Let $K$ be a field.

1. **Global dimension**

Let $A$ be a finite-dimensional $K$-algebra. The global dimension of $A$ is defined as

$$\text{gl.dim}(A) := \sup \{ \text{proj.dim}(M) \mid M \in \text{Mod}(A) \}.$$ 

By a theorem of Auslander we have

$$\text{gl.dim}(A) := \sup \{ \text{proj.dim}(M) \mid M \in \text{mod}(A) \}.$$ 

This implies that

$$\text{gl.dim}(A) := \sup \{ \text{proj.dim}(S) \mid S \in \text{mod}(A) \text{ simple} \}.$$ 

The dual definition using injective dimensions is equivalent to the above definition.

2. **Global dimension of factors of the path algebra of a quiver**

In this section we assume that $K$ is algebraically closed. Let $Q$ be a finite quiver. Let $J$ be the path ideal of the path algebra $KQ$, i.e. $J$ is generated by all paths of length at least one. Recall that an ideal $I$ in the path algebra $KQ$ is an admissible ideal if there exists some $n \geq 2$ such that

$$J^n \subseteq I \subseteq J^2.$$ 

It follows that $KQ/I$ is finite-dimensional. Since $K$ is algebraically closed, each finite-dimensional $K$-algebra is Morita equivalent to an algebra of the form $KQ/I$ for some quiver $Q$ and some admissible ideal $I$.

It is known that $\text{gl.dim}(KQ) \leq 1$, even if $KQ$ is infinite dimensional. Furthermore, if $Q$ has a loop, then $\text{gl.dim}(KQ/I) = \infty$ for all admissible ideals $I$. (This follows from Igusa and Lenzing’s validation of the No Loop Conjecture.)

Following Happel and Zacharia [HZ] we define

$$g(Q) := \sup \{ \text{gl.dim}(KQ/I) \mid I \text{ admissible in } KQ, \text{ gl.dim}(KQ/I) < \infty \}$$
and
\[ d(Q) := \sup\{\dim_K(KQ/I) \mid I \text{ admissible in } KQ, \text{ gl. dim}(KQ/I) < \infty\}. \]

As a matter of habit, I upgraded problems and questions in [HZ] to conjectures.

**Conjecture 2.1 ([HZ]).** \( g(Q) < \infty. \)

**Conjecture 2.2 ([HZ]).** \( d(Q) < \infty. \)

**Theorem 2.3 ([HZ]).** If \( d(Q) < \infty, \) then \( g(Q) < \infty. \)

**Conjecture 2.4 ([HZ]).** If \( g(Q) < \infty, \) then \( d(Q) < \infty. \)

One can refine the above conjectures by using
\[ g(Q, d) := \sup\{\text{gl. dim}(KQ/I) \mid I \text{ admissible in } KQ, \text{ gl. dim}(KQ/I) = d\} \]
and
\[ d(Q, d) := \sup\{\dim_K(KQ/I) \mid I \text{ admissible in } KQ, \text{ gl. dim}(KQ/I) = d\} \]
with \( d \geq 1. \)

**Conjecture 2.5 ([HZ]).** Assume that \( \text{gl. dim}(KQ/I) < \infty \) for some admissible ideal \( I. \) Then we have
\[ \text{gl. dim}(KQ/I) \leq \dim_K(KQ/I). \]

**Theorem 2.6 (Dlab, Ringel [DR1, DR2]).** Let \( Q \) be a quiver without loops. Then there exists an admissible ideal \( I \) such that \( \text{gl. dim}(KQ/I) \leq 2. \)

**Problem 2.7 ([HZ]).** Given a quiver \( Q \) and some \( d \geq 1. \) Find a sufficient and necessary condition on \( Q \) such that there exists an admissible ideal \( I \) with \( \text{gl. dim}(KQ/I) = d. \)

**Theorem 2.8 (Schofield [Sch]).** There is a function \( f : \mathbb{N} \to \mathbb{N} \) such \( \text{gl. dim}(A) \leq f(d) \) for all finite-dimensional \( K \)-algebras \( A \) with \( \dim_K(A) \leq d \) and \( \text{gl. dim}(A) < \infty. \)

Most of this section should have an analogue in the world of finite-dimensional \( K \)-algebras with \( K \) an arbitrary field.

### 3. Global Dimension and Derived Equivalence

If \( A \) and \( B \) are finite-dimensional \( K \)-algebras with
\[ D^b(\text{mod}(A)) \cong D^b(\text{mod}(B)), \]
then \( \text{gl. dim}(A) < \infty \) if and only if \( \text{gl. dim}(B) < \infty. \)

I learned the following two conjectures from Martin Kalck [K]. Being careful and aware of the lack of evidence, he proposed them as questions. I upgraded them boldly to conjectures.
Conjecture 3.1. Let $A$ and $B$ be finite-dimensional $K$-algebras with 
\[ D^b(\text{mod}(A)) \simeq D^b(\text{mod}(B)). \]
Assume that $\text{gl. dim}(A) < \text{gl. dim}(B)$. Then for each $n$ with $\text{gl. dim}(A) < n < \text{gl. dim}(B)$ there exists a finite-dimensional $K$-algebra $C$ with $\text{gl. dim}(C) = n$ and 
\[ D^b(\text{mod}(C)) \simeq D^b(\text{mod}(A)). \]

Conjecture 3.2. Let $A$ be a finite-dimensional $K$-algebra. There exists some $N_A \geq 0$ such that for each finite-dimensional $K$-algebra $B$ with 
\[ D^b(\text{mod}(A)) \simeq D^b(\text{mod}(B)) \]
we have $|\text{gl. dim}(A) - \text{gl. dim}(B)| \leq N_A$.

Let $T \in \text{mod}(A)$ be a classical tilting module, and let $B := \text{End}_A(T)^{\text{op}}$. Then $|\text{gl. dim}(A) - \text{gl. dim}(B)| \leq 1$ and 
\[ D^b(\text{mod}(A)) \simeq D^b(\text{mod}(B)), \]
see [H1, H2, HR]. So two algebras $A$ and $B$ which are derived equivalent via a sequence of classical tilting modules satisfy Conjecture 3.1.

Conjecture 3.3 (Kalck [K]). Let $K$ be algebraically closed. Let $X = X(p, \lambda)$ be a weighted projective line with weight sequence $p = (p_1, \ldots, p_t)$, and let $A$ be a finite-dimensional $K$-algebra with 
\[ D^b(\text{coh}(X)) \simeq D^b(\text{mod}(A)). \]
Then $\text{gl. dim}(A) \leq \max\{p_i \mid 1 \leq i \leq t\}$.

References


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