Let $K$ be a field, and let $A$ be a finite-dimensional $K$-algebra.

For an $A$-module $M \in \text{mod}(A)$ let $\Sigma(M)$ be the number of isomorphism classes of indecomposable direct summands of $M$.

Let $n$ be the number of isomorphism classes of simple $A$-module. We have $n = \Sigma(A)$.

Below we define rigid, selforthogonal, exceptional and tilting modules. This terminology varies from author to author.

1. Rigid modules

An $A$-module $M \in \text{mod}(A)$ is **rigid** if $\text{Ext}^1_A(M, M) = 0$. The following conjecture can be found in [K].

**Conjecture 1.1.** For each $d \geq 1$ there are only finitely many isomorphism classes of rigid $A$-modules of dimension $d$.

Using a geometric argument, Conjecture 1.1 can be proved provided the ground field $K$ is algebraically closed. It also shoudn’t be difficult to prove it in general. (It is clear for finite fields, and for infinite fields one can use Voigt’s Lemma.) It would be desirable to find a non-geometric proof.

There exist finite-dimensional $K$-algebras $A$ such that for each $m \geq 1$ there exists a rigid $A$-module $M \in \text{mod}(A)$ with $\Sigma(M) = m$, see [HIO]. Here is a related problem:

**Problem 1.2** (Iyama [I]). Find an algebra $A$ and an $A$-module

$$M := \bigoplus_{i \in I} M_i$$

with $I$ infinite, $M_i \in \text{mod}(A)$ is indecomposable for all $i$, and $M_i \not\cong M_j$ for all $i \neq j$ such that the following hold:
(i) $\text{Ext}^1_A(M, M) = 0$.
(ii) If $N \in \text{mod}(A)$ is indecomposable with $\text{Ext}^1_A(M, N) = 0$, then $N \cong M_i$ for some $i$.
(iii) If $N \in \text{mod}(A)$ is indecomposable with $\text{Ext}^1_A(N, M) = 0$, then $N \cong M_i$ for some $i$.

2. Selforthogonal modules

An $A$-module $M$ is selforthogonal if $\text{Ext}^i_A(M, M) = 0$ for all $i \geq 1$. The following conjecture can be found in [Ha].

**Conjecture 2.1.** Let $M \in \text{mod}(A)$ be selforthogonal. Then we have $\Sigma(M) \leq n$.

**Question 2.2** (Tachikawa). Let $A$ be selfinjective. Let $M \in \text{mod}(A)$ be selforthogonal. Does this imply that $M$ is projective?

The following conjecture is equivalent to the Nakayama Conjecture, compare [AR].

**Conjecture 2.3** (Auslander-Reiten). Let $M \in \text{mod}(A)$ be a selforthogonal generator-cogenerator of $\text{mod}(A)$. Then $M$ is projective.

The following conjecture is equivalent to the Generalized Nakayama Conjecture, compare [AR].

**Conjecture 2.4** (Auslander-Reiten). Let $M \in \text{mod}(A)$ be a selforthogonal generator of $\text{mod}(A)$. Then $M$ is projective.

3. Exceptional modules

An $A$-module $M \in \text{mod}(A)$ is called exceptional if $M$ is selforthogonal and if $\text{proj. dim}(M) < \infty$. The following conjecture can be found in [Ha].

**Conjecture 3.1.** Let $M \in \text{mod}(A)$ be exceptional. Then we have $\Sigma(M) \leq n$.

**Theorem 3.2** (Bongartz). Let $M \in \text{mod}(A)$ be exceptional with $\text{proj. dim}(M) \leq 1$. Then there exists a module $N$ such that $\Sigma(M \oplus N) = n$ and $\text{proj. dim}(M \oplus N) \leq 1$. In particular, we have $\Sigma(M) \leq n$.

4. Gorenstein symmetry conjecture

Recall that $A$ is a Gorenstein algebra or Iwanaga-Gorenstein algebra if $\text{proj. dim}(D(A_A)) < \infty$ and $\text{inj. dim}(A_A) < \infty$. The following conjecture can be found in [ARS].

**Conjecture 4.1** (Gorenstein Symmetry Conjecture). $\text{proj. dim}(D(A_A)) < \infty$ if and only if $\text{inj. dim}(A_A) < \infty$.

An $A$-module $M$ is coexceptional if $M$ is selforthogonal and if $\text{inj. dim}(M) < \infty$. The above conjecture can be reformulated as follows:

**Conjecture 4.2.** $D(A_A)$ is exceptional if and only if $A_A$ is coexceptional.
5. Tilting modules

A exceptional $A$-module $T \in \text{mod}(A)$ is a **tilting module** if there exists and exact sequence of the form

$$0 \to \mathcal{A}A \to T_0 \to T_1 \to \cdots \to T_m \to 0$$

with $T_i \in \text{add}(T)$ for all $1 \leq i \leq m$.

It is not hard to check that for tilting modules $T$ we have $\Sigma(T) = n$.

In the literature, tilting modules are sometimes called *generalized tilting modules*, whereas the term *tilting module* is used for tilting modules with projective dimension at most one. Tilting modules with projective dimension at most one are sometimes called *classical tilting modules*.

**Conjecture 5.1.** Let $M \in \text{mod}(A)$ be an exceptional $A$-module with $\Sigma(M) = n$. Then $M$ is a tilting module.

**Conjecture 5.2.** Assume that $D(A_A)$ is exceptional. Then $D(A_A)$ is a tilting module.

Direct summands of tilting modules are called **partial tilting modules**.

There are examples of exceptional modules which are not partial tilting modules, see [Ha].

A partial tilting module $M \in \text{mod}(A)$ with $\Sigma(M) = n - 1$ is an **almost complete tilting module**. An indecomposable $A$-module $C$ is a **complement** of an almost complete tilting module $M$ if $M \oplus C$ is a tilting module.

The following conjecture is equivalent to the Generalized Nakayama Conjecture, see [HaU].

**Conjecture 5.3.** Let $M \in \text{mod}(A)$ be a projective almost complete tilting module. Then $M$ has only finitely many complements up to isomorphism.

Here is a more general conjecture, see for example [HaU]:

**Conjecture 5.4.** Let $M \in \text{mod}(A)$ be an almost complete tilting module. Then $M$ has only finitely many complements up to isomorphism.

For more information on the background of the following conjecture we refer to [W].

**Conjecture 5.5** (Wakamatsu Tilting Conjecture). Let $T \in \text{mod}(A)$ such that the following hold:

(i) $T$ is exceptional.

(ii) There exists an exact sequence

$$0 \to \mathcal{A}A \to T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} T_2 \xrightarrow{f_2} T_3 \xrightarrow{f_3} \cdots$$

with $T_i \in \text{add}(T)$ and

$$\text{Im}(f_i) \subseteq \{ M \in \text{mod}(A) \mid \text{Ext}_A^j(M, T) = 0 \text{ for all } j \geq 1 \}$$

for all $i \geq 0$. 
Then $T$ is a tilting module.

References


