11. Übungsaufgaben Darstellungstheorie I, WS 06/07

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Let V be a module and let U be a submodule of V. The submodule U is **large** in V if $U \cap U' \neq 0$ for all non-zero submodules U' of V. The submodule U is **small** in V if $U + U' \subset V$ for all proper submodules U' of V.

1. Find these two definitions in some book or article. What is the oldest reference you can find?

2. Prove the following lemma:

Lemma 0.1. *Let U_1 and U_2 be submodules of a module V. If U_1 and U_2 are large in V, then $U_1 \cap U_2$ is large in V. If U_1 and U_2 are small in V, then $U_1 + U_2$ is small in V.*

3. Find non-trivial and instructive examples of large and small submodules. Are small submodules really "small", and are large submodules really "large"?

Classify the small submodules of $(K[T], T\cdot)$ and $N(\infty)$.

Find an example of a module V and a submodule U of V such that U is large and small in V.

4. Prove the following lemma:

Lemma 0.2. *For $1 \leq i \leq n$ let U_i be a submodule of a module V_i. Set $U = U_1 \oplus \cdots \oplus U_n$ and $V = V_1 \oplus \cdots \oplus V_n$. U is large in V if and only if U_i is large in V_i for all i. U is small in V if and only if U_i is small in V_i for all i.*

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Let $G = \mathbb{Z}_2$ be the group with two elements, and let K be a field.

5. Assume $\text{char}(K) \neq 2$. Show: Up to isomorphism there are exactly two simple $K[G]$-modules.

6. Assume $\text{char}(K) = 2$. Show: Up to isomorphism there are exactly two indecomposable $K[G]$-modules, and one of them is not simple.

7. Assume $\text{char}(K) = 2$. Construct an infinite number of 2-dimensional pairwise non-isomorphic representations of $K[G \times G]$.