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SIMPLY CONNECTED ALGEBRAS

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CONTENTS
1. The fundamental group 1
2. Simply connected and strongly simply connected algebras 2
3. Examples 3
References 4

Let \( K \) be an algebraically closed field, and let \( A \) be a finite-dimensional basic \( K \)-algebra. It follows that \( A \cong KQ/I \) with \( Q \) a quiver and \( I \) an admissible ideal in the path algebra \( KQ \). The algebra \( A \) is connected if \( Q \) is connected.

A pair \((Q,I)\) with \( Q \) a quiver and \( I \) an admissible ideal in \( KQ \) is a presentation of \( A \) if \( A \cong KQ/I \). The quiver \( Q \) is determined by \( A \) up to isomorphism.

1. The fundamental group

Let \((Q,I)\) be a presentation of \( A \). An element \( \rho \in KQ \) of the form

\[
\rho = \sum_{i=1}^{t} \lambda_i w_i
\]

is a relation for \((Q,I)\) if the following hold:

(i) \( 0 \neq \lambda_i \in K \);
(ii) \( w_i \) is a path in \( Q \) of length at least 2;
(iii) All \( w_i \) start in the same vertex;
(iv) All \( w_i \) end in the same vertex;
(v) \( \rho \in I \).

Such a relation

\[
\rho = \sum_{i=1}^{t} \lambda_i w_i
\]

is minimal if \( t \geq 2 \) and if for each non-empty proper subset \( J \subset \{1, \ldots, t\} \) we have

\[
\sum_{j \in J} \lambda_j w_j \notin I.
\]

For each arrow \( \alpha \in Q_1 \) let \( \alpha^- \) be its formal inverse with \( s(\alpha^-) := t(\alpha) \) and \( t(\alpha^-) = s(\alpha) \). Set \( Q_1^- := \{\alpha^- | \alpha \in Q_1\} \).
A walk in $Q$ is a sequence $w = (w_1, w_2, \ldots, w_m)$ with $w_i \in Q_1 \cup Q_1^-$ for all $i$, and $s(w_i) = t(w_{i+1})$ for all $1 \leq i \leq m - 1$. Set $s(w) := s(w_m)$ and $t(w) := t(w_1)$. In particular, each path in $Q$ can be seen as a walk. Let $1_x$ denote the trivial path at $x$. This also counts as a walk with $s(1_x) = t(1_x) = x$. For walks $u$ and $v$ with $s(u) = t(v)$, the composition $uv$ is also a walk. (The composition is defined in the obvious way.)

Let $\sim$ be the smallest equivalence relation on the set of all walks in $Q$ such that the following hold:

(i) If $\alpha : x \to y$ is an arrow in $Q$, then $(\alpha^-, \alpha) \sim 1_x$ and $(\alpha, \alpha^-) \sim 1_y$.
(ii) If

$$\rho = \sum_{i=1}^{t} \lambda_i w_i$$

is a minimal relation for $(Q, I)$, then $w_i \sim w_j$ for all $i, j$.
(iii) If $u \sim v$, then $ww' \sim wvw'$ for all $w, w'$ such that $ww'$ and $wvw'$ are walks.

We denote by $[u]$ the equivalence class of a walk $u$.

Let $x \in Q_0$. The set $\Pi_1(Q, I, x)$ of equivalence classes of all walks $w$ with $s(w) = t(w) = x$ is a group via $[u] \cdot [v] := [uv]$.

Assume that $A$ is connected. Then it is straightforward to check that $\Pi_1(Q, I, x)$ does not depend on the choice of $x$. Thus we set $\Pi_1(Q, I) := \Pi_1(Q, I, x)$ and call this the fundamental group of $(Q, I)$.

2. Simply connected and strongly simply connected algebras

2.1. Triangular algebras. Recall that $A$ is a triangular algebra if $Q$ does not have any oriented cycles.

2.2. Simply connected algebras. Assume that $A$ is a connected and triangular. Then $A$ is a simply connected algebra if for every presentation $(Q, I)$ of $A$ the fundamental group $\Pi_1(Q, I)$ is trivial.

2.3. Convex subalgebras. A full subquiver $Q'$ of $Q$ is a convex subquiver if any path in $Q$ with source and target in $Q'$ lies entirely in $Q'$.

For a presentation $(Q, I)$ of $A$ and a convex subquiver $Q'$ of $Q$ let $I' := KQ' \cap I$. Then $KQ'/I'$ is called a convex subalgebra of $A$. (This is an abuse of notation, since convex subalgebras of $A$ are in general not subalgebras of $A$. They are just isomorphic to subalgebras of $A$.)

2.4. Strongly simply connected algebras. Assume that $A$ is a connected and triangular. Then $A$ is strongly simply connected if every connected convex subalgebra of $A$ is simply connected.
2.5. **The separation property.** Let \((Q, I)\) be a presentation of \(A\). For each vertex \(x\) of \(Q\) let \(Q(x)\) be the full subquiver of \(Q\) obtained by deleting all those vertices of \(Q\) being a source of a path in \(Q\) with target \(x\). This includes the path \(1_x\) of length 0 associated with \(x\).

Each \(A\)-module can be seen as a representation of \(Q\). The **support** of an \(A\)-module \(M\) is the set of vertices \(x\) of \(Q\) such that \(1_xM \neq 0\).

For each vertex \(x\) of \(Q\), let \(P(x)\) be the indecomposable projective \(A\)-module associated with \(x\), and let \(R(x)\) be the radical of \(P(x)\). Then \(R(x)\) is **separated** if \(R(x)\) is a direct sum of pairwise non-isomorphic indecomposable modules whose supports are contained in pairwise different connected components of \(Q(x)\). The algebra \(A\) has the **separation property** if \(R(x)\) is separated for any vertex \(x\) of \(Q\). This property does not depend on the choice of the presentation \((Q, I)\) of \(A\).

**Proposition 2.1.** Assume that \(A\) is connected and triangular. If \(A\) has the separation property, then \(A\) is simply connected.

The following proposition makes it easy to check if a given basic algebra is strongly simply connected.

**Proposition 2.2 ([AL, S2]).** Assume that \(A\) is connected and triangular. The following are equivalent:

(i) \(A\) is strongly simply connected;

(ii) There exists a presentation \((Q, I)\) of \(A\) such that for each connected convex subquiver \(Q'\) of \(Q\) the algebra \(KQ'/(KQ' \cap I)\) has the separation property;

(iii) For all presentations \((Q, I)\) of \(A\) and each connected convex subquiver \(Q'\) of \(Q\) the algebra \(KQ'/(KQ' \cap I)\) has the separation property.

The class of tame strongly simply connected algebras has been studied intensively, see [DS] for some results in this direction.

For further reading on simply connected algebras we refer to [S1, S2].

3. **Examples**

3.1. A finite-dimensional connected path algebra \(KQ\) is simply connected if and only if \(Q\) is a tree.

3.2. The fundamental group \(\Pi_1(Q, I)\) does depend on the presentation of \(A\). The following example is taken from [AD]. Let \(Q\) be the quiver

\[
\begin{array}{c}
1 \\
\downarrow \delta \\
2 \\
\downarrow \gamma \\
3
\end{array}
\]

and let \(I = (\alpha \delta \gamma - \alpha \beta)\) and \(I' = (\alpha \beta)\). Then we have \(KQ/I \cong KQ/I'\), but \(\Pi_1(Q, I)\) is trivial, and \(\Pi_1(Q, I') \cong \mathbb{Z}\).
REFERENCES


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