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2-CALABI-YAU TILTED ALGEBRAS

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CONTENTS

1. 2-Calabi-Yau categories and cluster-tilting objects 1
2. 2-Calabi-Yau tilted algebras 1
3. Cluster-tilted algebras 2
References 2

1. 2-CALABI-YAU CATEGORIES AND CLUSTER-TILTING OBJECTS

Let $\mathcal{T}$ be a triangulated $K$-category such that all morphism spaces in $\mathcal{T}$ are finite-dimensional.

Then $\mathcal{T}$ is a 2-Calabi-Yau category if for all $M, N \in \mathcal{T}$ there is a functorial isomorphism

$$ \text{Ext}^1_\mathcal{T}(M, N) \cong D \text{Ext}^1_\mathcal{T}(N, M). $$

More generally, for $n \geq 0$, the triangulated category $\mathcal{T}$ is called an $n$-Calabi-Yau category if for all $M, N \in \mathcal{T}$ we have a functorial isomorphism

$$ \text{Hom}_\mathcal{T}(M, N) \cong D \text{Hom}_\mathcal{T}(N, M[n]). $$

An object $T$ in a 2-Calabi-Yau category $\mathcal{T}$ is a cluster-tilting object if the following hold:

(i) $\text{Ext}^1_\mathcal{T}(T, T) = 0$;
(ii) If $\text{Ext}^1_\mathcal{T}(T, M) = 0$ for some $M \in \mathcal{T}$, then $M \in \text{add}(T)$.

2. 2-CALABI-YAU TILTED ALGEBRAS

A finite-dimensional $K$-algebra $A$ is a 2-Calabi-Yau tilted algebra if $A \cong \text{End}_\mathcal{T}(T)^{\text{op}}$ for some cluster-tilting object $T$ in a 2-Calabi-Yau category $\mathcal{T}$.

**Theorem 2.1** (Keller, Reiten [KR1]). Let $\mathcal{T}$ be 2-Calabi-Yau, and let $T$ be a cluster-tilting object in $\mathcal{T}$. Then we have an equivalence of categories

$$ \text{Hom}_\mathcal{T}(T, -): \mathcal{T}/\text{add}(\tau_\mathcal{T}(T)) \to \text{mod}(\text{End}_\mathcal{T}(T)^{\text{op}}) $$

where $\tau_\mathcal{T}$ is the Auslander-Reiten translation in $\mathcal{T}$.

It is not known in general if the 2-Calabi-Yau tilted algebra $\text{End}_\mathcal{T}(T)^{\text{op}}$ determines $\mathcal{T}$, see [KR2] for some partial results.
For a finite-dimensional $K$-algebra $A$ let $\text{sub}(A)$ be the subcategory of $\text{mod}(A)$ with objects all modules $M$ which are isomorphic to submodules of some free $A$-module. If $A$ is 1-Iwanaga-Gorenstein, then $\text{sub}(A)$ is the Frobenius category of Gorenstein-projective $A$-modules. In particular, its stable category $\text{sub}(A)$ is a triangulated category.

**Theorem 2.2** (Keller, Reiten [KR1]). For a 2-Calabi-Yau tilted algebra $A$ the following hold:

(i) $A$ is a 1-Iwanaga-Gorenstein algebra.
(ii) $\text{gl. dim}(A) \leq 1$ or $\text{gl. dim}(A) = \infty$.
(iii) The triangulated category $\text{sub}(A)$ is 3-Calabi-Yau.

For further information on 2-Calabi-Yau tilted algebras we refer to Reiten’s excellent survey paper [R].

3. Cluster-tilted algebras

Let $Q$ be an acyclic quiver, and let $C_Q := D^b(\text{mod}(KQ))/\tau_{KQ}^{-1} \circ [1]$ be the cluster category associated with $Q$. Cluster categories were defined in [BMRRT]. Keller proved that $C_Q$ is a triangulated category with all morphism spaces finite-dimensional. Based on this, it is straightforward to check that $C_Q$ is a 2-Calabi-Yau category.

A finite-dimensional $K$-algebra $A$ is a cluster-tilted algebra if $A \cong \text{End}_{C_Q}(T)^{\text{op}}$ for some cluster-tilting object $T$ in some cluster category $C_Q$.

Obviously, cluster-tilted algebras are 2-Calabi-Yau tilted algebras. Cluster tilted algebras have been introduced and studied in [BMR].

**REFERENCES**


