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FINITISTIC DIMENSION CONJECTURE

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Contents

Let K be a field, and let A be a finite-dimensional K-algebra.

1. Finitistic dimension conjecture

Let

$$\text{fin.dim}(A) := \sup\{\text{proj.}\dim(M) \mid M \in \text{mod}(A), \text{proj.}\dim(M) < \infty\}$$

be the **finitistic dimension** of A. Thus this is the supremum of all projective dimensions of finite-dimensional A-modules with finite projective dimension.

Conjecture 1.1 (Finitistic Dimension Conjecture)**.** $\text{fin.dim.}(A) < \infty$.

The above conjecture has been confirmed for numerous classes of algebras. However, most classes of well studied algebras are defined by relatively easy relations like zero relations or commutativity relations. Examples with complicated overlapping relations involving scalars are too hard to handle. So despite more than 60 publications on this conjecture, there is not much evidence supporting it. For an overview we refer to [ZH].

Let

$$\text{fin.dim}'(A) := \sup\{\text{inj.}\dim(M) \mid M \in \text{mod}(A), \text{inj.}\dim(M) < \infty\}.$$

It is easy to construct examples of algebras A with $\text{fin.dim}'(A) \neq \text{fin.dim}(A)$.

Conjecture 1.2. *We have* $\text{fin.dim}(A) < \infty$ *if and only if* $\text{fin.dim}'(A) < \infty$.

2. Little versus big finitistic dimension

The **big finitistic dimenion** is defined by

$$\text{Fin.Dim}(A) := \sup\{\text{proj.}\dim(M) \mid M \in \text{Mod}(A), \text{proj.}\dim(M) < \infty\}$$

where the supremum is now taken over all A-modules with finite projective dimension.

There are examples of algebras A with $\text{fin.dim.}(A) \neq \text{Fin.Dim.}(A)$, see [S1].

3. Auslander conjecture

Conjecture 3.1 (Auslander)**.** *Let $M \in \mathrm{mod}(A)$. Then there exists some $n_M \geq 0$ such that the following holds: Suppose that for $N \in \mathrm{mod}(A)$ there exists some $n_{M,N} \geq 0$ such that $\mathrm{Ext}_A^i(M,N) = 0$ for all $i \geq s_{M,N}$. Then we have $\mathrm{Ext}_A^i(M,N) = 0$ for all $i \geq s_M$.*

Auslander proved the following: If Conjecture 3.1 holds for $A^e := A \otimes_K A^{\mathrm{op}}$, then Conjecture 1.1 holds for A. We refer to [H] for a proof.

There is a counterexample to Conjecture 3.1, see [S2].

References

[H] D. Happel, *Homological conjectures in representation theory of finite-dimensional algebras*, Unpublished note.

[S1] S. Smalø, *The supremum of the difference between the big and little finitistic dimensions is infinite*, Proc. Amer. Math. Soc. 126 (1998), no. 9, 2619–2622.

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