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WILD ALGEBRAS

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Let $K$ be a field, and let $A$ be a finite-dimensional $K$-algebra.
I will add more references in the future.

1. Tame algebras

Let $K$ be algebraically closed. The algebra $A$ is tame or of tame representation type if $A$ is not representation-finite, and if for each $d$ there exist finitely many $A$-$K[X]$-bimodules $M_1, \ldots, M_t$, which are free of finite rank as right $K[X]$-modules, such that (up to isomorphism) all but finitely many indecomposable $d$-dimensional $A$-modules are isomorphic to a module of the form $M_i \otimes_{K[X]} S$ with $S$ a simple $K[X]$-module. In this case, let $\mu(d)$ be the minimal number of such bimodules. (Recall that the simple $K[X]$-modules are of the form $S_\lambda := K[X]/(X - \lambda)$ with $\lambda \in K$, and $S_\lambda \cong S_\mu$ if and only if $\lambda = \mu$.)

2. Wild algebras

Most of this section is just a reformulation of [S]. The $K$-algebra $A$ is

- **wild** if there exists a faithful exact $K$-linear functor
  $$\text{mod}(K\langle x, y \rangle) \to \text{mod}(A)$$
  which respects isomorphism classes and indecomposables.
- **fully wild** a.k.a. **strictly wild** if there exists a fully faithful exact $K$-linear functor
  $$\text{mod}(K\langle x, y \rangle) \to \text{mod}(A).$$
- **Wild** if there exists a faithful exact $K$-linear functor
  $$\text{Mod}(K\langle x, y \rangle) \to \text{Mod}(A)$$
  which respects isomorphism classes and indecomposables.
• **Fully Wild** a.k.a. **Strictly Wild** if there exists a fully faithful exact $K$-linear functor
  \[ \text{Mod}(K\langle x, y \rangle) \to \text{Mod}(A). \]

• **controlled wild** if there exists a faithful exact $K$-linear functor
  \[ F : \text{mod}(K\langle x, y \rangle) \to \text{mod}(A) \]
  and a class $\mathcal{C}$ of modules in mod$(A)$ such that for all $M, N \in \text{mod}(K\langle x, y \rangle)$ we have
  \[ \text{Hom}_A(F(M), F(N)) = F(\text{Hom}_{K\langle x, y \rangle}(M, N)) \oplus \mathcal{C}(F(M), F(N)) \]
  where $\mathcal{C}(F(M), F(N))$ is the subspace of Hom$_A(F(M), F(N))$ consisting of all homomorphisms factoring through a finite direct sum of modules in $\mathcal{C}$.

• **endo-wild** if for each finite-dimensional $K$-algebra $B$ there exists some $M \in \text{mod}(A)$ with End$_A(M) \cong B$.

• **Endo-Wild** if for each $K$-algebra $B$ there exists some $M \in \text{mod}(A)$ with End$_A(M) \cong B$.

• **Corner endo-wild** if for each finite-dimensional $K$-algebra $B$ there exists some $M \in \text{mod}(A)$ and a nilpotent ideal $I$ of End$_A(M)$ with End$_A(M)/I \cong B$.

• **Corner Endo-Wild** if for each $K$-algebra $B$ there exists some $M \in \text{mod}(A)$ and a nilpotent ideal $I$ of End$_A(M)$ with End$_A(M)/I \cong B$.

One can show that $A$ is wild if and only if there exists an $A$-$K\langle X, Y \rangle$-bimodule $M$, which is free of finite rank as a right $K[X]$-module, such that the functor
  \[ M \otimes_{K\langle X, Y \rangle} - : \text{mod}(K\langle X, Y \rangle) \to \text{mod}(A) \]
respects isomorphism classes and indecomposables.

For any finitely generated $K$-algebra $B$ there exists a fully faithful exact $K$-linear functor
  \[ \text{mod}(B) \to \text{mod}(K\langle x, y \rangle). \]

**Theorem 2.1** (Drozd). Let $K$ be algebraically closed, and let $A$ be a finite-dimensional representation infinite $K$-algebra. Then $A$ is tame or wild, but not both.

The algebra $A$ has **enough large indecomposable modules** if for each infinite cardinal $\lambda$ there exists an indecomposable $A$-module of cardinality $\geq \lambda$.

### 3. Hierarchy of wild algebras

The proof of most of the following implications can be found in [S].

\[ \text{fully wild} \quad \downarrow \quad \text{leader wild} \quad \downarrow \quad \text{Fully Wild} \]
\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ \text{endo-wild} \quad \text{controlled wild} \quad \downarrow \quad \text{Endo-Wild} \]
\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ \text{Corner endo-wild} \quad \text{wild} \quad \text{Wild} \quad \text{Corner Endo-Wild} \]
\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]
\[ \text{rep. infinite} \leftrightarrow \text{enough large indec.} \]
4. Examples

All wild path algebras are fully wild.

The $m$-Kronecker algebra with $m \geq 3$ is fully wild and Fully Wild. The 2-Kronecker algebra is Fully Wild, but not wild, see Ringel [R1].

References


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