and δ_0, where 0 means the zero submodule of I_i, is a highest weight vector.

Let X be a submodule of I_i. We have a short exact sequence of Λ-modules

$$0 \to X \to I_i \xrightarrow{p} Y \to 0,$$

where Y is determined up to isomorphism by the isomorphism class of X, see [17, Lemma 1]. For $j \in [1, n]$ let m_j denote the multiplicity of S_j in the socle of Y, and let X_j be the unique submodule of I_i such that $X \subset X_j \subset I_i$ and X_j/X is isomorphic to $S_j^{\oplus m_j}$. Thus X_j is the pullback of p and the inclusion of $S_j^{\oplus m_j}$ into Y.

5.4. **Lemma.** *With the above notation, we have* $f_j^{\max}(\delta_X) = f_j^{(m_j)}(\delta_X) = \delta_{X_j}$.

Proof: By [17, Theorem 3 (ii)], we have that

$$f_j^k(\delta_X) = \int_{\mathfrak{f}=(X=X(1)\subset\cdots\subset X(k))} \delta_{X(k)}$$

where the integral is over the variety of flags $\mathfrak{f}$ of submodules of I_i such that $X(s)/X(s-1)$ is isomorphic to S_j for all $1 < s \leq k$. For $k > m_j$ this variety is empty by definition of m_j, hence $f_j^k(\delta_X) = 0$. For $k = m_j$, all flags $\mathfrak{f}$ have their last step equal to X_j. Moreover since $X_j/X \cong S_j^{\oplus m_j}$ this variety is isomorphic to the variety of complete flags in $\mathbb{C}^{m_j}$, whose Euler characteristic is $m_j!$. Hence $f_j^{m_j}(\delta_X) = m_j!\delta_{X_j}$, as claimed. □

5.5. **Lemma.** *Let* $k \in [-n,-1] \cup e(\mathbf{i})$ *and* $j = \theta^{-1}(k)$. *Then, if* $i = |i_k|$, *the module* $\mathrm{I}(j,\mathbf{i})$ *is a submodule of* I_i *and in* $L(\varpi_i)$ *there holds*

$$\delta_{\mathrm{I}(j,\mathbf{i})} = \begin{cases} f_{i_r}^{\max} f_{i_{r-1}}^{\max} \cdots f_{i_{k+1}}^{\max}(\delta_0) & \text{if } k \in e(\mathbf{i}), \\ f_{i_r}^{\max} f_{i_{r-1}}^{\max} \cdots f_{i_1}^{\max}(\delta_0) & \text{if } k \in [-n,-1]. \end{cases}$$

Proof: Recall that we defined the function θ in 1.5. If $k = -i \in [-n,-1]$, then $\mathrm{I}(j,\mathbf{i}) = I_i$. On the other hand in this case the product $f_{i_r}^{\max} f_{i_{r-1}}^{\max} \cdots f_{i_1}^{\max}$ maps δ_0 to the lowest weight vector of $L(\varpi_i)$, that is, to δ_{I_i}, as required.

If $k \in e(\mathbf{i})$, then $\mathrm{R}_Q(x(j))$ belongs to the τ-orbit of $\mathrm{R}_Q(x(k))$ by definition of θ, therefore to the τ-orbit of $D\mathrm{k}Q(-,i)$ see 1.3. It follows that $\mathrm{I}(j,\mathbf{i})$ is a submodule of I_i, see 1.2.

More precisely, $j = \theta^{-1}(k) = \min\{l \in [k+1,r] \mid i_l = i\}$. We conclude that in $L(\varpi_i)$ holds $f_{i_{j-1}}^{\max} \cdots f_{i_{k+1}}^{\max}(\delta_0) = \delta_0$ by 5.4, since the socle of I_i is S_i.

Now consider for $l \in [j-1,r]$ the $\widetilde{\Lambda}$-submodule $\widetilde{I}_{x(j)}(\leq l,\mathbf{i})$ of $D\widetilde{\Lambda}(-,x(j))$ with

$$\widetilde{I}_{x(j)}(\leq l,\mathbf{i})(x(m)) := \begin{cases} D\widetilde{\Lambda}(x(m),x(j)) & \text{if } m \leq l, \\ 0 & \text{else.} \end{cases}$$

With $I_{x(j)}(\leq l,\mathbf{i}) := F_\lambda \widetilde{I}_{x(j)}$ we see that $I_{x(j)}(\leq r,\mathbf{i}) = \mathrm{I}(j,\mathbf{i})$ is a submodule of I_i since $F_\lambda D\widetilde{\Lambda}(-,x(j)) \cong I_i$.

Using Lemma 5.4 we conclude that in $L(\varpi_i)$ we have

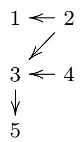
$$f_{i_l}^{\max}(\delta_{I_{x(j)}(\leq l-1,\mathbf{i})}) = \delta_{I_{x(j)}(\leq l,\mathbf{i})} \text{ for } l \in [j,r].$$

This yields that $\delta_{\mathrm{I}(j,\mathbf{i})}$ is the extremal vector of $L(\varpi_i)$ with weight $s_{i_r} s_{i_{r-1}} \cdots s_{i_{k+1}}(\varpi_i)$, and the lemma follows. □

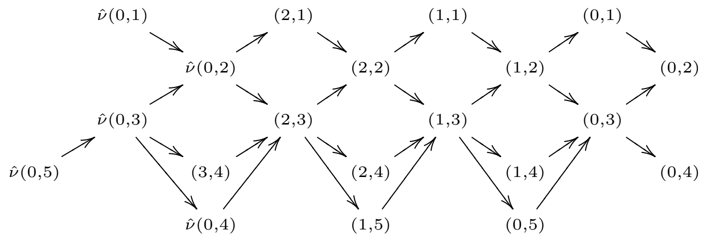
We can now finish the proof of Theorem 2. Using Lemma 4.5.1 and Lemma 5.5 we obtain that $\Delta(k,\mathbf{i}) = \delta_{\mathrm{I}(j,\mathbf{i})}$. But since $\mathrm{I}(j,\mathbf{i})$ is rigid, its orbit is open and we have $\delta_{\mathrm{I}(j,\mathbf{i})} = \rho_{\mathrm{I}(j,\mathbf{i})}$.

6. Examples

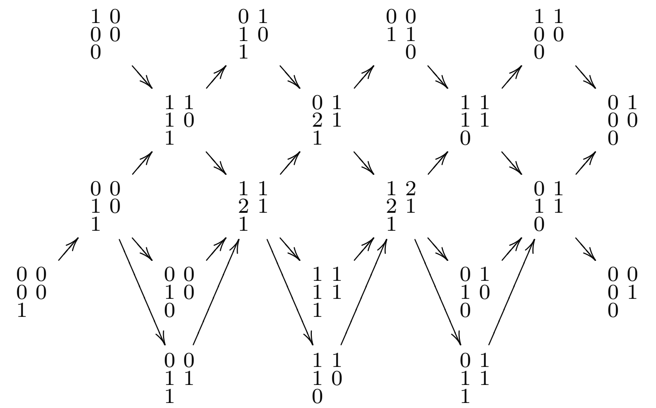
Our running example will be the Dynkin quiver Q of type D_5 with the following orientation:



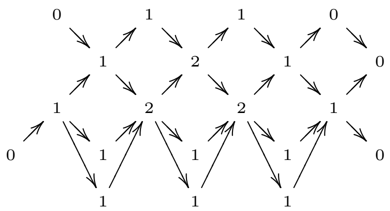
6.1. Γ_Q **for** D_5**.** We show the quiver A_Q of Γ_Q:

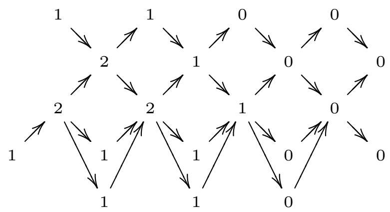


The isomorphism classes of the indecomposable representations of Q correspond to the vertices of A_Q. Each such representation is uniquely determined by its dimension vector:

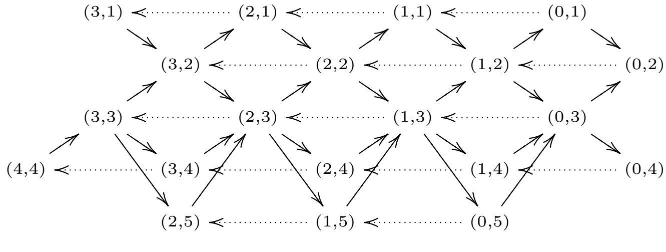


The dimension vector of the injective Γ_Q-module $D\Gamma_Q(-,(0,q))$ is given by the q-components of the corresponding dimension vectors. The dimension vector of $D\Gamma_Q(-,\tau^j(0,q))$ is obtained from this by "translation and cut-off". Here we show for example $\underline{\dim}\, D\Gamma_Q(-,(0,3))$ and $\underline{\dim}\, D\Gamma_Q(-,\tau(0,3))$:





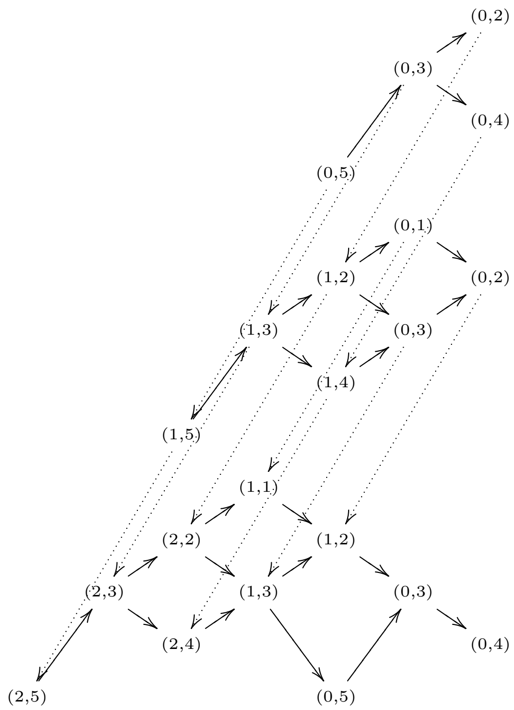
6.2. **The category** $\check{\Gamma}_Q$**.** We display here the quiver of the (graded) category $\check{\Gamma}_Q$ associated to a quiver of type D_5 with the same orientation as above. The arrows of degree one are dotted.



Recall that also the relations are easy to read off: For each arrow $\alpha\colon x \to y$ we have the corresponding mesh relation (2.1) from y to x if α is of degree 1. Otherwise, if x is not an "injective" vertex, (i.e. here if $x \notin \{(0,1),(0,3),(0,5)\}$) there are one or two paths of length 2 (and degree 1) from y to x. The first case occurs when y is a "projective" vertex (i.e. here if $y \in \{(3,2),(3,3),(2,5)\}$) and the unique path of length 2 from y to x is a zero-relation, otherwise the two paths form a commutativity relation.

6.3. **A projective-injective $\check{\Gamma}_Q$-module.** We display for $\check{\Gamma}_Q$ as in 6.2 the projective module $\check{\Gamma}_Q((0,5),-)$. Since $(0,5)$ is an injective vertex of A_Q we have $\check{\Gamma}_Q((0,5),-) \cong D\check{\Gamma}_Q(-,(0,\mu(5))$. Moreover, the projective modules $\check{\Gamma}_Q((j,5),-)$ for $0 \le j \le 2$ are easily found as submodules of $\check{\Gamma}_Q((0,5),-)$.

Each entry (n,q) represents a basis vector which corresponds to a (graded) simple composition factor of this type. The arrows indicate as usual the action of $\check{\Gamma}_Q$.

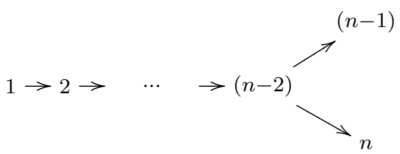


6.4. **Dimensions.** We include the result of some calculations of

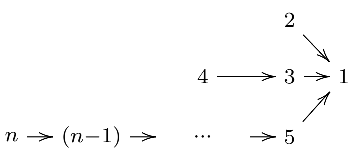
$$d(Q) := \dim \operatorname{End}_\Lambda(\mathrm{I}_Q)$$

for specific orientations. This can be done quite easily on a computer using the formula (3.4).

A_n: $1 \to 2 \to \cdots \to (n-1) \to n$ $\qquad d(A_n) = 2\binom{n}{5} + 7\binom{n}{4} + 9\binom{n}{3} + 5\binom{n}{2} + n,$

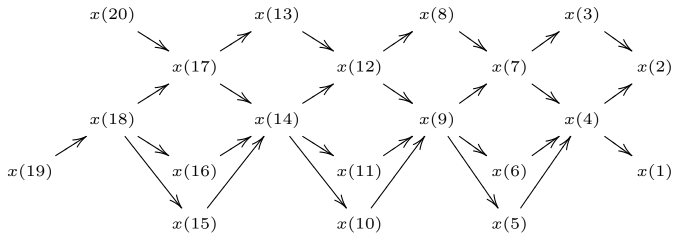
D_n: 

$$d(D_n) = 27\binom{n}{5} + 43\binom{n}{4} + 19\binom{n}{3} + 2\binom{n}{2},$$

E_n: 

$$d(E_n) = \begin{cases} 2444 & \text{if } n = 6, \\ 13130 & \text{if } n = 7, \\ 107114 & \text{if } n = 8. \end{cases}$$

6.5. **An adapted ordering on** $\mathrm{Obj}(\Gamma_Q)$**:**

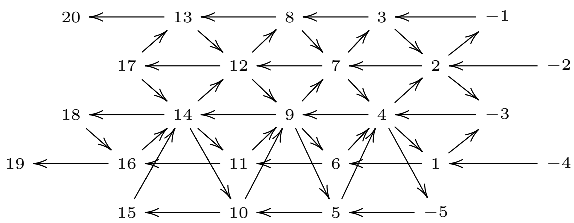


According to 1.3 we obtain for w_0 the following reduced expression which is adapted to Q:

$$\mathbf{i} = (4, 2, 1, 3, 5, 4, 2, 1, 3, 5, 4, 2, 1, 3, 5, 4, 2, 3, 4, 1)$$

6.6. **The quiver** $\widetilde{A}_{\mathbf{i}}$**.** For the adapted expression $\mathbf{i}$ from 6.5 we obtain

$\widetilde{A}_{\mathbf{i}} =$



The exchangeable vertices are $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16\}$

7. Appendix: Using the triangulated structure

7.1. **Proposition** (Freyd/Krause). [24, Appendix B] *Let* $\mathcal{D}$ *be a triangulated* k*-category with suspension functor* Σ.

- (a) *The category* $\mathcal{D}$-mod *is a Frobenius category. Thus the stable category* $\mathcal{D}$-$\underline{\mathrm{mod}}$ *is triangulated with suspension functor* $\Omega_{\mathcal{D}}^{-1}$, *the inverse of Heller's loop functor.*
- (b) *In* $\mathcal{D}$-$\underline{\mathrm{mod}}$ *we have a functorial isomorphism* $M^{\Sigma} \cong \Omega_{\mathcal{D}}^3 M$.

7.2. **Remarks.** (a) We may consider $\mathcal{D}$ as a $\mathcal{D}$-$\mathcal{D}$-bimodule, i.e. a functor $\mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to$ k-mod. Similarly, $D\mathcal{D}$ with $D\mathcal{D}(a, b) := \mathrm{Hom}_{\mathrm{k}}(\mathcal{D}(b, a), \mathrm{k})$ is a $\mathcal{D}$-$\mathcal{D}$-bimodule.

(b) Suppose that $\mathcal{D}$ admits Auslander-Reiten triangles with translate $\tau\colon \mathcal{D} \to \mathcal{D}$. In this case we set $\nu := \Sigma\tau$. Then the Auslander-Reiten formula $\mathcal{D}(x, \Sigma y) \cong \mathrm{Hom}_k(\mathcal{D}(y, \tau x), \mathrm{k})$ may be interpreted as an isomorphism of bimodules

$$D\mathcal{D} \cong \mathcal{D}^{\nu^{-1}} \quad \text{where } \mathcal{D}^{\nu^{-1}}(a, b) := \mathcal{D}(a, \nu b). \tag{7.1}$$

We conclude that

$$\mathcal{N}M := D\mathcal{D} \otimes_{\mathcal{D}} M \cong \mathcal{D}^{\nu^{-1}} \otimes_{\mathcal{D}} M \cong M^{\nu^{-1}}$$

is a Nakayama functor for $\mathcal{D}$-mod and ν^{-1} the corresponding Nakayama automorphism for $\mathcal{D}$, see for example [13, Section 2]. In this situation we will write

$$M^{(i)} := M^{\tau^i}.$$

(c) If moreover $\mathcal{D}$ is locally bounded, then $\mathcal{D}$-$\underline{\mathrm{mod}}$ admits Auslander-Reiten triangles with translation $\tau_{\mathcal{D}} = \Omega_{\mathcal{D}}^2\mathcal{N}$. With Proposition 7.1 we obtain the functorial isomorphisms

$$M^{(-1)} \cong \tau_{\mathcal{D}}\Omega_{\mathcal{D}}M \quad \text{and} \quad \tau_{\mathcal{D}}^{-3}M \cong M^{\Sigma\tau^3} \text{ (in } \mathcal{D}\text{-}\underline{\mathrm{mod}}). \tag{7.2}$$

In fact, we have $\tau_{\mathcal{D}}\Omega_D \cong \Omega^3\mathcal{N} \cong ?^{\Sigma\nu^{-1}} = ?^{\tau^{-1}}$ and $\tau_{\mathcal{D}}^{-3} = \Omega_{\mathcal{D}}^6\mathcal{N}^3 \cong ?^{\Sigma^{-2}\nu^3} \cong ?^{\Sigma\tau^3}$. (d) If we consider in this context *right* modules (i.e. contravariant functors), we get in $\underline{\text{mod}}$-$\mathcal{D}$ a functorial isomorphism $M^{\Sigma} \cong \Omega_{\mathcal{D}}^{-3}M$ and consequently

$$M^{(1)} \cong \tau_{\mathcal{D}}\Omega_{\mathcal{D}}M \quad \text{ and } \tau_{\mathcal{D}}^3 M \cong M^{\Sigma\tau^3} \text{ (in } \underline{\text{mod}}\text{-}\mathcal{D}).$$

7.3. **Derived categories.** It follows from Happel's description [20, I.5.6] of the derived category $\mathcal{D}^b(\mathrm{k}Q^{\mathrm{op}}) := \mathcal{D}^b(\mathrm{k}Q^{\mathrm{op}}\text{-mod})$ that we have a natural equivalence

$$\widetilde{\Lambda}\text{-inj} \cong \mathcal{D}^b(\mathrm{k}Q^{\mathrm{op}}).$$

In particular, $\widetilde{\Lambda}$-inj is a triangulated category which admits Auslander-Reiten triangles. The suspension functor resp. the Auslander-Reiten translate are

$$\Sigma I = I^{\hat{\nu}\tau} \quad \text{ resp. } \tau I = I^{\tau^{-1}},$$

see 2.2. In our situation, these functors are up to isomorphism determined by their effect on objects. We conclude that we have an isomorphism of bimodules

$$\widetilde{\Lambda}^{\hat{\nu}} \cong D\widetilde{\Lambda}, \tag{7.3}$$

see 7.2 (b).

In order to state our next result we introduce $\underline{\Gamma}_Q$, the full subcategory of the Auslander category Γ_Q which contains all objects except those of the form $\hat{\nu}(0,q)$ for $q \in Q_0$. We call $\underline{\Gamma}_Q$ the *stable Auslander category* of $\mathrm{k}Q$.

7.4. **Proposition.** *The category $\widetilde{\Lambda}$ is isomorphic to the repetitive category $\widehat{\underline{\Gamma}}_Q$ of the stable Auslander category of* $\mathrm{k}Q$.

Proof: Recall that by definition the objects of $\widehat{\underline{\Gamma}}_Q$ are of the form (z,x) with $z \in \mathbb{Z}$ and $x \in \mathrm{Obj}(\underline{\Gamma}_Q)$ and we have

$$\widehat{\underline{\Gamma}}_Q((z,x),(z',x')) = \begin{cases} \underline{\Gamma}_Q(x,x') & \text{if } z = z', \\ D\underline{\Gamma}_Q(x,x') & \text{if } z = z'-1, \\ 0 & \text{else.} \end{cases}$$

Here we define the dual $\underline{\Gamma}_Q$-$\underline{\Gamma}_Q$-bimodule $D\underline{\Gamma}_Q$ by $D\underline{\Gamma}_Q(x,y) := \mathrm{Hom}_{\mathrm{k}}(\underline{\Gamma}_Q(y,x),k)$, compare 7.2 (a). Now, by the bimodule isomorphism (7.3) we see that the assignation

$$(z,(i,q)) \mapsto \tau^i\hat{\nu}^{-z}(0,q)$$

induces an isomorphism $\widehat{\underline{\Gamma}}_Q \to \widetilde{\Lambda}$. □

7.5. **Conclusions.** (a) In our situation we note that in $\mathcal{D}^b(\mathrm{k}Q) \cong \mathcal{D}^b(\mathrm{k}Q^{\mathrm{op}})$ we have an isomorphism of functors

$$\Sigma^2 \cong \tau^{-h(Q)} \tag{7.4}$$

where $h(Q)$ is the Coxeter number of $|Q|$. In fact, both functors coincide on objects as one easily verifies in $\widetilde{\Lambda}$. In our quiver situation this is sufficient. From (7.2) we obtain immediately the remarkable functorial isomorphism

$$\tau_{\widetilde{\Lambda}}^6 \cong M^{(h(Q)-6)} \tag{7.5}$$

in $\widetilde{\Lambda}$-$\underline{\text{mod}} \cong \mathcal{D}^b(\mathrm{k}Q)$-$\underline{\text{mod}}$.

(b) The action of the infinite cyclic group $\langle\tau\rangle$ on $\widetilde{\Lambda}$ provides us with a *Galois covering*

$$F\colon \widetilde{\Lambda} \to \Lambda,$$

see for example [14]. We conclude that Λ-inj is the orbit category of $\widetilde{\Lambda}$-inj modulo the induced action of $\langle\tau\rangle$. Now, $\widetilde{\Lambda}\text{-inj} \cong \mathcal{D}^b(\mathrm{k}Q)$ is a triangulated category, and the hypothesis of [23] are obviously fulfilled. Thus Λ-inj is also a triangulated category with Auslander-Reiten triangles and the corresponding translation $\bar{\tau}$ is the identity. Note moreover that the induced suspension is isomorphic to the corresponding Nakayama automorphism, i.e. $\bar{\Sigma} \cong \bar{\nu}$. By applying (7.2) to $\mathcal{D} = (\Lambda\text{-inj})^{\mathrm{op}}$ we conclude that

$$\tau_\Lambda^3 M \cong M^{\bar{\nu}} \quad \text{and} \quad \tau_\Lambda^6 M \cong M$$

holds functorially in Λ-$\underline{\mathrm{mod}}$. In case ν is just a translation, see 2.3, we even have $\tau_\Lambda^3 M \cong M$. This is our interpretation of the proof for the 6-periodicity of τ_Λ in [1]. Even more directly by (7.2) we conclude that

$$\tau_\Lambda \Omega_\Lambda M \cong M$$

functorially. This means that the triangulated category Λ-$\underline{\mathrm{mod}}$ is of Calabi-Yau dimension 2.

(c) Since $\widetilde{\Lambda} \cong \widehat{\underline{\Gamma}}_Q$ we have by Happel's Theorem [20, II.4] $\widetilde{\Lambda}\text{-}\underline{\mathrm{mod}} \cong \mathcal{D}^b(\underline{\Gamma}_Q\text{-mod})$ as triangulated categories. If we consider the push-down functor

$$F_\lambda \colon \widetilde{\Lambda}\text{-}\underline{\mathrm{mod}} \to \Lambda\text{-}\underline{\mathrm{mod}}$$

associated to the Galois covering $F\colon \widetilde{\Lambda} \to \Lambda$ the subcategory of Λ-modules of the first kind (i.e. the subcategory of objects which are isomorphic to a push-down) is equivalent to the orbit category $\mathcal{D}^b(\underline{\Gamma}_Q\text{-mod})/\langle\tau\Sigma^{-1}\rangle$ via the above identifications and (7.2), see [11]. Here, Σ resp. τ are the suspension resp. the Auslander-Reiten translation in $\mathcal{D}^b(\underline{\Gamma}_Q\text{-mod})$.

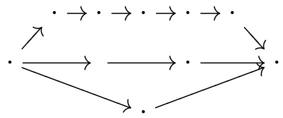
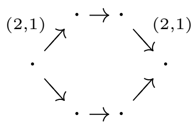
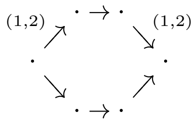
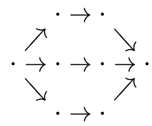
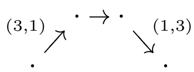
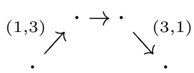
Now, for a Dynkin quiver Q with Coxeter number $h(Q) \leq 6$ the algebras $\underline{\Gamma}_Q$ are (quasi-) tilted. Thus, in these cases we find $\Lambda\text{-}\underline{\mathrm{mod}} \cong \mathcal{D}^b(\underline{\Gamma}_Q)/\langle\Sigma\tau^{-1}\rangle$ is a cluster category in the sense of [7].

7.6. **Remark.** Let H be a (basic, connected) finite-dimensional hereditary k-algebra of finite representation type. Then H is a species of type $\mathsf{A} - \mathsf{G}$ in the sense of Dlab and Ringel, see [9]. In this case we may also study the stable Auslander category $\underline{\Gamma}_H$. The same argument as in 7.3 and 7.4 shows that $\widehat{\underline{\Gamma}}_H\text{-inj} \cong \mathcal{D}^b(H^{\mathrm{op}})$. The Auslander-Reiten translate in $\mathcal{D}^b(H^{\mathrm{op}})$ induces an automorphism τ of $\widehat{\underline{\Gamma}}_H$. Thus we may consider the Galois covering $\widehat{\underline{\Gamma}}_H \to \widehat{\underline{\Gamma}}_H/\langle\tau\rangle$. So we are tempted to consider $\widehat{\underline{\Gamma}}_H/\langle\tau\rangle$ as the preprojective algebra of H. However, τ is now in general not determined by its effect on objects since $\mathrm{Out}_{\mathrm{k}}(H) = \mathrm{Aut}_k(H)/\mathrm{Inn}(H)$ is possibly non-trivial. Thus we are (for the moment) unable to compare $\widehat{\underline{\Gamma}}_H/\langle\tau\rangle$ with the possible choices for the preprojective algebra of H in the sense of Dlab and Ringel [10].

Anyway, if we denote by $|H|$ the (unoriented) diagram of H then we find the list which we present in Figure 1 of interest due to its similarity with the cluster types of $\mathbb{C}[N]$. In the case of Coxeter number $c(|H|) = 6$ we display the diagram of a canonical tubular algebra following Lenzing [25] and the corresponding extended affine root system in the sense of Saito [29]. In the case of G_2 there are two different diagrams of canonical algebras which produce derived equivalent algebras (for adequate choices of bimodules), the corresponding two root systems are isomorphic up to the marking. Note moreover, that in this case our previous calculations (7.5) predict that in the stable module category of the repetitive algebra $\widehat{\underline{\Gamma}}_H$-$\underline{\mathrm{mod}}$ the Auslander-Reiten translate should be 6-periodic. This is in fact the case for all (tubular) algebras which are derived equivalent to a canonical algebra with a diagram from our list.

$c(\|H\|)$	$\|H\|$	$\underline{\Gamma}_H$ is tilted of
3	A_2	A_1
4	A_3	A_3
	B_2	B_2
5	A_4	D_6

Figure 1.

$c(\|H\|)$	$\|H\|$	$\underline{\Gamma}_H$ is tilted of	root system
6	A_5		$\widetilde{\mathsf{E}}_8^{(1,1)}$
	B_3		$\widetilde{\mathsf{F}}_4^{(1,1)}$
	C_3		$\widetilde{\mathsf{F}}_4^{(2,2)}$
	D_4		$\widetilde{\mathsf{E}}_6^{(1,1)}$
	G_2		$\widetilde{\mathsf{G}}_2^{(3,1)}$
			$\widetilde{\mathsf{G}}_2^{(1,3)}$

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