April 4, 2016

WILD ALGEBRAS

JAN SCHRÖER

Contents

Let K be a field, and let A be a finite-dimensional K-algebra.

I will add more references in the future.

1. Tame algebras

Let K be algebraically closed. The algebra A is **tame** or of **tame representation type** if A is not representation-finite, and if for each d there exist finitely many A-$K[X]$-bimodules $M_1, \ldots, M_t$, which are free of finite rank as right $K[X]$-modules, such that (up to isomorphism) all but finitely many indecomposable d-dimensional A-modules are isomorphic to a module of the form $M_i \otimes_{K[X]} S$ with S a simple $K[X]$-module. In this case, let $\mu(d)$ be the minimal number of such bimodules. (Recall that the simple $K[X]$-modules are of the form $S_\lambda := K[X]/(X - \lambda)$ with $\lambda \in K$, and $S_\lambda \cong S_\mu$ if and only if $\lambda = \mu$.)

2. Wild algebras

Most of this section is just a reformulation of [S]. The K-algebra A is

- **wild** if there exists a faithful exact K-linear functor
$$\mathrm{mod}(K\langle x, y\rangle) \to \mathrm{mod}(A)$$
which respects isomorphism classes and indecomposables.
- **fully wild** a.k.a. **strictly wild** if there exists a fully faithful exact K-linear functor
$$\mathrm{mod}(K\langle x, y\rangle) \to \mathrm{mod}(A).$$
- **Wild** if there exists a faithful exact K-linear functor
$$\mathrm{Mod}(K\langle x, y\rangle) \to \mathrm{Mod}(A)$$
which respects isomorphism classes and indecomposables.

- **Fully Wild** a.k.a. **Strictly Wild** if there exists a fully faithful exact K-linear functor
$$\mathrm{Mod}(K\langle x,y\rangle) \to \mathrm{Mod}(A).$$
- **controlled wild** if there exists a faithful exact K-linear functor
$$F\colon \mathrm{mod}(K\langle x,y\rangle) \to \mathrm{mod}(A)$$
and a class $\mathcal{C}$ of modules in $\mathrm{mod}(A)$ such that for all $M,N \in \mathrm{mod}(K\langle x,y\rangle)$ we have
$$\mathrm{Hom}_A(F(M),F(N)) = F(\mathrm{Hom}_{K\langle x,y\rangle}(M,N)) \oplus \mathcal{C}(F(M),F(N))$$
where $\mathcal{C}(F(M),F(N))$ is the subspace of $\mathrm{Hom}_A(F(M),F(N))$ consisting of all homomorphisms factoring through a finite direct sum of modules in $\mathcal{C}$.
- **endo-wild** if for each finite-dimensional K-algebra B there exists some $M \in \mathrm{mod}(A)$ with $\mathrm{End}_A(M) \cong B$.
- **Endo-Wild** if for each K-algebra B there exists some $M \in \mathrm{Mod}(A)$ with $\mathrm{End}_A(M) \cong B$.
- **Corner endo-wild** if for each finite-dimensional K-algebra B there exists some $M \in \mathrm{mod}(A)$ and a nilpotent ideal I of $\mathrm{End}_A(M)$ with $\mathrm{End}_A(M)/I \cong B$.
- **Corner Endo-Wild** if for each K-algebra B there exists some $M \in \mathrm{Mod}(A)$ and a nilpotent ideal I of $\mathrm{End}_A(M)$ with $\mathrm{End}_A(M)/I \cong B$.

One can show that A is wild if and only if there exists an A-$K\langle X,Y\rangle$-bimodule M, which is free of finite rank as a right $K[X]$-module, such that the functor
$$M \otimes_{K\langle X,Y\rangle} -\colon \mathrm{mod}(K\langle X,Y\rangle) \to \mathrm{mod}(A)$$
respects isomorphism classes and indecomposables.

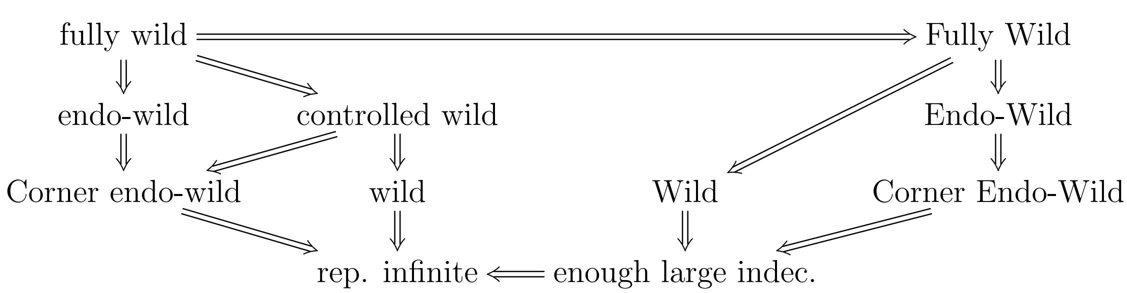
For any finitely generated K-algebra B there exists a fully faithful exact K-linear functor
$$\mathrm{mod}(B) \to \mathrm{mod}(K\langle x,y\rangle).$$

Theorem 2.1 (Drozd)**.** *Let K be algebraically closed, and let A be a finite-dimensional representation infinite K-algebra. Then A is tame or wild, but not both.*

The algebra A has **enough large indecomposable modules** if for each infinite cardinal λ there exists an indecomposable A-module of cardinality $\geq \lambda$.

3. Hierarchy of wild algebras

The proof of most of the following implications can be found in [S].



4. Examples

All wild path algebras are fully wild.

The m-Kronecker algebra with $m \geq 3$ is fully wild and Fully Wild. The 2-Kronecker algebra is Fully Wild, but not wild, see Ringel [R1].

References

[H] Y. Han, *Controlled wild algebras*, Proc. London Math. Soc. 83 (2001) 279–298.

[R1] C.M. Ringel, *Tame algebras are Wild*, Algebra Colloq. 6 (1999) 473–490.

[R2] C.M. Ringel, *Combinatorial representation theory, History and future*, In: D. Happel, Y.B. Zhang (Eds.), Representations of Algebras, vol. I, Proceedings of the Conference on ICRA IX, Beijing, 2000, Beijing Normal University Press, 2002, pp. 122–144.

[S] D. Simson, *On Corner type endo-wild algebras*, J. Pure Appl. Algebra 202 (2005), no. 1-3, 118–132.

Jan Schröer
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
Germany

E-mail address: schroer@math.uni-bonn.de