Construction problems in algebraic geometry and the Schottky problem

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Summary

In this thesis we study some problems concerning the topology and Hodge theory of smooth complex projective varieties and the geometry of theta divisors of principally polarized abelian varieties. The thesis contains four parts; the results have appeared in [74] and [76] and the preprints [75] and [77].

What are the possible Hodge numbers of a smooth complex projective variety? In the first part of this thesis, we construct enough varieties to show that many of the Hodge numbers can take all possible values satisfying the constraints given by Hodge theory. For example, the k-th cohomology group of a smooth complex projective variety in dimension $n \geq k+1$ can take arbitrary Hodge numbers if k is odd; the same result holds for k even as long as the middle Hodge number is larger than some quadratic bound in k. Our results answer questions of Kollár and Simpson formulated in [84].

The second part of this thesis is based on joint work with Tasin. We produce the first examples of smooth manifolds which admit infinitely many complex algebraic structures such that certain Chern numbers are unbounded. Our examples allow us to determine all Chern numbers of smooth complex projective varieties of dimensions ≥ 4 that are bounded by the underlying smooth manifold. Using bordism theory we also obtain an upper bound on the dimension of the space of linear combinations of Chern numbers with that property. Our results answer a question of Kotschick [45].

In the third part we study the Hodge structures of conjugate varieties X and X^σ, where X^σ is obtained from X by applying some field automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$ to the coefficients of the defining equations of X. We consider the K-algebra $H^{*,*}(X,K)$ of K-rational (p,p)-classes in Betti cohomology, where $K \subseteq \mathbb{C}$ denotes some subfield of the complex numbers. For all subfields $K \subseteq \mathbb{C}$ with $K \neq \mathbb{Q}$ and $K \neq \mathbb{Q}[\sqrt{-d}]$, $d \in \mathbb{N}$, we show that there are conjugate varieties X, X^σ with

$$H^{*,*}(X,K) \ncong H^{*,*}(X^\sigma,K).$$

This result is motivated by the Hodge conjecture, which predicts isomorphisms between $H^{*,*}(X,K)$ and $H^{*,*}(X^\sigma,K)$ for $K=\mathbb{Q}$ and $K=\mathbb{Q}[\sqrt{-d}]$. Concerning the topology of conjugate varieties, we produce in each birational equivalence class of dimension at least 10 two conjugate smooth complex projective varieties which are nonhomeomorphic. It follows that nonhomeomorphic con-

jugate varieties exist for all fundamental groups. This answers a question of Reed [67], who asked for simply connected examples.

In the fourth part of this thesis, we study the Schottky problem, which asks for criteria that decide whether a principally polarized abelian variety (ppav) (A,Θ) is isomorphic to the Jacobian $(J(C),\Theta_C)$ of a smooth projective curve C. By Riemann's theorem, the theta divisor Θ_C of the Jacobian of a smooth genus g curve can be written as the $(g-1)$-fold sum of an Abel–Jacobi embedded copy of C in $J(C)$, $\Theta_C = C+\dots+C$. We prove the following converse: let (A,Θ) be an indecomposable ppav with $\Theta = C+Y$, where C and Y are a curve and a codimension two subvariety in A respectively. Then C is smooth, (A,Θ) is isomorphic to the Jacobian of C and Y corresponds to a translate of the Brill–Noether locus $W_{g-2}(C)$. Slightly weaker versions of this result have previously been conjectured by Little [56] and Pareschi–Popa [60]. As an application, we deduce that an irreducible theta divisor is dominated by a product of curves if and only if the corresponding ppav is isomorphic to the Jacobian of a smooth curve. This solves a problem of Schoen [70].

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1 Introduction

This thesis consists of four parts, based on [74], [75], [76] and [77]. Each part constitutes a chapter[1] and contains a separate abstract and introduction. Moreover, each part can be read individually.

In this chapter we give a global introduction, point out the relations among the different parts of this thesis and present some supplementary material. For clarity, we divide the introduction into four sections.

1.1 Hodge numbers of algebraic varieties

Hodge theory is one of the most powerful tools in complex algebraic geometry. It relies on the Hodge decomposition

$$H^k(X,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X),$$

which holds for any Kähler manifold[2] X. Here,

$$H^{p,q}(X) \simeq H^q(X,\Omega_X^p)$$

corresponds to the subspace of $H^k(X,\mathbb{C})$ which (in de Rham cohomology) can be represented by closed (p,q)-forms.

The most basic invariants from Hodge theory are the Hodge numbers

$$h^{p,q}(X) := \dim H^{p,q}(X)$$

of an n-dimensional Kähler manifold X, where $0 \leq p,q \leq n$. Complex conjugation and Serre duality show that these numbers satisfy the Hodge symmetries

$$h^{p,q}(X) = h^{q,p}(X) = h^{n-p,n-q}(X). \tag{1.1}$$

Moreover, the Hard Lefschetz theorem implies

$$h^{p-1,q-1}(X) \leq h^{p,q}(X) \quad \text{for all} \quad p+q \leq n. \tag{1.2}$$

[1] Chapter 2 is based on [76], Chapter 3 is based on joint work with Tasin [77], Chapter 4 is based on [74] and Chapter 5 is based on [75].

[2] In this thesis, the term Kähler manifold refers to a compact connected complex manifold with Kähler metric.

The Hodge numbers of an n-dimensional Kähler manifold are usually assembled in the Hodge diamond as follows.

$$\begin{array}{ccccccccc} & & & & h^{n,n} & & & & \\ & & & h^{n,n-1} & & h^{n-1,n} & & & \\ & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\ & ⋰ & & & \vdots & & & \ddots & \\ h^{n,0} & & & & & & & & h^{0,n} \\ & \ddots & & & \vdots & & & ⋰ & \\ & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\ & & & h^{1,0} & & h^{0,1} & & & \\ & & & & h^{0,0} & & & & \end{array} \tag{1.3}$$

In order to get some ideas how the Hodge diamond of a smooth complex projective variety or a Kähler manifold might look like in practice, let us look at some examples.

The following table illustrates the Hodge diamond of a smooth projective curve of genus g:

$$\begin{array}{ccc} & 1 & \\ \mathrm{g} & & \mathrm{g} \\ & 1 & \end{array}$$

This describes all possible Hodge diamonds in dimension one; such a classification is open in all other dimensions.

Another interesting example is given by an n-dimensional abelian variety A. The Hodge numbers of such a variety are given by

$$h^{p,q}(A) = \binom{n}{p} \cdot \binom{n}{q}.$$

For instance, the following table shows the Hodge diamond of an abelian surface and an abelian threefold respectively.

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 3 & & \\ & 3 & & 9 & & 3 & \\ 1 & & 9 & & 9 & & 1 \\ & 3 & & 9 & & 3 & \\ & & 3 & & 3 & & \\ & & & 1 & & & \end{array}$$

The third family of examples we want to mention here are smooth degree d hypersurfaces $X_{d,n}$ in $\mathbb{P}^{n+1}$. By the Lefschetz hyperplane theorem,

$$h^{p,q}(X_{d,n}) = h^{p,q}(\mathbb{P}^{n+1})$$

for all $p+q \leq n-1$. Moreover, the Hodge numbers $h^{n-p,p}(X_{d,n})$ can be calculated, either via residues and Griffiths' work [90, Sec. 6], or via a certain Chern number calculation involving the Hirzebruch–Riemann–Roch formula for Ω_X^p, see (1.5) below. For instance, the surface $X_{d,2}$ has weight 2 Hodge numbers

$$h^{2,0}(X_{d,2}) = \binom{d-1}{3},$$

and

$$h^{1,1}(X_{d,2}) = \frac{1}{3} \cdot (2d^3 - 6d^2 + 7d).$$

Moreover, the threefold $X_{d,3}$ has weight 3 Hodge numbers

$$h^{3,0}(X_{d,3}) = \binom{d-1}{4},$$

and

$$h^{2,1}(X_{d,3}) = \frac{1}{24} \cdot (11d^4 - 50d^3 + 85d^2 - 70d + 24).$$

The following illustrates the special case of the Hodge diamonds of $X_{10,2}$ and $X_{10,3}$ respectively.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 84 & & 490 & & 84 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 126 & & 2826 & & 2826 & & 126 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

In Chapter 2 we study the question which collections of natural numbers $(h^{p,q})_{p,q}$, satisfying the Hodge symmetries (1.1) and the Lefschetz conditions (1.2), can actually be realized by a smooth complex projective variety. For some partial results in dimensions two and three we refer to [6, 14, 37, 61, 69].

In his survey article on the construction problem in Kähler geometry [84], Simpson raises many aspects of the construction problem for Hodge numbers. For instance, Kollár and Simpson ask whether the outer Hodge numbers are always dominated by the middle Hodge numbers, which seemingly agrees with all known examples, such as those given above. More specifically [84, p. 9]: is it possible to realize a vector

$$(h^{k,0}, \dots, h^{0,k}) \tag{1.4}$$

of weight k Hodge numbers with large numbers at the end and small numbers in the middle?

The Hodge numbers of a smooth complex projective variety are known to reflect many of its geometric properties. For instance, if

$$h^{2,0}(X) \leq h^{1,0}(X) - 2,$$

then X fibers over a smooth curve of genus ≥ 2 by Castelnuovo–de Francis' lemma; generalizations concerning fibrations over higher dimensional bases were given by Catanese [10]. Only recently, Lazarsfeld–Popa [51] and Lombardi [57] found many more inequalities among the Hodge numbers of large classes of irregular complex projective varieties. Most (but not all) of these inequalities involve the outer Hodge numbers $h^{p,0}$.

Besides determining the Betti numbers, the Hodge numbers may also restrict the ring structure of $H^*(X,\mathbb{C})$, hence the topology of the underlying smooth manifold. Indeed, the Hodge decomposition is compatible with the cup product which induces a linear map

$$H^k(X,\mathbb{C}) \otimes H^m(X,\mathbb{C}) \longrightarrow H^{k+m}(X,\mathbb{C}).$$

The kernel of this map has therefore dimension at least

$$\sum_{\substack{p+q=k \\ r+s=m}} \max(0, h^{p,q} \cdot h^{r,s} - h^{p+r,q+s}).$$

For instance, if a variety X has large $h^{2,0}$, whereas $h^{1,1}$ and $h^{4,0}$ are both small, then the linear map

$$H^2(X,\mathbb{C}) \otimes H^2(X,\mathbb{C}) \longrightarrow H^4(X,\mathbb{C}),$$

induced by the cup product, has a large kernel.

By the Hirzebruch–Riemann–Roch formula, the Euler characteristics

$$\chi^p(X) := \chi(X,\Omega_X^p) = \sum_i (-1)^i h^{p,i}(X) \tag{1.5}$$

can be expressed in terms of Chern numbers of X. Since Chern numbers tend to satisfy certain inequalities, the relations among Hodge and Chern numbers are one source of potential inequalities among the Hodge numbers of smooth complex projective varieties.

For instance, using the Bogomolov–Miyaoka–Yau inequality, we observed in [72] that

$$h^{1,1}(S) > h^{2,0}(S) \tag{1.6}$$

for all Kähler surfaces S, see also [76, Prop. 22]. Combining a similar approach in dimension four with Kollár–Matsusaka's theorem, we also proved [72] that the third Betti number b_3 of a smooth complex projective fourfold with $b_2 = 1$ can be bounded from above in terms of b_4, see also [76, Prop. 32]. These results show that the known constraints which Hodge theory puts on the Hodge and Betti numbers of a smooth complex projective variety are not complete.

Chapter 2 contains several main results on the construction problem for Hodge numbers. The first one answers Kollár–Simpson's question about the realizability of certain weight k Hodge numbers (1.4) by a variety.

Theorem 1.1.1 (Theorem 2.1.1)**.** *Fix $k \geq 1$ and let $(h^{p,q})_{p+q=k}$ be a symmetric collection of natural numbers. If $k = 2m$ is even, we assume*

$$h^{m,m} \geq m \cdot \lfloor (m+3)/2 \rfloor + \lfloor m/2 \rfloor^2 .$$

Then in each dimension $n \geq k+1$ there exists a smooth complex projective variety whose Hodge structure of weight k realizes the given Hodge numbers.

The examples which realize the given weight k Hodge numbers in Theorem 1.1.1 have dimension $n \geq k+1$. At least for $k = 2$, this assumption on the dimension is necessary by (1.6). However, if the $h^{p,q}$ in Theorem 1.1.1 are even and $h^{k,0} = 0$, then a similar result as in Theorem 1.1.1 also holds for the k-th cohomology group in dimension $n = k$, see Corollary 2.5.3. Theorem 1.1.1 might be surprising, as it is known [89, Rem. 10.20] that a very general integral polarized Hodge structure of weight ≥ 2 (not of K3 type) cannot be realized by a variety.

The second main result solves the construction problem for large subcollections of Hodge numbers of the whole Hodge diamond.

Theorem 1.1.2 (Theorem 2.1.3)**.** *Fix $n \geq 1$ and let $(h^{p,q})_{p+q<n}$ be a collection of natural numbers with $h^{p,q} = h^{q,p}$, $h^{p-1,q-1} \leq h^{p,q}$ and $h^{0,0} = 1$. Suppose that the following two additional conditions are satisfied.*

1. *For $p < n/2$, the primitive numbers $l^{p,p} := h^{p,p} - h^{p-1,p-1}$ satisfy*

$$l^{p,p} \geq p \cdot (n^2 - 2n + 5)/4.$$

2. *The outer numbers $h^{k,0}$ vanish either for all $k = 1, \dots, n-3$, or for all $k \neq k_0$ for some $k_0 \in \{1, \dots, n-1\}$.*

Then there exists an n-dimensional smooth complex projective variety X with

$$h^{p,q}(X) = h^{p,q},$$

for all p and q with $p+q<n$.

For instance, Theorem 1.1.2 implies that any given collection of natural numbers which lies neither on the boundary nor on the horizontal middle axis of (1.3) and which satisfies the Hodge symmetries (1.1) and the Lefschetz conditions (1.2) can be realized by a smooth complex projective variety as long as the primitive (p,p)-type Hodge numbers are bounded from below by some constant which depends only on p and n and not on the given collection $(h^{p,q})_{p+q<n}$. In this result, we can additionally choose one Hodge number $h^{k,0}$ to be arbitrary; all other outer Hodge numbers $h^{p,0}$ with $p \neq \{0,k,n\}$ vanish in our examples.

Theorem 1.1.2 has interesting consequences concerning possible universal inequalities[3] among the Hodge numbers of smooth complex projective varieties.

Corollary 1.1.3 (Corollary 2.10.2)**.** *Any universal inequality among the Hodge numbers below the horizontal middle axis in (1.3) of n-dimensional smooth complex projective varieties is a consequence of the Lefschetz conditions (1.2).*

Corollary 1.1.3 implies for instance that the Lefschetz conditions (1.2) are the only universal inequalities which hold in all sufficiently large dimensions at the same time.

A vector of natural numbers $(b_0,\dots,b_{2n}) \in \mathbb{N}^{2n}$ is called vector of formal Betti numbers (in dimension n), if

$$b_0 = 1, \quad b_k = b_{2n-k}, \quad \text{and} \quad b_{2k+1} \equiv 0 \mod 2,$$

for all k. Theorem 1.1.2 implies that under a mild lower bound on the primitive even degree Betti numbers, the Betti numbers b_k with $k \neq n$ of any formal vector of Betti numbers in dimension n can be realized by a variety.

Corollary 1.1.4 (Corollary 2.1.4)**.** *Let $(b_0,\dots,b_{2n})$ be a vector of formal Betti numbers with*

$$b_{2k} - b_{2k-2} \geq k \cdot (n^2 - 2n + 5)/8 \quad \textit{for all } k < n/2.$$

Then there exists an n-dimensional smooth complex projective variety X with $b_k(X) = b_k$ for all $k \neq n$.

Corollary 1.1.4 implies for instance that in even dimensions, the construction problem for the odd Betti numbers is solvable without any additional assumptions.

For the proof of Theorems 1.1.1 and 1.1.2 we establish a method which allows us to manipulate single Hodge numbers below the horizontal middle

[3]The term "universal inequality" underlines that we are looking for inequalities which hold for all smooth complex projective varieties.

axis and away from the boundary of (1.3) in a very efficient way, see Section 2.4.2. Our method uses the Lefschetz hyperplane theorem and so we are not able to control the weight n Hodge numbers of our n-dimensional examples.

Motivated by work of Cynk and Hulek [17], we establish another construction method which is based on a careful resolution of certain quotient singularities. This allows us to produce n-dimensional examples with interesting weight n Hodge numbers. For instance, we prove that for any $i = 0, \ldots, \lfloor n/2 \rfloor$, there is an n-dimensional smooth complex projective variety X such that

$$h^{n-i,i}(X) = h^{i,n-i}(X)$$

is arbitrarily large, whereas $h^{p,q}(X) = 0$ for all other $p \neq q$, see Theorem 2.8.1. Taking products of these examples with projective spaces yields the following.

Corollary 1.1.5 (Corollary 2.10.3)**.** *Any universal inequality among the Hodge numbers away from the vertical middle axis in (1.3) of n-dimensional smooth complex projective varieties is a consequence of the Lefschetz conditions (1.2).*

The above corollary determines all universal inequalities among the Hodge numbers $h^{p,q}$ with $p \neq q$ in fixed dimension n. This should be compared to Corollary 1.1.3, which is the mirrored statement.

Combining all our constructed examples with recent work of Roulleau and Urzúa [69], we are able to determine all possible dominations among two Hodge numbers in a fixed dimension.

Corollary 1.1.6 (Corollary 2.9.1)**.** *Suppose there are $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$ such that for all smooth complex projective varieties X of dimension n:*

$$\lambda_1 h^{r,s}(X) + \lambda_2 \geq h^{p,q}(X). \tag{1.7}$$

Then $\lambda_1 \geq 1$ and (1.7) is either a consequence of the Lefschetz conditions (1.2), or $n = 2$ and it is a consequence of (1.6).

Let us explain why in our approach to Theorems 1.1.1 and 1.1.2, lower bounds on the primitive (p,p)-type Hodge numbers $l^{p,p}$ are necessary. The reason comes from the existence of the cycle class map

$$\mathrm{cl}_p : \mathrm{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H^{2p}(X, \mathbb{Q}), \tag{1.8}$$

whose image is contained in the group of rational Hodge classes

$$H^{p,p}(X, \mathbb{Q}) := H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q}).$$

It follows for instance that the Picard number $\rho(X)$ is bounded from above by $h^{1,1}(X)$. More generally, if $h^{p,p}(X)$ is small then the image of the codimension

p cycles of X in cohomology is small. In our constructions we need to introduce one additional line bundle for each Hodge number that we want to manipulate. Certain intersection products of these line bundles are nonzero in cohomology, which explains the bound on $l^{p,p}$ in our approach; we do not know if such bounds are necessary in general.

Using a rather ad hoc implementation of the Godeaux–Serre construction, we are able to prove that at least for the weight 2 Hodge structure, the optimal bound $h^{1,1} \geq 1$ can be reached. That is, any weight two Hodge numbers $(h^{2,0}, h^{1,1}, h^{0,2})$ with $h^{2,0} = h^{0,2}$ and $h^{1,1} \geq 1$ can be realized by a smooth complex projective variety of dimension $n \geq 3$, see Theorem 2.7.1. However, the method used in that proof is not very flexible and cannot easily be generalized to Hodge structures of higher weight.

As mentioned above, the lower bound on $h^{p,p}$ in Theorem 1.1.1 stems from the existence of certain algebraic classes in $H^{2p}(X, \mathbb{Q})$. These classes form a rational sub-Hodge structure. Taking the orthogonal complement, it follows that any symmetric vector $(h^{k,0}, \dots, h^{0,k})$ of natural numbers can be realized by the Hodge numbers of some rational sub-Hodge structure

$$V \subseteq H^k(X, \mathbb{Q}),$$

where X is a smooth complex projective variety of dimension $n \geq k + 1$, see Corollary 2.5.1. This statement is the main result of Arapura's paper [3], which was written after the preprint version of [76] appeared.

1.2 Chern numbers of algebraic structures

The Hodge numbers are in general not topological invariants of the underlying smooth manifold [48]. However, due to the Hodge decomposition, the Hodge numbers of a smooth complex algebraic variety are bounded from above by the Betti numbers and so they are determined up to finite ambiguity by the underlying smooth manifold.

Similarly, the Chern numbers of a smooth complex projective variety are in general not determined by the underlying smooth manifold [46, 47]. However, in contrast to the case of Hodge numbers, it was not known whether the Chern numbers are determined up to finite ambiguity by the underlying smooth manifold. This boundedness question was raised by Kotschick in [45].

For instance, the Chern numbers c_n and $c_1 c_{n-1}$ are linear combinations of Hodge numbers [54], hence bounded by the underlying smooth manifold. In particular, the Chern numbers of smooth complex projective surfaces are bounded by the underlying smooth manifold. Moreover, Kotschick observed that the Chern numbers of minimal smooth projective three- and fourfolds of general type are also bounded by the underlying smooth manifold.

In a recent preprint [9], Cascini and Tasin use the above boundedness result and the minimal model program in dimension three to prove that many smooth complex projective threefolds of general type have all their Chern numbers bounded by the underlying smooth manifold.

In [42], Kollár proved that a smooth manifold with second Betti number $b_2 = 1$ carries at most finitely many different deformation equivalence classes of complex algebraic structures. Since Chern numbers are deformation invariants, it follows that the Chern numbers of a smooth complex projective variety with $b_2 = 1$ are determined by the underlying smooth manifold up to finite ambiguity.

Kollár's result does not generalize to varieties with arbitrary second Betti number. Indeed, Freedman and Morgan [28] gave an example of a smooth 8-manifold carrying infinitely many complex algebraic structures such that some of its Chern classes are unbounded; however, the Chern numbers of their examples are indeed bounded.

Chapter 3 is based on joint work with Tasin [77]. We produce the first examples of smooth manifolds such that certain Chern numbers with respect to all possible complex algebraic structures are unbounded. Our construction works in all complex dimensions at least four; it is flexible enough allowing us to determine all partitions $\mathfrak{m}$ of n such that the Chern number $c_{\mathfrak{m}}$ in complex dimension $n \geq 4$ is bounded by the underlying smooth manifold.

Theorem 1.2.1 (Theorem 3.1.1)**.** *In complex dimension* 4*, the Chern numbers* c_4*,* c_1c_3 *and* c_2^2 *of a smooth complex projective variety are the only Chern numbers* $c_{\mathfrak{m}}$ *which are determined up to finite ambiguity by the underlying smooth manifold. In complex dimension* $n \geq 5$*, only* c_n *and* c_1c_{n-1} *are determined up to finite ambiguity by the underlying smooth manifold.*

For instance, we find that for fourfolds, c_1^4 and $c_1^2c_2$ are not bounded by the underlying smooth 8-manifold. This might be surprising, as we recall that at least the Chern numbers of minimal fourfolds of general type are known to be bounded. This compares to our result as all of our examples are of negative Kodaira dimension.

In view of Theorem 1.2.1, very few Chern numbers of smooth complex projective varieties are determined up to finite ambiguity by the underlying smooth manifold. This changes considerably if we are asking for all linear combinations of Chern numbers with that property. Indeed, the Euler characteristics χ^p (see (1.5) above) as well as the Pontryagin numbers in even complex dimension are linear combinations of Chern numbers which are bounded by the underlying smooth manifold.

When studying linear combinations of Chern numbers, it is most convenient to work with the rational complex cobordism ring $\Omega_*^U \otimes \mathbb{Q}$. Its degree n part $\Omega_n^U \otimes \mathbb{Q}$ is the group of rational bordism classes of stably almost complex manifolds of real dimension $2n$. This group is known to be generated by smooth complex projective varieties. The Chern numbers in complex dimension n are well-defined linear forms on $\Omega_n^U \otimes \mathbb{Q}$, which yield in fact a basis of the dual space of $\Omega_n^U \otimes \mathbb{Q}$, see [86, p. 117].

Due to the work of Novikov and Milnor [86, p. 128], $\Omega_*^U \otimes \mathbb{Q}$ is a polynomial ring with one generator in each degree. Moreover, a (stably almost) complex manifold X of real dimension $2n$ can be taken as generator in degree n if and only if its Milnor number $s_n(X)$ is nonzero.

In Section 3.6, we consider a sequence $(\alpha_n)_{n\geq 1}$ of smooth complex projective varieties, given by $\alpha_1 = \mathbb{P}^1$, $\alpha_2 = \mathbb{P}^2$ and

$$\alpha_n := \mathbb{P}(\mathcal{O}_A(1) \oplus \mathcal{O}_A^{n-3}),$$

where A denotes an abelian surface with ample line bundle $\mathcal{O}_A(1)$ and α_n is the projectivization of the rank $n-2$ vector bundle $\mathcal{O}_A(1) \oplus \mathcal{O}_A^{n-3}$ on A. Using Lemma 2.3 in [73], one computes $s_n(\alpha_n) \neq 0$ and so we have a sequence of ring generators:

$$\Omega_*^U \otimes \mathbb{Q} = \mathbb{Q}[\alpha_1, \alpha_2, \ldots].$$

Using this presentation, we consider the ideal

$$\mathcal{I}^* := \langle \alpha_1\alpha_k \mid k \geq 3 \rangle$$

in $\Omega_*^U \otimes \mathbb{Q}$, generated by all $\alpha_1\alpha_k$ with $k \geq 3$. The degree n part of this ideal is denoted by $\mathcal{I}^n$.

Our second main result in Chapter 3 is as follows.

Theorem 1.2.2 (Theorem 3.6.1). *Any linear combination of Chern numbers in dimension n, which on smooth complex projective varieties is bounded by the underlying smooth manifold vanishes on $\mathcal{I}^n$.*

By Theorem 1.2.2, any linear combination of Chern numbers in dimension n which on smooth complex projective varieties is bounded by the underlying smooth manifold descends to the quotient

$$(\Omega_n^U \otimes \mathbb{Q})/\mathcal{I}^n. \tag{1.9}$$

Denoting by $p(n)$ the number of partitions of n by positive integers, we therefore obtain the following.

Corollary 1.2.3 (Corollary 3.6.3). *In dimension $n \geq 4$, the space of rational linear combinations of Chern numbers which on smooth complex projective varieties are bounded by the underlying smooth manifold is a quotient of the dual space of (1.9); its dimension is therefore at most*

$$\dim(\Omega_n^U \otimes \mathbb{Q}) - \dim(\mathcal{I}^n) = p(n) - p(n-1) + \left\lfloor \frac{n+1}{2} \right\rfloor.$$

In order to compare the above upper bound with the known lower bound, given by the Euler characteristics χ^p and the Pontryagin numbers in even complex dimensions, we consider the ideal

$$\mathcal{J}^* := \langle \alpha_{2k+1} \mid k \geq 1 \rangle + \langle \alpha_1 \alpha_{2k} \mid k \geq 2 \rangle$$

in $\Omega_*^U \otimes \mathbb{Q}$. We explain in Section 3.6 that the degree n part $\mathcal{J}^n$ is the kernel of the span of the Euler characteristics χ^p and Pontryagin numbers. That is, the dual space of the quotient

$$(\Omega_n^U \otimes \mathbb{Q})/\mathcal{J}^n$$

is naturally isomorphic to the span of the Euler characteristics and the Pontryagin numbers in dimension n.

We note that

$$\mathcal{I}^4 = \mathcal{J}^4.$$

By Theorem 1.2.2, any linear combination of Chern numbers which on smooth complex projective fourfolds is bounded by the underlying smooth manifold is therefore a linear combination of the Euler characteristics χ^p and the Pontryagin numbers.

Conversely, the inclusion

$$\mathcal{I}^* \subseteq \mathcal{J}^*$$

is proper for all $n \neq 4$, and so the problem of determining all bounded linear combinations remains open in all dimensions $n \geq 3$ other than $n = 4$.

1.3 Hodge structures of conjugate varieties

Let X denote a smooth complex projective variety. For a field automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$ of the complex numbers, we consider the conjugate variety X^σ, defined by the base change

$$\begin{tikzcd} X^\sigma \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "\sigma^*"] & \mathrm{Spec}(\mathbb{C}). \end{tikzcd}$$

That is, X^σ is the smooth variety whose defining equations in some projective space are given by applying σ to the coefficients of the equations of X. Algebraically defined invariants, such as étale cohomology or the algebraic fundamental group coincide on X and X^σ. Conversely, Serre [78] produced the first examples of conjugate varieties with different topological fundamental groups; many more examples of nonhomeomorphic conjugate varieties were given later [1, 7, 62, 83].

In 2009, Charles proved the following.

Theorem 1.3.1 (Charles [12])**.** *There exist conjugate smooth complex projective varieties with distinct real cohomology algebras.*

Charles' result might be surprising, as the ℓ-adic and hence also the complex cohomology algebras of conjugate smooth complex projective varieties are isomorphic.

The cycle class maps (1.8) fit together to yield a homomorphism of graded $\mathbb{Q}$-algebras

$$\mathrm{cl}_* : \mathrm{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H^{2*}(X, \mathbb{Q}).$$

Although the target of the above map cannot be computed algebraically, its kernel is still an algebraic invariant of X. In order to see this it suffices to note that $\ker(\mathrm{cl}_*)$ is a rational subspace and

$$\ker(\mathrm{cl}_*) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = \ker(\mathrm{cl}_* \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$$

can be computed via the cycle class map in étale cohomology with coefficients in $\mathbb{Q}_\ell$.

Since $\mathrm{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\ker(\mathrm{cl}_*)$ can be computed algebraically, the same holds for (the isomorphism type of) the image $\mathrm{im}(\mathrm{cl}_*)$. The Hodge conjecture identifies the latter with the algebra of rational (p,p)-classes, which is a priori a highly transcendental invariant of X.

Hodge Conjecture. *Let X be a smooth complex projective variety. Then*

$$\mathrm{im}(\mathrm{cl}_*) = H^{*,*}(X, \mathbb{Q}) := H^{*,*}(X) \cap H^{2*}(X, \mathbb{Q}).$$

The Hodge conjecture implies that the isomorphism type of $H^{*,*}(-, \mathbb{Q})$ coincides on conjugate varieties. This implication of the Hodge conjecture goes back to Deligne; its validity or falsity might be easier to check because it is a purely Hodge theoretic statement which does not refer anymore to algebraic cycles.

The above discussion motivates the investigation of the K-algebra

$$H^{*,*}(X, K) := H^{2*}(X, K) \cap H^{*,*}(X)$$

of K-rational (p,p)-classes in Betti cohomology, where $K \subseteq \mathbb{C}$ denotes some subfield of the complex numbers. If $K = \mathbb{Q}[\sqrt{-d}]$ is an imaginary quadratic extension of $\mathbb{Q}$, then $H^{*,*}(X,K)$ is obtained from $H^{*,*}(X,\mathbb{Q})$ by extension of scalars and so the Hodge conjecture predicts

$$H^{*,*}(X,K) \simeq H^{*,*}(X^\sigma,K), \tag{1.10}$$

for $K = \mathbb{Q}$ or $K = \mathbb{Q}[\sqrt{-d}]$ and all $\sigma \in \mathrm{Aut}(\mathbb{C})$.

In Chapter 4 we prove that for all remaining subfields $K \subseteq \mathbb{C}$, the isomorphism in (1.10) may fail.

Theorem 1.3.2 (Theorem 4.1.3)**.** *Let $K \subseteq \mathbb{C}$ be a subfield, different from $\mathbb{Q}$ and different from any imaginary quadratic extension $\mathbb{Q}[\sqrt{-d}]$ of $\mathbb{Q}$. Then there exist conjugate smooth complex projective varieties whose graded algebras of K-rational (p,p)-classes are not isomorphic.*

Theorem 1.3.2 is already interesting for $K = \mathbb{C}$, as it shows that the complex Hodge structure on the complex cohomology algebra of varieties is not an algebraic invariant. This contrasts the fact that as bigraded ring (and not as $\mathbb{C}$-algebra),

$$\bigoplus_{p,q} H^{p,q}(X) = \bigoplus_{p,q} H^q(X,\Omega_X^p)$$

is clearly an algebraic invariant of X.

The proof of Theorem 1.3.2 is divided into two parts. Firstly, if $K \subseteq \mathbb{C}$ in Theorem 1.3.2 is different from $\mathbb{R}$ and $\mathbb{C}$, then we prove that there are conjugate smooth complex projective varieties X and X^σ whose groups of K-rational (p,p)-classes have different dimensions

$$H^{p,p}(X,K) \ncong H^{p,p}(X^\sigma,K), \tag{1.11}$$

see Theorem 4.1.5.

In order to explain the idea of the proof of that statement, let us first look at an elliptic curve E. Such a curve can be embedded as a plane curve in $\mathbb{P}^2$ with affine equation

$$\left\{y^2 = 4x^3 - g_2x - g_3\right\}. \tag{1.12}$$

This description is useful if we want to compute the conjugate curve E^σ: we simply apply σ to the coefficients g_2 and g_3.

Calculating the Hodge structure of E is equivalent to finding an element in the upper half plane $\tau \in \mathbb{H}$ with

$$E \simeq \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}).$$

The j-invariant $j(E) := j(\tau)$ is a modular form on $\mathbb{H}$ which determines the isomorphism type of E uniquely. In terms of the affine equation (1.12),

$$j(\tau) = 1728 \cdot \frac{g_2^3}{g_2^3 - 27g_3^2}.$$

Hence, the conjugate curve E^σ has j-invariant $j(E^\sigma) = \sigma(j(E))$. It follows that all elliptic curves E with transcendental j-invariant lie in the same $\mathrm{Aut}(\mathbb{C})$-orbit.

In order to prove (1.11), one could now try to use products of elliptic curves whose j-invariants are algebraically independent over $\mathbb{Q}$ and prove that among such products, there are always two examples whose groups of K-rational (p,p)-classes have different dimensions. This approach works well for special classes of subfields $K \subseteq \mathbb{C}$. In general, difficulties arise since it is very hard to control explicitly for which elements in the upper half plane, the corresponding j-invariants are transcendental over $\mathbb{Q}$. We circumvent these difficulties by the use of abelian surfaces. Their moduli are parametrized by Riemann's second order theta constants and we are able to prove the necessary (and elementary) transcendence results for these modular forms.

It is worth noting that in order to prove (1.11) for all subfields $K \subseteq \mathbb{C}$ different from $\mathbb{Q}$, $\mathbb{Q}[\sqrt{-d}]$, $\mathbb{R}$ and $\mathbb{C}$, the use of varieties defined over transcendental extensions of $\mathbb{Q}$ is necessary. Indeed, there are only countably many varieties defined over $\overline{\mathbb{Q}}$, and so there is a countably generated subfield $K_0 \subseteq \mathbb{C}$ such that

$$H^{p,p}(X, K_0) \otimes_{K_0} \mathbb{C} \simeq H^{p,p}(X, \mathbb{C}),$$

for all smooth complex projective varieties X that can be defined over $\overline{\mathbb{Q}}$, see Remark 4.3.5. Since the Hodge numbers are algebraic invariants of X, it follows that

$$H^{p,p}(X, K_0) \simeq H^{p,p}(X^\sigma, K_0),$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C})$ and all smooth complex projective varieties X that can be defined over $\overline{\mathbb{Q}}$.

The second part of the proof of Theorem 1.3.2 deals with $K = \mathbb{R}$ or $K = \mathbb{C}$. For such K, the algebras of K-rational (p,p)-classes of conjugate varieties X and X^σ are isomorphic in each degree and so it is really the ring structure that matters. Here we use the Charles–Voisin method [12, 91], which we briefly recall in the following.

In [12], Charles starts with an abelian variety $A \subseteq \mathbb{P}^N$ with special endomorphisms and considers the blow-up X of

$$A \times A \times \mathbb{P}^N$$

along the graphs of certain morphisms between two factors, such as some endomorphism $f : A \longrightarrow A$. For certain field automorphisms $\sigma \in \mathrm{Aut}(\mathbb{C})$, the abelian varieties A and A^σ are isomorphic and σ maps the graph of the given endomorphism f to the graph of the conjugate endomorphism f^σ. That is, X and X^σ are both blow-ups of $A \times A \times \mathbb{P}^N$, but the endomorphism f whose graph was blown-up may have changed. The key point, already used in Voisin's solution of the Kodaira problem [91], is the fact that the real cohomology algebra of X encodes the action of the endomorphism f^* on $H^*(A, \mathbb{R})$. Roughly speaking, if f^* and $(f^\sigma)^*$ act differently on $H^*(A, \mathbb{R})$, then $H^*(X, \mathbb{R})$ and $H^*(X^\sigma, \mathbb{R})$ are nonisomorphic, hence Charles' result in Theorem 1.3.1.

In the situation of Theorem 1.3.2, difficulties arise because we cannot use the whole cohomology algebra of an abelian variety A, whose explicit structure is used in Charles' and Voisin's work. In fact, we replace the abelian variety in Charles' approach by certain simply connected surfaces with special automorphisms; the construction of these surfaces is inspired by some constructions from Chapter 2. After this replacement, we are able to implement the Charles–Voisin method in our situation.

The examples we construct via the Charles–Voisin method are simply connected smooth complex projective varieties X and X^σ defined over cyclotomic number fields. For instance, one pair of examples X and X^σ is defined over $\mathbb{Q}[\zeta_{12}]$ and satisfies

$$H^{*,*}(X, \mathbb{Q}[\sqrt{3}]) \ncong H^{*,*}(X^\sigma, \mathbb{Q}[\sqrt{3}]),$$

although the dimensions of the above algebras coincide in each degree.

Applying the Lefschetz hyperplane theorem, we are able to cut down the dimension of our examples to any $n \geq 4$. We also analyze the multilinear intersection forms

$$H^2(X, \mathbb{R})^{\otimes n} \longrightarrow \mathbb{R} \quad \text{and} \quad H^2(X^\sigma, \mathbb{R})^{\otimes n} \longrightarrow \mathbb{R},$$

given by cup product and evaluation on the corresponding fundamental classes. We prove that these multilinear intersection forms are not (weakly) isomorphic in our examples, see Theorem 4.1.6. It follows that we have produced the first known nonhomeomorphic conjugate varieties that are simply connected. This answers a question of Reed [67].

Once the existence of simply connected nonhomeomorphic conjugate varieties is settled, it is natural to ask for other fundamental groups as well. A natural generalization of that question asks for nonhomeomorphic conjugate varieties in a given birational equivalence class. We are able to answer this question in sufficiently high dimensions.

Theorem 1.3.3 (Theorem 4.1.7)**.** *Any birational equivalence class of complex projective varieties in dimension* ≥ 10 *contains conjugate smooth complex projective varieties whose even-degree real cohomology algebras* $H^{2*}(-,\mathbb{R})$ *are nonisomorphic.*

Theorem 1.3.3 implies immediately:

Corollary 1.3.4 (Corollary 4.1.8)**.** *Let* G *be the fundamental group of a smooth complex projective variety. Then there exist conjugate smooth complex projective varieties with fundamental group* G*, but nonisomorphic even-degree real cohomology algebras.*

1.4 Geometry of theta divisors and the Schottky problem

We have explained in Section 1.1 above that the Hodge numbers, or equivalently, the complex Hodge structures, of a smooth complex projective variety determine several of its geometric properties. One obtains of course finer invariants if instead of the complex Hodge structure, one considers the integral Hodge structure together with a suitable polarization.

For special classes of varieties, such datum actually determines the isomorphism type of the variety uniquely. The most prominent such example is the Torelli theorem for curves[4]. It states that the isomorphism class of a smooth curve C is uniquely determined by its Jacobian $(J(C),\Theta_C)$, which is the principally polarized abelian variety (ppav) associated to the integral weight one Hodge structure on $H^1(C,\mathbb{Z})$, together with the polarization that is induced by the cup product and Poincaré duality. By Riemann's theorem (see (1.15) below), the theta divisor Θ_C is irreducible, which is equivalent to saying that $(J(C),\Theta_C)$ is indecomposable, see [8, p. 75].

By the Torelli theorem,

$$C \longmapsto (J(C),\Theta_C)$$

gives rise to an injective map from the moduli stack of smooth projective genus g curves to the moduli stack of ppav of dimension g. Chapter 5 studies the Schottky problem, which asks to describe the image of that map. That is, given an indecomposable ppav (A,Θ), how can we decide whether it is the Jacobian of a curve?

[4]If not mentioned otherwise, the term "curve" refers here and in the following to an irreducible complete variety of dimension one over $\mathbb{C}$.

Fixing a point on C, we obtain the Abel–Jacobi embedding $C \longrightarrow J(C)$ whose image is denoted by $W_1(C)$. For $1 \le d \le g-1$, the d-th Brill–Noether locus

$$W_d(C) = W_1(C) + \cdots + W_1(C)$$

is the d-fold sum of $W_1(C)$ in $J(C)$. Poincaré's formula computes the cohomology class of $W_d(C)$ [4, p. 25]:

$$[W_d(C)] = \frac{1}{(g-d)!} \cdot [\Theta_C]^{g-d}. \tag{1.13}$$

The most famous characterization of Jacobians among all indecomposable ppavs is a partial converse of Poincaré's formula, due to Matsusaka and Hoyt [36, 58]. It asserts that an indecomposable ppav (A, Θ) is isomorphic to a Jacobian of a smooth curve if and only if there is a curve $C \subseteq A$ with minimal class

$$[C] = \frac{1}{(g-1)!} \cdot [\Theta]^{g-1}. \tag{1.14}$$

Moreover, if (1.14) holds, C is automatically smooth and (A, Θ) is isomorphic to $(J(C), \Theta_C)$.

Closely related to the Poincaré formula (1.13) is Riemann's theorem. It asserts that the theta divisor Θ_C of a smooth genus g curve C can be written as the $(g-1)$-fold sum of the Abel–Jacobi embedded copy $W_1(C)$ of C. That is,

$$\Theta_C = W_{g-1}(C). \tag{1.15}$$

Riemann's theorem implies that

$$\Theta_C = W_1(C) + W_{g-2}(C)$$

has a curve summand $W_1(C)$. The main result of Chapter 5 is the following converse of that statement.

Theorem 1.4.1 (Theorem 5.1.1)**.** *Let (A, Θ) be an indecomposable ppav of dimension g. Suppose that there is a curve C and a codimension two subvariety Y in A such that*

$$\Theta = C + Y.$$

Then C is smooth and there is an isomorphism $(A, \Theta) \simeq (J(C), \Theta_C)$ which identifies C and Y with translates of $W_1(C)$ and $W_{g-2}(C)$ respectively.

Our proof uses Welters' method [95] to reduce Theorem 1.4.1 to Matsusaka–Hoyt's criterion mentioned above. A crucial ingredient in our proof is Ein–Lazarsfeld's theorem [25], saying that the theta divisor of an indecomposable ppav is normal with at most rational singularities.

Theorem 1.4.1 has been conjectured by Pareschi and Popa (see Section 5.5), in connection with their study of generic vanishing sheaves, associated to subvarieties of ppavs in [60]. Following Pareschi–Popa, a coherent sheaf $\mathcal{F}$ on an abelian variety A is a generic vanishing sheaf or a GV-sheaf, if for all i its i-th cohomological support locus

$$S^i(\mathcal{F}) := \left\{L \in \operatorname{Pic}^0(A) \mid H^i(A, \mathcal{F} \otimes L) \neq 0\right\}$$

has codimension $\geq i$ in $\operatorname{Pic}^0(A)$, see [60, p. 212].

A subvariety Z of a ppav (A, Θ) is called GV-subvariety if the twisted ideal sheaf $\mathcal{I}_Z(\Theta) = \mathcal{I}_Z \otimes \mathcal{O}_A(\Theta)$ is a GV-sheaf. Pareschi and Popa proved that the Brill–Noether locus $W_d(C)$ inside the Jacobian of a smooth curve is a GV-subvariety. Besides Brill–Noether loci, there is only one more example of a GV-subvariety of dimension $1 \leq d \leq g-2$, where $g = \dim(A)$, known: the Fano surface of lines inside the intermediate Jacobian of a smooth cubic threefold is a 2-dimensional GV-subvariety in a 5-dimensional indecomposable ppav that is not isomorphic to the Jacobian of a smooth curve [16].

Pareschi–Popa conjectured that these are all examples of geometrically nondegenerate[5] GV-subvarieties of dimension d in g-dimensional ppavs with $1 \leq d \leq g-2$, see Conjecture 5.5.2. They proved their conjecture for $d = 1$ and $d = g-2$. We use Theorem 1.4.1 and the results in [19] and [60] to prove Pareschi–Popa's conjecture for nondegenerate subvarieties with curve summands.

Theorem 1.4.2 (Theorem 5.1.2)**.** *Let (A, Θ) be an indecomposable ppav, and let $Z \subsetneq A$ be a geometrically nondegenerate subvariety of dimension d. Suppose that the following holds:*

1. *$Z = Y + C$ has a curve summand $C \subseteq A$,*
2. *the twisted ideal sheaf $\mathcal{I}_Z(\Theta)$ is a GV-sheaf.*

Then C is smooth and there is an isomorphism $(A, \Theta) \simeq (J(C), \Theta_C)$ which identifies C, Y and Z with translates of $W_1(C)$, $W_{d-1}(C)$ and $W_d(C)$ respectively.

[5] A subvariety of an abelian variety A is geometrically nondegenerate if and only if it meets all subvarieties $W \subseteq A$ of complementary dimension, see Section 5.2. Brill Noether loci $W_d(C)$ as well as the Fano surface of lines of a smooth cubic threefold have this property.

If Z has codimension one, then it is a GV-subvariety of (A,Θ) if and only if it is a translate of Θ. This explains that Theorem 1.4.1 is a special case of Theorem 1.4.2. However, in our proof, Theorem 1.4.2 is in fact a consequence of Theorem 1.4.1. The key ingredient here is a result of Pareschi and Popa [60] which implies that any geometrically nondegenerate GV-subvariety Z of a ppav (A,Θ) is a summand of Θ. That is,

$$\Theta = Z + W$$

for some subvariety W of A. If in this situation Z has a curve summand, Theorem 1.4.1 applies and so we can use Debarre's theorem [19] for the precise determination of C, Y and Z in Theorem 1.4.2.

Our original motivation for Chapter 5 is the study of varieties X which admit a dominant rational map from a product of curves,

$$C_1 \times \cdots \times C_n \dashrightarrow X.$$

A variety which admits such a dominant rational map is called DPC. Examples of DPC varieties include unirational varieties, abelian varieties and Fermat hypersurfaces $\{x_0^k + \cdots + x_n^k = 0\} \subseteq \mathbb{P}^n$ of arbitrary degree k. Conversely, answering a question of Grothendieck, Serre [80] constructed the first example of a smooth complex projective variety which is not DPC.

Later, Deligne [20, Sec. 7] and Schoen [70] found a Hodge theoretic obstruction, which allowed them to show that a sufficiently ample and very general hypersurface in any smooth complex projective variety of dimension ≥ 3 is not DPC. For instance, this result includes the very general hypersurface $X_{d,n}$ in $\mathbb{P}^{n+1}$, $n \geq 2$, of degree

$$d \geq \max(n+2, 5).$$

This condition on d excludes Fano hypersurfaces of arbitrary dimension as well as Calabi-Yau hypersurfaces of dimension two (i.e. K3 surfaces). In fact, it is not known whether the very general projective K3 surface is DPC, although special families, such as Kummer surfaces or isotrivial elliptic K3 surfaces are easily seen to be DPC.

Apart from K3 surfaces, another interesting class of varieties where Delgine–Schoen's Hodge theoretic obstruction does not apply is the case of theta divisors of indecomposable ppav. Clearly, the theta divisor of the Jacobian of a smooth curve is DPC by Riemann's theorem. Conversely, Schoen found that his Hodge theoretic obstruction does not prevent the general theta divisor from being DPC. This led him to ask [70, Sec. 7.4] whether there are theta divisors which are not DPC.

As an easy corollary of Theorem 1.4.1, we obtain a complete answer to Schoen's question.

Corollary 1.4.3 (Corollary 5.1.3)**.** *Let (A, Θ) be an indecomposable ppav. The theta divisor Θ is DPC if and only if (A, Θ) is isomorphic to the Jacobian of a smooth curve.*

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2 On the construction problem for Hodge numbers

Abstract. For any symmetric collection $(h^{p,q})_{p+q=k}$ of natural numbers, we construct a smooth complex projective variety X whose weight k Hodge structure has Hodge numbers $h^{p,q}(X) = h^{p,q}$; if $k = 2m$ is even, then we have to impose that $h^{m,m}$ is bigger than some quadratic bound in m. Combining these results for different weights, we solve the construction problem for the truncated Hodge diamond under two additional assumptions. Our results lead to a complete classification of all nontrivial dominations among Hodge numbers of Kähler manifolds.

2.1 Introduction

For a Kähler manifold X, the Hodge decomposition gives an isomorphism

$$H^k(X,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X). \tag{2.1}$$

As a refinement of the Betti numbers of X, one therefore defines the (p,q)-th Hodge number $h^{p,q}(X)$ of X to be the dimension of $H^{p,q}(X)$. This way one can associate to each n-dimensional Kähler manifold X its collection of Hodge numbers $h^{p,q}(X)$ with $0 \le p,q \le n$. Complex conjugation and Serre duality show that such a collection of Hodge numbers $(h^{p,q})_{p,q}$ in dimension n needs to satisfy the Hodge symmetries

$$h^{p,q} = h^{q,p} = h^{n-p,n-q}. \tag{2.2}$$

Moreover, as a consequence of the Hard Lefschetz Theorem, the Lefschetz conditions

$$h^{p,q} \ge h^{p-1,q-1} \quad \text{for all} \quad p+q \le n \tag{2.3}$$

This chapter is based on [76]; some minor changes are made as follows. Overlaps of the published article [76] with results of the authors Part III Essay [72] are indicated in this chapter; the corresponding results are cited from [72] and [76]. Moreover, Corollaries 2.9.1 and 2.10.1 are not contained in [76].

hold. Given these classical results, the construction problem for Hodge numbers asks which collections of natural numbers $(h^{p,q})_{p,q}$, satisfying (2.2) and (2.3), actually arise as Hodge numbers of some n-dimensional Kähler manifold. In his survey article on the construction problem in Kähler geometry [84], C. Simpson explains our lack of knowledge on this problem. Indeed, even weak versions where instead of all Hodge numbers one only considers small subcollections of them are wide open; for some partial results in dimensions two and three we refer to [6, 14, 37, 61]. Let us also mention the recent progress of Roulleau and Urzúa [69] on the geography problem for surfaces, which appeared after the article [76] on which this chapter is based was written.

This part of the thesis provides three main results on the above construction problem in the category of smooth complex projective varieties, which is stronger than allowing arbitrary Kähler manifolds. We present them in the following three subsections respectively.

2.1.1 The construction problem for weight k Hodge structures

It follows from Griffiths transversality that a general integral weight k ($k \geq 2$) Hodge structure (not of K3 type) cannot be realized by a smooth complex projective variety, see [89, Remark 10.20]. This might lead to the expectation that general weight k Hodge numbers can also not be realized by smooth complex projective varieties. Our first result shows that this expectation is wrong. This answers a question in [84].

Theorem 2.1.1. *Fix $k \geq 1$ and let $(h^{p,q})_{p+q=k}$ be a symmetric collection of natural numbers. If $k = 2m$ is even, we assume*

$$h^{m,m} \geq m \cdot \lfloor (m+3)/2 \rfloor + \lfloor m/2 \rfloor^2 .$$

Then in each dimension $n \geq k+1$ there exists a smooth complex projective variety whose Hodge structure of weight k realizes the given Hodge numbers.

The examples which realize given weight k Hodge numbers in the above theorem have dimension $n \geq k+1$. However, if we assume that the outer Hodge number $h^{k,0}$ vanishes and that the remaining Hodge numbers are even, then we can prove a version of Theorem 2.1.1 also in dimension $n = k$, see Corollary 2.5.3 in Section 2.5.

Since any smooth complex projective variety contains a hyperplane class, it is clear that some kind of bound on $h^{m,m}$ in Theorem 2.1.1 is necessary. For $m = 1$, for instance, the bound provided by the above Theorem is $h^{1,1} \geq 2$. In Section 2.7 we will show that in fact the optimal bound $h^{1,1} \geq 1$ can be reached.

That is, we will show (Theorem 2.7.1) that any natural numbers $h^{2,0}$ and $h^{1,1}$ with $h^{1,1} \geq 1$ can be realised as weight two Hodge numbers of some smooth complex projective variety. For $m \geq 2$, we do not know whether the bound on $h^{m,m}$ in Theorem 2.1.1 is optimal or not.

2.1.2 The construction problem for the truncated Hodge diamond

Given Theorem 2.1.1 one is tempted to ask for solutions to the construction problem for collections of Hodge numbers which do not necessarily correspond to a single cohomology group. In order to explain our result on this problem, we introduce the following notion: An n-dimensional formal Hodge diamond is a table

$$\begin{array}{ccccccccc} & & & & h^{n,n} & & & & \\ & & & h^{n,n-1} & & h^{n-1,n} & & & \\ & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\ & \iddots & & & \vdots & & & \ddots & \\ h^{n,0} & & & & & & & & h^{0,n} \\ & \ddots & & & \vdots & & & \iddots & \\ & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\ & & & h^{1,0} & & h^{0,1} & & & \\ & & & & h^{0,0} & & & & \end{array} \tag{2.4}$$

of natural numbers $h^{p,q}$, satisfying the Hodge symmetries (2.2), the Lefschetz conditions (2.3) and the connectivity condition $h^{0,0} = h^{n,n} = 1$. The $h^{p,q}$ are referred to as Hodge numbers and the sum over all $h^{p,q}$ with $p + q = k$ as k-th Betti number b_k of this formal diamond; the vector $(b_0, \ldots, b_{2n})$ is called a vector of formal Betti numbers. Finally, for $p + q \leq n$, the primitive (p, q)-th Hodge number of the above diamond is defined via

$$l^{p,q} := h^{p,q} - h^{p-1,q-1}.$$

Definition 2.1.2. *A truncated n-dimensional formal Hodge diamond is a formal Hodge diamond (2.4) as above where the horizontal middle axis, i.e. the row of Hodge numbers $h^{p,q}$ with $p + q = n$, is omitted.*

We note that for a Kähler manifold X its truncated Hodge diamond together with all holomorphic Euler characteristics $\chi(X, \Omega_X^p)$, where $p = 0, \ldots, \lfloor n/2 \rfloor$, is equivalent to giving the whole Hodge diamond. It is shown in [48] that a linear combination of Hodge numbers can be expressed in terms of Chern numbers if and only if it is a linear combination of these Euler characteristics. Therefore, the Hodge numbers of the truncated Hodge diamond form a complement to the space of Hodge numbers which are determined by Chern numbers, cf. [48] where the Hodge numbers in dimension n are regarded as linear forms on the weight n part of a certain graded ring.

Our second main result solves the construction problem for the truncated Hodge diamond under two additional assumptions:

Theorem 2.1.3. *Suppose we are given a truncated n-dimensional formal Hodge diamond whose Hodge numbers $h^{p,q}$ satisfy the following two additional assumptions:*

1. *For $p < n/2$, the primitive Hodge numbers $l^{p,p}$ satisfy*

$$l^{p,p} \geq p \cdot (n^2 - 2n + 5)/4.$$

2. *The outer Hodge numbers $h^{k,0}$ vanish either for all $k = 1, \dots, n-3$, or for all $k \neq k_0$ for some $k_0 \in \{1, \dots, n-1\}$.*

Then there exists an n-dimensional smooth complex projective variety whose truncated Hodge diamond coincides with the given one.

Theorem 2.1.3 has several important consequences. For instance, for the union of $h^{n-2,0}$ and $h^{n-1,0}$ with the collection of all Hodge numbers which neither lie on the boundary, nor on the horizontal or vertical middle axis of (2.4), the construction problem is solvable without any additional assumptions. That is, the corresponding subcollection of any n-dimensional formal Hodge diamond can be realized by a smooth complex projective variety. The number of Hodge numbers we omit in this statement from the whole diamond (2.4) grows linearly in n, whereas the number of all entries of (2.4) grows quadratically in n. In this sense, Theorem 2.1.3 yields very good results on the construction problem in high dimensions.

Theorem 2.1.3 deals with Hodge structures of different weights simultaneously. This enables us to extract from it results on the construction problem for Betti numbers. Indeed, the following corollary rephrases Theorem 2.1.3 in terms of Betti numbers.

Corollary 2.1.4. *Let $(b_0, \dots, b_{2n})$ be a vector of formal Betti numbers with*

$$b_{2k} - b_{2k-2} \geq k \cdot (n^2 - 2n + 5)/8 \quad \text{for all } k < n/2.$$

Then there exists an n-dimensional smooth complex projective variety X with $b_k(X) = b_k$ for all $k \neq n$.

This corollary says for instance that in even dimensions, the construction problem for the odd Betti numbers is solvable without any additional assumptions.

2.1.3 Universal inequalities and Kollár–Simpson's domination relation

Following Kollár–Simpson [84, p. 9], we say that a Hodge number $h^{r,s}$ dominates $h^{p,q}$ in dimension n, if there exist positive constants $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$ such that for all n-dimensional smooth complex projective varieties X, the following holds:

$$\lambda_1 \cdot h^{r,s}(X) + \lambda_2 \geq h^{p,q}(X). \tag{2.5}$$

Moreover, such a domination is called nontrivial if $(0,0) \neq (p,q) \neq (n,n)$, and if (2.5) does not follow from the Hodge symmetries (2.2) and the Lefschetz conditions (2.3).

In [84] it is speculated that the middle Hodge numbers should probably dominate the outer ones. In our third main theorem of this part of the thesis, we classify all nontrivial dominations among Hodge numbers in any given dimension. As a result we see that the above speculation is accurate precisely in dimension two.

Theorem 2.1.5. *The Hodge number $h^{1,1}$ dominates $h^{2,0}$ nontrivially in dimension two and this is the only nontrivial domination in dimension two. Moreover, there are no nontrivial dominations among Hodge numbers in any dimension different from two.*

Firstly, as an easy consequence of the classification of surfaces and the Bogomolov–Miyaoka–Yau inequality, we observed in [72] that

$$h^{1,1}(S) > h^{2,0}(S) \tag{2.6}$$

holds for all Kähler surfaces S. That is, the middle degree Hodge number $h^{1,1}$ indeed dominates $h^{2,0}$ nontrivially in dimension two.

Secondly, in addition to Theorem 2.1.3, the proof of Theorem 2.1.5 will rely on the following result, see Theorem 2.8.1 in Section 2.8: For all $a > b$ with $a + b \leq n$, there are n-dimensional smooth complex projective varieties whose primitive Hodge numbers $l^{p,q}$ satisfy $l^{a,b} >> 0$ and $l^{p,q} = 0$ for all other $p > q$.

Theorem 2.1.5 deals with universal inequalities of the form (2.5). Apart from one exception, all such inequalities follow from the Lefschetz conditions. The exception concerns the nontrivial domination of $h^{2,0}$ by $h^{1,1}$ in dimension two, but Theorem 2.1.5 leaves open the determination of the sharp constants λ_1 and λ_2 in that domination. We will use Roulleau–Urzúa's recent result [69] on the geography problem for surfaces, to fill this gap, see Corollary 2.9.1.

In Section 2.10 we deduce from the main results of this part of the thesis further progress on the analogous problem for inequalities in higher dimensions (Corollaries 2.10.2, 2.10.3 and 2.10.4). For instance, we will see that any universal inequality among the Hodge numbers of smooth complex projective varieties which holds in all sufficiently large dimensions at the same time is a consequence of the Lefschetz conditions.

The problem of determining all universal inequalities among Hodge numbers of smooth complex projective varieties in a fixed dimension n remains open for all $n \geq 3$.

2.1.4 Some negative results

By(2.6), the constraints which classical Hodge theory puts on the Hodge numbers of Kähler manifolds are not complete in dimension two. Indeed, given weight two Hodge numbers can in general not be realized by a surface – by Theorem 2.1.1 (resp. Theorem 2.7.1) they can however be realized by higher dimensional varieties.

In Section 2.11 of this thesis we will prove a similar issue in dimension three: a threefold with $h^{1,1} = 1$ and $h^{3,0} \geq 2$ (such as any complete intersection of sufficiently high degree in projective space) satisfies

$$h^{2,1} \leq 12^6 \cdot h^{3,0}.$$

Here the bound 12^6 is certainly not optimal. Moreover, the middle Hodge number $h^{2,1}$ is bounded by some multiple of the outer Hodge number $h^{3,0}$ and not the other way around. Looking at the blow-up of a sufficiently high degree complete intersection curve in suitable threefolds shows that the above inequality does not hold for $h^{1,1} \geq 2$.

Further results which demonstrate similar issues in dimensions four can be found in Section 12 of [76]. These results are already contained in the authors Part III essay [72] and so they are not inlcuded here. For instance, using Kollár–Matsusaka's theorem [50, p. 239] we proved that the third Betti number b_3 of a 4-dimensional Kähler manifold with $b_2 = 1$ is bounded from above in terms of b_4. This cannot be explained with classical Hodge theory, which shows that even for the Betti numbers of smooth complex projective varieties, the known constraints are not complete.

2.1.5 Notation and conventions

The natural numbers $\mathbb{N} := \mathbb{Z}_{\geq 0}$ include zero. All Kähler manifolds are compact and connected, if not mentioned otherwise. A variety is a separated integral scheme of finite type over $\mathbb{C}$. Using the GAGA principle [79], we usually identify a smooth projective variety with its corresponding analytic space, which is a Kähler manifold. If not mentioned otherwise, cohomology means de Rham or Betti cohomology with coefficients in $\mathbb{C}$; the cup product on cohomology will be denoted by $\wedge$.

With a group action $G \times Y \to Y$ on a variety Y, we always mean a group action by automorphisms from the left. For any subgroup $\Gamma \subseteq G$, the fixed point set of the induced Γ-action on Y will be denoted by

$$\mathrm{Fix}_Y(\Gamma) \coloneqq \{y \in Y \mid g(y) = y \text{ for all } g \in \Gamma\}. \tag{2.7}$$

This fixed point set has a natural scheme structure. If $\Gamma = \langle \phi \rangle$ is cyclic, then we will frequently write $\mathrm{Fix}_Y(\Gamma) = \mathrm{Fix}_Y(\phi)$ for this fixed point set (or scheme).

2.2 Outline of our construction methods

The starting point of our constructions is the observation that there are finite group actions $G \times T \to T$, where T is a product of hyperelliptic curves, such that the G-invariant cohomology of T is essentially concentrated in a single (p,q)-type, see Section 2.3.2. In local holomorphic charts, G acts by linear automorphisms. Thus, by the Chevalley–Shephard–Todd Theorem, T/G is smooth if and only if G is generated by quasi-reflections, that is, by elements whose fixed point set is a divisor on T. Unfortunately, it turns out that in our approach this strong condition can rarely be met. We therefore face the problem of a possibly highly singular quotient T/G.

One way to deal with this problem is to pass to a smooth model of T/G. However, only the outer Hodge numbers $h^{k,0}$ are birational invariants [48]. Therefore, there will be in general only very little relation between the cohomology of the smooth model and the G-invariant cohomology of T. Nevertheless, we will find in Section 2.8 examples T/G which admit smooth models whose cohomology is, apart from (a lot of) additional (p,p)-type classes, indeed given by the G-invariants of T. We will overcome technical difficulties by a general inductive approach which is inspired by work of Cynk–Hulek [17], see Proposition 2.8.3.

In Theorems 2.1.1 and 2.1.3 we need to construct examples with bounded $h^{p,p}$ and so the above method does not work anymore. Instead, we will use the following lemma, known as the Godeaux–Serre construction, cf. [5, 81]:

Lemma 2.2.1. *Let G be a finite group whose action on a smooth complex projective variety Y is free outside a subset of codimension $> n$. Then Y/G contains an n-dimensional smooth complex projective subvariety whose cohomology below degree n is given by the G-invariant classes of Y.*

Proof. A general n-dimensional G-invariant complete intersection subvariety $Z \subseteq Y$ is smooth by Bertini's theorem. For a general choice of Z, the G-action on Z is free and so Z/G is a smooth subvariety of Y/G which by the

Lefschetz hyperplane theorem, applied to $Z \subseteq Y$, has the property we want in the Lemma. $\square$

Roughly speaking, the construction method which we develop in Section 2.4 (Proposition 2.4.2) and which is needed in Theorems 2.1.1 and 2.1.3 works now as follows. Instead of a single group action, we will consider a finite number of finite group actions $G_i \times T_i \to T_i$, indexed by $i \in I$. Blowing up all T_i simultaneously in a large ambient space Y, we are able to construct a smooth complex projective variety $\tilde{Y}$ which admits an action of the product $G = \prod_{i \in I} G_i$ that is free outside a subset of large codimension and so Lemma 2.2.1 applies. Moreover, the G-invariant cohomology of $\tilde{Y}$ will be given in terms of the G_i-invariant cohomology of the T_i. This is a quite powerful method since it allows us to apply Lemma 2.2.1 to a finite number of group actions simultaneously – even without assuming that the group actions we started with are free away from subspaces of large codimension.

2.3 Hyperelliptic curves and group actions

2.3.1 Basics on hyperelliptic curves

In this section, following mostly [82, pp. 214], we recall some basic properties of hyperelliptic curves, see also [87]. In order to unify our discussion, hyperelliptic curves of genus 0 and 1 will be $\mathbb{P}^1$ and elliptic curves respectively.

For $g \geq 0$, let $f \in \mathbb{C}[x]$ be a degree $2g+1$ polynomial with distinct roots. Then, a smooth projective model X of the affine curve Y given by

$$\left\{y^2 = f(x)\right\} \subseteq \mathbb{C}^2$$

is a hyperelliptic curve of genus g. Although Y is smooth, its projective closure has for $g > 1$ a singularity at ∞. The hyperelliptic curve X is therefore explicitly given by the normalization of this projective closure. It turns out that X is obtained from Y by adding one additional point at ∞. This additional point is covered by an affine piece, given by

$$\left\{v^2 = u^{2g+2} \cdot f\left(u^{-1}\right)\right\}, \text{ where } x = u^{-1} \text{ and } y = v \cdot u^{-g-1}.$$

On an appropriate open cover of X, local holomorphic coordinates are given by x, y, u and v respectively. Moreover, the smooth curve X has genus g and a basis of $H^{1,0}(X)$ is given by the differential forms

$$\omega_i := \frac{x^{i-1}}{y} \cdot dx,$$

where $i = 1, \dots, g$.

Let us now specialize to the situation where $f(x) = x^{2g+1} + 1$ and denote the corresponding hyperelliptic curve of genus g by C_g. It follows from the explicit description of the two affine pieces of C_g that this curve carries an automorphism ψ_g of order $2g+1$ given by

$$(x,y) \longmapsto (\zeta \cdot x, y) \quad \text{and} \quad (u,v) \longmapsto (\zeta^{-1} \cdot u, \zeta^g \cdot v),$$

where ζ denotes a primitive $(2g+1)$-th root of unity. Similarly,

$$(x,y) \longmapsto (x,-y) \quad \text{and} \quad (u,v) \longmapsto (u,-v),$$

defines an involution which we denote by multiplication with -1. Moreover, it follows from the above description of $H^{1,0}(C_g)$ that the ψ_g-action on $H^{1,0}(C_g)$ has eigenvalues $\zeta, \dots, \zeta^g$, whereas the involution acts by multiplication with -1 on $H^{1,0}(C_g)$.

Any smooth curve can be embedded into $\mathbb{P}^3$. For the curve C_g, we fix the explicit embedding which is given by

$$[1 : x : y : x^{g+1}] = [u^{g+1} : u^g : v : 1].$$

Obviously, the involution as well as the order $(2g+1)$-automorphism ψ_g of $C_g \subseteq \mathbb{P}^3$ extend to $\mathbb{P}^3$ via

$$[1 : 1 : -1 : 1] \quad \text{and} \quad [1 : \zeta : 1 : \zeta^{g+1}]$$

respectively.

2.3.2 Group actions on products of hyperelliptic curves

Let

$$T := C_g^k$$

be the k-fold product of the hyperelliptic curve C_g with automorphism ψ_g defined in Section 2.3.1. For $a \geq b$ with $a + b = k$, we define for each $i = 1,2,3$ a subgroup $G^i(a,b,g)$ of $\mathrm{Aut}(T)$ whose elements are called automorphisms of the i-th kind. The subgroup of automorphisms of the first kind is given by

$$G^1(a,b,g) := \left\{ \psi_g^{j_1} \times \cdots \times \psi_g^{j_{a+b}} \mid j_1 + \cdots + j_a - j_{a+1} - \cdots - j_{a+b} \equiv 0 \mod (2g+1) \right\}.$$

In order to define the automorphisms of the second kind, let us consider the group $\mathrm{Sym}(a) \times \mathrm{Sym}(b) \times \mu_2^{a+b}$, where $\mu_2 = \{1, -1\}$ is the multiplicative group on two elements. An element (σ,τ,ϵ), where $\sigma \in \mathrm{Sym}(a)$, $\tau \in \mathrm{Sym}(b)$ and $\epsilon = (\epsilon_1,\ldots,\epsilon_{a+b})$ is a vector of signs $\epsilon_i \in \{1,-1\}$, acts on T via

$$(x_1,\ldots,x_a,y_1,\ldots,y_b) \longmapsto \left(\epsilon_1 \cdot x_{\sigma(1)},\ldots,\epsilon_a \cdot x_{\sigma(a)},\epsilon_{a+1}\cdot y_{\tau(1)},\ldots,\epsilon_{a+b}\cdot y_{\tau(b)}\right).$$

Here, multiplication with -1 means that we apply the involution $-1 \in \mathrm{Aut}(C_g)$. We define

$$G^2(a,b,g) \subseteq \mathrm{Sym}(a) \times \mathrm{Sym}(b) \times \mu_2^{a+b}$$

to be the index four subgroup consisting of those elements (σ,τ,ϵ) which satisfy

$$\mathrm{sign}(\sigma)\cdot \epsilon_1 \cdot \cdots \cdot \epsilon_a = 1 \quad \text{and} \quad \mathrm{sign}(\tau)\cdot \epsilon_{a+1}\cdot \cdots \cdot \epsilon_{a+b} = 1,$$

where sign denotes the signum of the corresponding permutation. Via the above action of $\mathrm{Sym}(a) \times \mathrm{Sym}(b) \times \mu_2^{a+b}$ on T, the group $G^2(a,b,g)$ is a finite subgroup of $\mathrm{Aut}(T)$.

Finally, $G^3(a,b,g)$ is trivial, if $a \neq b$ and if $a = b$, then it is generated by the automorphism which interchanges the two factors of $T = C_g^a \times C_g^a$.

Definition 2.3.1. *The group $G(a,b,g)$ is the subgroup of $\mathrm{Aut}(T)$ which is generated by the union of $G^i(a,b,g)$ for $i = 1,2,3$.*

Automorphisms of different kinds do in general not commute with each other. However, it is easy to see that each element in $G(a,b,g)$ can be written as a product $\phi_1 \circ \phi_2 \circ \phi_3$ such that ϕ_i lies in $G^i(a,b,g)$. Therefore, $G(a,b,g)$ is a finite group which naturally acts on the cohomology of T.

Lemma 2.3.2. *If $a > b$, then the $G(a,b,g)$-invariant cohomology of T is a direct sum*

$$V^{a,b} \oplus V^{b,a} \oplus \left(\bigoplus_{p=0}^{k} V^{p,p}\right),$$

where $V^{a,b} = \overline{V^{b,a}}$ is a g-dimensional space of (a,b)-classes and $V^{p,p} \simeq V^{k-p,k-p}$ is a space of (p,p)-classes of dimension $\min(p+1,b+1)$, where $p \leq k/2$ is assumed.

Proof. We denote the fundamental class of the j-th factor of T by $\Omega_j \in H^{1,1}(T)$. Moreover, we pick for $j = 1,\ldots,k$ a basis $\omega_{j1},\ldots,\omega_{jg}$ of $(1,0)$-classes of the j-th factor of T in such a way that

$$\psi_g^* \omega_{jl} = \zeta^l \omega_{jl}$$

for a fixed $(2g+1)$-th root of unity ζ. Then the cohomology ring of T is generated by the Ω_j's, ω_{jl}'s and their conjugates. Moreover, the involution on the j-th curve factor of T acts on ω_{jl} and $\overline{\omega_{jl}}$ by multiplication with -1 and leaves Ω_j invariant.

Suppose that we are given a $G(a,b,g)$-invariant class which contains the monomial

$$\Omega_{i_1} \wedge \cdots \wedge \Omega_{i_s} \wedge \omega_{j_1 l_1} \wedge \cdots \wedge \omega_{j_r l_r} \wedge \overline{\omega_{j_{r+1}\, l_{r+1}}} \wedge \cdots \wedge \overline{\omega_{j_t\, l_t}} \tag{2.8}$$

nontrivially. Since the product of a $(1,0)$- and a $(0,1)$-class of the i-th curve factor is a multiple of Ω_i, and since classes of degree 3 vanish on curves, we may assume that the indices $i_1, \ldots, i_s, j_1, \ldots, j_t$ are pairwise distinct. Therefore, application of a suitable automorphism of the first kind shows $t = 0$ if $s \geq 1$ and $t = a + b$ if $s = 0$. In the latter case, suppose that there are indices i_1 and i_2 with either $i_1, i_2 \leq r$ or $i_1, i_2 > r$, such that $j_{i_1} \leq a$ and $j_{i_2} > a$ holds. Then, application of a suitable automorphism of the first kind yields $l_{i_1} + l_{i_2} = 0$ in $\mathbb{Z}/(2g+1)\mathbb{Z}$, which contradicts $1 \leq l_i \leq g$. This shows

$$\{j_1, \ldots, j_r\} = \{1, \ldots, a\} \quad \text{or} \quad \{j_1, \ldots, j_r\} = \{a+1, \ldots, a+b\}\,.$$

By applying suitable automorphisms of the first kind once more, one obtains $l_1 = \cdots = l_t$. Thus, we have just shown that a $G(a,b,g)$-invariant class of T is either a polynomial in the Ω_j's, or a linear combination of

$$\omega_l := \omega_{1l} \wedge \cdots \wedge \omega_{al} \wedge \overline{\omega_{a+1\, l}} \wedge \cdots \wedge \overline{\omega_{a+b\, l}}, \tag{2.9}$$

or their conjugates, where $l = 1, \ldots, g$. Note that ω_l is of (a,b)-type whereas any polynomial in the Ω_j's is a sum of (p,p)-type classes. Moreover, by the definition of $G^1(a,b,g)$ and $G^2(a,b,g)$, both groups act trivially on ω_l and $\overline{\omega}_l$. Since $a > b$, the group $G^3(a,b,g)$ is trivial and so it follows that each ω_l and $\overline{\omega}_l$ is $G(a,b,g)$-invariant. Therefore, the span of $\omega_1, \ldots, \omega_g$ yields a g-dimensional space $V^{a,b}$ of $G(a,b,g)$-invariant (a,b)-classes. Its conjugate $V^{b,a} := \overline{V^{a,b}}$ is spanned by the $G(a,b,g)$-invariant (b,a)-classes $\overline{\omega}_1, \ldots, \overline{\omega}_g$.

Next, we define $V^{p,p}$ to consist of all $G(a,b,g)$-invariant homogeneous degree p polynomials in $\Omega_1, \ldots, \Omega_{a+b}$. Application of a suitable automorphism of the second kind shows that any element Θ in $V^{p,p}$ is a polynomial in the elementary symmetric polynomials in $\Omega_1, \ldots, \Omega_a$ and $\Omega_{a+1}, \ldots, \Omega_{a+b}$. By standard facts about symmetric polynomials, it follows that Θ can be written as a polynomial in

$$\sum_{j=1}^{a} {\Omega_j}^i \quad \text{and} \quad \sum_{j=a+1}^{a+b} {\Omega_j}^i$$

for $i \geq 0$. Since Ω_j^2 vanishes for all j, we see that a basis of $V^{p,p}$ is given by the elements

$$\left(\Omega_1 + \cdots + \Omega_a\right)^{p-i} \wedge \left(\Omega_{a+1} + \cdots + \Omega_{a+b}\right)^i ,$$

where $0 \le p-i \le a$ and $0 \le i \le b$. Using $a > b$, this concludes the Lemma by an easy counting argument. $\square$

Lemma 2.3.3. *If $a = b$, then the $G(a,b,g)$-invariant cohomology of T is a direct sum $\bigoplus_{p=0}^{k} V^{p,p}$, where $V^{p,p} \simeq V^{k-p,k-p}$ is a space of (p,p)-classes whose dimension is given by $\lfloor p/2 \rfloor + 1$, if $p < a$, and by $\lfloor p/2 \rfloor + g + 1$, if $p = a$.*

Proof. We use the same notation as in the proof of Lemma 2.3.2 and put $b := a$. Suppose that we are given a $G(a,a,g)$-invariant cohomology class on T which contains the monomial (2.8) nontrivial. This monomial is then necessarily $G^1(a,a,g)$-invariant and the same arguments as in Lemma 2.3.2 show that it is either a monomial in the Ω_j's, or it coincides with one of the ω_l's and their conjugates, defined in (2.9).

For each $l = 1, \dots, g$, the classes ω_l and $\overline{\omega_l}$ are invariant under the action of $G^1(a,a,g)$ and $G^2(a,a,g)$. Moreover, the generator of $G^3(a,a,g)$ interchanges the two factors of $T = C_g^a \times C_g^a$. Its action on cohomology therefore maps ω_l to $(-1)^a \cdot \overline{\omega_l}$. This shows that a linear combination of the ω_l's and their conjugates is $G(a,a,g)$-invariant if and only if it is a linear combination of the classes

$$\omega_l + (-1)^a \cdot \overline{\omega_l}, \tag{2.10}$$

where $l = 1, \dots, g$. This yields a g-dimensional space of $G(a,a,g)$-invariant (a,a)-classes.

It remains to study which homogeneous polynomials in the Ω_j's are $G(a,a,g)$-invariant. As in the proof of Lemma 2.3.2, one shows that any such polynomial of degree p is necessarily a linear combination of

$$\Omega(p-i,i) := (\Omega_1 + \dots + \Omega_a)^{p-i} \wedge (\Omega_{a+1} + \dots + \Omega_{2a})^i,$$

where $0 \le p-i \le a$ and $0 \le i \le a$. The above monomials are clearly invariant under the action of $G^1(a,a,g)$ and $G^2(a,a,g)$. Moreover, the generator of $G^3(a,a,g)$ interchanges the two factors of T and hence its action on cohomology maps $\Omega(p-i,i)$ to $\Omega(i,p-i)$. We are therefore reduced to linear combinations of

$$\Omega(i,p-i) + \Omega(p-i,i),$$

where $0 \le i \le p-i \le a$. Such linear combinations are certainly $G(a,a,g)$-invariant. If $p \le a$, then the condition on the index i means $0 \le i \le p/2$. It follows that for $p \le a$, the space of those $G(a,a,g)$-invariant (p,p)-classes which are given by polynomials in the Ω_j's has dimension $\lfloor p/2 \rfloor + 1$. Combining this with our previous observation that the classes in (2.10) span a g-dimensional space of $G(a,a,g)$-invariant (a,a)-classes, this concludes the Lemma. $\square$

For later applications, we will also need the following:

Lemma 2.3.4. *For all $a \geq b$ there exists some $N > 0$ and an embedding of $G(a,b,g)$ into $\mathrm{GL}(N+1)$ such that a $G(a,b,g)$-equivariant embedding of C_g^{a+b} into $\mathbb{P}^N$ exists. Moreover, C_g^{a+b} contains a point which is fixed by $G(a,b,g)$.*

Proof. For the first statement, we use the embedding of C_g into $\mathbb{P}^3$, constructed in Section 2.3.1. This yields an embedding of C_g^{a+b} into $(\mathbb{P}^3)^{a+b}$. From the explicit description of that embedding, it follows that the action of $G(a,b,g)$ on C_g^{a+b} extends to an action on $(\mathbb{P}^3)^{a+b}$ which is given by first multiplying homogeneous coordinates with some roots of unity and then permuting these in some way. Using the Segre map, we obtain for some large N an embedding of $G(a,b,g)$ into $\mathrm{GL}(N+1)$ together with a $G(a,b,g)$-equivariant embedding

$$C_g^{a+b} \hookrightarrow \mathbb{P}^N.$$

This proves the first statement in the Lemma.

For the second statement, note that the point ∞ of C_g is fixed by both, ψ_g as well as the involution. Thus, ∞ yields a point on the diagonal of C_g^{a+b} which is fixed by $G(a,b,g)$. □

2.4 Group actions on blown-up spaces

2.4.1 Cohomology of blow-ups

Let Y be a Kähler manifold, T a submanifold of codimension r and let

$$\pi : \tilde{Y} := Bl_T(Y) \longrightarrow Y$$

be the blow-up of Y along T. Then the exceptional divisor $j : E \hookrightarrow \tilde{Y}$ of this blow-up is a projective bundle of rank $r-1$ over T and we denote the dual of the tautological line bundle on E by $\mathcal{O}_E(1)$. Then the Hodge structure on $\tilde{Y}$ is given by the following theorem, see [89, p. 180].

Theorem 2.4.1. *We have an isomorphism of Hodge structures*

$$H^k(Y,\mathbb{Z}) \oplus \left(\bigoplus_{i=0}^{r-2} H^{k-2i-2}(T,\mathbb{Z}) \right) \longrightarrow H^k\left(\tilde{Y},\mathbb{Z}\right),$$

where on $H^{k-2i-2}(T,\mathbb{Z})$, the natural Hodge structure is shifted by $(i+1,i+1)$. On $H^k(Y,\mathbb{Z})$, the above morphism is given by π^* whereas on $H^{k-2i-2}(T,\mathbb{Z})$ it is given by $j_* \circ h^i \circ \pi|_E^*$, where h denotes the cup product with $c_1(\mathcal{O}_E(1)) \in H^2(E,\mathbb{Z})$ and j_* is the Gysin morphism of the inclusion $j : E \hookrightarrow \tilde{Y}$.

We will need the following property of the ring structure of $H^*(\tilde{Y},\mathbb{Z})$. Note that the first Chern class of $\mathcal{O}_E(1)$ coincides with the pullback of

$$-[E] \in H^2(\tilde{Y},\mathbb{Z})$$

to E. For a class $\alpha \in H^{k-2i-2}(T,\mathbb{Z})$, this implies:

$$(j_* \circ h^i \circ \pi|_E^*)(\alpha) = j_*(j^*(-[E])^i \wedge \pi|_E^*(\alpha)) = (-[E])^i \wedge j_*(\pi|_E^*(\alpha)), \quad (2.11)$$

where we used the projection formula.

2.4.2 Key construction

Let I be a finite nonempty set, and let $i_0 \in I$. Suppose that for each $i \in I$, we are given a representation

$$G_i \to GL(V_i)$$

of a finite group G_i on a finite dimensional complex vector space V_i. Further, assume that the induced G_i-action on $\mathbb{P}(V_i)$ restricts to an action on a smooth subvariety $T_i \subseteq \mathbb{P}(V_i)$ and that there is a point $p_{i_0} \in T_{i_0}$ which is fixed by G_{i_0}. Then we have the following key result.

Proposition 2.4.2. *For any $n > 0$, there exists some complex vector space V and pairwise disjoint embeddings of T_i into $Y := T_{i_0} \times \mathbb{P}(V)$, such that the blow-up $\tilde{Y}$ of Y along all T_i with $i \neq i_0$ inherits an action of $G := \prod_{i\in I} G_i$ which is free outside a subset of codimension $> n$. Moreover, $\tilde{Y}/G$ contains an n-dimensional smooth complex projective subvariety X whose primitive Hodge numbers are, for all $p+q<n$, given by*

$$l^{p,q}(X) = \dim\left(H^{p,q}(T_{i_0})^{G_{i_0}}\right) + \sum_{i\neq i_0} \dim\left(H^{p-1,q-1}(T_i)^{G_i}\right).$$

Proof. The product

$$G := \prod_{i\in I} G_i$$

acts naturally on the direct sum $\bigoplus_{i\in I} V_i$. We pick some $k >> 0$. Then

$$V := (\bigoplus_{i\in I} V_i) \oplus (\bigoplus_{g\in G} g\cdot \mathbb{C}^k)$$

inherits a linear G-action where $h \in G$ acts on the second factor by sending $g \cdot \mathbb{C}^k$ canonically to $(h\cdot g)\cdot \mathbb{C}^k$. Then we obtain G-equivariant inclusions

$$T_i \hookrightarrow \mathbb{P}(V_i) \hookrightarrow \mathbb{P}(V),$$

where for $j \neq i$, the group G_j acts via the identity on T_i and $\mathbb{P}(V_i)$. The product

$$Y := T_{i_0} \times \mathbb{P}(V)$$

inherits a G-action via the diagonal, where for $i \neq i_0$ elements of G_i act trivially on T_{i_0}.

Using the base point $p_{i_0} \in T_{i_0}$, we obtain for all $i \in I$ disjoint inclusions

$$T_i \hookrightarrow Y,$$

and we denote the blow-up of Y along the union of all T_i with $i \neq i_0$ by $\tilde{Y}$. Since $p_{i_0} \in T_{i_0}$ is fixed by G, the G-action maps each T_i to itself and hence lifts to $\tilde{Y}$.

We want to prove that the G-action on $\tilde{Y}$ is free outside a subset of codimension $> n$. For k large enough, the G-action on Y certainly has this property. Hence, it suffices to check that the induced G-action on the exceptional divisor E_j above $T_j \subseteq Y$ is free outside a subset of codimension $> n$.

For $|I| = 1$, this condition is empty. For $|I| \geq 2$, we fix an index $j \in I$ with $j \neq i_0$. Then it suffices to show that for a given nontrivial element $\phi \in G$ the fixed point set $\mathrm{Fix}_{E_j}(\phi)$ has codimension $> n$ in E_j. If $t_j \in T_j$ is not fixed by ϕ, then the fiber of $E_j \to T_j$ above t_j is moved by ϕ and hence disjoint from $\mathrm{Fix}_{E_j}(\phi)$. Conversely, if t_j is fixed by ϕ, then ϕ acts on the normal space

$$\mathcal{N}_{T_j,t_j} = T_{Y,t_j}/T_{T_j,t_j}$$

via a linear automorphism and the projectivization of this vector space is the fiber of $E_j \to T_j$ above t_j. The tangent space T_{Y,t_j} equals

$$T_{T_{i_0},p_{i_0}} \oplus (L^* \otimes (V/L)),$$

where L is the line in V which corresponds to the image of t_j under the projection $Y \to \mathbb{P}(V)$. Since $\phi \neq \mathrm{id}$, it follows for large k that the fixed point set of ϕ on the fiber of E_j above t_j has codimension $> n$. Hence, $\mathrm{Fix}_{E_j}(\phi)$ has codimension $> n$ in E_j, as we want.

As we have just shown, the G-action on $\tilde{Y}$ is free outside a subset of codimension $> n$. Hence, by Lemma 2.2.1, the quotient $\tilde{Y}/G$ contains an n-dimensional smooth complex projective subvariety X whose cohomology below the middle degree is given by the G-invariants of $\tilde{Y}$. In order to calculate the dimension of the latter, we first note that for all $i \in I$, the divisor E_i on $\tilde{Y}$ is preserved by G. Since $\mathcal{O}_{E_i}(-1)$ is given by the restriction of $\mathcal{O}_{\tilde{Y}}(E_i)$ to E_i, it follows that $c_1(\mathcal{O}_{E_i}(1))$ is G-invariant. For $p+q<n$, the primitive (p,q)-th Hodge number of X is by Theorem 2.4.1 therefore given by:

$$l^{p,q}(X) = \dim(H^{p,q}(Y)^G) - \dim(H^{p-1,q-1}(Y)^G) + \sum_{i \neq i_0} \dim\left(H^{p-1,q-1}(T_i)^{G_i}\right),$$

where $H^*(-)^G$ denotes G-invariant cohomology. Since any automorphism of projective space acts trivially on its cohomology, the Künneth formula implies

$$\dim(H^{p,q}(Y)^G) - \dim(H^{p-1,q-1}(Y)^G) = \dim\left(H^{p,q}(T_{i_0})^{G_{i_0}}\right).$$

This finishes the proof of Proposition 2.4.2. □

2.5 Proof of Theorem 2.1.1

Proof of Theorem 2.1.1. Fix $k \geq 1$ and let $(h^{p,q})_{p+q=k}$ be a symmetric collection of natural numbers. In the case where $k = 2m$ is even, we additionally assume

$$h^{m,m} \geq m \cdot \left(m - \left\lfloor \frac{m}{2} \right\rfloor + 1\right) + \left\lfloor \frac{m}{2} \right\rfloor^2.$$

Then we want to construct for $n > k$ an n-dimensional smooth complex projective variety X with the above Hodge numbers on $H^k(X, \mathbb{C})$.

Let us consider the index set $I := \{0, \ldots, \lfloor (k-1)/2 \rfloor\}$ and put $i_0 := 0$. Then, for all $i \in I$, we consider the $(k-2i)$-fold product

$$T_i := (C_{h^{k-i,i}})^{k-2i},$$

where $C_{h^{k-i,i}}$ denotes the hyperelliptic curve of genus $h^{k-i,i}$, defined in Section 2.3.1. On T_i we consider the action of

$$G_i := G(k-2i\,, 0\,,\, h^{k-i,i}),$$

defined in Section 2.3.2.

By Lemma 2.3.4, we may apply the construction method of Section 2.4.2 to the set of data (T_i, G_i, I, i_0). Thus, by Proposition 2.4.2, there exists an n-dimensional smooth complex projective variety X whose primitive Hodge numbers are for $p + q < n$ given by

$$l^{p,q}(X) = \dim\left(H^{p,q}(T_{i_0})^{G_{i_0}}\right) + \sum_{i \neq i_0} \dim\left(H^{p-1,q-1}(T_i)^{G_i}\right).$$

Lemma 2.3.2 says that for $p > q$, the only G_i-invariant (p, q)-classes on T_i are of type $(k-2i, 0)$. Therefore, $l^{p,q}(X)$ vanishes for $p > q$ and $p + q < n$ in all but the following cases:

$$l^{k,0}(X) = \dim\left(H^{k,0}(T_{i_0})^{G_{i_0}}\right) = h^{k,0},$$

and

$$l^{k-2i+1,1}(X) = \dim\left(H^{k-2i,0}(T_i)^{G_i}\right) = h^{k-i,i},$$

for all $1 \leq i < k/2$. Using the formula

$$h^{k-i,i}(X) = \sum_{s=0}^{i} l^{k-i-s,i-s}(X),$$

we deduce for $0 \leq i < k/2$:

$$h^{k-i,i}(X) = h^{k-i,i}.$$

Thus, if k is odd, then the Hodge symmetries imply that the Hodge structure on $H^k(X,\mathbb{C})$ has Hodge numbers $(h^{k,0},\ldots,h^{0,k})$.

We are left with the case where $k = 2m$ is even. Since blowing-up a point increases $h^{m,m}$ by one and leaves $h^{p,q}$ with $p \neq q$ unchanged, it suffices to prove

$$h^{m,m}(X) = m \cdot \left(m - \left\lfloor \frac{m}{2} \right\rfloor + 1\right) + \left\lfloor \frac{m}{2} \right\rfloor^2 .$$

As we have seen:

$$\begin{aligned} h^{m,m}(X) &= \sum_{s=0}^{m} l^{s,s}(X) && (2.12)\\ &= \sum_{s=0}^{m} \left(\dim\left(H^{s,s}(T_0)^{G_0}\right) + \sum_{0<i<k/2} \dim\left(H^{s-1,s-1}(T_i)^{G_i}\right) \right). && (2.13) \end{aligned}$$

By Lemma 2.3.2, we have $\dim\left(H^{s,s}(T_i)^{G_i}\right) = 1$ for all $0 \leq s \leq 2\cdot\dim(T_i)$ and so

$$h^{m,m}(X) = m + 1 + \sum_{s=0}^{m-1} \sum_{0<i<k/2} \dim\left(H^{s,s}(T_i)^{G_i}\right).$$

Since T_i has dimension $2(m-i)$, we see that

$$\sum_{s=0}^{m-1} \dim\left(H^{s,s}(T_i)^{G_i}\right) = \begin{cases} m, & \text{if } 2(m-i) > m-1, \\ 2(m-i)+1, & \text{if } 2(m-i) \leq m-1. \end{cases}$$

Hence

$$h^{m,m}(X) = m + 1 + \sum_{i=1}^{\lfloor m/2 \rfloor} m + \sum_{i=\lfloor m/2 \rfloor + 1}^{m-1} (2(m-i)+1),$$

and it is straightforward to check that this simplifies to

$$h^{m,m}(X) = m \cdot \lfloor (m+3)/2 \rfloor + \lfloor m/2 \rfloor^2 .$$

This finishes the proof of Theorem 2.1.1. □

The examples constructed above have the following consequence.

Corollary 2.5.1. *Let $(h^{k,0},\dots,h^{0,k})$ be a symmetric collection of natural numbers. Then there is a smooth complex projective variety X of dimension $n \geq k+1$ and a rational sub-Hodge structure $V \subseteq H^k(X,\mathbb{Q})$ with*

$$h^{k-i,i}(V_{\mathbb{C}}) = h^{k-i,i}$$

for all i.

Proof. If k is odd, we take the n-dimensional example X from Theorem 2.1.1 and put $V = H^k(X,\mathbb{Q})$.

If $k = 2m$ is even, then we replace the given $h^{m,m}$ by a sufficiently high number $h^{m,m} + l$ such that Theorem 2.1.1 yields a n-dimensional example X with these weight k Hodge numbers. In order to find the rational sub-Hodge structure $V \subseteq H^k(X,\mathbb{Q})$ we are looking for, it suffices to prove that there is a Tate-type sub-Hodge structure

$$W \subseteq H^{2m}(X,\mathbb{Q})$$

of dimension l; V is then given by the orthogonal complement $V := W^{\perp}$.

In order to prove the existence of W, it suffices to see that $H^{m,m}(X)$ is generated by algebraic classes. Up to the classes introduced by blow-ups of points, $H^{m,m}(X)$ is by (2.13) generated by the images of $H^{s,s}(T_0)^{G_0}$ and $H^{s-1,s-1}(T_i)^{G_i}$ under certain algebraic correspondences. By Lemma 2.3.2, $H^{s,s}(T_0)^{G_0}$ and $H^{s-1,s-1}(T_i)^{G_i}$ are one-dimensional, generated by the power of a G_0- respectively G_i-invariant ample class. This concludes Corollary 2.5.1. □

Remark 2.5.2. *Corollary 2.5.1 is not stated in [76]. We mention it here because it is the main result of Arapura's paper [3], which was written after the preprint version of [76] appeared on the arXiv.*

In Theorem 2.1.1 we have only dealt with Hodge structures below the middle degree. Under stronger assumptions, the following corollary of Theorem 2.1.1 deals with Hodge structures in the middle degree. We will use this corollary in the proof of Theorem 2.1.5 in Section 2.9.

Corollary 2.5.3. *Let $(h^{n,0},\dots,h^{0,n})$ be a symmetric collection of even natural numbers such that $h^{n,0} = 0$. If $n = 2m$ is even, then we additionally assume*

$$h^{m,m} \geq 2\cdot(m-1)\cdot\lfloor(m+2)/2\rfloor + 2\cdot\lfloor(m-1)/2\rfloor^2 .$$

Then there exists an n-dimensional smooth complex projective variety X whose Hodge structure of weight n realizes the given Hodge numbers.

Proof. For $n = 1$ we may put $X = \mathbb{P}^1$ and for $n = 2$ the blow-up of $\mathbb{P}^2$ in $h^{1,1} - 1$ points does the job. It remains to deal with $n \geq 3$. Here, by Theorem 2.1.1 there exists an $(n-1)$-dimensional smooth complex projective variety Y whose Hodge decomposition on $H^{n-2}(Y, \mathbb{C})$ has Hodge numbers

$$(\frac{1}{2} \cdot h^{n-1,1}, \dots, \frac{1}{2} \cdot h^{1,n-1}).$$

By the Künneth formula, the product $X := Y \times \mathbb{P}^1$ has Hodge numbers

$$h^{p,q}(X) = h^{p,q}(Y) + h^{p-1,q-1}(Y).$$

Using the Hodge symmetries on Y, Corollary 2.5.3 follows. □

2.6 Proof of Theorem 2.1.3

In this section we prove Theorem 2.1.3; we will follow the same lines as in the proof of Theorem 2.1.1 in Section 2.5.

Proof of Theorem 2.1.3. Given a truncated n-dimensional formal Hodge diamond whose Hodge numbers (resp. primitive Hodge numbers) are denoted by $h^{p,q}$ (resp. $l^{p,q}$). Suppose that one of the following two additional conditions holds:

(C1) The number $h^{k,0}$ vanishes for all $k \neq k_0$ for some $k_0 \in \{1, \dots, n-1\}$.

(C2) The number $h^{k,0}$ vanishes for all $k = 1, \dots, n-3$.

We will construct universal constants $C(p,n)$ such that under the additional assumption $l^{p,p} \geq C(p,n)$ for all $1 \leq p < n/2$, an n-dimensional smooth complex projective variety X with the given truncated Hodge diamond exists. Then Theorem 2.1.3 follows as soon as we have shown $C(p,n) \leq p \cdot (n^2 - 2n + 5)/4$.

Since blowing-up a point on X increases the primitive Hodge number $l^{1,1}(X)$ by one and leaves the remaining primitive Hodge numbers unchanged, it suffices to deal with the case where $l^{1,1} = C(1,n)$ is minimal.

To explain our construction, let us for each $r \geq s > 0$ with $2 < r + s < n$ consider the $(r+s-2)$-fold product

$$T_{r,s} := (C_{l^{r,s}})^{r+s-2},$$

where $C_{l^{r,s}}$ is the hyperelliptic curve of genus $l^{r,s}$, constructed in Section 2.3.1. On $T_{r,s}$ we consider the group action of

$$G_{r,s} := G(r-1, s-1, l^{r,s}),$$

defined in Section 2.3.2.

At this point we need to distinguish between the above cases (C1) and (C2). We begin with (C1) and consider the index set

$$I := \{(r,s) :\ r \geq s > 0, \quad n > r+s > 2\} \cup \{i_0\},$$

and put

$$T_{i_0} := (C_{l^{k_0,0}})^{k_0} \quad \text{and} \quad G_{i_0} := G(k_0, 0, l^{k_0,0}).$$

By Lemma 2.3.4, we may apply the construction method of Section 2.4.2 to the set of data (T_i, G_i, I, i_0). Thus, Proposition 2.4.2 yields an n-dimensional smooth complex projective variety X whose primitive Hodge numbers $l^{p,q}(X)$ with $p+q<n$ are given by

$$l^{p,q}(X) = \dim\left(H^{p,q}(T_{i_0})^{G_{i_0}}\right) + \sum_{(r,s)\in I\setminus\{i_0\}} \dim\left(H^{p-1,q-1}(T_{r,s})^{G_{r,s}}\right). \tag{2.14}$$

If $p>q$, then Lemmas 2.3.2 and 2.3.3 say that

$$\dim\left(H^{p-1,q-1}(T_{r,s})^{G_{r,s}}\right) = \begin{cases} 0 & \text{if } (r,s) \neq (p,q), \\ l^{p,q} & \text{if } (r,s) = (p,q). \end{cases} \tag{2.15}$$

Moreover,

$$\dim\left(H^{p,q}(T_{i_0})^{G_{i_0}}\right) = \begin{cases} 0 & \text{if } (k_0,0) \neq (p,q), \\ l^{p,q} & \text{if } (k_0,0) = (p,q). \end{cases} \tag{2.16}$$

In (2.14), the summation condition $(r,s) \in I \setminus \{i_0\}$ means $r \geq s > 0$ and $n > r+s > 2$. It therefore follows from (2.15) and (2.16) that $l^{p,q}(X) = l^{p,q}$ holds for all $p>q$ with $p+q<n$. By the Hodge symmetries on X, $l^{p,q}(X) = l^{p,q}$ then follows for all $p \neq q$ with $p+q<n$.

Next, for $p=q$, one extracts from (2.14) an explicit formula of the form

$$l^{p,p}(X) = l^{p,p} + C_1(p,n),$$

where $C_1(p,n)$ is a constant which only depends on p and n. Replacing $l^{p,p}$ by $l^{p,p} - C_1(p,n)$ in the above argument then shows that in case (C1), an n-dimensional smooth complex projective variety with the given truncated Hodge diamond exists as long as

$$l^{p,p} \geq C_1(p,n)$$

holds for all $1 \leq p < n/2$.

In order to find a rough estimation for $C_1(p,n)$, we deduce from Lemmas 2.3.2 and 2.3.3 the following inequalities

$$\dim\left(H^{p,p}(T_{i_0})^{G_{i_0}}\right) \leq 1 \quad \text{for all } p,$$

and

$$\dim\left(H^{p-1,p-1}(T_{r,s})^{G_{r,s}}\right) \le \begin{cases} p & \text{if } (r,s) \neq (p,p), \\ p + l^{p,p} & \text{if } (r,s) = (p,p). \end{cases}$$

Using these estimates, (2.14) gives

$$C_1(p,n) \le 1 + \sum_{\substack{r \ge s > 0 \\ n > r+s > 2}} p, \tag{2.17}$$

where we used that $(r,s) \in I \setminus \{i_0\}$ is equivalent to $r \ge s > 0$ and $n > r+s > 2$. If we write $\lfloor x \rfloor$ for the floor function of x, then (2.17) gives explicitly:

$$C_1(p,n) \le p \cdot n \cdot \left\lfloor \frac{n-1}{2} \right\rfloor - p \cdot \left\lfloor \frac{n-1}{2} \right\rfloor \cdot \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1 \right).$$

If n is odd, then the above right-hand-side equals $p \cdot (n-1)^2/4$ and if n is even, then it is given by $p \cdot n(n-2)/4$. Hence,

$$C_1(p,n) \le p \cdot (n-1)^2/4.$$

Let us now turn to case (C2). Here we consider the same index set I as above, and for all $i \neq i_0$ we also define T_i and G_i as above. However, for $i = i_0$, we put

$$T_{i_0} := (C_{l^{n-1,0}})^{n-1} \times (C_{l^{n-2,0}})^{n-2}$$

and

$$G_{i_0} := G(n-1, 0, l^{n-1,0}) \times G(n-2, 0, l^{n-2,0}).$$

By Lemma 2.3.4, there exist integers N_1 and N_2 such that G_{i_0} admits an embedding into $GL(N_1+1) \times GL(N_2+1)$ in such a way that an G_{i_0}-equivariant embedding of T_{i_0} into $\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}$ exists. Using the Segre map, we obtain for $N > 0$ an embedding of G_{i_0} into $\mathrm{GL}(N+1)$ and an G_{i_0}-equivariant embedding of T_{i_0} into $\mathbb{P}^N$. Moreover, by Lemma 2.3.4, T_{i_0} contains a point p_{i_0} which is fixed by G_{i_0}. Hence, the construction method of Section 2.4.2 can be applied to the above set of data. Therefore, Proposition 2.4.2 yields an n-dimensional smooth complex projective variety X whose primitive Hodge numbers $l^{p,q}(X)$ are given by formula (2.14).

For $p > q$ and $p+q < n$, the G_{i_0}-invariant cohomology of T_{i_0} is trivial whenever (p,q) is different from $(n-2,0)$ and $(n-1,0)$. Moreover, for $(p,q) = (n-1,0)$ it has dimension $l^{n-1,0}$ and for $(p,q) = (n-2,0)$ its dimension equals $l^{n-2,0}$. Thus, (2.14) and the Hodge symmetries on X yield $l^{p,q}(X) = l^{p,q}$ for all $p \neq q$ with $p+q < n$. Moreover, as in case (C1), we obtain

$$l^{p,p}(X) = l^{p,p} + C_2(p,n),$$

where $C_2(p,n)$ is a constant in p and n which can be estimated by

$$C_2(p,n) \le p + 1 + \sum_{\substack{r \ge s > 0 \\ n > r+s > 2}} p,$$

where we used that $H^{p,p}(T_{i_0})^{G_{i_0}}$ has dimension $p+1$. Our estimation for $C_1(p,n)$ shows

$$C_2(p,n) \le p \cdot (n-1)^2/4 + p.$$

Then, for $l^{p,p} \ge C_2(p,n)$, we may replace $l^{p,p}$ by $l^{p,p} - C_2(p,n)$ in the above argument and obtain an n-dimensional smooth complex projective variety with the given truncated Hodge diamond.

Let us now define

$$C(p,n) := \max\left(C_1(p,n), C_2(p,n)\right). \tag{2.18}$$

Then in both cases, (C1) and (C2), a variety with the desired truncated Hodge diamond exists if $l^{p,p} \ge C(p,n)$. Moreover, $C(p,n)$ can roughly be estimated by

$$C(p,n) \le p \cdot \frac{n^2 - 2n + 5}{4}.$$

This finishes the proof of Theorem 2.1.3. □

Remark 2.6.1. *As we have seen in the above proof, we may replace the given lower bound on $l^{p,p}$ in assumption 1 of Theorem 2.1.3 by the smaller constant $C(p,n)$, defined in (2.18).*

2.7 Special weight 2 Hodge structures

In this section we show that for weight two Hodge structures, the lower bound $h^{1,1} \ge 2$ in Theorem 2.1.1 can be replaced by the optimal lower bound $h^{1,1} \ge 1$. Our proof uses an ad hoc implementation of the Godeaux-Serre construction. The examples we construct here compare nicely to the results in Section 2.11. However, since the methods of this section are not used elsewhere in this thesis, the reader can easily skip this section.

Theorem 2.7.1. *Let $h^{2,0}$ and $h^{1,1}$ be natural numbers with $h^{1,1} \ge 1$. Then in each dimension ≥ 3 there exists a smooth complex projective variety X with*

$$h^{2,0}(X) = h^{2,0} \quad \textit{and} \quad h^{1,1}(X) = h^{1,1}.$$

Proof. Since blowing-up a point increases $h^{1,1}$ by one and leaves $h^{2,0}$ unchanged, in order to prove Theorem 2.7.1, it suffices to construct for given g in each dimension $n > 2$ a smooth complex projective variety X with $h^{2,0}(X) = g$ and $h^{1,1}(X) = 1$.

We fix some large integers N_1 and N_2 and consider $T := C_g^2$ together with the subgroups $G^1(2,0,g)$ and $G^2(2,0,g)$ of $\mathrm{Aut}(T)$, defined in Section 2.3.2. For $j = 1,\dots,N_1$, we denote a copy of T^{N_2} by A_j and we put

$$A := A_1 \times \cdots \times A_{N_1}.$$

That is, A is a $(2 \cdot N_1 \cdot N_2)$-fold product of C_g, but we prefer to think of A to be an N_1-fold product of T^{N_2}, where the j-th factor is denoted by A_j.

Next, we explain the construction of a certain subgroup G of automorphisms of A. This group is generated by five finite subgroups $G_1,\dots,G_5$ in $\mathrm{Aut}(A)$. The first subgroup of $\mathrm{Aut}(A)$ is given by

$$G_1 := G^1(2,0,g)^{\times N_1},$$

where $G^1(2,0,g)$ acts on each A_j via the diagonal action. The second one is

$$G_2 := G^1(2,0,g)^{\times N_2},$$

acting on A via the diagonal action. The third one is given by

$$G_3 := G^2(2,0,g),$$

acting on each A_j as well as on A via the diagonal action. The fourth group of automorphisms of A equals

$$G_4 := \mathrm{Sym}(N_1),$$

which acts on A via permutation of the A_j's. Finally, we put

$$G_5 := \mathrm{Sym}(N_2),$$

which permutes the T-factors of each A_j and acts on A via the diagonal action.

Suppose we are given some elements $\phi_i \in G_i$. Then, ϕ_3 commutes with ϕ_4 and ϕ_5, and $\phi_3 \circ \phi_1 = \phi_1' \circ \phi_3$, respectively $\phi_1 \circ \phi_3 = \phi_3 \circ \phi_1''$ as well as $\phi_3 \circ \phi_2 = \phi_2' \circ \phi_3$, respectively $\phi_2 \circ \phi_3 = \phi_3 \circ \phi_2''$ holds for some $\phi_i', \phi_i'' \in G_i$, where $i = 1,2$. Similar relations can be checked for all products $\phi_i \circ \phi_j$ and so we conclude that each element ϕ in the group $G \subseteq \mathrm{Aut}(A)$, which is generated by $G_1,\dots,G_5$, can be written in the form

$$\phi = \phi_1 \circ \phi_2 \circ \phi_3 \circ \phi_4 \circ \phi_5,$$

where ϕ_i lies in G_i.

Suppose that the fixed point set $\mathrm{Fix}_A(\phi)$ contains an irreducible component whose codimension is less than

$$\min\left(N_1/2, 2N_2\right).$$

Since ϕ is just some permutation of the $2N_1N_2$ curve factors of A, followed by automorphisms of each factor, we deduce that ϕ needs to fix more than

$$2N_1N_2 - \min(N_1, 4N_2)$$

curve factors. If ϕ_4 were nontrivial, then ϕ would fix at most $2(N_1-2)N_2$ curve factors, and if ϕ_5 were nontrivial, then ϕ would fix at most $2N_1(N_2-2)$ curve factors. Thus, $\phi_4 = \phi_5 = \mathrm{id}$. If ϕ_3 were nontrivial, then its action on a single factor $T = C_g^2$ cannot permute the two curve factors. Thus, ϕ_3 is just multiplication with -1 on each curve factor. This cannot be canceled with automorphisms in $G^1(2,0,g)$, since the latter is a cyclic group of order $2g+1$. Therefore, $\phi_3 = \mathrm{id}$ follows as well.

Since ϕ fixes more than $2N_1N_2 - N_1$ curve factors, we see that $\phi = \phi_1 \circ \phi_2$ needs to be the identity on at least one A_{j_0}. Since ϕ_2 acts on each A_j in the same way, it lies in $G_1 \cap G_2$ and so we may assume $\phi_2 = \mathrm{id}$. Finally, any nontrivial automorphism in G_1 has a fixed point set of codimension $\geq 2N_2$. This is a contradiction.

For N_1 and N_2 large enough, it follows that the G-action on A is free outside a subset of codimension $> n$. Then, by Lemma 2.2.1, A/G contains a smooth n-dimensional subvariety X whose cohomology below degree n is given by the G-invariants of A.

To conclude Theorem 2.7.1, it remains to show $h^{2,0}(X) = g$ and $h^{1,1}(X) = 1$. For this purpose, we denote the fundamental class of the j-th curve factor of A by

$$\Omega_j \in H^{1,1}(A).$$

Moreover, we pick for $j = 1, \ldots, 2N_1N_2$ a basis $\omega_{j1}, \ldots, \omega_{jg}$ of $(1,0)$-classes of the j-th curve factor of A in such a way that

$$\psi_g^* \omega_{jl} = \zeta^l \omega_{jl},$$

for a fixed $(2g+1)$-th root of unity ζ holds. Then the cohomology ring of A is generated by the Ω_j's, ω_{jl}'s and their conjugates.

Suppose that we are given a G-invariant $(1,1)$-class which contains $\omega_{is} \wedge \overline{\omega_{jr}}$ nontrivially. Then application of a suitable automorphism in G_1 shows that after relabeling $A_1, \ldots, A_{N_1}$, we may assume $1 \leq i, j \leq 2N_2$. Moreover, it follows that i and j have the same parity, since otherwise $r+s$ is zero modulo $2g+1$, which contradicts $1 \le r, s \le g$. Finally, application of a suitable element in G_2 shows $i = j$. Since $\omega_{is} \wedge \overline{\omega_{ir}}$ is a multiple of Ω_i, it follows that our G-invariant $(1,1)$-class is of the form

$$\lambda_1 \cdot \Omega_1 + \cdots + \lambda_{2N_1N_2} \cdot \Omega_{2N_1N_2}.$$

Since G acts transitively on the curve factors of A, this class is G-invariant if and only if $\lambda_1 = \cdots = \lambda_{2N_1N_2}$. This proves $h^{1,1}(X) = 1$.

It remains to show $h^{2,0}(X) = g$. Therefore, we define for $l = 1, \ldots, g$ the $(2,0)$-class

$$\omega_l := \sum_{i=1}^{N_1N_2} \omega_{2i-1\,l} \wedge \omega_{2i\,l}$$

and claim that these form a basis of the G-invariant $(2,0)$-classes of A. Clearly, they are linearly independent and it is easy to see that they are G-invariant.

Conversely, suppose that a G-invariant class contains $\omega_{il_1} \wedge \omega_{jl_2}$ nontrivially. Then, application of a suitable element in G_1 shows that $l_1 \pm l_2$ is zero modulo $2g+1$. This implies $l_1 = l_2$. Therefore, our G-invariant $(2,0)$-class is of the form

$$\sum_{ijl} \lambda_{ijl} \cdot \omega_{il} \wedge \omega_{jl}.$$

For fixed $l = 1, \ldots, g$, we write $\lambda_{ij} = \lambda_{ijl}$ and note that

$$\sum_{ij} \lambda_{ij} \cdot \omega_{il} \wedge \omega_{jl}$$

is also G-invariant. We want to show that this class is a multiple of ω_l. To that end we apply suitable elements of G_1 to see that the above $(2,0)$-class is a sum of $(2,0)$-classes of the factors $A_1, \ldots, A_{N_1}$. Since this sum is invariant under the permutation of the factors $A_1, \ldots, A_{N_1}$, it suffices to consider the class

$$\sum_{i,j=1}^{2N_2} \lambda_{ij} \cdot \omega_{il} \wedge \omega_{jl}$$

on A_1, which is invariant under the induced G_2- and G_5-action on A_1. In this sum we may assume $\lambda_{ij} = 0$ for all $i \ge j$ and application of a suitable element in G_2 shows that the above class is given by

$$\sum_{i=1}^{N_2} \lambda_{2i-1\,2i} \cdot \omega_{2i-1\,l} \wedge \omega_{2i\,l}.$$

Finally, application of elements of G_5 proves that our class is a multiple of

$$\sum_{i=1}^{N_2} \omega_{2i-1\,l} \wedge \omega_{2i\,l}.$$

This finishes the proof of $h^{2,0}(X) = g$ and thereby establishes Theorem 2.7.1. $\square$

Remark 2.7.2. *The above construction does not generalize to higher degrees – at least not in the obvious way.*

2.8 Primitive Hodge numbers away from the vertical middle axis

In this section we produce examples whose primitive Hodge numbers away from the vertical middle axis of the Hodge diamond (2.4) are concentrated in a single (p,q)-type. These examples will then be used in the proof of Theorem 2.1.5 in Section 2.9. Our precise result is as follows:

Theorem 2.8.1. *For $a > b \geq 0$, $n \geq a+b$ and $c \geq 1$, there exists an n-dimensional smooth complex projective variety whose primitive (p,q)-type cohomology has dimension $(3^c-1)/2$ if $p = a$ and $q = b$, and vanishes for all other $p > q$.*

In comparison with Theorem 2.1.3, the advantage of Theorem 2.8.1 is that it also controls the Hodge numbers $h^{p,q}$ with $p \neq q$ and $p+q = n$. These numbers lie in the horizontal middle row of the Hodge diamond (2.4) and so they were excluded in the statement of Theorem 2.1.3.

Using an iterated resolution of $(\mathbb{Z}/3\mathbb{Z})$-quotient singularities whose local description is given in Section 2.8.1, we explain an inductive construction method in Section 2.8.2. Using this construction, Theorem 2.8.1 will easily follow in Section 2.8.3. Our approach is inspired by Cynk–Hulek's construction of rigid Calabi-Yau manifolds [17].

2.8.1 Local resolution of $\mathbb{Z}/3\mathbb{Z}$-quotient singularities

Fix a primitive third root of unity ξ and choose affine coordinates $(x_1,\dots,x_n)$ on $\mathbb{C}^n$. For an open ball $Y \subseteq \mathbb{C}^n$ centered at 0 and for some $r \geq 0$, we consider the automorphism $\phi : Y \to Y$ given by

$$(x_1,\dots,x_n) \longmapsto (\xi \cdot x_1,\dots,\xi \cdot x_r, \xi^2 \cdot x_{r+1},\dots,\xi^2 \cdot x_n).$$

Let Y' be the blow-up of Y in the origin with exceptional divisor $E' \subseteq Y'$. Then ϕ lifts to an automorphism $\phi' \in \mathrm{Aut}(Y')$ and we define Y'' to be the blow-up of Y' along $\mathrm{Fix}_{Y'}(\phi')$. The exceptional divisor of this blow-up is denoted by $E'' \subseteq Y''$ and ϕ' lifts to an automorphism $\phi'' \in \mathrm{Aut}(Y'')$. In this situation, we have the following lemma.

Lemma 2.8.2. *The fixed point set of ϕ'' on Y'' equals E''. Moreover:*

1. *If $r = 0$ or $r = n$, then $E'' \simeq E' \simeq \mathbb{P}^{n-1}$. Otherwise, $E' \simeq \mathbb{P}^{n-1}$ and E'' is a disjoint union of $\mathbb{P}^{r-1} \times \mathbb{P}^{n-r}$ and $\mathbb{P}^{r} \times \mathbb{P}^{n-r-1}$.*

2. *The quotient Y''/ϕ'' is smooth and admits local holomorphic coordinates $(z_1, \dots, z_n)$ where each z_j comes from a ϕ-invariant meromorphic function on Y, explicitly given by a quotient of two monomials in $x_1, \dots, x_n$.*

Proof. This Lemma is proven by a calculation, similar to that in [41, pp. 84-87], where the case $n = 2$ is carried out.

The automorphism ϕ' acts on the exceptional divisor $E' \simeq \mathbb{P}^{n-1}$ of $Y' \to Y$ as follows:

$$[x_1 : \dots : x_n] \longmapsto [\xi \cdot x_1 : \dots : \xi \cdot x_r : \xi^2 \cdot x_{r+1} : \dots : \xi^2 \cdot x_n].$$

Hence, if $r = 0$ or $r = n$, then $\mathrm{Fix}_{Y'}(\phi')$ equals E'. Since this is a smooth divisor on Y', the blow-up $Y'' \to Y'$ is an isomorphism and the quotient Y''/ϕ'' is smooth. Moreover, $E' \simeq E''$ is covered by n charts $U_1, \dots, U_n$ such that on U_i, coordinates are given by

$$\left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, x_i, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right). \tag{2.19}$$

The quotient Y''/ϕ'' is then covered by $U_1/\phi'', \dots, U_n/\phi''$. Coordinate functions on U_i/ϕ'' are given by the following ϕ-invariant rational functions on Y:

$$\left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, x_i^3, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right).$$

This proves the Lemma for $r = 0$ or $r = n$.

If $0 < r < n$, then $\mathrm{Fix}_{Y'}(\phi')$ equals the disjoint union of $E'_1 \simeq \mathbb{P}^{r-1}$ and $E'_2 \simeq \mathbb{P}^{n-r-1}$, sitting inside E'. The exceptional divisor E' is still covered by the n-charts $U_1, \dots, U_n$, defined above. Moreover, the charts $U_1, \dots, U_r$ cover E'_1 and $U_{r+1}, \dots, U_n$ cover E'_2. Fix a chart U_i with coordinate functions $(z_1, \dots, z_n)$. If $i \le r$, then ϕ' acts on $r - 1$ of these coordinates by the identity and on the remaining coordinates by multiplication with ξ. Conversely, if $i > r$, then ϕ' acts on $n - r - 1$ coordinates by the identity and on the remaining coordinates by multiplication with ξ^2. We are therefore in the situation discussed in the previous paragraph and the Lemma follows by an application of that result in dimension $n - r + 1$ and $r + 1$ respectively. $\square$

2.8.2 Inductive approach

In this section we explain a general construction method which will allow us to prove Theorem 2.8.1 by induction on the dimension in Section 2.8.3.

For natural numbers $a \neq b$ and $c \geq 0$, let $\mathcal{S}_c^{a,b}$ denote the family of pairs (X, ϕ), consisting of a smooth complex projective variety X of dimension $a+b$ and an automorphism $\phi \in \operatorname{Aut}(X)$ of order 3^c, such that properties (P1)–(P5) below hold. Here, ζ denotes a fixed primitive 3^c-th root of unity and $g := (3^c - 1)/2$:

(P1) The Hodge numbers $h^{p,q}$ of X are given by $h^{a,b} = h^{b,a} = g$ and $h^{p,q} = 0$ for all other $p \neq q$.

(P2) The action of ϕ on $H^{a,b}(X)$ has eigenvalues $\zeta, \dots, \zeta^g$.

(P3) The group $H^{p,p}(X)$ is for all $p \geq 0$ generated by algebraic classes which are fixed by the action of ϕ.

(P4) The set $\operatorname{Fix}_X\left(\phi^{3^{c-1}}\right)$ can be covered by local holomorphic charts such that ϕ acts on each coordinate function by multiplication with some power of ζ.

(P5) For $0 \leq l \leq c-1$, the cohomology of $\operatorname{Fix}_X\left(\phi^{3^l}\right)$ is generated by algebraic classes which are fixed by the action of ϕ.

For $0 \leq l \leq c-1$, we have obvious inclusions

$$\operatorname{Fix}_X\left(\phi^{3^l}\right) \subseteq \operatorname{Fix}_X\left(\phi^{3^{c-1}}\right).$$

It therefore follows from (P4) that $\operatorname{Fix}_X\left(\phi^{3^l}\right)$ can be covered by local holomorphic coordinates on which ϕ^{3^l} acts by multiplication with some power of ζ^{3^l}. In particular, $\operatorname{Fix}_X\left(\phi^{3^l}\right)$ is smooth for all $0 \leq l \leq c-1$; its cohomology is of (p,p)-type, since it is generated by algebraic classes by (P5). We also remark that condition (P3) implies that each variety in $\mathcal{S}_c^{a,b}$ satisfies the Hodge conjecture. Finally, note that $(X, \phi) \in \mathcal{S}_c^{a,b}$ is equivalent to $(X, \phi^{-1}) \in \mathcal{S}_c^{b,a}$.

The inductive approach to Theorem 2.8.1 is now given by the following.

Proposition 2.8.3. *Let* $(X_1, \phi_1^{-1}) \in \mathcal{S}_c^{a_1,b_1}$ *and* $(X_2, \phi_2) \in \mathcal{S}_c^{a_2,b_2}$*. Then*

$$(X_1 \times X_2) / \langle \phi_1 \times \phi_2 \rangle$$

admits a smooth model X *such that the automorphism* $\operatorname{id} \times \phi_2$ *on* $X_1 \times X_2$ *induces an automorphism* $\phi \in \operatorname{Aut}(X)$ *with* $(X, \phi) \in \mathcal{S}_c^{a,b}$*, where* $a = a_1 + a_2$ *and* $b = b_1 + b_2$*.*

Proof. We define the subgroup

$$G := \langle \phi_1 \times \mathrm{id}, \mathrm{id} \times \phi_2 \rangle$$

of $\mathrm{Aut}(X_1 \times X_2)$. For $i = 1, \ldots, c$ we consider the element

$$\eta_i := (\phi_1 \times \phi_2)^{3^{c-i}}$$

of order 3^i in G. This element generates a cyclic subgroup

$$G_i := \langle \eta_i \rangle \subseteq G,$$

and we obtain a filtration

$$0 = G_0 \subset G_1 \subset \cdots \subset G_c = \langle \phi_1 \times \phi_2 \rangle ,$$

such that each quotient G_i/G_{i-1} is cyclic of order three, generated by the image of η_i.

By definition, G acts on

$$Y_0 := X_1 \times X_2.$$

Using the assumptions that (X_1, ϕ_1^{-1}) and (X_2, ϕ_2) satisfy (P1)–(P3), it is easily seen (and we will give the details later in this proof) that the $\langle \phi_1 \times \phi_2 \rangle$-invariant cohomology of Y_0 has Hodge numbers $h^{a,b} = h^{b,a} = g$ and $h^{p,q} \neq 0$ for all other $p \neq q$. The strategy of the proof of Proposition 2.8.3 is now as follows.

We will construct inductively for $i = 1, \ldots, c$ smooth models Y_i of Y_0/G_i, fitting into the following diagram:

$$\begin{tikzcd}[column sep=small]
& Y''_{c-1} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & Y''_1 \arrow[dl] \arrow[dr] & & Y''_0 \arrow[dl] \arrow[dr] & \\
Y_c & & Y_{c-1} & & Y_2 & & Y_1 & & Y_0.
\end{tikzcd} \tag{2.20}$$

Here, $Y''_{i-1} \to Y_i$ will be a $3:1$ cover, branched along a smooth divisor, and $Y''_i \to Y_i$ will be the composition $Y''_i \to Y'_i \to Y_i$ of two blow-down maps. This way we obtain a smooth model

$$X := Y_c$$

of $Y_0/\langle \phi_1 \times \phi_2 \rangle$. At each stage of our construction, the group G will act (in general non-effectively) and we will show that each blow-up and each triple quotient changes the $\langle \phi_1 \times \phi_2 \rangle$-invariant cohomology only by algebraic classes which are fixed by the G-action. Since $\langle \phi_1 \times \phi_2 \rangle$ acts trivially on X, it follows that $H^*(X,\mathbb{C})$ is generated by $\langle\phi_1\times\phi_2\rangle$-invariant classes on Y_0 together with algebraic classes which are fixed by the action of G. Hence, X satisfies (P1). We then define $\phi\in\operatorname{Aut}(X)$ via the action of $\mathrm{id}\times\phi_2\in G$ on Y_c and show carefully that the technical conditions (P2)–(P5) are met by (X,ϕ).

In the following, we give the details of the approach outlined above.

We begin with the explicit construction of diagram (2.20). Firstly, let Y_0' be the blow-up of Y_0 along $\operatorname{Fix}_{Y_0}(\eta_1)$. Since G is an abelian group, its action on Y_0 restricts to an action on $\operatorname{Fix}_{Y_0}(\eta_1)$ and so it lifts to an action on the blow-up Y_0'. This allows us to define Y_0'' via the blow-up of Y_0' along $\operatorname{Fix}_{Y_0'}(\eta_1)$. Again, G lifts to Y_0'' since it is abelian. Using this action, we define

$$Y_1 \coloneqq Y_0''/\langle\eta_1\rangle\,,$$

where by abuse of notation, $\langle\eta_1\rangle$ denotes the subgroup of $\operatorname{Aut}(Y_0'')$ which is generated by the action of $\eta_1\in G$.

We claim that Y_1 is a smooth model of $Y_0/\langle\eta_1\rangle$. To see this, we define

$$U_0 \coloneqq Y_0\smallsetminus\operatorname{Fix}_{Y_0}(\eta_1)$$

and note that the preimage of this set under the blow-down maps

$$Y_0''\longrightarrow Y_0'\longrightarrow Y_0$$

gives Zariski open subsets

$$U_0'\subseteq Y_0' \quad\text{and}\quad U_0''\subseteq Y_0'',$$

both isomorphic to U_0. The group G acts on these subsets and so

$$U_1\coloneqq U_0''/\langle\eta_1\rangle$$

is a Zariski open subset in Y_1 which is isomorphic to the Zariski open subset

$$U_0/\langle\eta_1\rangle\subseteq Y_0/\langle\eta_1\rangle\,.$$

The latter is smooth since η_1 acts freely on U_0 and so it remains to see that Y_1 is smooth at points of the complement of $U_1\subseteq Y_1$. To see this, note that by (P4),

$$\operatorname{Fix}_{Y_0}(\eta_1)=\operatorname{Fix}_{X_1}(\phi_1^{3^{c-1}})\times\operatorname{Fix}_{X_2}(\phi_2^{3^{c-1}})$$

inside Y_0 can be covered by local holomorphic coordinates on which $\phi_1\times\phi_2$ acts by multiplication with some powers of ζ. On these coordinates, η_1 acts by multiplication with some powers of a third root of unity. The local considerations of Lemma 2.8.2 therefore apply and we deduce that Y_1 is indeed a smooth model of Y_0/G_1.

Since G is abelian, the G-action on Y_0'' descends to a G-action on Y_1. The subgroup $G_1 \subseteq G$ acts trivially on Y_1 and the induced G/G_1-action on Y_1 is effective. Also note that G_i acts freely on $U_0 \subseteq Y_0$ and so G_i/G_1 acts, for $2 \le i \le c$, freely on the Zariski open subset $U_1 \subseteq Y_1$. By (P4), the complement of U_0 in Y_0 can be covered by local holomorphic coordinates on which G acts by multiplication with some roots of unity on each coordinate. It therefore follows from the second statement in Lemma 2.8.2 that the complement of U_1 in Y_1 can also be covered by local holomorphic coordinates in which G acts by multiplication with some roots of unity on each coordinate. This shows that we can repeat the above construction inductively.

We obtain for $i \in \{1, \dots, c\}$ smooth models

$$Y_i := Y_{i-1}''/\langle \eta_i \rangle$$

of Y_0/G_i on which G acts (non-effectively). The smooth model Y_i contains a Zariski open subset

$$U_i \simeq U_0/\langle \eta_i \rangle$$

on which G_l/G_i acts freely for all $i+1 \le l \le c$; explicitly, $U_i := U_{i-1}''/\langle \eta_i \rangle$, where $U_{i-1}'' \subseteq Y_{i-1}''$ is isomorphic to U_{i-1}. The complement of U_i is covered by local holomorphic coordinates on which G acts by multiplication with some roots of unity on each coordinate.

Y_i'' is then defined via the two-fold blow-up

$$Y_i'' \longrightarrow Y_i' \longrightarrow Y_i, \tag{2.21}$$

where one blows up the fixed point set of the action of η_{i+1} on Y_i and Y_i' respectively. The preimage of U_i in Y_i' and Y_i'' gives Zariski open subsets

$$U_i' \subseteq Y_i' \quad \text{and} \quad U_i'' \subseteq Y_i'',$$

which are both isomorphic to U_i. Since G is abelian, the G-action on Y_i induces actions on Y_i' and Y_i'' and these actions restrict to actions on $U_i \simeq U_i' \simeq U_i''$. The complement of U_i' in Y_i' (resp. U_i'' in Y_i'') is by Lemma 2.8.2 covered by local holomorphic coordinates on which G acts by multiplication with some roots of unity on each coordinate. Using the local considerations in Lemma 2.8.2, it follows that $Y_{i+1} = Y_i''/\langle \eta_{i+1} \rangle$ is a smooth model of Y_0/G_{i+1} which has the above stated properties. This finishes the inductive construction of diagram (2.20).

Our next aim is to compute the cohomology of Y_c. Since G_c acts trivially on Y_c, we may as well compute the G_c-invariant cohomology of Y_c. This point of view has the advantage that it allows an inductive approach, since for $i = 0, \dots, c-1$, the G_c-invariant cohomology of Y_i is easier to compute than its ordinary cohomology.

Before we can carry out these calculations, we have to study the action of arbitrary subgroups $\Gamma \subseteq G$ on Y_i, Y_i' and Y_i''. Since G is an abelian group, it follows that it acts on the fixed point sets $\mathrm{Fix}_{Y_i}(\Gamma)$, $\mathrm{Fix}_{Y_i'}(\Gamma)$ and $\mathrm{Fix}_{Y_i''}(\Gamma)$, defined in (2.7). These actions have the following important properties, where as usual, cohomology means singular cohomology with coefficients in $\mathbb{C}$ (see our conventions in Section 2.1.5).

Lemma 2.8.4. *Let $\Gamma \subseteq G$ be a subgroup which is not contained in G_i. Then $\mathrm{Fix}_{Y_i}(\Gamma)$, $\mathrm{Fix}_{Y_i'}(\Gamma)$ and $\mathrm{Fix}_{Y_i''}(\Gamma)$ are smooth, their G-actions restrict to actions on each irreducible component and their G_c-invariant cohomology is generated by G-invariant algebraic classes.*

Note that the assumption $\Gamma \notin G_i$ is equivalent to saying that the action of Γ is nontrivial on each of the spaces Y_i, Y_i' and Y_i''.

Proof of Lemma 2.8.4. To begin with, we want to verify the Lemma for

$$\mathrm{Fix}_{Y_0}(\Gamma),$$

where $\Gamma \subseteq G$ is nontrivial. Recall that $Y_0 = X_1 \times X_2$ and that each element in Γ is of the form $\phi_1^j \times \phi_2^k$. The fixed point set of such an element is then given by

$$\mathrm{Fix}_{Y_0}(\phi_1^j \times \phi_2^k) = \mathrm{Fix}_{X_1}(\phi_1^j) \times \mathrm{Fix}_{X_2}(\phi_2^k).$$

The intersection of sets of the above form is still of the above form and so

$$\mathrm{Fix}_{Y_0}(\Gamma) = \mathrm{Fix}_{X_1}(\phi_1^j) \times \mathrm{Fix}_{X_2}(\phi_2^k),$$

for some natural numbers j and k. Since (X_1, ϕ_1^{-1}) and (X_2, ϕ_2) satisfy (P4), it follows that $\mathrm{Fix}_{Y_0}(\Gamma)$ is smooth. Also, G acts trivially on $H^0(\mathrm{Fix}_{Y_0}(\Gamma), \mathbb{C})$ by (P5) and so the G-action restricts to an action on each irreducible component of $\mathrm{Fix}_{Y_0}(\Gamma)$.

Since Γ is not the trivial group, we now assume without loss of generality that j is not divisible by 3^c. Since (X, ϕ_1^{-1}) satisfies (P5), the cohomology of $\mathrm{Fix}_{X_1}(\phi_1^j)$ is then generated by $\langle\phi_1\rangle$-invariant algebraic classes. The G_c-invariant cohomology of $\mathrm{Fix}_{Y_0}(\Gamma)$ is therefore generated by products of these algebraic classes with $\langle\phi_2\rangle$-invariant classes on $\mathrm{Fix}_{X_2}(\phi_2^k)$. Since (X_2, ϕ_2) satisfies (P1)–(P3) and (P5), the latter are, regardless whether k is divisible by 3^c or not, given by $\langle\phi_2\rangle$-invariant algebraic classes. This shows that the G_c-invariant cohomology of $\mathrm{Fix}_{Y_0}(\Gamma)$ is generated by G-invariant algebraic classes, as we want.

Using induction, let us now assume that the Lemma is true for $\mathrm{Fix}_{Y_i}(\Gamma)$ for some $i \geq 0$ and for all $\Gamma \notin G_i$. Blowing-up $\mathrm{Fix}_{Y_i}(\eta_{i+1})$ on Y_i, we obtain the following diagram:

$$\begin{tikzcd} \mathrm{Fix}_{Y_i'}(\Gamma) \arrow[r, hook] \arrow[d] & Y_i' \arrow[d] \\ \mathrm{Fix}_{Y_i}(\Gamma) \arrow[r, hook] & Y_i \end{tikzcd}$$

and we denote the exceptional divisor of the blow-up $Y_i' \to Y_i$ by $E_i' \subseteq Y_i'$.

Let us first prove that $\mathrm{Fix}_{Y_i'}(\Gamma)$ is smooth and that G acts on its irreducible components. To see this, note that away from E_i', the blow-down map $Y_i' \to Y_i$ is an isomorphism onto its image. Since $\mathrm{Fix}_{Y_i}(\Gamma)$ is smooth, it is then clear that the intersection of $\mathrm{Fix}_{Y_i'}(\Gamma)$ with $Y_i' \smallsetminus E_i'$ is smooth. Also, G acts on the irreducible components of $\mathrm{Fix}_{Y_i'}(\Gamma)$ which are not contained in E_i', since the analogous statement holds for the components of $\mathrm{Fix}_{Y_i}(\Gamma)$. On the other hand, E_i' can be covered by local holomorphic coordinates on which G acts by multiplication with roots of unity. In each of these charts, $\mathrm{Fix}_{Y_i'}(\Gamma)$ corresponds to a linear subspace on which G acts. We conclude that $\mathrm{Fix}_{Y_i'}(\Gamma)$ is smooth and that G acts on each of its irreducible components.

Next, let P be an irreducible component of $\mathrm{Fix}_{Y_i'}(\Gamma)$. We have to prove the following

Claim 2.8.5. *The G_c-invariant cohomology of P is generated by G-invariant algebraic classes.*

Proof. Let us denote the image of P in Y_i by Z. Then Z is contained in $\mathrm{Fix}_{Y_i}(\Gamma)$ and the proof of the claim is divided into two cases.

In the first case, we suppose that Z is not contained in the intersection

$$\mathrm{Fix}_{Y_i}(\langle \Gamma, \eta_{i+1} \rangle) = \mathrm{Fix}_{Y_i}(\Gamma) \cap \mathrm{Fix}_{Y_i}(\eta_{i+1}). \tag{2.22}$$

In this case, P is the strict transform of Z in Y_i'. Conversely, if $\tilde{Z} \subseteq \mathrm{Fix}_{Y_i}(\Gamma)$ is any irreducible component, not contained in (2.22), then its strict transform in Y_i' is contained in $\mathrm{Fix}_{Y_i'}(\Gamma)$. Hence, Z is in fact an irreducible component of $\mathrm{Fix}_{Y_i}(\Gamma)$. This implies that $\mathrm{Fix}_Z(\eta_{i+1})$ consists of irreducible components of (2.22) and so $\mathrm{Fix}_Z(\eta_{i+1})$ is smooth by induction. Moreover, the strict transform P of Z in Y_i' can be identified with the blow-up of Z along $\mathrm{Fix}_Z(\eta_{i+1})$. We denote the exceptional divisor of this blow-up by D and obtain natural maps

$$f : D \hookrightarrow P \quad \text{and} \quad g : D \to \mathrm{Fix}_Z(\eta_{i+1}),$$

where f denotes the inclusion and g the projection map respectively. Using Theorem 2.4.1 and (2.11), we see that the cohomology of P is generated (as a $\mathbb{C}$-module) by pull-back classes of Z together with products

$$[D']^j \wedge f_*(g^*(\alpha)),$$

where D' is an irreducible component of D, j is some natural number and α is a cohomology class on $\mathrm{Fix}_Z(\eta_{i+1})$.

The image $g(D')$ is an irreducible component of $\mathrm{Fix}_Z(\eta_{i+1})$. By induction, G acts on $g(D')$ and hence also on D', the projectivization of the normal bundle of $g(D')$ in Z. This implies that $[D'] \in H^*(P,\mathbb{C})$ is a G-invariant algebraic class. Moreover, the G_c-invariant cohomology of Z as well as the G_c-invariant cohomology of $\mathrm{Fix}_Z(\eta_{i+1})$ is generated by G-invariant algebraic classes by induction. It therefore follows from the above description of $H^*(P,\mathbb{C})$ that the G_c-invariant cohomology of P is indeed generated by G-invariant algebraic classes.

It remains to deal with the case where the image Z of P in Y_i is contained in (2.22). In this case, around each point of Z there are local holomorphic coordinates $(z_1,\dots,z_n)$ on which G acts by multiplication with some roots of unity. In these local coordinates, the fixed point set of η_{i+1} corresponds to the vanishing set of certain coordinate functions. After relabeling these coordinate functions if necessary, we may therefore assume that locally, $\mathrm{Fix}_{Y_i}(\eta_{i+1})$ corresponds to $\{z_m = \dots = z_n = 0\}$ for some $m \le n$. This yields local homogeneous coordinates

$$(z_1,\dots,z_{m-1},[z_m:\dots:z_n]) \tag{2.23}$$

along the exceptional divisor E_i' of $Y_i' \to Y_i$. After relabeling of the first $m-1$ coordinates if necessary, we may assume that Γ acts trivially on $z_1,\dots,z_{k-1}$ and nontrivially on $z_k,\dots,z_{m-1}$ for some $1 \le k \le m-1$. After relabeling $z_m,\dots,z_n$ if necessary, we may then assume that in the homogeneous coordinates (2.23), P corresponds to $\{z_k = \dots = z_h = 0\}$ for some $m \le h \le n$. Here, each element $\gamma \in \Gamma$ acts trivially on $[z_{h+1}:\dots:z_n]$, that is, γ acts by multiplication with the same root of unity on $z_{h+1},\dots,z_n$.

The above local description shows that $P \to Z$ is a PGL-subbundle of the PGL-bundle $E_i'|_Z \to Z$; explicit bundle charts for P are given by

$$(z_1,\dots,z_{k-1},[z_{h+1}:\dots:z_n]),$$

as above. The exceptional divisor E_i' carries the line bundle $\mathcal{O}_{E_i'}(1)$ and we denote its restriction to P by $\mathcal{O}_P(1)$. The cohomology of P is then generated (as a $\mathbb{C}$-module) by products of pull-back classes on the base Z with powers of $c_1(\mathcal{O}_P(1))$. The line bundle $\mathcal{O}_{E_i'}(1)$ on the exceptional divisor E_i' is isomorphic to the restriction of the line bundle $\mathcal{O}_{Y_i'}(-E_i')$ on Y_i'. The first Chern class of the latter line bundle is G-invariant since G acts on E_i'. It follows that $c_1(\mathcal{O}_P(1))$ is a G-invariant algebraic cohomology class on P.

In the above local coordinates $(z_1,\dots,z_n)$ on Y_i, Z is given by

$$\{z_k = \dots = z_n = 0\}.$$

The latter set is in fact the fixed point set of $\langle \Gamma, \eta_{i+1} \rangle$ in this local chart and so it follows that Z is an irreducible component of (2.22). By induction, the G_c-invariant cohomology of Z is therefore generated by G-invariant algebraic classes. By the above description of $H^*(P,\mathbb{C})$, we conclude that the G_c-invariant cohomology of P is generated by G-invariant algebraic classes, as we want. This finishes the proof of our claim. $\square$

Altogether, we see that the Lemma 2.8.4 holds for $\mathrm{Fix}_{Y_i'}(\Gamma)$. Repeating the above argument, we then deduce the same assertion for $\mathrm{Fix}_{Y_i''}(\Gamma)$.

Next, let Γ be a subgroup of G, not contained in G_{i+1}. We denote by

$$p_i : Y_i'' \longrightarrow Y_{i+1}$$

the quotient map. Then,

$$p_i^{-1}(\mathrm{Fix}_{Y_{i+1}}(\Gamma)) = \left\{ y \in Y_i'' \mid g(y) \in \left\{ y, \eta_{i+1}(y), \eta_{i+1}^2(y) \right\} \text{ for all } g \in \Gamma \right\}.$$

If this set is contained in $\mathrm{Fix}_{Y_i''}(\eta_{i+1})$, then it is given by $\mathrm{Fix}_{Y_i''}(\langle \Gamma, \eta_{i+1} \rangle)$. The restriction of p_i to $\mathrm{Fix}_{Y_i''}(\eta_{i+1})$ is an isomorphism onto its image and so we deduce that in this case, $\mathrm{Fix}_{Y_{i+1}}(\Gamma)$ satisfies the Lemma.

Conversely, if $p_i^{-1}(\mathrm{Fix}_{Y_{i+1}}(\Gamma))$ is not contained in $\mathrm{Fix}_{Y_i''}(\eta_{i+1})$, then we pick some

$$y \in p_i^{-1}(\mathrm{Fix}_{Y_{i+1}}(\Gamma)) \quad \text{with} \quad y \notin \mathrm{Fix}_{Y_i''}(\eta_{i+1}).$$

Since η_{i+1} acts trivially on Y_{i+1} and since we are interested in $\mathrm{Fix}_{Y_{i+1}}(\Gamma)$, we assume without loss of generality that η_{i+1} is contained in Γ. Then, Γ acts transitively on $\{y, \eta_{i+1}(y), \eta_{i+1}^2(y)\}$. This gives rise to a short exact sequence

$$1 \longrightarrow H \longrightarrow \Gamma \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow 1,$$

where $H \subseteq \Gamma$ acts trivially on y and where $g \in \Gamma$ is mapped to $j + 3\mathbb{Z}$ if and only if $g(y) = \eta_{i+1}^j(y)$. Recall that $G \simeq \mathbb{Z}/3^c\mathbb{Z} \times \mathbb{Z}/3^c\mathbb{Z}$, and so $\Gamma \simeq \mathbb{Z}/3^k\mathbb{Z} \times \mathbb{Z}/3^m\mathbb{Z}$ for some $k, m \geq 0$. In the above short exact sequence, η_{i+1} is mapped to a generator in $\mathbb{Z}/3\mathbb{Z}$ and so η_{i+1} cannot be a multiple of 3 in Γ. That is,

$$\Gamma \simeq \langle \eta_{i+1} \rangle \times \langle \gamma \rangle,$$

for some $\gamma \in \Gamma$. Since η_{i+1} acts trivially on Y_{i+1}, one easily deduces

$$\mathrm{Fix}_{Y_{i+1}}(\Gamma) = \mathrm{Fix}_{Y_{i+1}}(\gamma) = \bigcup_{j=0}^{2} p_i \left(\mathrm{Fix}_{Y_i''}(\gamma \circ \eta_{i+1}^j) \right). \tag{2.24}$$

The irreducible components of $\mathrm{Fix}_{Y_{i+1}}(\Gamma)$ are therefore of the form $p_i(Z)$ where Z is an irreducible component of

$$\bigcup_{j=0}^{2} \mathrm{Fix}_{Y_i''}(\gamma \circ \eta_{i+1}^j).$$

As we have already proven the Lemma on Y_i'', we know that the G-action on Y_i'' restricts to an action on Z. In particular,

$$p_i(Z) = Z / \langle \eta_{i+1} \rangle .$$

Since the abelian group G acts on Z, it also acts on the above quotient.

For the moment we assume that $p_i(Z)$ is smooth. Its cohomology is then given by the η_{i+1}-invariant classes on Z. Since η_{i+1} is contained in G_c, it follows that the G_c-invariant cohomology of $p_i(Z)$ is given by the G_c-invariant cohomology of Z. Since we know the Lemma on Y_i'', the latter is generated by G-invariant algebraic classes, as we want.

It remains to see that $\mathrm{Fix}_{Y_{i+1}}(\Gamma)$ is smooth. In the local holomorphic charts which cover the complement of U_{i+1} in Y_{i+1}, this fixed point set is given by linear subspaces which are clearly smooth. It therefore suffices to prove that the fixed point set of Γ on U_{i+1} is smooth. By (2.24), the latter is given by

$$\mathrm{Fix}_{U_{i+1}}(\Gamma) = \left(\bigcup_{j=0}^{2} \mathrm{Fix}_{U_i''}(\gamma \circ \eta_{i+1}^j) \right) / \langle \eta_{i+1} \rangle .$$

Since we know the Lemma already on Y_i'', the set $\mathrm{Fix}_{U_i''}(\gamma \circ \eta_{i+1}^j)$ is smooth and η_{i+1} acts on it. This action is free of order three since G_{i+1}/G_i acts freely on U_i''. Therefore,

$$\mathrm{Fix}_{U_i''}(\gamma \circ \eta_{i+1}^j) / \langle \eta_{i+1} \rangle$$

is smooth for all j. The smoothness of $\mathrm{Fix}_{U_{i+1}}(\Gamma)$ follows since

$$\mathrm{Fix}_{U_i''}(\gamma \circ \eta_{i+1}^{j_1}) \cap \mathrm{Fix}_{U_i''}(\gamma \circ \eta_{i+1}^{j_2}) = \varnothing$$

holds for $j_1 \not\equiv j_2 \pmod 3$. This concludes Lemma 2.8.4 by induction on i. $\square$

Via diagram (2.20), we have constructed a smooth model

$$X := Y_c$$

of $Y_0 / \langle \phi_1 \times \phi_2 \rangle$. The group G acts on X and the automorphism $\phi \in \mathrm{Aut}(X)$ which we have to construct in Proposition 2.8.3 is simply given by the action of $\mathrm{id} \times \phi_2 \in G$ on X. This automorphism has order 3^c since this is true on the

Zariski open subset $U_c \subseteq X$. By Lemma 2.8.4, the pair (X, ϕ) satisfies (P5); it remains to show that (X, ϕ) satisfies (P1)–(P4).

The cohomology of X. Using Lemma 2.8.4, we are now able to read off the cohomology of X from diagram (2.20). Indeed, the cohomology of Y_i'' is given by the cohomology of Y_i (via pullbacks) plus some classes which are introduced by blowing up $\mathrm{Fix}_{Y_i}(\eta_{i+1})$ on Y_i and $\mathrm{Fix}_{Y_i'}(\eta_{i+1})$ on Y_i' respectively. By Lemma 2.8.4, these blown-up loci are smooth and their G_c-invariant cohomology is generated by G-invariant algebraic classes. Moreover, G acts on each irreducible component of the blown-up locus and so G acts on each irreducible component of the exceptional divisors of the blow-ups. In particular, the corresponding divisor classes in cohomology are G-invariant. It follows that the G_c-invariant cohomology of Y_i'' is given by the G_c-invariant cohomology of Y_i plus some G-invariant algebraic classes. Also, since η_{i+1} is contained in G_c, the quotient map $Y_i'' \to Y_{i+1}$ induces an isomorphism on G_c-invariant cohomology. It follows inductively that the G_c-invariant cohomology of X – which coincides with the whole cohomology of X – is given by the G_c-invariant cohomology of Y_0 plus G-invariant algebraic classes.

Let us now calculate the G_c-invariant cohomology of Y_0. For $i = 1, 2$, there is by assumption on (X_i, ϕ_i) a basis $\omega_{i1}, \ldots, \omega_{ig}$ of $H^{a_i, b_i}(X_i)$ with

$$\phi_1^*(\omega_{1j}) = \zeta^{-j}\omega_{1j} \text{ and } \phi_2^*(\omega_{2j}) = \zeta^{j}\omega_{2j}. \tag{2.25}$$

This shows that for $j = 1, \ldots, g$, the following linearly independent (a, b)-classes on Y_0 are G_c-invariant:

$$\omega_j := \omega_{1j} \wedge \omega_{2j}.$$

Since (X_1, ϕ_1^{-1}) and (X_2, ϕ_2) satisfy (P1), (P2) and (P3), it follows that apart from the above (a, b)-classes (and their complex conjugates), all G_c-invariant classes on Y_c are generated by products of algebraic classes on X_1 and X_2. These products are G-invariant by (P3). Finally, ϕ acts on ω_j by multiplication with ζ^j. Altogether, we have just shown that (X, ϕ) satisfies (P1), (P2) and (P3).

Charts around $\mathrm{Fix}_X\left(\phi^{3^{c-1}}\right)$. By our construction, there are holomorphic charts which cover the complement of U_c in Y_c, such that ϕ acts on each coordinate function by multiplication with some power of ζ. Therefore, in order to show that (X, ϕ) satisfies (P4), it remains to see that around points of

$$W_c := \mathrm{Fix}_{Y_c}\left(\phi^{3^{c-1}}\right) \cap U_c,$$

the same holds true.

Let us first prove that the preimage of W_c under the $3^c : 1$ étale covering $\pi : U_0 \to U_c$ coincides with the following set:

$$W_0 := \left(\left(\mathrm{Fix}_{X_1}\left(\phi_1^{3^{c-1}}\right) \times X_2\right) \cup \left(X_1 \times \mathrm{Fix}_{X_2}\left(\phi_2^{3^{c-1}}\right)\right)\right) \cap U_0.$$

Clearly, $W_0 \subseteq \pi^{-1}(W_c)$. Conversely, suppose that $(x_1, x_2) \in \pi^{-1}(W_c)$. Then there exists a natural number $1 \leq k \leq 3^c$ with

$$x_1 = \phi_1^k(x_1) \text{ and } \phi_2^{3^{c-1}}(x_2) = \phi_2^k(x_2).$$

If x_1 is not fixed by $\phi_1^{3^{c-1}}$, then 3^{c-1} does not lie in the mod 3^c orbit of k. That is, k is divisible by 3^c and we deduce that x_2 is fixed by $\phi_2^{3^{c-1}}$. This shows $(x_1, x_2) \in W_0$, as we want.

Since $\pi : U_0 \to U_c$ is an étale covering, local holomorphic charts on U_0 give local holomorphic charts on U_c. Around each point

$$x \in \left(\mathrm{Fix}_{X_1}\left(\phi_1^{3^{c-1}}\right) \times X_2\right) \cap U_0$$

we may by assumptions on (X_1, ϕ_1^{-1}) choose local holomorphic coordinates $(z_1, \dots, z_n)$, such that $\phi_1^{-1} \times \mathrm{id}$ acts on each z_j by multiplication with some power of ζ. Moreover, the images of $\phi_1^{-1} \times \mathrm{id}$ and $\mathrm{id} \times \phi_2$ in the quotient G/G_c coincide and so the action of $\phi_1^{-1} \times \mathrm{id}$ on X actually coincides with the automorphism ϕ. This shows that $(z_1, \dots, z_n)$ give local holomorphic coordinates around $\pi(x)$ on which ϕ acts by multiplication with some powers of ζ.

The case

$$x \in \left(X_1 \times \mathrm{Fix}_{X_2}\left(\phi_2^{3^{c-1}}\right)\right) \cap U_0$$

is done similarly and so we conclude that (P4) holds for (X, ϕ). This finishes the proof of Proposition 2.8.3. □

2.8.3 Proof of Theorem 2.8.1

Proof of Theorem 2.8.1. For $a > b \geq 0$, $n \geq a + b$ and $c \geq 1$, we need to construct an n-dimensional smooth complex projective variety $Z_c^{a,b,n}$ whose primitive (p,q)-type cohomology has dimension $(3^c - 1)/2$ if $p = a$ and $q = b$, and vanishes for all other $p > q$. Suppose that we have already settled the case when $n = a + b$. Then, for $n > a + b$, the product

$$Z_c^{a,b,n} := Z_c^{a,b,a+b} \times \mathbb{P}^{n-a-b}$$

has the desired properties. In order to prove Theorem 2.8.1, it therefore suffices to show that the set $\mathcal{S}_c^{a,b}$, defined in Section 2.8.2, is nonempty for all $a > b \geq 0$ and $c \geq 1$. We will prove the latter by induction on $a + b$.

We put $g = (3^c - 1)/2$ and consider the hyperelliptic curve C_g with automorphism ψ_g from Section 2.3.1. It is then straightforward to check that

$$(C_g, \psi_g) \in \mathcal{S}_c^{1,0}. \tag{2.26}$$

Indeed, it is clear that (C_g, ψ_g) satisfies (P1)–(P3) in the definition of $\mathcal{S}_c^{1,0}$. Moreover, the complement of the point $\infty \in C_g$ is given by the affine curve $y^2 = x^{2g+1} + 1$ and ψ_g acts by multiplication with a primitive 3^c-th root of unity ζ on x. For all $0 \le l \le c-1$, the fixed point set $\mathrm{Fix}_{C_g}\left(\psi_g^{3^l}\right)$ is therefore given by the points $(x,y) = (0,\pm 1)$ and ∞. These points are ψ_g-invariant and so their cohomology is generated by ψ_g-invariant algebraic classes, which shows that (P5) holds. It remains to establish (P4). That is, we need to find suitable holomorphic coordinates around the three fixed points of $\psi_g^{3^{c-1}}$. Differentiating the affine equation $y^2 = x^{2g+1}+1$ gives $2y \cdot dy = (2g+1)x^{2g} \cdot dx$. This shows that dx spans the cotangent space at $(0,\pm 1)$ and so x is a local coordinate function near $(0,\pm 1)$. The automorphism ψ_g acts on this function by multiplication with ζ, as we want in (P4). In order to find a suitable coordinate function around ∞, we use the coordinates (u,v), introduced in Section 2.3.1. In these coordinates, the curve C_g is given by the equation $v^2 = u + u^{2g+2}$ and ∞ corresponds to the point $(u,v) = (0,0)$. Around this point, the function v yields a coordinate function on which ψ_g acts via multiplication with ζ^g, see Section 2.3.1. This establishes (2.26) and hence settles the case $a+b=1$.

Let now $a > b$ with $a+b > 1$. If $b = 0$, then by induction, the sets $\mathcal{S}_c^{1,0}$ and $\mathcal{S}_c^{a-1,0}$ are nonempty and so Proposition 2.8.3 yields an element in $\mathcal{S}_c^{a,0}$, as desired. If $b \ge 1$, then $\mathcal{S}_c^{a,b-1}$ is nonempty by induction. Also, $\mathcal{S}_c^{0,1}$ is nonempty since it contains (C_g, ψ_g^{-1}) by (2.26). Application of Proposition 2.8.3 then yields an element in $\mathcal{S}_c^{a,b}$, as we want. This concludes Theorem 2.8.1. □

Remark 2.8.6. *The variety in $\mathcal{S}_c^{a,b}$ which the above proof produces inductively is easily seen to be a smooth model of the quotient of C_g^{a+b} by the group action of $G^1(a,b,g)$, defined in Section 2.3.2.*

2.9 Proof of Theorem 2.1.5

Proof of Theorem 2.1.5. To begin with, let us recall that we have proven in [72] that for all Kähler surfaces S,

$$h^{1,1}(S) > h^{2,0}(S), \tag{2.27}$$

see also [76, Prop. 22]. Therefore, $h^{1,1}$ dominates $h^{2,0}$ in dimension two.

Conversely, let us suppose that the Hodge number $h^{r,s}$ dominates $h^{p,q}$ non-trivially in dimension n. That is, there are positive constants $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$ such that for all n-dimensional smooth complex projective varieties X, the following holds:

$$\lambda_1 \cdot h^{r,s}(X) + \lambda_2 \ge h^{p,q}(X). \tag{2.28}$$

By the Hodge symmetries (2.2), we may assume $r \geq s$, $p \geq q$, $r + s \leq n$ and $1 \leq p+q \leq n$. The nontriviality of the above domination then means that (2.28) does not follow from the Lefschetz conditions (2.3). In order to prove Theorem 2.1.5, it now remains to show $n = 2$, $r = s = 1$ and $p = 2$.

Suppose that $r + s < n$. Since (2.28) does not follow from the Lefschetz conditions (2.3), Theorem 2.1.3 (or Corollary 2.10.2 below) shows $p + q = n$. Using the Lefschetz hyperplane theorem and the Hirzebruch–Riemann–Roch formula, we see however that a smooth hypersurface $V_d \subseteq \mathbb{P}^{n+1}$ of degree d satisfies $h^{r,s}(V_d) \leq 1$, whereas $h^{p,q}(V_d)$ tends to infinity if d does. This is a contradiction and so $r + s = n$ holds.

Suppose that $r \neq s$. Then, considering a blow-up of $\mathbb{P}^n$ in sufficiently many distinct points proves $p \neq q$. Since $p \neq q$ and $r \neq s$, we may then use certain examples from Theorem 2.8.1 to deduce that (2.28) follows from the Lefschetz conditions (2.3). This contradicts the nontriviality of our given domination. Hence, $r = s$ and in particular $n = 2r$ is even.

Suppose that $p = q$. Considering again a blow-up of $\mathbb{P}^n$ in sufficiently many distinct points then proves $\lambda_1 \geq 1$ and so (2.28) follows from the Lefschetz conditions. This contradicts the nontriviality of (2.28) and so it proves $p \neq q$.

Suppose that $p + q < n$. Using sufficiently high-degree hyperplane sections of n-dimensional examples from Theorem 2.1.3, one proves that there is a sequence of $(n-1)$-dimensional smooth complex projective varieties $(Y_j)_{j\geq 1}$ such that $h^{r-1,r-1}(Y_j)$ is bounded whereas $h^{p,q}(Y_j)$ tends to infinity if j does. (Note that we used $p \neq q$ here.) Since $n = 2r$, we have $h^{r-1,r-1}(Y_j) = h^{r,r}(Y_j)$ by the Hodge symmetries. Therefore, the sequence of n-dimensional smooth complex projective varieties

$$\left(Y_j \times \mathbb{P}^1\right)_{j\geq 1}$$

has bounded $h^{r,r}$ but unbounded $h^{p,q}$. This is a contradiction and hence shows $p + q = n$.

Next, using Corollary 2.5.3 from Section 2.5, it follows that $p = 2r$ and $q = 0$ holds. By what we have shown so far we are thus left with the case where $n = 2r = 2s$, $p = 2r$ and $q = 0$. In order to finish the proof of Theorem 2.1.5, it therefore suffices to show $r = 1$. For a contradiction, we assume that $r \geq 2$. By Theorem 2.8.1 there exists a $(2r-1)$-dimensional smooth complex projective variety Y with $h^{2r-1,0}(Y) = h^{0,2r-1}(Y) = 1$ and $h^{p,q}(Y) = 0$ for all other $p \neq q$. Since $r \geq 2$, this implies for a smooth curve C_g of genus g:

$$h^{2r,0}(Y \times C_g) = g \quad \text{and} \quad h^{r,r}(Y \times C_g) = 2 \cdot h^{r-1,r-1}(Y).$$

Hence, $(Y \times C_g)_{g\geq 1}$ is a sequence of $2r$-dimensional smooth complex projective varieties such that $h^{r,r}$ is constant whereas $h^{2r,0}$ tends to infinity if g does. This is the desired contradiction and hence shows $r = 1$. This finishes the proof of Theorem 2.1.5. $\square$

The next result combines Theorem 2.1.5 with a very recent result concerning the geography of surfaces [69].

Corollary 2.9.1. *Suppose there are $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$ such that for all smooth complex projective varieties X of dimension n:*

$$\lambda_1 h^{r,s}(X) + \lambda_2 \geq h^{p,q}(X). \tag{2.29}$$

Then $\lambda_1 \geq 1$ and (2.29) is either a consequence of the Lefschetz conditions (2.3), or $n = 2$ and it is a consequence of (2.27).

Proof. By Theorem 2.1.5, it suffices to prove that any universal inequality of the form

$$\lambda_1 h^{1,1}(S) + \lambda_2 \geq h^{2,0}(S),$$

with $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$, which holds for all smooth complex projective surfaces S satisfies $\lambda_1 \geq 1$. As we will see in the following, this follows easily from Roulleau–Urzúa's work. Indeed, they prove [69, Thm. 1.1] that for any $r \in [2,3]$ there are simply connected smooth complex projective surfaces S of general type such that the quotient of Chern numbers $c_1^2(S)/c_2(S)$ is arbitrarily close to r. In particular, there is a sequence S_n of simply connected smooth complex projective surfaces with

$$c_1^2(S_n) = (3 - \epsilon_n) c_2(S_n), \tag{2.30}$$

where ϵ_n tends to 0 and $c_2(S_n)$ tends to infinity for $n \to \infty$.

Since S_n is simply connected, we have

$$c_2(S_n) = 2 + 2h^{2,0}(S_n) + h^{1,1}(S_n),$$

and

$$c_1^2(S_n) = 10 + 10h^{2,0}(S_n) - h^{1,1}(S_n).$$

By (2.30), this yields

$$4 + 4h^{2,0}(S_n) - 4h^{1,1}(S_n) = -\epsilon_n(2 + 2h^{2,0}(S_n) + h^{1,1}(S_n)).$$

Hence,

$$h^{2,0}(S_n) = \frac{4 - \epsilon_n}{4 + 2\epsilon_n} \cdot h^{1,1}(S_n) - 1. \tag{2.31}$$

The given universal inequality $h^{2,0} \leq \lambda_1 h^{1,1} + \lambda_2$ thus implies

$$\frac{4-\epsilon_n}{4+2\epsilon_n} \cdot h^{1,1}(S_n) - 1 \leq \lambda_1 h^{1,1}(S_n) + \lambda_2. \tag{2.32}$$

Since S_n is simply connected,

$$b_2(S_n) = c_2(S_n) - 2$$

tends to infinity if n does. By (2.27), $b_2(S_n) < 3 \cdot h^{1,1}(S_n)$, and so $h^{1,1}(S_n)$ tends to infinity for $n \to \infty$. For $n \to \infty$, inequality (2.32) therefore implies:

$$\lambda_1 \geq 1.$$

This finishes the proof of Corollary 2.9.1. □

Remark 2.9.2. *One could of course strengthen Kollár–Simpson's domination relation between Hodge numbers by requiring that (2.5) holds for all n-dimensional Kähler manifolds X. However, since (2.27) holds for all Kähler surfaces, it is immediate that Theorem 2.1.5 and Corollary 2.9.1 remain true for this stronger domination relation.*

2.10 Inequalities among Hodge numbers

It is a difficult problem to determine all universal inequalities among Hodge numbers in a fixed dimension. However, in dimension two, one can use recent work of Roulleau and Urzúa [69] to solve this problem.[1]

Corollary 2.10.1. *Any universal integral linear inequality among the Hodge numbers of smooth complex projective surfaces is a consequence of $h^{1,1} \geq h^{2,0}+1$.*

Proof. Any integral linear inequality among the Hodge numbers of surfaces can be written in the form

$$\lambda_1 h^{1,0} + \lambda_2(h^{1,1} - 1) \geq \lambda_3 + \lambda_4 h^{2,0},$$

with $\lambda_i \in \mathbb{Z}$. For the corollary, it suffices to prove $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_3 \leq 0$ and $\lambda_2 \geq \lambda_4$.

[1] B. Totaro pointed out to us that instead of [69], one could also use ball quotients associated to Kottwitz lattices. The main point being that these surfaces have finite covering spaces T with $b_1(T) = 0$ (by a result of Rapoport–Zink [66]), $c_1^2(T)/c_2(T) = 3$ and $c_2(T)$ arbitrarily large.

Looking at the product of $\mathbb{P}^1$ with a smooth projective curve of sufficiently high genus proves $\lambda_1 \geq 0$. The blow-up of $\mathbb{P}^2$ in sufficiently many points proves $\lambda_2 \geq 0$. The projective plane $\mathbb{P}^2$ proves $\lambda_3 \leq 0$.

It remains to prove $\lambda_2 \geq \lambda_4$. This follows from the examples constructed in [69]. Indeed, let us consider the sequence S_n of simply connected surfaces that we have already used in the proof of Corollary 2.9.1. By (2.31), we obtain

$$\lambda_2 \cdot (h^{1,1}(S_n) - 1) \geq \lambda_3 + \lambda_4 \cdot \left(\frac{4-\epsilon_n}{4+2\epsilon_n} \cdot h^{1,1}(S_n) - 1 \right).$$

Recall that for $n \to \infty$, ϵ_n and $h^{1,1}(S_n)$ tend to zero and infinity respectively. The above inequality therefore implies $\lambda_2 \geq \lambda_4$, as we want. □

The remaining results in this section are consequences of the main theorems of this chapter; they are again contained in the published article [76].

Corollary 2.10.2. *Any universal inequality among the Hodge numbers below the horizontal middle axis in (2.4) of n-dimensional smooth complex projective varieties is a consequence of the Lefschetz conditions (2.3).*

Proof. Assume that we are given a universal inequality between the Hodge numbers of the truncated Hodge diamond of smooth complex projective n-folds. In terms of the primitive Hodge numbers $l^{p,q}$, this means that for all natural numbers p and q with $0 < p + q < n$ there are real numbers $\lambda_{p,q}$ and a constant $C \in \mathbb{R}$ such that

$$\sum_{0<p+q<n} \lambda_{p,q} \cdot l^{p,q}(X) \geq C \tag{2.33}$$

holds for all smooth n-folds X. Using the Hodge symmetries (2.2), we may further assume that $\lambda_{p,q} = \lambda_{q,p}$ holds for all p and q. If we put $X = \mathbb{P}^n$, then we see $C \leq 0$. Moreover, for any natural numbers p and q with $0 < p + q < n$, there exists by Theorem 2.1.3 a smooth complex projective variety X with $l^{p,q}(X) >> 0$, whereas (modulo the Hodge symmetries) all remaining primitive Hodge numbers of its truncated Hodge diamond are bounded from above, by n^3 say. This proves $\lambda_{p,q} \geq 0$. That is, the universal inequality (2.33) is a consequence of the Lefschetz conditions (2.3), as we want. □

As an immediate consequence of the above corollary, we note the following.

Corollary 2.10.3. *Any universal inequality among the Hodge numbers of smooth complex projective varieties which holds in all sufficiently large dimensions at the same time is a consequence of the Lefschetz conditions.*

In the same way we deduced Corollary 2.10.2 from Theorem 2.1.3, one deduces the following from Theorem 2.8.1:

Corollary 2.10.4. *Any universal inequality among the Hodge numbers away from the vertical middle axis in (2.4) of n-dimensional smooth complex projective varieties is a consequence of the Lefschetz conditions (2.3).*

Corollary 2.10.2 implies that in dimension n, the Betti numbers b_k with $k \neq n$ do not satisfy any universal inequalities, other than the Lefschetz conditions

$$b_k \geq b_{k-2} \quad \text{for all } k \leq n. \tag{2.34}$$

Using products of high degree hypersurfaces with projective spaces, one can easily deduce that in fact any universal inequality among the Betti numbers of smooth complex projective varieties in any given dimension is a consequence of the Lefschetz conditions, see [72] and [76, Prop. 27].

2.11 Threefolds with $h^{1,1} = 1$

Here we show that in dimension three, the constraints which classical Hodge theory puts on the Hodge numbers of smooth complex projective varieties are not complete. Our results apply to threefolds with $h^{1,1} = 1$ and $h^{3,0} \geq 2$, such as any sufficiently high degree complete intersection threefold in a smooth projective variety with $h^{1,1} = 1$. Smooth projective varieties with $h^{1,1} = 1$ and arbitrary $h^{2,0}$ were constructed in Theorem 2.7.1.

Proposition 2.11.1. *Let X be a smooth complex projective threefold with Hodge numbers $h^{p,q} := h^{p,q}(X)$. If $h^{1,1} = 1$ and $h^{3,0} \geq 2$, then*

$$h^{1,0} = 0, \quad h^{2,0} < h^{3,0} \quad \textit{and} \quad h^{2,1} < 12^6 \cdot h^{3,0}.$$

Proof. The first two assertions are proven in the authors Part III essay [72], see also [76, Prop. 28]. Here we will only prove $h^{2,1} < 12^6 \cdot h^{3,0}$, which is not contained in [72].

The assumption $h^{1,1} = 1$ implies that the canonical class of X is a multiple of an ample class. Therefore, the assumption $h^{3,0} \geq 2$ ensures that K_X is ample and so Yau's inequality holds [96]:

$$c_1c_2(X) \leq \frac{3}{8}c_1^3(X). \tag{2.35}$$

Moreover, the Riemann–Roch formula in dimension three says

$$c_1c_2(X) = 24\chi(X, \mathcal{O}_X). \tag{2.36}$$

Since K_X is ample, Fujita's conjecture predicts that $6 \cdot K_X$ is very ample, cf. [50, p. 252]. Although this conjecture is still open, Lee proves in [53] that $10 \cdot K_X$ is very ample. Thus, the following argument due to Catanese–Schneider [11] applies: Firstly, the linear series $|10 \cdot K_X|$ embeds X into some $\mathbb{P}^N$ and hence $\Omega_X(20 \cdot K_X)$ is a quotient of $\Omega_{\mathbb{P}^N}(2)$ restricted to X. Since the latter is globally generated, it is nef and hence $\Omega_X(20 \cdot K_X)$ is nef.

Secondly, by [22, Cor. 2.6], any Chern number of a nef bundle F on an n-dimensional smooth complex projective variety X is bounded from above by $c_1^n(F)$. In our situation, this yields

$$c_3(\Omega_X^1(20 \cdot K_X)) \le c_1^3(\Omega_X^1(20 \cdot K_X)). \tag{2.37}$$

A standard computation gives

$$c_3(\Omega_X^1(20 \cdot K_X)) = -8\,400 \cdot c_1^3(X) - 20 \cdot c_1 c_2(X) - c_3(X)$$

and

$$c_1^3(\Omega_X^1(20 \cdot K_X)) = -61^3 \cdot c_1^3(X).$$

Together with Yau's inequality (2.35), this yields in (2.37)

$$1\,748\,588 \cdot c_1 c_2(X) \le 3 \cdot c_3(X). \tag{2.38}$$

By the Riemann–Roch formula (2.36), this inequality is in fact one between the Hodge numbers of threefolds with ample canonical bundle. In our case, $h^{1,1} = 1$ and $h^{1,0} = 0$ yield:

$$6\,994\,346 + 6\,994\,346 \cdot h^{2,0} + 3 \cdot h^{2,1} \le 6\,994\,349 \cdot h^{3,0}.$$

Thus, a rough estimation yields

$$h^{2,1} < 12^6 \cdot h^{3,0}.$$

This concludes the proof of the Proposition. □

Remark 2.11.2. *Instead of using [22], but still relying on [53], Chang–Lopez prove in [15] that there is a computable constant $C > 0$ such that*

$$C \cdot c_1 c_2(X) \le c_3(X),$$

for all threefolds X with ample canonical bundle. Computing C explicitly shows that it is about four times smaller then the analogous constant which appears in (2.38). However, since the explicit extraction of C is slightly tedious and since this constant is still far from being realistic, we did not try to carry this out here.

3 Algebraic structures with unbounded Chern numbers

Abstract. We determine all Chern numbers of smooth complex projective varieties of dimension ≥ 4 which are determined up to finite ambiguity by the underlying smooth manifold. We also give an upper bound on the dimension of the space of linear combinations of Chern numbers with that property and prove its optimality in dimension four.

3.1 Introduction

To each n-dimensional complex manifold X and for each partition $\mathfrak{m}$ of n, one can associate a Chern number $c_{\mathfrak{m}}(X)$. In 1954, Hirzebruch [34, Problem 31] asked which linear combinations of Chern and Hodge numbers are topological invariants of smooth algebraic varieties. Recently, this problem has been solved by Kotschick [46, 47] for what concerns the Chern numbers and by Kotschick and the author [48] in full generality.

Generalizing the Hirzebruch problem, Kotschick asks which Chern numbers of smooth complex projective varieties are determined up to finite ambiguity by the underlying smooth manifold [45, pp. 522]. Such a boundedness statement is known for c_n and c_1c_{n-1} in arbitrary dimension n, since these Chern numbers can be expressed in terms of Hodge numbers [54] and so they are bounded by the Betti numbers. The first nontrivial instance of Kotschick's boundedness question concerns therefore the Chern number c_1^3 in dimension 3. In a recent preprint [9], Cascini and Tasin show that in many cases this number is indeed bounded by the topology of the smooth projective threefold.

Conversely, there are no known examples of a smooth manifold such that the set of Chern numbers with respect to all possible complex algebraic structures is known to be unbounded. In this chapter we produce such examples in dimensions ≥ 4; our result is as follows.

Theorem 3.1.1. *In complex dimension* 4*, the Chern numbers* c_4*,* c_1c_3 *and* c_2^2 *of a smooth complex projective variety are the only Chern numbers* $c_{\mathfrak{m}}$ *which*

This chapter is based on joint work with Tasin [77].

are determined up to finite ambiguity by the underlying smooth manifold. In complex dimension $n \geq 5$, only c_n and c_1c_{n-1} are determined up to finite ambiguity by the underlying smooth manifold.

The dimension four case of the above theorem might be surprising. Indeed, it was observed by Kotschick that the Chern numbers of a minimal smooth projective fourfold of general type are bounded by the underlying smooth manifold, see Remark 3.4.3 below. Based on an MMP approach, similar to the one given in [9] for threefolds, one might expect that this boundedness statement holds more generally for all fourfolds of general type, which is the largest class in the Kodaira classification. This compares to Theorem 3.1.1 as the examples we are using there are of negative Kodaira dimension.

By Theorem 3.1.1, only very few Chern numbers of high dimensional smooth complex projective varieties are bounded by the underlying smooth manifold. This changes considerably if we are asking for all linear combinations of Chern numbers with that property. Indeed, the space of such linear combinations contains the Euler characteristics $\chi^p = \chi(X, \Omega_X^p)$, as well as all Pontryagin numbers in even complex dimensions. In dimension four, the Euler characteristics χ^p and Pontryagin numbers span a space of codimension one in the space of all Chern numbers. Therefore, Theorem 3.1.1 implies:

Corollary 3.1.2. *Any linear combination of Chern numbers which on smooth complex projective fourfolds is determined up to finite ambiguity by the underlying smooth manifold is a linear combination of the Euler characteristics χ^p and the Pontryagin numbers.*

Using bordism theory, we provide in Corollary 3.6.3 a nontrivial upper bound on the dimension of the space of linear combinations of Chern numbers which are determined up to finite ambiguity by the underlying smooth manifold. Our upper bound is in general bigger than the known lower bound; determining all bounded linear combinations therefore remains open in all dimensions $n \geq 3$ other than $n = 4$.

It was known for some time that the boundedness question for Chern numbers behaves differently in the non-Kähler setting. Indeed, LeBrun showed [52] that there is a smooth 6-manifold with infinitely many (non-Kähler) complex structures such that c_1c_2 is unbounded, which cannot happen for complex Kähler structures. In Corollary 3.5.1 we use products with LeBrun's examples and Theorem 3.1.1 to conclude that in complex dimension $n \geq 4$, the topological Euler number c_n is the only Chern number which on complex manifolds is bounded by the underlying smooth manifold.

Theorem 3.1.1 is based on the existence of certain projective bundles over threefolds which admit infinitely many different algebraic structures. An important observation here is that the Chern numbers of the base do not matter too much. To obtain unbounded Chern numbers for the projective bundles it is enough to have a three-dimensional base with unbounded first Chern class, its Chern numbers may well be independent of the complex structures chosen. This is in contrast to Kotschick's work [47], where bundles over surfaces with varying signatures are used, see also Remark 3.4.2.

3.2 Dolgachev surfaces

We recall here some basic properties of Dolgachev surfaces. For a detailed treatment see [23, 30] and [29, Sec. I.3].

Let $S \subseteq \mathbb{P}^2 \times \mathbb{P}^1$ be a generic element of the linear series $|\mathcal{O}(3,1)|$. That is, S is isomorphic to the blow-up of $\mathbb{P}^2$ at the nine intersection points of two generic degree three curves and the second projection $\pi : S \longrightarrow \mathbb{P}^1$ is an elliptic fibration with irreducible fibres. For each odd integer $q \geq 3$, the Dolgachev surface S_q is realised applying logarithmic transformations of order 2 and q at two smooth fibres of π. The surface S_q comes with an elliptic fibration $\pi_q : S_q \longrightarrow \mathbb{P}^1$, which away from the two multiple fibers is isomorphic to the one of S. For a proof of the following proposition, see [29, Sec. I.3] and the references therein.

Proposition 3.2.1. *The Dolgachev surface S_q is a simply connected algebraic surface with*

1. $h^{2,0}(S_q) = 0$ *and* $b_2(S_q) = 10$,
2. $c_1^2(S_q) = 0$ *and* $c_2(S_q) = 12$,
3. $c_1(S_q) = (q-2)G_q$, *where* $G_q \in H^2(S_q, \mathbb{Z})$ *is a nonzero primitive class,*
4. *the intersection pairing on* $H^2(S_q, \mathbb{Z})$ *is odd of type* $(1,9)$.

Proposition 3.2.1 has two important consequences that we will use in this chapter. Firstly, since $h^{1,0}(S_q) = h^{2,0}(S_q) = 0$, it follows that the first Chern class is an isomorphism

$$c_1 : \mathrm{Pic}(S_q) \xrightarrow{\sim} H^2(S_q, \mathbb{Z}).$$

Hence, every element of $H^2(S_q, \mathbb{Z})$ can be represented by a holomorphic line bundle.

Secondly, let us denote the smooth manifold which underlies S_q by M_q. By item 4 in Proposition 3.2.1, Wall's theorem [93] implies the existence of a smooth h-cobordism W_q between M_3 and M_q.

Although we will not need this here, let us mention that the homeomorphism type of M_q does not depend on q by Freedman's classification theorem of simply connected 4-manifolds. However, generalizing a result of Donaldson, Friedman–Morgan showed [29] that M_q and $M_{q'}$ are never diffeomorphic for $q \neq q'$.

3.3 Chern numbers of projective bundles

In this section we systematically treat the Chern numbers of projective bundles. Most of the results are taken from the author's Bachelor thesis [71] and we will make precise indications where this is the case. We formulate and use our results for holomorphic vector bundles over complex manifolds, but they hold more generally for arbitrary complex vector bundles over stably almost complex manifolds.

Let B be a complex manifold of dimension $n+1-k$ and let E be a holomorphic vector bundle of rank k on B. The Segre class of E is the inverse of its total Chern class; we denote it by

$$\alpha := (1 + c_1(E) + \cdots + c_k(E))^{-1} \in H^*(B, \mathbb{Z}).$$

The degree $2k$-component of α is denoted by $\alpha_k \in H^{2k}(B, \mathbb{Z})$.

For $\mathfrak{a} = (a_1, \dots, a_p) \in \mathbb{N}^p$, we denote its weight by $|\mathfrak{a}| = \sum a_i$. With this notation in mind, we put

$$f(\mathfrak{a}) := \sum_{\mathfrak{d} \in \mathbb{N}^p} \left(\prod_{i=1}^{p} \binom{k - d_i}{k - a_i} c_{d_i}(E) \right) \alpha_{(|\mathfrak{a}| - |\mathfrak{d}| - (k-1))}, \tag{3.1}$$

where $\mathfrak{d} = (d_1, \dots, d_p)$, and where we use the convention $\binom{a}{b} = 0$, if $b < 0$ or $a < b$. The above definition yields a cohomology class in $H^{2(|\mathfrak{a}| - (k-1))}(B, \mathbb{Q})$, which can actually be shown to be integral. Its definition in [71] is motivated by the following result.

Proposition 3.3.1. *Let $\mathfrak{m} = (m_1, \dots, m_p)$ be a partition of $n = \dim(\mathbb{P}(E))$. Then the $\mathfrak{m}$-th Chern number of the projective bundle $\mathbb{P}(E)$ is given by*

$$c_{\mathfrak{m}}(\mathbb{P}(E)) = \sum_{j_1, \dots, j_p} c_{j_1}(B) \cdot \dots \cdot c_{j_p}(B) \cdot f(m_1 - j_1, \dots, m_p - j_p),$$

where the right hand side is identified with its evaluation on the fundamental class of B.

Proof. A complete proof is given in [71] and [77], we repeat it here for the convenience of the reader. Let $\pi : \mathbb{P}(E) \longrightarrow B$ be the projection morphism and T_π be the tangent bundle along the fibres of π, that is, $T_\pi = \ker(\pi_*)$, where $\pi_* : T_{\mathbb{P}(E)} \longrightarrow \pi^* T_X$. By the Whitney formula, the total Chern classes are related by

$$c(\mathbb{P}(E)) = c(T_\pi) \cdot \pi^* c(B).$$

If $\mathcal{O}_E(-1)$ denotes the tautological bundle of $\mathbb{P}(E)$, then we have the exact sequence

$$0 \longrightarrow \mathcal{O}_E(-1) \longrightarrow \pi^* E \longrightarrow T_\pi \otimes \mathcal{O}_E(-1) \longrightarrow 0.$$

It follows that the total Chern classes of T_π and $\pi^* E \otimes \mathcal{O}_E(1)$ coincide. Hence,

$$c(T_\pi) = \sum_{i=0}^{k} \pi^* c_i(E)(1+y)^{k-i},$$

where $y = c_1(\mathcal{O}_E(1))$. From now on, we may ignore the pull-back map π^* in the computations. Setting $b_i := c_i(B)$ and $e_i := c_i(E)$, we can write

$$c(\mathbb{P}(E)) = \left(\sum_{j\geq 0} b_j\right)\left(\sum_{i\geq 0} e_i(1+y)^{k-i}\right),$$

and so

$$c(\mathbb{P}(E)) = \sum_{i,j,l\geq 0} \binom{k-i}{l} e_i b_j y^l.$$

The $\mathfrak{m}$-th Chern number is hence given by

$$c_{\mathfrak{m}}(\mathbb{P}(E)) = \prod_{t=1}^{p} \sum_{i_t+j_t+l_t=m_t} \binom{k-i_t}{l_t} e_{i_t} b_{j_t} y^{l_t},$$

where $i_t, j_t, l_t \geq 0$, and where we identify the right hand with its evaluation on the fundamental class of $\mathbb{P}(E)$.

Reordering, we can write

$$c_{\mathfrak{m}}(\mathbb{P}(E)) = \sum_{j_1,\ldots,j_p} b_{j_1}\cdots b_{j_p} \sum_{i_1,\ldots,i_p} \left(\prod_{t=1}^{p} \binom{k-i_t}{k+j_t-m_t} e_{i_t}\right) y^{\sum_{t=1}^{p} l_t},$$

where we are assuming that $l_t = m_t - j_t - i_t \geq 0$ for $t = 1, \ldots, p$.

For any $0 \leq m \leq n$ and any $\omega \in H^{2(n-m)}(B, \mathbb{Z})$, the product ωy^m coincides with the top-degree component of $\omega \alpha y^{k-1}$, see [73, Lem. 2.2]. This simplifies the above expression of the $\mathfrak{m}$-th Chern number of $\mathbb{P}(E)$ to

$$c_{\mathfrak{m}}(\mathbb{P}(E)) = \sum_{j_1,\ldots,j_p} b_{j_1}\cdots b_{j_p} \sum_{i_1,\ldots,i_p} \left(\prod_{t=1}^{p} \binom{k-i_t}{k+j_t-m_t} e_{i_t}\right) \alpha y^{k-1},$$

where on the right hand side only the term in cohomological degree $2n$ is considered. The statement follows since on any fibre of π the class y^{k-1} evaluates to 1. $\square$

Proposition 3.3.1 reduces the computation of Chern numbers of projective bundles to the computation of $f(\mathfrak{a})$ defined in (3.1). It is easy to see that $f(\mathfrak{a})$ is invariant under permutations of $(a_1, \dots, a_p)$. Moreover, $f(\mathfrak{a})$ is possibly nonzero only for $k-1 \leq |\mathfrak{a}| \leq n$ and $0 \leq a_i \leq k$, and a simple argument shows $f(\mathfrak{a}) = 0$ for $a_i = k$. For small values of $|\mathfrak{a}|$, we are actually able to compute $f(\mathfrak{a})$ explicitly as follows.

Lemma 3.3.2. *Let σ_i be the elementary symmetric polynomial of degree i in $a_1, \dots, a_p$ and denote by $e_i := c_i(E)$ the i-th Chern class of E. Then,*

1. $f(\mathfrak{a}) = \prod_{i=1}^p \binom{k}{a_i}$ *, if* $|\mathfrak{a}| = k-1$,
2. $f(\mathfrak{a}) = 0$ *, if* $|\mathfrak{a}| = k$,
3. $f(\mathfrak{a}) = \prod_{i=1}^p \binom{k}{a_i} \cdot (\sigma_2 - k) \cdot \left(\frac{1}{k^2} e_1^2 - \frac{2}{k(k-1)} e_2\right)$ *, if* $|\mathfrak{a}| = k+1$.

Proof. The first assertion is immediate from the definitions and the second assertion is proven in [71] by a computation, the third statement is not contained in [71]. We will give an alternative proof of the second statement and a complete proof of item 3.

For any line bundle L on B, $\mathbb{P}(E)$ and $\mathbb{P}(E \otimes L)$ are isomorphic. For $|\mathfrak{a}| = k$ the expression $f(\mathfrak{a})$ has cohomological degree two and so it is a multiple of e_1. Specializing the base manifold B to an elliptic curve, Proposition 3.3.1 shows that for any line bundle L on B, $f(\mathfrak{a})$ is invariant under replacing E by $E \otimes L$. The claim follows because no nontrivial multiple of e_1 has this property.

It remains to prove (3). Since $|\mathfrak{a}| = k+1$, we have

$$f(\mathfrak{a}) = \sum_{|\mathfrak{d}|=0} \left(\prod_{i=1}^p \binom{k-d_i}{k-a_i} e_{d_i} \right) \alpha_2 + \sum_{|\mathfrak{d}|=1} \left(\prod_{i=1}^p \binom{k-d_i}{k-a_i} e_{d_i} \right) \alpha_1 + \sum_{|\mathfrak{d}|=2} \left(\prod_{i=1}^p \binom{k-d_i}{k-a_i} e_{d_i} \right) \alpha_0,$$

which gives

$$f(\mathfrak{a}) = \prod_{i=1}^p \binom{k}{a_i} \left(\alpha_2 + \sum_{s=1}^p \frac{a_s}{k} e_1 \alpha_1 + \sum_{s=1}^p \frac{a_s(a_s-1)}{k(k-1)} e_2 \alpha_0 + \sum_{1 \leq s < t \leq p} \frac{a_s a_t}{k^2} e_1^2 \alpha_0 \right).$$

Noting that

$$\alpha_1 = -e_1 \quad \text{and} \quad \alpha_2 = e_1^2 - e_2,$$

we can compute $f(\mathfrak{a})$ to

$$\prod_{i=1}^p \binom{k}{a_i} \left(\left(\sum_{1 \leq s < t \leq p} a_s a_t - \sum_{s=1}^p a_s k + k^2 \right) \frac{e_1^2}{k^2} + \left(\sum_{s=1}^p a_s(a_s-1) - k(k-1) \right) \frac{e_2}{k(k-1)} \right).$$

Now it is easy to conclude using $\sum_{s=1}^p a_s^2 = \sigma_1^2 - 2\sigma_2$ and $\sigma_1 = |\mathfrak{a}| = k+1$. $\square$

In the construction of our examples, we will need the following easy estimate, which proves positivity of the constant appearing in $f(\mathfrak{a})$ for $|\mathfrak{a}| = k+1$.

Lemma 3.3.3. *Let $k \geq 2$ be an integer. For any partition $\mathfrak{a} = (a_1, \ldots, a_p)$ of $k+1$ with $0 \leq a_i \leq k$ for all i, the expression*

$$\prod_{i=1}^{p} \binom{k}{a_i} \cdot (\sigma_2 - k) \tag{3.2}$$

from Lemma 3.3.2 is nonnegative; it is positive if additionally $a_i < k$ for all i.

Proof. The product $\prod_{i=1}^{p} \binom{k}{a_i}$ is positive since $0 \leq a_i \leq k$ for all i. It thus suffices to consider

$$\sum_{i<j} a_i a_j - k. \tag{3.3}$$

Here we may ignore all a_i that are zero. After reordering, we may therefore assume $1 \leq a_1 \leq a_2 \leq \cdots \leq a_p \leq k$.

If $p = 2$, then

$$a_1 \cdot a_2 - k = a_1(k+1-a_1) - k$$

is a negatively curved quadratic equation in a_1 with zeros at $a_1 = k$ and $a_1 = 1$ and so the assertion follows because $a_1 = 1$ implies $a_2 = k$.

If $p \geq 3$, then

$$\sum_{i<j} a_i a_j \geq \sum_{i=2}^{p} a_1 a_i + a_p a_{p-1} \geq \sum_{i=2}^{p} a_i + a_1 = k+1 > k.$$

Thus, (3.3) is positive, which finishes the prove of the lemma. □

3.4 Proof of Theorem 3.1.1

In the notation of Section 3.2, for any odd integer $q \geq 3$ we have a smooth h-cobordism W_q between M_3 and M_q which induces an isomorphism

$$H^2(S_3, \mathbb{Z}) \simeq H^2(S_q, \mathbb{Z}).$$

Using this isomorphism we fix a class

$$\omega \in H^2(S_3, \mathbb{Z}) \simeq H^2(S_q, \mathbb{Z}),$$

of positive square. Since the intersection pairing on S_3 has type $(1, 9)$, it follows that the orthogonal complement of ω is negative definite. Hence, $G_q^2 = 0$ implies

$$\omega \cdot G_q \neq 0$$

for all q. Via the first Chern class, each S_q carries a unique holomorphic line bundle L_q with $c_1(L_q) = \omega$.

Let C be a smooth curve of genus $g \geq 0$ and consider the threefold

$$Y_q := S_q \times C.$$

This threefold carries the holomorphic vector bundle

$$E_q := (\mathrm{pr}_1^*(L_q) \otimes \mathrm{pr}_2^* \mathcal{O}_C(1)) \oplus \mathcal{O}_{Y_q}^{\oplus r} \tag{3.4}$$

of rank $r+1$, where $\mathcal{O}_C(1)$ denotes some degree one line bundle on C. The projectivization

$$X_q := \mathbb{P}(E_q)$$

is a smooth complex projective variety of dimension $n := r+3$.

Proposition 3.4.1. *If $n \geq 3$, then the oriented diffeomorphism class of the smooth manifold which underlies X_q is independent of q. If $n = 4$, then the Chern numbers $c_1^4(X_q)$ and $c_1^2c_2(X_q)$ are unbounded in q. If $n \geq 5$, then the $\mathfrak{m}$'s Chern number $c_{\mathfrak{m}}(X_q)$ is unbounded in q for all partitions $\mathfrak{m} = (m_1, \dots, m_p)$ of n with $1 \leq m_i \leq n-2$ for all i.*

Proof. We first prove the assertion concerning the diffeomorphism type of the manifold which underlies X_q; this part of the proof follows an argument used in [45] and [47].

Fix an odd integer $q \geq 3$ and consider the h-cobordism W_q. It follows from the exponential sequence for smooth functions that complex line bundles on W_q are classified by $H^2(W_q, \mathbb{Z})$. Hence, we can find a complex line bundle $\mathbb{L}$ on W_q with

$$c_1(\mathbb{L}) = \omega \in H^2(S_3, \mathbb{Z}) \simeq H^2(W_q, \mathbb{Z}).$$

Since the isomorphism $H^2(S_3, \mathbb{Z}) \simeq H^2(S_q, \mathbb{Z})$ is induced by W_q, it follows that the restriction of $\mathbb{L}$ to each of the boundary components of W_q coincides with the complex line bundle which underlies the holomorphic line bundle L_3 resp. L_q on S_3 resp. S_q.

Let us first consider the case $C \simeq \mathbb{P}^1$. The product $W_q \times \mathbb{P}^1$ is a simply connected h-cobordism between $M_3 \times \mathbb{P}^1$ and $M_q \times \mathbb{P}^1$. It carries the complex vector bundle

$$\mathbb{E} := \left(\mathrm{pr}_1^* \mathbb{L} \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}}^1(1)\right) \oplus \underline{\mathbb{C}}^{\oplus r}.$$

The restrictions of this bundle to the boundary components of $W_q \times \mathbb{P}^1$ coincide with the complex vector bundle which underlies the holomorphic vector bundle in (3.4). Hence, the projectivization $\mathbb{P}(\mathbb{E})$ is a simply connected h-cobordism between the simply connected oriented $2n$-manifolds which underly X_3 and

X_q. It thus follows from the h-cobordism theorem [85] that these smooth $2n$-manifolds are orientation-preserving diffeomorphic, as we claimed.

The above argument proves the first assertion in the proposition for $g = 0$. For $g \geq 1$, one can use the s-cobordism theorem [40]. More precisely, since $\pi_1(M_q \times C) = \pi_1(C)$ and since the Whitehead group $\mathrm{Wh}(\pi_1(C))$ is trivial [26, Thm. 1.11], the s-cobordism theorem applies and we can conclude as before.

In order to prove the second assertion, we use the computational tools given in Proposition 3.3.1 and Lemma 3.3.2 together with the positivity result in Lemma 3.3.3. Note that it suffices to compute $c_{\mathfrak{m}}(X_q)$ modulo all terms that do not depend on q. For ease of notation, we identify cohomology classes on S_q via pullback with classes on Y_q. Using this notation, and fixing a point $c \in C$, we obtain

$$\begin{aligned}
c_1(Y_q) &= c_1(S_q) + (2-2g)\cdot[S_q \times c],\\
c_2(Y_q) &= c_2(S_q) + (2-2g)\cdot c_1(S_q)\cdot[S_q \times c],\\
c_3(Y_q) &= (2-2g)\cdot c_2(S_q)\cdot[S_q \times c].
\end{aligned}$$

In the above formulas, only $c_1(S_q) = (q-2)G_q$ depends on q.

In the notation of Proposition 3.3.1 and Lemma 3.3.2, the rank of E_q is denoted by $k = r+1$. Recall that for any partition $\mathfrak{a}$ of $r+i$ the class $f(\mathfrak{a})$ is a cohomology class in $H^{2i}(Y_q)$. By Lemma 3.3.2, this class is always independent of q, and it vanishes if additionally $i = 1$. For any partition $\mathfrak{m} = (m_1, \dots, m_p)$ of $n = r+3$ with $m_i \geq 1$ for all i, the $\mathfrak{m}$-th Chern number of X_q is computed in Proposition 3.3.1. Using Lemma 3.3.2, we obtain

$$c_{\mathfrak{m}}(X_q) = c_1(Y_q)\cdot\sum_{\mathfrak{j}} f(m_1 - j_1, \dots, m_p - j_p) + O(1), \tag{3.5}$$

where $\mathfrak{j} = (j_1, \dots, j_p)$ runs through all partitions of 1 by nonnegative integers, and where $O(1)$ denotes a term which does not depend on q. Here we used that $f(\mathfrak{a})$ is independent of q and that it vanishes if $\mathfrak{a}$ has weight $|\mathfrak{a}| = k$. In particular, the formula for $c_{\mathfrak{m}}(X_q)$ has no nontrivial contribution by terms of the form $c_1(Y_q)^2 \cdot f(\mathfrak{a})$ or $c_2(Y_q)\cdot f(\mathfrak{a})$. Moreover, we used $c_1(Y_q)^3 = 0$ and $c_1(Y_q)c_2(Y_q) \in O(1)$, which follows from $c_1(S_q)^2 = 0$ and the fact that $c_2(S_q)$ does not depend on q.

By construction of E_q, we have $c_2(E_q) = 0$ and

$$c_1(E_q) = \omega + [S_q \times c].$$

This implies

$$c_1(Y_q)\cdot c_1(E_q)^2 = 2(q-2)G_q\cdot\omega\cdot[S_q \times c] + O(1).$$

This number is unbounded in q since $G_q \cdot \omega$ is nonzero for all q. It follows from Lemmas 3.3.2 and 3.3.3 that (3.5) is unbounded in q as long as one of the partitions

$$\mathfrak{a} := (m_1 - j_1, \dots, m_p - j_p)$$

that appears in (3.5) satisfies $m_i - j_i < k = n - 2$.

If $n > 4$, then this condition is equivalent to $m_i \leq n - 2$ for all i.

If $n = 4$, then the above condition is only satisfied for c_1^4 and $c_1^2 c_2$, as we want in the proposition. □

Proof of Theorem 3.1.1. Recall that the Chern numbers c_n and $c_1 c_{n-1}$ are linear combinations of Hodge numbers [54, Prop. 2.3], which on Kähler manifolds are bounded in terms of the Betti numbers of the underlying smooth manifold. Therefore, if $n \geq 5$, the theorem follows from Proposition 3.4.1.

In complex dimension $n = 4$, the second Pontryagin number is given by

$$p_2 = c_2^2 - 2c_1c_3 + 2c_4. \tag{3.6}$$

This number depends only on the underlying oriented smooth 8-manifold; changing the orientation changes p_2 by a sign. Since c_1c_3 and c_4 are already known to be bounded by the underlying smooth manifold, the same conclusion holds for c_2^2. By Proposition 3.4.1, c_1^4 and $c_1^2c_2$ are unbounded, which finishes the proof of Theorem 3.1.1. □

Remark 3.4.2. *It easily follows from item 2 in Lemma 3.3.2 that the Chern numbers of a projective bundle over any surface remain bounded while changing the algebraic structure of the base. This explains why in our approach we had to use a base of dimension at least three.*

Remark 3.4.3. *The examples used in the proof of Theorem 3.1.1 are ruled and so they have negative Kodaira dimension. This compares to an observation of Kotschick which implies that in dimensions three and four the Chern numbers of a minimal projective manifold of general type are bounded by the underlying smooth manifold. Using the Miyaoka–Yau inequality, this was proven by Kotschick [45, p. 522 and p. 525] under the stronger assumption of ample canonical class. His argument applies because the inequality used holds more generally for arbitrary minimal projective manifolds of general type [88, 97].*

Remark 3.4.4. *Kollár [42, Thm. 4.2.3] proved that on a smooth manifold with* $b_2 = 1$*, the set of deformation equivalence classes of algebraic structures is finite, hence the Chern numbers are bounded. Conversely, it was observed by Friedman and Morgan [28] that the self-product of a Dolgachev surface yields an example of a smooth* 8*-manifold where the set of deformation equivalence* *classes of algebraic structures is infinite because the order of divisibility of the canonical class can become arbitrarily large. The Chern numbers of these examples are however bounded.*

3.5 Some applications

The following corollary combines Theorem 3.1.1 with LeBrun's examples [52].

Corollary 3.5.1. *In complex dimension $n \geq 4$, the topological Euler number c_n is the only Chern number which on complex manifolds is bounded by the underlying smooth manifold.*

Proof. The Chern number c_n is clearly bounded by the underlying topological space.

Conversely, LeBrun [52] showed that there is a sequence $(Y_m)_{m\geq 1}$ of complex structures on the 6-manifold $S^2 \times M$, where M denotes the 4-manifold which underlies a complex K3 surface, such that $c_1c_2(Y_m)$ is unbounded, whereas $c_1^3(Y_m)$ and $c_3(Y_m)$ are both bounded. It follows by induction on n that

$$Y_m \times (\mathbb{P}^1)^{n-3}$$

has unbounded c_1c_{n-1}. One also checks that $c_2^2(Y_m \times \mathbb{P}^1)$ is unbounded. This finishes the proof of Corollary 3.5.1 by Theorem 3.1.1. $\square$

It is not known whether on complex manifolds c_1^3 is bounded by the underlying smooth manifold. As in the case of smooth complex projective varieties, c_1^3 is the only Chern number where unboundedness remains open.

The next two corollaries generalize an observation of Kotschick [47, Rem. 20], asserting that the Chern number c_1^n in dimension $n \geq 3$ does not lie in the span of the Euler characteristics χ^p.

Corollary 3.5.2. *A Chern number $c_{\mathfrak{m}}$ lies in the span of the Euler characteristics χ^p and the Pontryagin numbers if and only if*

$$c_{\mathfrak{m}} \in \{c_1c_{n-1}, c_n\} \quad \text{or} \quad c_{\mathfrak{m}} \in \left\{c_2^2, c_1c_3, c_4\right\}.$$

Proof. The assertion is clear for $n \leq 2$, and it follows for $n = 3$ because the space of the Euler characteristics χ^p is spanned by c_1c_2 and c_3, and there are no Pontryagin numbers. If $n \geq 4$, then it follows immediately from Theorem 3.1.1 and the fact that c_1c_{n-1} and c_n lie in the span of the Euler characteristics χ^p, and c_2^2 lies in the span of the Euler characteristics and Pontryagin numbers in dimension four. $\square$

Corollary 3.5.3. *The Chern numbers c_1c_{n-1} and c_n are the only Chern numbers that lie in the span of the χ^p's. No Chern number in even complex dimensions lies in the span of the Pontryagin numbers.*

Proof. The fact that c_1c_{n-1} and c_n are the only Chern numbers that lie in the span of the χ^p's, follows from Corollary 3.5.2 and the observation that in dimension $n = 4$, the span of the Euler characteristics χ^p has a basis given by c_4, c_1c_3 and $3c_2^2 + 4c_1^2c_2 - c_1^4$, and so it does not contain c_2^2.

The assertion about the Pontryagin numbers in dimension $n = 2$ follows from $p_1 = c_1^2 - 2c_2$. For $n \geq 4$, it suffices by Corollary 3.5.1 to show that c_n is not a Pontryagin number. This follows for example from [46, Thm. 5] and the fact that the signature is not a multiple of c_n. □

3.6 On the space of bounded linear combinations

In this section we give an upper bound on the dimension of the space of linear combinations of Chern numbers of smooth complex projective varieties that are bounded by the underlying smooth manifold. For this purpose we determine the complex cobordism classes of the manifolds X_q constructed in Section 3.4 in terms of suitable generators of $\Omega_*^U \otimes \mathbb{Q}$. This approach is based on the fact that in complex dimension n, the Chern numbers are complex cobordism invariants which form the dual space of $\Omega_n^U \otimes \mathbb{Q}$, see [86, p. 117].

Consider the elements $\alpha_1 := \mathbb{P}^1$, $\alpha_2 := \mathbb{P}^2$ and

$$\alpha_n := \mathbb{P}(\mathcal{O}_A(1) \oplus \mathcal{O}_A^{n-2}),$$

where A denotes an abelian surface and $\mathcal{O}_A(1)$ denotes some ample line bundle on A. It follows from Lemma 2.3 in [73] that the Milnor number $s_n(\alpha_n)$ is nonzero. By the structure theorem of J.W. Milnor and S.P. Novikov [86, p. 128], $(\alpha_n)_{n\geq 1}$ is therefore a sequence of generators of the complex cobordism ring with rational coefficients . That is,

$$\Omega_*^U \otimes \mathbb{Q} \simeq \mathbb{Q}[\alpha_1, \alpha_2, \dots].$$

Using this presentation, we consider the graded ideal

$$\mathcal{I}^* := \langle \alpha_1\alpha_k \mid k \geq 3\rangle$$

in $\Omega_*^U \otimes \mathbb{Q}$. Denoting the degree n-part of this ideal by $\mathcal{I}^n$, the main result of this section is the following.

Theorem 3.6.1. *Any linear combination of Chern numbers in dimension n, which on smooth complex projective varieties is bounded by the underlying smooth manifold vanishes on $\mathcal{I}^n$.*

Proof. For $n \geq 4$, let us consider the bundle E_q on Y_q of rank $n-2$ and the corresponding n-dimensional projective bundle $X_q := \mathbb{P}(E_q)$ from Section 3.4. By Proposition 3.4.1, the smooth manifold which underlies X_q does not depend on q. Theorem 3.6.1 therefore follows from Proposition 3.6.2 below. □

Proposition 3.6.2. *Let $n \geq 4$ and let $X_q := \mathbb{P}(E_q)$ be as in Section 3.4. Then there is an unbounded function $g_n(q)$ in q such that the following identity holds in Ω_*^U:*

$$X_q = g_n(q) \cdot \alpha_1 \alpha_{n-1} + O(1),$$

where $O(1)$ denotes terms that are bounded when $q \to \infty$.

Proof. Let $\mathfrak{m}$ be a partition of n. By (3.5) and since $c_1(Y_q) = c_1(S_q) + O(1)$, we have

$$c_{\mathfrak{m}}(X_q) = \sum_{|\mathfrak{j}|=1} c_1(S_q) \cdot f(m_1 - j_1, \ldots, m_p - j_p) + O(1),$$

where $\mathfrak{j} = (j_1, \ldots, j_p)$ runs through all partitions of 1 by nonnegative integers.

We claim that up to the bounded summand $O(1)$, the Chern number $c_{\mathfrak{m}}(X_q)$ is a multiple of $c_{\mathfrak{m}}(\alpha_1 \alpha_{n-1})$. To see this, let us consider the product $B := \mathbb{P}^1 \times A$ together with the vector bundle $\mathrm{pr}_2^* \mathcal{O}_A(1) \oplus \mathcal{O}_B^{n-3}$. The projectivization

$$\mathbb{P}(\mathrm{pr}_2^* \mathcal{O}_A(1) \oplus \mathcal{O}_B^{n-3})$$

has class $\alpha_1 \alpha_{n-1}$ in Ω_*^U. By Proposition 3.3.1 we find

$$c_{\mathfrak{m}}(\mathbb{P}(\mathrm{pr}_2^* \mathcal{O}_A(1) \oplus \mathcal{O}_B^{n-3})) = f(m_1, \ldots, m_p) + \sum_{|\mathfrak{j}|=1} c_1(B) \cdot f(m_1 - j_1, \ldots, m_p - j_p),$$

because $c_i(A) = 0$ for all $i \geq 1$. In the above calculation, $f(m_1, \ldots, m_p)$ is a cohomology class of degree 6 which is actually a pullback from the second factor of B and hence vanishes. This establishes the existence of $g_n(q)$ in Proposition 3.6.2; its unboundedness follows from Proposition 3.4.1 since $n \geq 4$. □

By Theorem 3.6.1, any linear combination of Chern numbers in dimension n which on smooth complex projective varieties is bounded by the underlying smooth manifold descends to the quotient

$$(\Omega_n^U \otimes \mathbb{Q}) / \mathcal{I}^n. \tag{3.7}$$

Denoting by $p(n)$ the number of partitions of n by positive natural numbers, we therefore get the following.

Corollary 3.6.3. *In dimension $n \geq 4$, the space of rational linear combinations of Chern numbers which on smooth complex projective varieties are bounded by the underlying smooth manifold is a quotient of the dual space of (3.7); its dimension is therefore at most*

$$\dim(\Omega_n^U \otimes \mathbb{Q}) - \dim(\mathcal{I}^n) = p(n) - p(n-1) + \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Proof. We need to show that

$$\dim(\mathcal{I}^n) = p(n-1) - \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Clearly

$$\dim \langle \alpha_1 \alpha_k \mid k \geq 1 \rangle_n = p(n-1),$$

and we have to subtract the number of partitions of $n-1$ by 1 and 2, which is $\left\lfloor \frac{n+1}{2} \right\rfloor$. This concludes the corollary. □

Finally, let us compare the upper bound from Corollary 3.6.3 with the lower bound which is given by all Euler characteristics χ^p and all Pontryagin numbers in even complex dimension. For this purpose, consider the ideal

$$\mathcal{J}^* := \langle \alpha_{2k+1} \mid k \geq 1 \rangle + \langle \alpha_1 \alpha_{2k} \mid k \geq 2 \rangle$$

in $\Omega_*^U \otimes \mathbb{Q}$ which is generated by all α_{2k+1} with $k \geq 1$ and all $\alpha_1 \alpha_{2k}$ where $k \geq 2$. It is easily seen that the Euler characteristics χ^p as well as the Pontryagin numbers vanish on $\mathcal{J}^*$. By [48, Cor. 4], the signature is the only linear combination of Pontryagin numbers which is contained in the span of the Euler characteristics χ^p. A simple dimension count therefore shows that the Euler characteristics and Pontryagin numbers in dimension n form the dual space of

$$(\Omega_n^U \otimes \mathbb{Q}) / \mathcal{J}^n.$$

We note that the inclusion $\mathcal{I}^n \subseteq \mathcal{J}^n$ is proper for all $n \geq 3$ with the exception of $n = 4$, where equality holds.

4 Hodge structures of conjugate varieties

Abstract. For any subfield $K \subseteq \mathbb{C}$, not contained in an imaginary quadratic extension of $\mathbb{Q}$, we construct conjugate varieties whose algebras of K-rational (p,p)-classes are not isomorphic. This compares to the Hodge conjecture which predicts isomorphisms when K is contained in an imaginary quadratic extension of $\mathbb{Q}$; additionally, it shows that the complex Hodge structure on the complex cohomology algebra is not invariant under the $\mathrm{Aut}(\mathbb{C})$-action on varieties. In our proofs, we find simply connected conjugate varieties whose multilinear intersection forms on $H^2(-,\mathbb{R})$ are not (weakly) isomorphic. Using these, we detect nonhomeomorphic conjugate varieties for any fundamental group and in any birational equivalence class of dimension ≥ 10.

4.1 Introduction

For a smooth complex projective variety X and an automorphism σ of $\mathbb{C}$, the conjugate variety X^σ is defined via the fiber product diagram

$$\begin{tikzcd} X^\sigma \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "\sigma^*"] & \mathrm{Spec}(\mathbb{C}). \end{tikzcd}$$

To put it another way, X^σ is the smooth variety whose defining equations in some projective space are given by applying σ to the coefficients of the equations of X. As abstract schemes – but in general not as schemes over $\mathrm{Spec}(\mathbb{C})$ – X and X^σ are isomorphic. This has several important consequences for the singular cohomology of conjugate varieties.

Pull-back of forms induces a σ-linear isomorphism between the algebraic de Rham complexes of X and X^σ. This induces an isomorphism of complex

This chapter is based on [74].

Hodge structures

$$H^*(X,\mathbb{C}) \otimes_\sigma \mathbb{C} \xrightarrow{\sim} H^*(X^\sigma,\mathbb{C}), \tag{4.1}$$

where $\otimes_\sigma\mathbb{C}$ means that the tensor product is taken over $\mathbb{C}$, which maps to $\mathbb{C}$ via σ, see [13]. In particular, Hodge and Betti numbers of conjugate varieties coincide.

The singular cohomology with $\mathbb{Q}_\ell$-coefficients coincides on smooth complex projective varieties with ℓ-adic étale cohomology. Since étale cohomology does not depend on the structure morphism to $\operatorname{Spec}(\mathbb{C})$, we obtain isomorphisms of graded $\mathbb{Q}_\ell$-, resp. $\mathbb{C}$-algebras,

$$H^*(X,\mathbb{Q}_\ell) \xrightarrow{\sim} H^*(X^\sigma,\mathbb{Q}_\ell) \;\text{ and }\; H^*(X,\mathbb{C}) \xrightarrow{\sim} H^*(X^\sigma,\mathbb{C}), \tag{4.2}$$

depending on an embedding $\mathbb{Q}_\ell \subseteq \mathbb{C}$. Since the latter isomorphism is $\mathbb{C}$-linear, it is not induced by (4.1).

Only recently, Charles discovered that there are however aspects of singular cohomology which are not invariant under conjugation:

Theorem 4.1.1 (Charles [12])**.** *There exist conjugate smooth complex projective varieties with distinct real cohomology algebras.*

4.1.1 Algebras of K-rational (p,p)-classes

For any subfield $K \subseteq \mathbb{C}$, we denote the space of K-rational (p,p)-classes on X by

$$H^{p,p}(X,K) \coloneqq H^{p,p}(X) \cap H^{2p}(X,K);$$

the corresponding graded K-algebra is denoted by $H^{*,*}(X,K)$. The Hodge conjecture predicts that $H^{*,*}(X,\mathbb{Q})$ is generated by algebraic cycles. Since each algebraic cycle $Z \subseteq X$ induces a canonical cycle $Z^\sigma \subseteq X^\sigma$ and vice versa, the Hodge conjecture implies

Conjecture 4.1.2. *The graded $\mathbb{Q}$-algebra $H^{*,*}(-,\mathbb{Q})$ is conjugation invariant.*

Apart from the (few) cases where the Hodge conjecture is known, and apart from Deligne's result [21] which settles Conjecture 4.1.2 for abelian varieties, the above conjecture remains wide open, see [13, 92].

The above consequence of the Hodge conjecture motivates the investigation of potential conjugation invariance of $H^{*,*}(-,K)$ for an arbitrary field of coefficients $K \subseteq \mathbb{C}$. If $K = \mathbb{Q}(iw)$ with $w^2 \in \mathbb{N}$ is an imaginary quadratic extension of $\mathbb{Q}$, then the real part, as well as $1/w$ times the imaginary part of a $\mathbb{Q}(iw)$-rational (p,p)-class is $\mathbb{Q}$-rational. Hence,

$$H^{*,*}(-,\mathbb{Q}(iw)) \simeq H^{*,*}(-,\mathbb{Q}) \otimes_\mathbb{Q} \mathbb{Q}(iw).$$

It follows that the Hodge conjecture predicts the conjugation invariance of $H^{*,*}(-,K)$, when K is contained in an imaginary quadratic extension of $\mathbb{Q}$. In this chapter, we are able to settle all remaining cases:

Theorem 4.1.3. *Let $K \subseteq \mathbb{C}$ be a subfield, not contained in an imaginary quadratic extension of $\mathbb{Q}$. Then there exist conjugate smooth complex projective varieties whose graded algebras of K-rational (p,p)-classes are not isomorphic.*

By Theorem 4.1.3, there are conjugate smooth complex projective varieties X, X^σ with

$$H^{*,*}(X,\mathbb{C}) \ncong H^{*,*}(X^\sigma,\mathbb{C}).$$

This shows the following:

Corollary 4.1.4. *The complex Hodge structure on the complex cohomology algebra of smooth complex projective varieties is not invariant under the* $\mathrm{Aut}(\mathbb{C})$*-action on varieties.*

Corollary 4.1.4 is in contrast to (4.1) and (4.2) which show that the complex Hodge structure in each degree, as well as the $\mathbb{C}$-algebra structure of $H^*(-,\mathbb{C})$ are $\mathrm{Aut}(\mathbb{C})$-invariant. The above corollary also shows that there is no embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ which guarantees that the isomorphism (4.2), induced by isomorphisms between ℓ-adic étale cohomologies, respects the complex Hodge structures.

Theorem 4.1.3 will follow from Theorems 4.1.5 and 4.1.6 below. Firstly, if K is different from $\mathbb{R}$ and $\mathbb{C}$, then Theorem 4.1.3 follows from

Theorem 4.1.5. *Let $K \subseteq \mathbb{C}$ be a subfield, not contained in an imaginary quadratic extension of $\mathbb{Q}$. If K is different from $\mathbb{R}$ and $\mathbb{C}$, then there exist for any $p \geq 1$ and in any dimension $\geq p+1$ conjugate smooth complex projective varieties X, X^σ with*

$$H^{p,p}(X,K) \ncong H^{p,p}(X^\sigma,K).$$

It is worth noting that Theorem 4.1.5 does not remain true if one restricts to smooth complex projective varieties that can be defined over $\overline{\mathbb{Q}}$, see Remark 4.3.5.

Next, the case $K = \mathbb{R}$ in Theorem 4.1.3 follows from the case where $K = \mathbb{C}$ since

$$H^{*,*}(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq H^{*,*}(X,\mathbb{C})$$

holds; so it remains to deal with $K = \mathbb{C}$. As the isomorphism type of the $\mathbb{C}$-vector space $H^{p,p}(-,\mathbb{C})$ coincides on conjugate varieties, we now really need to make use of the algebra structure of $H^{*,*}(-,\mathbb{C})$. Remarkably, it turns out that it suffices to use only a very little amount of the latter, namely the symmetric multilinear intersection form

$$H^{1,1}(X,\mathbb{C})^{\otimes n} \longrightarrow H^{2n}(X,\mathbb{C}),$$

where $n = \dim(X)$. We explain our result, Theorem 4.1.6 below, in the next subsection.

4.1.2 Multilinear intersection forms on $H^{1,1}(-,K)$ and $H^2(-,K)$

We say that two symmetric K-multilinear forms $V^{\otimes n} \to K$ and $W^{\otimes n} \to K$ on two given K-vector spaces V and W are (weakly) isomorphic if there exists a K-linear isomorphism $V \simeq W$ which respects the given multilinear forms (up to a multiplicative constant). If K is closed under taking n-th roots, then weakly isomorphic intersection forms are already isomorphic.

For a smooth complex projective variety X of dimension n, cup product defines symmetric multilinear forms

$$H^{1,1}(X,K)^{\otimes n} \longrightarrow H^{2n}(X,K) \simeq K \quad \text{and} \quad H^{2}(X,K)^{\otimes n} \longrightarrow H^{2n}(X,K) \simeq K,$$

where $H^{2n}(X,K) \simeq K$ is the canonical isomorphism that is induced by integrating de Rham classes over X. The weak isomorphism types of the above multilinear forms are determined by the isomorphism types of the graded K-algebras $H^{*,*}(X,K)$ and $H^{2*}(X,K)$ respectively.

By the Lefschetz theorem, the Hodge conjecture is true for $(1,1)$-classes and so it is known that the isomorphism type of the intersection form on $H^{1,1}(-,\mathbb{Q})$ is conjugation invariant. Additionally, it follows from (4.2) that the isomorphism types of the intersection forms on $H^2(-,\mathbb{Q}_\ell)$ and $H^2(-,\mathbb{C})$ are invariant under conjugation. Our result, which settles the case $K = \mathbb{C}$ in Theorem 4.1.3, contrasts these positive results:

Theorem 4.1.6. *There exist in any dimension ≥ 4 simply connected conjugate smooth complex projective varieties whose $\mathbb{R}$-multilinear intersection forms on $H^2(-,\mathbb{R})$, as well as $\mathbb{C}$-multilinear intersection forms on $H^{1,1}(-,\mathbb{C})$, are not weakly isomorphic.*

The examples we will construct in the proof of Theorem 4.1.6 in Section 4.6 are defined over cyclotomic number fields. For instance, one series of examples is defined over $\mathbb{Q}[\zeta_{12}]$; their complex $(1,1)$-classes are spanned by $\mathbb{Q}[\sqrt{3}]$-rational ones. This yields examples X, X^σ such that the intersection forms on the equidimensional vector spaces $H^{1,1}(X,\mathbb{Q}[\sqrt{3}])$ and $H^{1,1}(X^\sigma,\mathbb{Q}[\sqrt{3}])$ are not weakly isomorphic, see Corollary 4.6.3.

It follows from Theorem 4.1.6 that the even-degree real cohomology algebra $H^{2*}(-,\mathbb{R})$, as well as the subalgebra $SH^2(-,\mathbb{R})$ which is generated by $H^2(-,\mathbb{R})$, is not invariant under conjugation. Since Charles's examples have dimension ≥ 12 and fundamental group $\mathbb{Z}^8$, Theorem 4.1.6 generalizes Theorem 4.1.1 in several different directions. Another generalization of Theorem 4.1.1, namely Theorem 4.1.7 below, is explained in the following subsection.

4.1.3 Applications to conjugate varieties with given fundamental group.

Conjugate varieties are homeomorphic in the Zariski topology but in general not in the analytic one. Historically, this was first observed by Serre in [78], who constructed conjugate varieties whose fundamental groups are infinite but nonisomorphic. The first nonhomeomorphic conjugate varieties with finite fundamental group were constructed by Abelson [1]. His construction however only works for nonabelian finite groups which satisfy some strong cohomological condition.

Other examples of conjugate varieties which are not homeomorphic (or, weaker: not deformation equivalent) are constructed in [7, 12, 24, 62, 83]. Again, the fundamental groups of these examples are of special shapes. In particular, our conjugate varieties in Theorem 4.1.6 are the first known nonhomeomorphic examples which are simply connected. This answers a question, posed more than 15 years ago by D. Reed in [67]. Reed's question was our initial motivation to study conjugate varieties and leads us to the more general problem of determining those fundamental groups for which nonhomeomorphic conjugate varieties exist. Since the fundamental group of smooth varieties is a birational invariant, the problem of detecting nonhomeomorphic conjugate varieties in a given birational equivalence class refines this problem. Building upon the examples we will construct in the proof of Theorem 4.1.6, we will be able to prove the following:

Theorem 4.1.7. *Any birational equivalence class of complex projective varieties in dimension* ≥ 10 *contains conjugate smooth complex projective varieties whose even-degree real cohomology algebras are nonisomorphic.*

Theorem 4.1.7 implies immediately:

Corollary 4.1.8. *Let* G *be the fundamental group of a smooth complex projective variety. Then there exist conjugate smooth complex projective varieties with fundamental group* G*, but nonisomorphic even-degree real cohomology algebras.*

In Theorem 4.8.1 in Section 4.8 we show that the examples in Theorem 4.1.7 can be chosen to have nonisotrivial deformations. This is in contrast to the observation that the previously known nonhomeomorphic conjugate varieties tend to be rather rigid, cf. Remark 4.8.3.

4.1.4 Constructions and methods of proof.

Using products of special surfaces with projective space, we will prove Theorem 4.1.5 in Section 4.3. The key idea is to construct real curves in the moduli space of abelian surfaces, respectively Kummer K3 surfaces, on which $\dim(H^{1,1}(-,K))$ is constant. Using elementary facts about modular forms, we then prove that each of our curves contains a transcendental point, i.e. a point whose coordinates are algebraically independent over $\mathbb{Q}$. The action of $\mathrm{Aut}(\mathbb{C})$ being transitive on the transcendental points of our moduli spaces, Theorem 4.1.5 follows as soon as we have seen that our assumptions on K ensure the existence of two real curves as above on which $\dim(H^{p,p}(-,K))$ takes different (constant) values.

For the proof of Theorem 4.1.6 in Section 4.6 we use the Charles–Voisin method [12, 91], see Section 4.4. We start with simply connected surfaces $Y \subseteq \mathbb{P}^N$ with special automorphisms, constructed in Section 4.5. Then we blow-up five smooth subvarieties of $Y \times Y \times \mathbb{P}^N$, e.g. the graphs of automorphisms of Y. In order to keep the dimensions low, we then pass to a complete intersection subvariety T of this blow-up. If $\dim(T) \geq 4$, then the cohomology of T encodes the action of the automorphisms on $H^2(Y,\mathbb{R})$ and $H^{1,1}(Y,\mathbb{C})$. The latter can change under the $\mathrm{Aut}(\mathbb{C})$-action, which will be the key ingredient in our proofs.

In order to prove Theorem 4.1.7 in Section 4.7, we start with a smooth complex projective variety Z of dimension ≥ 10, representing a given birational equivalence class. From our previous results, we will be able to pick a four-dimensional variety T and an automorphism σ of $\mathbb{C}$ with $Z \simeq Z^\sigma$, such that T and T^σ have nonisomorphic even-degree real cohomology algebras. Since T is four-dimensional, we can embed it into the exceptional divisor of the blow-up $\hat{Z}$ of Z in a point and define $W = Bl_T(\hat{Z})$. Then, $W^\sigma = Bl_{T^\sigma}(\hat{Z}^\sigma)$ is birational to $Z^\sigma \simeq Z$. Moreover, we will be able to arrange that $b_2(T)$ is larger than $b_4(Z)+4$. This will allow us to show that any isomorphism between $H^{2*}(W,\mathbb{R})$ and $H^{2*}(W^\sigma,\mathbb{R})$ induces an isomorphism between $H^{2*}(T,\mathbb{R})$ and $H^{2*}(T^\sigma,\mathbb{R})$. Theorem 4.1.7 will follow.

4.1.5 Conventions.

All Kähler manifolds are compact and connected, if not mentioned otherwise. A variety is a separated integral scheme of finite type over $\mathbb{C}$. Using the

GAGA principle [79], we usually identify a smooth projective variety with its corresponding analytic space, which is a Kähler manifold.

4.2 Preliminaries

4.2.1 Cohomology of blow-ups

In this subsection we recall important properties about the cohomology of blow-ups, which we will use (tacitly) throughout Sections 4.4, 4.6 and 4.7. Some of these results were already mentioned in Section 2.4.1 of Chapter 2, we repeat them here to ensure that each chapter is self contained.

Let $Y \subseteq X$ be Kähler manifolds and let $\tilde{X} = Bl_Y(X)$ be the blow-up of X in Y with exceptional divisor $D \subseteq \tilde{X}$. We then obtain a commutative diagram

$$\begin{tikzcd} D \arrow[r, "j"] \arrow[d, "p"] & \tilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, "i"] & X, \end{tikzcd}$$

where i denotes the inclusion of Y into X and j denotes the inclusion of the exceptional divisor D into $\tilde{X}$. Let r denote the codimension of Y in X, then we have the following, see [89, p. 180].

Theorem 4.2.1. *There is an isomorphism of integral Hodge structures*

$$H^k(X,\mathbb{Z}) \oplus \left(\bigoplus_{i=0}^{r-2} H^{k-2i-2}(Y,\mathbb{Z}) \right) \xrightarrow{\sim} H^k\left(\tilde{X},\mathbb{Z}\right),$$

where on $H^{k-2i-2}(Y,\mathbb{Z})$, the natural Hodge structure is shifted by $(i+1,i+1)$. On $H^k(X,\mathbb{Z})$, the above morphism is given by π^*. On $H^{k-2i-2}(Y,\mathbb{Z})$ it is given by $j_* \circ h^i \circ p^*$, where h denotes the cup product with $c_1(\mathcal{O}_D(1)) \in H^2(D,\mathbb{Z})$ and j_* is the Gysin morphism of the inclusion $j : D \hookrightarrow \tilde{X}$.

By the above lemma, each cohomology class of $\tilde{X}$ is a sum of pullback classes from X and push forward classes from D. The ring structure on $H^*(\tilde{X},\mathbb{Z})$ is therefore uncovered by the following lemma.

Lemma 4.2.2. *Let $\alpha, \beta \in H^*(D,\mathbb{Z})$ and $\eta \in H^*(X,\mathbb{Z})$. Then,*

$$\pi^*(\eta) \cup j_*(\alpha) = j_*(p^*(i^*\eta) \cup \alpha) \quad \text{and} \quad j_*(\alpha) \cup j_*(\beta) = -j_*(h \cup \alpha \cup \beta),$$

where $h = c_1(\mathcal{O}_D(1)) \in H^2(D,\mathbb{Z})$.

Proof. Note first that j satisfies the projection formula in cohomology. That is,

$$j_*(\omega_1 \cup j^*\omega_2) = (j_*\omega_1) \cup \omega_2,$$

for all $\omega_1 \in H^*(D, \mathbb{Z})$ and $\omega_2 \in H^*(\widetilde{X}, \mathbb{Z})$, which can easily be seen on the level of homology.

Using $i \circ p = \pi \circ j$, the first assertion in Lemma 4.2.2 follows immediately from the projection formula for j.

For the second assertion, one first proves

$$j_*(\alpha) \cup j_*(\beta) = j_*(1) \cup j_*(\alpha \cup \beta) \tag{4.3}$$

by realizing that the dual statement in homology holds. Note that

$$j_*(1) = c_1(\mathcal{O}_{\tilde{X}}(D)).$$

Moreover, the restriction of $\mathcal{O}_{\tilde{X}}(D)$ to D is isomorphic to $\mathcal{O}_D(-1)$. This implies $-h = j^*(j_*(1))$ and so the projection formula for j yields:

$$-j_*(h \cup \alpha \cup \beta) = j_*(1) \cup j_*(\alpha \cup \beta).$$

This concludes the proof by (4.3). □

4.2.2 Eigenvalues of conjugate endomorphisms

Let X be a smooth complex projective variety with endomorphism f and let σ be an automorphism of $\mathbb{C}$. Via base change, f induces an endomorphism f^σ of X^σ. If an explicit embedding of X into some projective space $\mathbb{P}^N$ with homogeneous coordinates $z = [z_0 : \cdots : z_N]$ is given, then f^σ is determined by

$$f^\sigma(\sigma(z))) = \sigma(f(z))$$

for all $z \in X$, where σ acts on each homogeneous coordinate simultaneously. On cohomology, we obtain linear maps

$$f^* : H^{p,q}(X) \longrightarrow H^{p,q}(X) \quad \text{and} \quad (f^\sigma)^* : H^{p,q}(X^\sigma) \longrightarrow H^{p,q}(X^\sigma).$$

These maps commute with the σ-linear isomorphism

$$H^{p,q}(X) \xrightarrow{\sim} H^{p,q}(X^\sigma)$$

induced by (4.1). This observation proves:

Lemma 4.2.3. *The set of eigenvalues of $(f^\sigma)^*$ on $H^{p,q}(X^\sigma)$ is given by the σ-conjugate of the set of eigenvalues of f^* on $H^{p,q}(X)$.*

4.2.3 The j-invariant of elliptic curves

Recall that the j-invariant of an elliptic curve E with affine Weierstrass equation $y^2 = 4x^3 - g_2 x - g_3$ equals

$$j(E) = 1728 \cdot \frac{g_2^3}{g_2^3 - 27g_3^2}.$$

Two elliptic curves are isomorphic if and only if their j-invariants coincide. From the above formula, we deduce $j(E^\sigma) = \sigma(j(E))$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. For an element τ in the upper half plane $\mathbb{H}$, we use the notation

$$E_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \quad \text{and} \quad j(\tau) := j(E_\tau). \tag{4.4}$$

Then, j induces an isomorphism between any fundamental domain of the action of the modular group $SL_2(\mathbb{Z})$ on $\mathbb{H}$ and $\mathbb{C}$. Moreover, j is holomorphic on $\mathbb{H}$ with a cusp of order one at $i \cdot \infty$.

4.2.4 Kummer K3 surfaces and theta constants

Let $M \in M_2(\mathbb{C})$ be a symmetric matrix whose imaginary part is positive definite. Then,

$$A_M := \mathbb{C}^2/(\mathbb{Z}^2 + M\mathbb{Z}^2)$$

is a principally polarized abelian surface. The associated Kummer K3 surface $K3(A_M)$ is the quotient of the blow-up of A_M at its 16 2-torsion points by the involution $\cdot(-1)$. Equivalently, $K3(A_M)$ is the blow-up of $A_M/(-1)$ at its 16 singular points.

Let L_M be a line bundle on A_M which induces the principal polarization on A_M. The linear series $|L_M^{\otimes 2}|$ then defines a morphism $A_M \longrightarrow \mathbb{P}^3$. This morphism induces an isomorphism of $A_M/(-1)$ with a degree four hypersurface

$$\{F_M = 0\} \subseteq \mathbb{P}^3.$$

The coefficients of F_M are given by homogeneous degree 12 expressions in the coordinates of Riemann's second order theta constant $\Theta_2(M) \in \mathbb{P}^3$, see [31] and also [68, Example 1.1]. This constant is defined as

$$\Theta_2(M) := [\Theta_2[0,0](M) : \Theta_2[1,0](M) : \Theta_2[0,1](M) : \Theta_2[1,1](M)]. \tag{4.5}$$

Here, for $\delta \in \{0,1\}^2$, the complex number $\Theta_2[\delta](M)$ denotes the Fourier series

$$\Theta_2[\delta](M) := \sum_{n \in \mathbb{Z}^2} e^{2\pi i \cdot Q_M(n+\delta/2)}, \tag{4.6}$$

where $Q_M(z)$ is the quadratic form $z^t M z$, associated to M.

The above discussion allows us to calculate conjugates of $K3(A_M)$ explicitly.

Lemma 4.2.4. *If $\sigma(\Theta_2(M)) = \Theta_2(M')$ holds for some automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$, then*

$$K3(A_M)^\sigma \simeq K3(A_{M'}).$$

Proof. As mentioned above, the coefficients of F_M and $F_{M'}$ are polynomial expressions in the coordinates of $\Theta_2(M)$ and $\Theta_2(M')$ respectively. The action of σ therefore maps the polynomial F_M to $F_{M'}$ and hence $\{F_M = 0\}$ to $\{F_{M'} = 0\}$. Moreover, this action maps the 16 singular points of $\{F_M = 0\}$ to the 16 singular points of $\{F_{M'} = 0\}$. The lemma follows from the above description of $K3(A_M)$ and $K3(A_{M'})$ as smooth models of $\{F_M = 0\}$ and $\{F_{M'} = 0\}$ respectively. □

Remark 4.2.5. *The linear series $|L_M^{\otimes 3}|$ defines an embedding of A_M into $\mathbb{P}^8$. It is in principle possible to use this embedding in order to calculate conjugates A_M^σ of A_M. In the preceding section we only presented the analogous (easier) calculation for the associated Kummer K3 surface which will suffice for our purposes.*

4.3 Proof of Theorem 4.1.5

Proof of Theorem 4.1.5. Let us fix a subfield $K \subseteq \mathbb{C}$, different from $\mathbb{R}$ and $\mathbb{C}$, which is not contained in any imaginary quadratic extension of $\mathbb{Q}$. We then need to construct for any $p \geq 1$ and in any dimension $n \geq p+1$ conjugate smooth complex projective varieties X, X^σ with $H^{p,p}(X,K) \ncong H^{p,p}(X^\sigma,K)$. After taking products with $\mathbb{P}^{n-2}$, it clearly suffices to settle the case $p = 1$ and $n = 2$.

We denote by $K_{\mathbb{R}} := K \cap \mathbb{R}$ the maximal real subfield of K. The proof of Theorem 4.1.5 for $p = 1$ and $n = 2$ is now divided into four different cases. Cases 1 and 2 deal with $K_{\mathbb{R}} \neq \mathbb{Q}$; in Cases 3 and 4 we settle $K_{\mathbb{R}} = \mathbb{Q}$.

In Cases 1–3 we will consider for $\tau \in \mathbb{H}$ the elliptic curve E_τ with associated j-invariant $j(\tau)$ from (4.4), and use the following

Lemma 4.3.1. *Let $L \subset \mathbb{C}$ be a subfield. Then we have for any $a, b \in \mathbb{R}_{>0}$,*

$$\dim(H^{1,1}(E_{ia} \times E_{ib}, L)) = \begin{cases} 2, & \text{if } a/b \notin L \text{ and } a \cdot b \notin L, \\ 3, & \text{if } a/b \in L \text{ and } a \cdot b \notin L, \text{ or if } a/b \notin L \text{ and } a \cdot b \in L, \\ 4, & \text{if } a/b \in L \text{ and } a \cdot b \in L. \end{cases}$$

Proof. For $j = 1, 2$, we denote the holomorphic coordinate on the j-th factor of $E_{ia} \times E_{ib}$ by $z_j = x_j + iy_j$. Then there are basis elements

$$\alpha_1, \beta_1 \in H^1(E_{ia}, \mathbb{Z}) \ \text{ and } \ \alpha_2, \beta_2 \in H^1(E_{ib}, \mathbb{Z}),$$

such that

$$dz_1 = \alpha_1 + ia \cdot \beta_1 \in H^{1,0}(E_{ia}) \quad \text{and} \quad dz_2 = \alpha_2 + ib \cdot \beta_2 \in H^{1,0}(E_{ib}).$$

We deduce that the following four $(1,1)$-classes form a basis of $H^{1,1}(E_{ia} \times E_{ib})$:

$$\alpha_1 \cup \beta_1, \quad \alpha_2 \cup \beta_2, \quad \alpha_1 \cup \alpha_2 + ab \cdot \beta_1 \cup \beta_2 \quad \text{and} \quad \alpha_1 \cup \beta_2 + (a/b) \cdot \alpha_2 \cup \beta_1.$$

The lemma follows. □

Case 1: $K_{\mathbb{R}}$ is uncountable.

The restriction of the j-invariant to $i \cdot \mathbb{R}_{\geq 1}$ is injective. Since $K_{\mathbb{R}}$ is uncountable, it follows that there is some $\lambda \geq 1$ in $K_{\mathbb{R}}$ such that $j(i\lambda)$ is transcendental.

By assumptions, $K_{\mathbb{R}}$ is different from $\mathbb{R}$. The additive action of $K_{\mathbb{R}}$ on $\mathbb{R}$ has therefore more than one orbit and so $\mathbb{R}_{\geq 1} \smallsetminus K_{\mathbb{R}}$ is uncountable. As above, it follows that there is some $\mu \in \mathbb{R}_{\geq 1} \smallsetminus K_{\mathbb{R}}$ such that $j(i\mu)$ is transcendental. Hence, there is some $\sigma \in \mathrm{Aut}(\mathbb{C})$ with $\sigma(j(i\lambda)) = j(i\mu)$. Since $j(i) = 1$, it follows from the discussion in Section 4.2.3 that

$$X := E_{i\lambda} \times E_i \quad \text{with} \quad X^\sigma \simeq E_{i\mu} \times E_i.$$

Since $\lambda \in K$ and $\mu \notin K$, it follows from Lemma 4.3.1 that $H^{1,1}(X, K)$ and $H^{1,1}(X^\sigma, K)$ are not equidimensional. This concludes Case 1.

Case 2: $K_{\mathbb{R}}$ is countable and $K_{\mathbb{R}} \neq \mathbb{Q}$.

Here we will need the following lemma.

Lemma 4.3.2. *Let $\lambda \in \mathbb{R}_{>0}$ be irrational, and let $U \subseteq \mathbb{R}_{>0}$ be an uncountable subset. Then there is some $\mu \in U$ such that $j(\mu)$ and $j(\lambda\mu)$ are algebraically independent over $\mathbb{Q}$.*

Proof. For a contradiction, suppose that $j(\mu)$ and $j(\lambda\mu)$ are algebraically dependent over $\mathbb{Q}$ for all $\mu \in U$. Since the polynomial ring in two variables over $\mathbb{Q}$ is countable, whereas U is uncountable, we may assume that $j(\mu)$ and $j(\lambda\mu)$ satisfy the same polynomial relation for all $\mu \in U$. Any uncountable subset of $\mathbb{R}$ contains an accumulation point. Hence, the identity theorem yields a polynomial relation between the holomorphic functions $j(\tau)$ and $j(\lambda\tau)$ in the variable $\tau \in \mathbb{H}$. That is,

$$\sum_{l=0}^{n} c_l(j(\tau)) \cdot j(\lambda\tau)^l = 0,$$

where $c_l(j(\tau))$ is a polynomial in $j(\tau)$ which is nontrivial for $l = n$. We may assume that n is the minimal integer such that a polynomial relation as above exists. The modular form $j(\tau)$ does not satisfy any nontrivial polynomial relation since it has a pole of order one at $i\infty$. Thus, $n \geq 1$. For $k \in \mathbb{Z}$, we have $j(\tau) = j(\tau + k)$ and so the above identity yields

$$\sum_{l=0}^{n} c_l(j(\tau)) \cdot \left(j(\lambda\tau)^l - j(\lambda\tau + \lambda k)^l\right) = 0,$$

for all $k \in \mathbb{Z}$. Since λ is irrational, $\lambda\tau$ and $\lambda\tau + \lambda k$ do not lie in the same $SL_2(\mathbb{Z})$ orbit and so $j(\lambda\tau) - j(\lambda\tau + \lambda k)$ is nonzero for all $k \in \mathbb{Z}$. Thus,

$$\sum_{l=1}^{n} c_l(j(\tau)) \cdot \sum_{h=0}^{l-1} j(\lambda\tau)^h j(\lambda\tau + \lambda k)^{l-1-h} = 0.$$

If we now choose a sequence of integers $(k_m)_{m\geq 1}$ such that λk_m tends to zero modulo $\mathbb{Z}$, then the above identity tends to the identity

$$\sum_{l=1}^{n} c_l(j(\tau)) \cdot l \cdot j(\lambda\tau)^{l-1} = 0.$$

This contradicts the minimality of n. Lemma 4.3.2 follows. □

Since $K_\mathbb{R}$ is countable, it follows that for any $t > 0$,

$$U_t := \left\{\mu \in \mathbb{R}_{\geq 1} \mid t\mu^2 \notin K\right\}$$

is uncountable. By assumptions in Case 2, $K_\mathbb{R}$ contains a positive irrational number λ. Additionally, we pick a positive irrational number $\lambda' \notin K$.

Then, by Lemma 4.3.2, there are elements $\mu \in U_\lambda$ and $\mu' \in U_{\lambda'}$ such that $j(i\mu)$ and $j(i\lambda\mu)$, as well as $j(i\mu')$ and $j(i\lambda'\mu')$, are algebraically independent over $\mathbb{Q}$. It follows that for some $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$X := E_{i\lambda\mu} \times E_{i\mu} \quad \text{with} \quad X^\sigma \simeq E_{i\lambda'\mu'} \times E_{i\mu'}.$$

Since $\lambda \in K$ and $\lambda\mu^2, \lambda', \lambda'\mu'^2 \notin K$, it follows from Lemma 4.3.1 that $H^{1,1}(X, K)$ and $H^{1,1}(X^\sigma, K)$ are not equidimensional. This concludes Case 2.

Case 3: K is uncountable and $K_\mathbb{R} = \mathbb{Q}$.

Since K is uncountable, there are elements $\tau, \tau' \in \mathbb{H}$ with $\tau, \overline{\tau'} \in K$ such that $j(\tau)$ and $j(\tau')$ are algebraically independent over $\mathbb{Q}$. Also, there are positive real numbers $\mu, \mu' \in \mathbb{R}_{>0}$ with $\mu\mu', \mu/\mu' \notin K_\mathbb{R} = \mathbb{Q}$ such that $j(i\mu)$ and $j(i\mu')$ are algebraically independent over $\mathbb{Q}$. For some $\sigma \in \mathrm{Aut}(\mathbb{C})$, we then have

$$X := E_\tau \times E_{\tau'} \quad \text{with} \quad X^\sigma \simeq E_{i\mu} \times E_{i\mu'}.$$

Since $\tau, \overline{\tau'} \in K$, the space $H^{1,1}(X, K)$ is at least three-dimensional. Conversely, $H^{1,1}(X^\sigma, K)$ is two-dimensional by Lemma 4.3.1. This concludes Case 3.

Case 4: K is countable and $K_{\mathbb{R}} = \mathbb{Q}$.

This case is slightly more difficult; instead of products of elliptic curves, we will use Kummer K3 surfaces and their theta constants, see Section 4.2.4. We begin with the definition of certain families of such surfaces. For $t = t_1 + it_2 \in \mathbb{C}$ with $t_1 \neq 0$ and $\mu \in \mathbb{R}_{>0}$, we consider the symmetric matrix

$$M(\mu,t) := i\frac{\mu}{2t_1} \cdot \begin{pmatrix} 2t_1 & 1 \\ 1 & |t|^2 \end{pmatrix}.$$

For a suitable choice of $t \in \mathbb{C}$, the matrix $-iM(\mu,t)$ is positive definite for all $\mu > 0$ and so the abelian surface $A_{M(\mu,t)}$ as well as its associated Kummer K3 surface exist. For such t, we have the following lemma, where $\hat{A}$ denotes the dual of the abelian surface A.

Lemma 4.3.3. *Let $L \subseteq \mathbb{C}$ be a subfield, let $\mu > 0$ and let $t = t_1 + it_2 \in \mathbb{C}$ such that $-i \cdot M(\mu,t)$ is positive definite. If t_1, $|t|^2$ and $\det(M(\mu,t))$ do not lie in L, then*

$$\dim(H^{1,1}(K3(\hat{A}_{M(\mu,t)}), L)) = \begin{cases} 17, & \text{if } \ (|t|^2 + 2t_1 \cdot L) \cap L = \varnothing, \\ 18, & \text{otherwise.} \end{cases}$$

Proof. Fix $t \in \mathbb{C}$ and $\mu > 0$ such that $-i \cdot M(\mu,t)$ is positive definite and assume that t_1, $|t|^2$ and $\det(M(\mu,t))$ do not lie in L. The rational degree two Hodge structure of a Kummer surface $K3(A)$ is the direct sum of 16 divisor classes with the degree two Hodge structure of A. It therefore remains to investigate the dimension of $H^{1,1}(\hat{A}_{M(\mu,t)}, L)$.

We denote the holomorphic coordinates on $\mathbb{C}^2$ by $z = (z_1, z_2)$, where

$$z_j = x_j + iy_j.$$

The cohomology of $\hat{A}_{M(\mu,t)}$ is given by the homology of $A_{M(\mu,t)}$ and so

$$\alpha_1 = dx_1,\ \alpha_2 = dx_2,\ \alpha_3 = \mu/(2t_1) \cdot (2t_1 dy_1 + dy_2),\ \alpha_4 = \mu/(2t_1) \cdot \left(dy_1 + |t|^2 dy_2\right)$$

form a basis of $H^1(\hat{A}_{M(\mu,t)}, \mathbb{Q})$. Next, $H^{1,1}(\hat{A}_{M(\mu,t)})$ has basis

$$dz_1 \cup d\overline{z}_1, \quad dz_1 \cup d\overline{z}_2, \quad dz_2 \cup d\overline{z}_1 \quad \text{and} \quad dz_2 \cup d\overline{z}_2.$$

This basis can be expressed in terms of $\alpha_j \cup \alpha_k$, where $1 \leq j < k \leq 4$. Applying the Gauß algorithm then yields the following new basis of $H^{1,1}(\hat{A}_{M(\mu,t)})$:

$$\begin{aligned}
\Omega_1 &:= \alpha_2 \cup \alpha_4 + \alpha_1 \cup \alpha_3, \\
\Omega_2 &:= \alpha_1 \cup \alpha_4 - |t|^2 \cdot \alpha_1 \cup \alpha_3, \\
\Omega_3 &:= \alpha_2 \cup \alpha_3 - 2t_1 \cdot \alpha_1 \cup \alpha_3, \\
\Omega_4 &:= \alpha_3 \cup \alpha_4 - \det(M(\mu,t)) \cdot \alpha_1 \cup \alpha_2.
\end{aligned}$$

From this description it follows that if a linear combination $\sum \lambda_i \Omega_i$ is L-rational, then all λ_i lie in L. Moreover, since $\det(M(\mu,t)) \notin L$, the coefficient λ_4 needs to vanish.

Since $t_1, |t|^2 \notin L$, neither Ω_2 nor Ω_3 is L-rational. We conclude that

$$H^{1,1}(\hat{A}_{M(\mu,t)}, L)$$

is two-dimensional if $|t|^2 + 2t_1 \cdot l_1 = l_2$ has a solution $l_1, l_2 \in L$, and it is one-dimensional otherwise. The lemma follows. □

In the following we will stick to parameters t that are contained in a sufficiently small neighborhood of $1/3 + 3i$. For such t, the matrix $-i \cdot M(\mu,t)$ is positive definite. The reason for the explicit choice of the base point $1/3 + 3i$ is due to the fact that it slightly simplifies the proof of the subsequent lemma. In order to state it, we call a point in $\mathbb{P}^3$ transcendental if its coordinates in some standard affine chart are algebraically independent over $\mathbb{Q}$. Equivalently, $z \in \mathbb{P}^3$ is transcendental if and only if $P(z) \neq 0$ for all nontrivial homogeneous polynomials P with rational coefficients. That is, the transcendental points of $\mathbb{P}^3$ are those which lie in the complement of the (countable) union of hypersurfaces which can be defined over $\mathbb{Q}$. It is important to note that $\mathrm{Aut}(\mathbb{C})$ acts transitively on this set of points.

Lemma 4.3.4. *There is a neighborhood $V \subseteq \mathbb{C}$ of $1/3 + 3i$, such that for all $t = t_1 + it_2 \in V$ with 1, t_1 and $|t|^2$ linearly independent over $\mathbb{Q}$, the following holds. Any uncountable subset $U \subseteq \mathbb{R}_{>0}$ contains a point $\mu \in U$ with:*

1. *The matrix $-i \cdot M(\mu,t)$ is positive definite.*
2. *The determinant of $M(\mu,t)$ is not rational.*
3. *The theta constant $\Theta_2(M(\mu,t))$ is a transcendental point of $\mathbb{P}^3$.*

Proof. We define the quadratic form

$$Q(z) := 2t_1 z_1^2 + 2z_1 z_2 + |t|^2 z_2^2,$$

where $z = (z_1, z_2) \in \mathbb{R}^2$. For $\delta \in \{0,1\}^2$, the homogeneous coordinate

$$\Theta_2[\delta](M(\mu,t))$$

of the theta constant $\Theta_2(M(\mu,t))$ is then given by

$$\Theta_2[\delta](M(\mu,t)) = \sum_{n \in \mathbb{Z}^2} \exp\left(-\frac{\pi\mu}{t_1} \cdot Q(n + \delta/2)\right), \tag{4.7}$$

see (4.6). At the point $t = 1/3 + 3i$, we have

$$Q(z)|_{t=1/3+3i} = \frac{2}{3} \cdot (z_1 + 3z_2/2)^2 + \frac{137}{18} \cdot z_2^2.$$

This shows that there is a neighborhood V of $1/3 + 3i$ such that $-i \cdot M(\mu, t)$ is positive definite for all $t \in V$ and all $\mu > 0$. For such t, the function in (4.7) is a modular form in the variable $i \cdot \mu \in \mathbb{H}$, see [27].

Let us now pick some $t \in V$ with 1, t_1 and $|t|^2$ linearly independent over $\mathbb{Q}$. Then $-i \cdot M(\mu, t)$ is positive definite and so $\det(M(\mu, t))$ is a nonzero multiple of μ^2. After possibly removing countably many points of U, we may therefore assume

$$\det(M(\mu, t)) \notin \mathbb{Q}$$

for all $\mu \in U$.

For a contradiction, we now assume that there is no $\mu \in U$ such that $\Theta_2(M(\mu, t))$ is a transcendental point of $\mathbb{P}^3$. Since the polynomial ring in four variables over $\mathbb{Q}$ is countable, we may then assume that there is one homogeneous polynomial P with $P(\Theta_2(M(\mu, t))) = 0$ for all $\mu \in U$. Since $U \subseteq \mathbb{R}_{>0}$ is uncountable, it contains an accumulation point. Then the identity theorem yields

$$P(\Theta_2(M(-i\tau, t))) = 0, \tag{4.8}$$

where the left hand side is considered as holomorphic function in $\tau \in \mathbb{H}$.

For $\tau \to i\infty$, the modular form $\Theta_2[\delta](M(-i\tau, t))$ from (4.7) is dominated by the summand where the exponent $Q(n)$ with $n \in \mathbb{N}^2 + \delta$ is minimal. After possibly shrinking V, these minima $n_\delta \in \mathbb{N}^2 + \delta$ of $Q(n)$ are given as follows:

$$n_{0,0} = (0,0),\ n_{1,0} = \pm(1/2, 0),\ n_{0,1} = \pm(-1, 1/2) \ \text{ and } \ n_{1,1} = \pm(-1/2, 1/2).$$

Noting that $Q(n_{0,0})$ vanishes, we conclude that for $\tau \to i\infty$, the monomial

$$\Theta_2[0,0](M)^h \cdot \Theta_2[1,0](M)^j \cdot \Theta_2[0,1](M)^k \cdot \Theta_2[1,1](M)^l,$$

where we wrote $M = M(-i\tau, t)$, is dominated by the summand

$$2 \cdot \exp\left(\frac{\pi i \tau}{t_1} \cdot (j \cdot Q(n_{1,0}) + k \cdot Q(n_{0,1}) + l \cdot Q(n_{1,1}))\right).$$

The left hand side in (4.8) is then dominated by those summands for which

$$j \cdot Q(n_{1,0}) + k \cdot Q(n_{0,1}) + l \cdot Q(n_{1,1})$$

is minimal. We will therefore arrive at a contradiction as soon as we have seen that this summand is unique. That is, it suffices to see that $Q(n_{1,0})$, $Q(n_{0,1})$

and $Q(n_{1,1})$ are linearly independent over $\mathbb{Q}$. In order to see the latter, we calculate

$$Q(n_{1,0}) = t_1/2,\ Q(n_{0,1}) = |t|^2/4 + 2t_1 - 1 \text{ and } Q(n_{1,1}) = |t|^2/4 + t_1/2 - 1/2.$$

The claim is now obvious since 1, t_1 and $|t|^2$ are linearly independent over $\mathbb{Q}$ by assumptions. This finishes the proof of the lemma. □

We are now able to conclude Case 4. Let V be the neighborhood of $\frac{1}{3}+3i$ from Lemma 4.3.4. Since $K_{\mathbb{R}} = \mathbb{Q}$ and since K is not contained in any imaginary quadratic extension of $\mathbb{Q}$, we may pick some $t = t_1 + it_2 \in K \cap V$ which is not quadratic over $\mathbb{Q}$. Then t_1 is not rational since otherwise $(t - t_1)^2$ would lie in $K_{\mathbb{R}} = \mathbb{Q}$, which yielded a quadratic relation for t over $\mathbb{Q}$. It follows that 1, $t + \bar{t} = 2t_1$ and $t \cdot \bar{t} = |t|^2$ are linearly independent over $\mathbb{Q}$, as otherwise $\bar{t}$ would lie in K and so $t + \bar{t} = 2t_1 \in K_{\mathbb{R}} = \mathbb{Q}$ were rational. Hence, the assumptions of Lemma 4.3.4 are satisfied and so there is some $\mu \in \mathbb{R}_{>0}$ such that the pair (μ, t) satisfies (1)–(3) in Lemma 4.3.4.

Next, we consider $t' = t'_1 + 3i \in V$ with 1, t'_1 and t'^2_1 linearly independent over $\mathbb{Q}$. Since V is a neighborhood of $1/3 + 3i$, there are uncountably many values for t'_1 such that t' has the above property. We claim that we can choose t'_1 within this uncountable set such that additionally

$$2t'_1\lambda_1 = \lambda_2 + |t'|^2 \tag{4.9}$$

has no solution $\lambda_1, \lambda_2 \in K$. In order to prove this, suppose that t'_1 is a solution of (4.9) for some $\lambda_1, \lambda_2 \in K$. Since $|t'|^2$ is a real number, it follows that t'_1 lies in the set of quotients x/y where x and y are imaginary parts of some elements of K. Since K is countable, so is the latter set. Our claim follows since we can choose t'_1 within an uncountable set. That is, we have just shown that there is a point $t' = t'_1 + 3i \in V$ with 1, t'_1 and $|t'|^2$ linearly independent over $\mathbb{Q}$ such that additionally, (4.9) has no solution in K. Then again the assumptions of Lemma 4.3.4 are met and so there is some $\mu' \in \mathbb{R}_{>0}$ such that the pair (μ', t') satisfies (1)–(3) in Lemma 4.3.4.

Since (μ, t) and (μ', t') satisfy Lemma 4.3.4, $\Theta_2(M(\mu, t))$ and $\Theta_2(M(\mu', t'))$ are transcendental points of $\mathbb{P}^3$. Because $\mathrm{Aut}(\mathbb{C})$ acts transitively on such points it follows that there is some automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$ with

$$\sigma(\Theta_2(M(\mu, t))) = \Theta_2(M(\mu', t')).$$

As the functor $A \longmapsto \hat{A}$ on the category of abelian varieties commutes with the $\mathrm{Aut}(\mathbb{C})$-action, it follows from Lemma 4.2.4 that

$$X := K3(\hat{A}_{M(\mu,t)}) \text{ with } X^\sigma \simeq K3(\hat{A}_{M(\mu',t')}).$$

By our choices, t_1, $|t|$ and $\det(M(\mu,t))$ lie in $\mathbb{R}\smallsetminus\mathbb{Q}$ and the same holds for the pair (μ',t'). Since $K_{\mathbb{R}}=\mathbb{Q}$, it follows that (μ,t) as well as (μ',t') satisfy the assumptions of Lemma 4.3.3. Since (4.9) has no solution in K, whereas

$$2t_1\lambda_1=\lambda_2+|t|^2$$

has the solution $\lambda_1=t$ and $\lambda_2=t^2$ in K, it follows from Lemma 4.3.3 that $H^{1,1}(X,K)$ and $H^{1,1}(X^\sigma,K)$ are not equidimensional. This concludes Case 4 and hence finishes the proof of Theorem 4.1.5. □

Remark 4.3.5. *Theorem 4.1.5 does not remain true if one restricts to smooth complex projective varieties which can be defined over $\overline{\mathbb{Q}}$. Indeed, for each smooth complex projective variety X there is a finitely generated extension K_X of $\mathbb{Q}$ such that for all $p\geq 0$ the group $H^{p,p}(X,\mathbb{C})$ is generated by K_X-rational classes. As there are only countably many varieties over $\overline{\mathbb{Q}}$, it follows that there is an extension K_0 of $\mathbb{Q}$ which is generated by countably many elements such that for each smooth complex projective variety X over $\overline{\mathbb{Q}}$ and for each $p\geq 0$, the dimension of $H^{p,p}(X,K_0)$ equals $h^{p,p}(X)$. The above claim follows, since $h^{p,p}(X)$ is invariant under conjugation.*

4.4 The Charles–Voisin construction

In this section we carry out a variant of a general construction method due to Charles and Voisin [12, 91]. The proofs of Propositions 4.4.1 and 4.4.2 below will then be the technical heart of the proof of Theorem 4.1.6 in Section 4.6.

We start with a smooth complex projective surface Y with $b_1(Y)=0$ and automorphisms $f,f'\in\operatorname{Aut}(Y)$. Then we pick an embedding

$$i:Y\hookrightarrow\mathbb{P}^N$$

and assume that f^* and f'^* fix the pullback i^*h of the hyperplane class h in $H^2(\mathbb{P}^N,\mathbb{Z})$.

For a general choice of points u, v, w and t of $\mathbb{P}^N$ and y of Y, the following smooth subvarieties of $Y\times Y\times\mathbb{P}^N$ are disjoint:

$$Z_1:=Y\times y\times u,\quad Z_2:=\Gamma_{\mathrm{id}_Y}\times v,\quad Z_3:=\Gamma_f\times w,\quad Z_4:=\Gamma_{f'}\times t,\quad Z_5:=y\times\Gamma_i, \tag{4.10}$$

where Γ denotes the graph of a morphism. The blow-up

$$X:=Bl_{Z_1\cup\cdots\cup Z_5}\left(Y\times Y\times\mathbb{P}^N\right)$$

of $Y \times Y \times \mathbb{P}^N$ along the union $Z_1 \cup \cdots \cup Z_5$ is a smooth complex projective variety. Since $b_1(Y) = 0$ and $\dim(Y) = 2$, it follows from the description of the cohomology of blow-ups, see Section 4.2.1, that the cohomology algebra of X is generated by degree two classes.

Next, let σ be any automorphism of $\mathbb{C}$. Then the automorphisms f and f' of Y induce automorphisms f^σ and f'^σ of Y^σ. Since conjugation commutes with blow-ups, we have

$$X^\sigma = Bl_{Z_1^\sigma \cup \cdots \cup Z_5^\sigma}\left(Y^\sigma \times Y^\sigma \times \mathbb{P}^N\right),$$

where we identified $\mathbb{P}^N$ with its conjugate $\mathbb{P}^{N\sigma}$, and where

$$Z_1^\sigma = Y^\sigma \times y^\sigma \times u^\sigma,\ \ Z_2^\sigma = \Gamma_{\mathrm{id}_{Y^\sigma}} \times v^\sigma,\ \ Z_3^\sigma = \Gamma_{f^\sigma} \times w^\sigma,\ \ Z_4^\sigma = \Gamma_{f'^\sigma} \times t^\sigma,\ \ Z_5^\sigma = y^\sigma \times \Gamma_{i^\sigma}.$$

Here u^σ, v^σ, w^σ and t^σ are points on $\mathbb{P}^N$, $y^\sigma \in Y^\sigma$, and $i^\sigma : Y^\sigma \hookrightarrow \mathbb{P}^N$ is the inclusion, induced by i. The pullback of the hyperplane class via i^σ is denoted by $i^{\sigma *} h^\sigma$.

In the next proposition, we will assume that the surface Y has the following properties.

(A1) There exist elements $\alpha, \beta \in H^{1,1}(Y, \mathbb{Q})$ with $\alpha^2 = \beta^2 = 0$ and $\alpha \cup \beta \neq 0$.

(A2) The sets of eigenvalues of f^* and f'^* on $H^2(Y, \mathbb{C})$ are distinct.

Then, for a smooth complete intersection subvariety

$$T \subseteq X,$$

with $\dim(T) \geq 4$, the following holds.

Proposition 4.4.1. *Suppose that (A1) and (A2) hold, and let $K \subseteq \mathbb{C}$ be a subfield. Then any weak isomorphism between the K-multilinear intersection forms on $H^2(T, K)$ and $H^2(T^\sigma, K)$ induces an isomorphism of graded K-algebras*

$$\psi : H^*(Y, K) \xrightarrow{\sim} H^*(Y^\sigma, K),$$

with the following two properties:

(P1) In degree two, ψ maps $i^* h$ to a multiple of $i^{\sigma *} h^\sigma$.

(P2) The isomorphism ψ commutes with the induced actions of f and f', i.e.

$$\psi \circ f^* = (f^\sigma)^* \circ \psi \quad \text{and} \quad \psi \circ (f')^* = (f'^\sigma)^* \circ \psi.$$

Proposition 4.4.1 has an analog for isomorphisms between intersection forms on $H^{1,1}(-, K)$. In order to state it, we need the following variant of (A2):

(A3) The sets of eigenvalues of f^* and f'^* on $H^{1,1}(Y,\mathbb{C})$ are distinct and $\mathrm{Aut}(\mathbb{C})$-invariant.

Note that f^* and f'^* are defined on integral cohomology and so their sets of eigenvalues on $H^2(Y,\mathbb{C})$ – but not on $H^{1,1}(Y,\mathbb{C})$ – are automatically $\mathrm{Aut}(\mathbb{C})$-invariant. For this reason, we did not have to impose this additional condition in (A2).

Proposition 4.4.2. *Suppose that (A1) and (A3) hold, and let $K \subseteq \mathbb{C}$ be a subfield which is stable under complex conjugation. Then any weak isomorphism between the K-multilinear intersection forms on $H^{1,1}(T,K)$ and $H^{1,1}(T^\sigma,K)$ induces an isomorphism of graded K-algebras*

$$\psi : H^{*,*}(Y,K) \xrightarrow{\sim} H^{*,*}(Y^\sigma,K),$$

which satisfies (P1) and (P2) of Proposition 4.4.1.

Remark 4.4.3. *The assumption (A1) in the above propositions is only needed if* $\dim(T) = 4$.

In the following two subsections we prove Propositions 4.4.1 and 4.4.2 respectively; important steps will be similar to arguments of Charles [12] and Voisin [91].

4.4.1 Proof of Proposition 4.4.1

Proof of Proposition 4.4.1. Suppose that there is a K-linear isomorphism

$$\phi' : H^2(T,K) \xrightarrow{\sim} H^2(T^\sigma,K), \tag{4.11}$$

which induces a weak isomorphism between the respective multilinear intersection forms.

By the Lefschetz hyperplane theorem, the natural maps

$$H^k(X,K) \longrightarrow H^k(T,K) \quad \text{and} \quad H^k(X^\sigma,K) \longrightarrow H^k(T^\sigma,K) \tag{4.12}$$

are isomorphisms for $k < n$ and injective for $k = n$, where $n := \dim(T)$. Using this we will identify classes on X and X^σ of degree $\le n$ with classes on T and T^σ respectively.

We denote by $SH^2(-,K)$ the subalgebra of $H^*(-,K)$ that is generated by $H^2(-,K)$. Its quotient by all elements of degree $\ge r+1$ is denoted by $SH^2(-,K)^{\le r}$. Since $\dim(T) \ge 4$, we obtain from (4.12) canonical isomorphisms

$$SH^2(X,K)^{\le 4} \xrightarrow{\sim} SH^2(T,K)^{\le 4} \quad \text{and} \quad SH^2(X^\sigma,K)^{\le 4} \xrightarrow{\sim} SH^2(T^\sigma,K)^{\le 4}.$$

Claim 4.4.4. *The isomorphism ϕ' from (4.11) induces a unique isomorphism*

$$\phi : SH^2(X,K)^{\leq 4} \xrightarrow{\sim} SH^2(X^\sigma,K)^{\leq 4}$$

of graded K-algebras.

Proof. In degree two, we define ϕ to coincide with ϕ' from (4.11). Since the respective algebras are generated in degree two, this determines ϕ uniquely as homomorphism of K-algebras; we have to check that it is well-defined though. In order to see the latter, let $\alpha_1,\dots,\alpha_r$ and $\beta_1,\dots,\beta_r$ be elements in $H^2(T,K)$. Then we have to prove:

$$\sum_i \alpha_i \cup \beta_i = 0 \quad \Rightarrow \quad \sum_i \phi'(\alpha_i) \cup \phi'(\beta_i) = 0.$$

Let us assume that $\sum_i \alpha_i \cup \beta_i = 0$. Since ϕ' induces a weak isomorphism between the corresponding intersection forms, this implies

$$\sum_i \phi'(\alpha_i) \cup \phi'(\beta_i) \cup \eta = 0 \;\text{ in }\; H^{2n}(T^\sigma,K),$$

for all $\eta \in SH^2(T^\sigma,K)^{2n-4}$. The class $\sum_i \phi'(\alpha_i)\cup\phi'(\beta_i)\cup\eta$ lies in $SH^2(T^\sigma,K)$ and hence it is a pullback of a class on X. Therefore, the above condition is equivalent to saying that

$$\sum_i \phi'(\alpha_i) \cup \phi'(\beta_i) \cup \eta \cup [T^\sigma] = 0 \;\text{ in }\; H^{2N+8}(X^\sigma,K),$$

for all $\eta \in SH^2(X^\sigma,K)^{2n-4}$. Since the cohomology of X is generated by degree two classes, Poincaré duality shows

$$\sum_i \phi'(\alpha_i) \cup \phi'(\beta_i) \cup [T^\sigma] = 0 \;\text{ in }\; H^{2N-2n+12}(X^\sigma,K).$$

Since $[T^\sigma]$ is the $(N+4-n)$-th power of some hyperplane class on X^σ, the Hard Lefschetz theorem implies

$$\sum_i \phi'(\alpha_i) \cup \phi'(\beta_i) = 0 \;\text{ in }\; H^4(X^\sigma,K),$$

as we wanted. Similarly, one proves that ϕ'^{-1} induces a well-defined inverse of ϕ. This finishes the proof of the claim. □

From now on, we will work with the isomorphism ϕ of K-algebras from Claim 4.4.4 instead of the weak isomorphism of intersection forms ϕ' from (4.11).

To describe the degree two cohomology of X, we denote by $D_i \subseteq X$ the exceptional divisor above Z_i and we denote by h the pullback of the hyperplane class of $\mathbb{P}^N$ to X. Then, by Theorem 4.2.1:

$$H^2(X,K) = \left(\bigoplus_{i=1}^{5}[D_i]\cdot K\right) \oplus H^2(Y\times Y,K) \oplus h\cdot K. \tag{4.13}$$

Similarly, we denote by $D_i^\sigma \subseteq X^\sigma$ the conjugate of D_i by σ and we denote by h^σ the pullback of the hyperplane class of $\mathbb{P}^N$ to X^σ. This yields:

$$H^2(X^\sigma,K) = \left(\bigoplus_{i=1}^{5}[D_i^\sigma]\cdot K\right) \oplus H^2(Y^\sigma\times Y^\sigma,K) \oplus h^\sigma\cdot K. \tag{4.14}$$

Next, we pick a base point $0 \in Y$ and consider the projections

$$Y\times Y \longrightarrow Y\times 0 \quad \text{and} \quad Y\times Y\longrightarrow 0\times Y.$$

Using pullbacks, this allows us to view $H^*(Y\times 0,K)$ and $H^*(0\times Y,K)$ as subspaces of $H^*(Y\times Y,K)$. By assumption, the first Betti number of Y vanishes and so we have a canonical identity

$$H^2(Y\times Y,K) = H^2(Y\times 0,K)\oplus H^2(0\times Y,K), \tag{4.15}$$

of subspaces of $H^2(X,K)$. A similar statement holds on X^σ.

Claim 4.4.5. *The isomorphism ϕ respects the decompositions in (4.13) and (4.14), that is:*

$$\phi(H^2(Y\times Y,K)) = H^2(Y^\sigma\times Y^\sigma,K), \tag{4.16}$$

$$\phi([D_i]\cdot K) = [D_i^\sigma]\cdot K \quad \text{for all } i=1,\dots,5, \tag{4.17}$$

$$\phi(h\cdot K) = h^\sigma\cdot K. \tag{4.18}$$

Proof. In order to prove (4.16), we define S to be the linear subspace of $H^2(X,K)$ which is spanned by all classes whose square is zero. By the ring structure of the cohomology of blow-ups (cf. Lemma 4.2.2), S is contained in $H^2(Y\times Y,K)$. Furthermore, let S^2 be the subspace of $H^4(X,K)$ which is given by products of elements in S. By assumption (A1), this subspace contains $H^4(Y\times 0,K)$ and $H^4(0\times Y,K)$. By the ring structure of the cohomology of X, it then follows that $H^2(Y\times Y,K)$ in (4.13) is equal to the linear subspace of $H^2(X,K)$ that is spanned by those classes whose square lies in S^2.

By Lefschetz's theorem on $(1,1)$-classes, the cohomology of Y^σ also satisfies (A1). Hence, $H^2(Y^\sigma\times Y^\sigma,K)$ inside $SH^2(X^\sigma,K)^{\le 4}$ has a similar intrinsic description as we have found for $H^2(Y\times Y,K)$ inside $SH^2(X,K)^{\le 4}$. This proves (4.16).

It remains to prove (4.17) and (4.18). For this, we consider for $i = 1, \ldots, 5$ the following kernels:

$$F_i := \ker\left(\cup[D_i] : H^2(Y \times Y, K) \longrightarrow H^4(X, K)\right). \tag{4.19}$$

Using Theorem 4.2.1 and Lemma 4.2.2, we obtain the following lemma, which is the analogue of Charles's Lemma 7 in [12].

Lemma 4.4.6. *Using the identification (4.15), the kernels $F_i \subseteq H^2(Y \times Y, K)$ are given as follows:*

$$F_1 = \left\{(0, \beta) : \beta \in H^2(Y, K)\right\}, \tag{4.20}$$
$$F_2 = \left\{(\beta, -\beta) : \beta \in H^2(Y, K)\right\}, \tag{4.21}$$
$$F_3 = \left\{(f^*\beta, -\beta) : \beta \in H^2(Y, K)\right\}, \tag{4.22}$$
$$F_4 = \left\{(f'^*\beta, -\beta) : \beta \in H^2(Y, K)\right\}, \tag{4.23}$$
$$F_5 = \left\{(\beta, 0) : \beta \in H^2(Y, K)\right\}. \tag{4.24}$$

In addition to the above lemma, we have as in [12] the following.

Lemma 4.4.7. *Let $\alpha \in H^2(Y \times Y, K)$ be a nonzero class. Then the images of*

$$\cup\alpha, \cup h, \cup[D_1], \ldots, \cup[D_5] : H^2(Y \times Y, K) \longrightarrow H^4(X, K)$$

are in direct sum, $\cup h$ is injective and

$$\dim(\ker(\cup\alpha)) < b_2(Y). \tag{4.25}$$

Proof. Apart from (4.25), the assertions in Lemma 4.4.7 are immediate consequences of the ring structure of the cohomology of blow-ups, see Theorem 4.2.1 and Lemma 4.2.2.

In order to proof (4.25), we write

$$\alpha = \alpha_1 + \alpha_2$$

according to the decomposition (4.15). Without loss of generality, we assume $\alpha_1 \neq 0$. Then, $\cup\alpha$ restricted to $H^2(0 \times Y, K)$ is injective. Moreover, by Poincaré duality there is some $\beta_1 \in H^2(Y \times 0, K)$ with

$$\beta_1 \cup \alpha_1 \neq 0$$

Then, $\beta_1 \cup \alpha$ is nontrivial and does not lie in the image of $\cup\alpha$ restricted to $H^2(0 \times Y, K)$. Thus, $\dim(\mathrm{im}(\cup\alpha)) > b_2(Y)$ and (4.25) follows. $\square$

Of course, the obvious analogues of Lemma 4.4.6 and 4.4.7 hold on X^σ.

Note the following elementary fact from linear algebra. If a finite number of linear maps $l_1, \ldots, l_r$ between two vector spaces have images in direct sum, then the kernel of a linear combination $\sum \lambda_i l_i$ is given by intersection of all $\ker(l_i)$ with $\lambda_i \neq 0$.

By Lemma 4.4.6, each F_i has dimension $b_2(Y)$ and hence the above linear algebra fact together with Lemma 4.4.7 shows that there is a permutation $\rho \in \mathrm{Sym}(5)$ with

$$\phi([D_i] \cdot K) = [D^\sigma_{\rho(i)}] \cdot K.$$

We are now able to prove (4.18). For some real numbers $a_0, \ldots, a_5$ and for some class $\beta^\sigma \in H^2(Y^\sigma \times Y^\sigma, K)$ we have

$$\phi(h) = a_0 h^\sigma + \sum_{j=1}^{5} a_j [D^\sigma_j] + \beta^\sigma.$$

For $i = 1, \ldots, 4$, the cup product $h \cup [D_i]$ vanishes and hence

$$a_0 h^\sigma \cup [D^\sigma_{\rho(i)}] + \sum_{j=1}^{5} a_j [D^\sigma_j] \cup [D^\sigma_{\rho(i)}] + \beta^\sigma \cup [D^\sigma_{\rho(i)}] = 0.$$

Since the cup product $[D^\sigma_j] \cup [D^\sigma_k]$ vanishes for $j \neq k$, we deduce

$$a_0 h^\sigma \cup [D^\sigma_{\rho(i)}] + a_{\rho(i)} [D^\sigma_{\rho(i)}]^2 + \beta^\sigma \cup [D^\sigma_{\rho(i)}] = 0$$

for all $i = 1, \ldots, 4$. From Theorem 4.2.1, it follows that $a_{\rho(i)}$ vanishes for all $i = 1, \ldots, 4$.

If i is such that $\rho(i) \in \{1, \ldots, 4\}$, then

$$h^\sigma \cup [D^\sigma_{\rho(i)}] = 0 \quad \text{and so} \quad \beta^\sigma \cup [D^\sigma_{\rho(i)}] = 0.$$

By Lemma 4.4.6, the intersection $\bigcap_{j \neq k} F_j$ is zero for each $k = 1, \ldots, 5$. Since the same holds on X^σ, we deduce that β^σ vanishes. Hence,

$$\phi(h) = a_0 h^\sigma + a_{\rho(5)} [D^\sigma_{\rho(5)}].$$

In $H^4(X, K)$ we have the identity

$$h \cup [D_5] = (i^* h) \cup [D_5] \in H^2(Y \times Y) \cup [D_5],$$

and similarly on X^σ. Since (4.16) is already proven, we deduce

$$a_0 h^\sigma \cup [D^\sigma_{\rho(5)}] + a_{\rho(5)} [D^\sigma_{\rho(5)}]^2 \in H^2(Y^\sigma \times Y^\sigma) \cup [D^\sigma_{\rho(5)}].$$

This implies $a_{\rho(5)} = 0$. Since ϕ is an isomorphism, $a_0 \neq 0$ follows, which proves (4.18).

It remains to prove (4.17). That is, we need to see that $\rho \in \mathrm{Sym}(5)$ is the identity. This will be achieved by a similar argument as in [12, Lem. 11].

Note that $h \cup [D_i]$ as well as $h^\sigma \cup [D_i^\sigma]$ vanish for $i \neq 5$ and are nontrivial for $i = 5$. Since (4.18) is already proven, $\rho(5) = 5$ follows.

By assumption on Y, f^* and f'^* fix i^*h. Therefore, the intersection

$$F_2 \cap F_3 \cap F_4$$

is nontrivial. Conversely, $F_1 \cap F_i = 0$ for all $i = 2, 3, 4$. Since analogue statements hold on X^σ, we obtain $\rho(1) = 1$.

Next, we use that $F_i \oplus F_j = H^2(Y \times Y, K)$ for all $i = 1, 5$ and $j = 2, 3, 4$. This allows us to define for $2 \leq j, k \leq 4$ endomorphisms $g_{j,k}$ of F_1 via the following composition:

$$g_{j,k} : F_1 \hookrightarrow F_5 \oplus F_j \xrightarrow{\mathrm{pr}_1} F_5 \hookrightarrow F_1 \oplus F_k \xrightarrow{\mathrm{pr}_1} F_1.$$

There is a canonical identification between F_1 and $H^2(Y, K)$. Using Lemma 4.4.6, a straightforward calculation then shows:

$$g_{3,2} = f^*, \quad g_{4,2} = f'^*, \quad g_{4,3} = (f' \circ f^{-1})^*, \quad g_{j,j} = \mathrm{id} \quad \text{and} \quad g_{j,k} = g_{k,j}^{-1}, \tag{4.26}$$

for all $2 \leq j, k \leq 4$.

Analogue to (4.19), we define

$$F_i^\sigma := \ker\left(\cup[D_i^\sigma] : H^2(Y^\sigma \times Y^\sigma, K) \longrightarrow H^4(X^\sigma, K)\right).$$

These subspaces are described by the corresponding statements of Lemma 4.4.6. Thus, the above construction yields for any $2 \leq j, k \leq 4$ endomorphisms $g_{j,k}^\sigma$ of F_1^σ. Using the canonical identification of F_1^σ with $H^2(Y^\sigma, K)$, these endomorphisms are given by

$$g_{3,2}^\sigma = (f^\sigma)^*, \quad g_{4,2}^\sigma = (f'^\sigma)^*, \quad g_{4,3}^\sigma = (f' \circ f^{-1})^{\sigma *}, \quad g_{j,j}^\sigma = \mathrm{id} \quad \text{and} \quad g_{j,k}^\sigma = (g_{k,j}^\sigma)^{-1}, \tag{4.27}$$

for all $2 \leq j, k \leq 4$.

Since ϕ maps $[D_1]$ to a multiple of $[D_1^\sigma]$, it follows that the restriction of ϕ to F_1 induces a K-linear isomorphism

$$\psi : F_1 = H^2(Y, K) \xrightarrow{\sim} H^2(Y^\sigma, K) = F_1^\sigma. \tag{4.28}$$

Since ϕ maps F_i isomorphically to $F_{\rho(i)}^\sigma$, the above isomorphism satisfies

$$\psi \circ g_{j,k} = g_{\rho(j),\rho(k)}^\sigma \circ \psi \tag{4.29}$$

for all $2 \le j, k \le 4$.

We now denote the eigenvalues of $g_{j,k}$ by $\mathrm{Eig}(g_{j,k})$, and similarly for $g^{\sigma}_{j,k}$. Since f and f' are automorphisms, it follows from (A2) and (4.26) that $\mathrm{Eig}(g_{3,2})$ and $\mathrm{Eig}(g_{4,2})$ are distinct $\mathrm{Aut}(\mathbb{C})$-invariant sets of roots of unity. By Lemma 4.2.3 and since $g_{j,k} = g^{-1}_{k,j}$, we deduce:

$$\mathrm{Eig}(g_{3,2}) = \mathrm{Eig}(g_{2,3}) = \mathrm{Eig}(g^{\sigma}_{3,2}) = \mathrm{Eig}(g^{\sigma}_{2,3}),$$
$$\mathrm{Eig}(g_{4,2}) = \mathrm{Eig}(g_{2,4}) = \mathrm{Eig}(g^{\sigma}_{4,2}) = \mathrm{Eig}(g^{\sigma}_{2,4}).$$

Since $g_{4,3} = g_{2,3} \circ g_{4,2}$ and $g_{3,4} = g_{2,4} \circ g_{3,2}$, it also follows that each of the sets $\mathrm{Eig}(g_{3,4})$, $\mathrm{Eig}(g_{4,3})$, $\mathrm{Eig}(g^{\sigma}_{3,4})$ and $\mathrm{Eig}(g^{\sigma}_{4,3})$ is distinct from $\mathrm{Eig}(g_{2,3})$ and $\mathrm{Eig}(g_{4,2})$. Therefore, (4.29) implies that ρ respects the subsets $\{2,3\}$ and $\{2,4\}$. Hence, $\rho = \mathrm{id}$, as we wanted. This finishes the proof of Claim 4.4.5. □

Since $b_1(Y) = 0$ and $\dim(Y) = 2$, the cohomology algebra $H^*(0 \times Y, K)$ is a subalgebra of $SH^2(X, K)^{\le 4}$. Restriction of ϕ therefore extends the K-linear isomorphism ψ from (4.28) to an isomorphism

$$\psi : H^*(Y, K) \xrightarrow{\sim} H^*(Y^{\sigma}, K) \tag{4.30}$$

of graded K-algebras which we denote with the same letter. Since ρ in the proof of Claim 4.4.5 is the identity, it follows from (4.26), (4.27) and (4.29) that ψ satisfies (P2).

In order to prove (P1), we note that

$$\ker\big(\cup[D_5] : F_1 \oplus h \cdot K \longrightarrow H^4(X, K)\big) = (i^*h - h) \cdot K,$$

where $i^*h \in F_1 = H^2(0 \times Y, K)$. A similar statement holds on X^{σ}. Since ϕ maps F_1 to F^{σ}_1, $[D_5] \cdot K$ to $[D^{\sigma}_5] \cdot K$ and $h \cdot K$ to $h^{\sigma} \cdot K$, it follows that ϕ maps $i^*h \cdot K$ to $i^{\sigma *}h^{\sigma} \cdot K$. This finishes the proof of Proposition 4.4.1. □

4.4.2 Proof of Proposition 4.4.2

Proof of Proposition 4.4.2. As in the proof of Proposition 4.4.1, we use (4.12) in order to identify classes of degree $\le n$ on T with classes on X. Further, $SH^{1,1}(-, K)$ denotes the subalgebra of $H^*(-, K)$ that is generated by $H^{1,1}(-, K)$; its quotient by elements of degree $\ge r + 1$ is denoted by

$$SH^{1,1}(-, K)^{\le r}.$$

Let us now suppose that there is a K-linear isomorphism

$$\phi' : H^{1,1}(T, K) \xrightarrow{\sim} H^{1,1}(T^{\sigma}, K), \tag{4.31}$$

which induces a weak isomorphism between the respective intersection forms. Then we have the following analogue of Claim 4.4.4 in the proof of Proposition 4.4.1:

Claim 4.4.8. *The isomorphism from (4.11) induces a unique isomorphism*

$$\phi : SH^{1,1}(X,K)^{\leq 4} \xrightarrow{\sim} SH^{1,1}(X^\sigma,K)^{\leq 4}$$

of graded K-algebras.

Proof. As in the proof of Claim 4.4.4, this claim reduces to showing the following: Suppose we have K-rational $(1,1)$-classes $\alpha_1,\ldots,\alpha_r$ and $\beta_1,\ldots,\beta_r$ on T such that

$$\sum_i \phi'(\alpha_i) \cup \phi'(\beta_i) \cup \eta \cup [T^\sigma] = 0 \quad \text{in} \quad H^{2N+8}(X^\sigma,K), \tag{4.32}$$

for all $\eta \in SH^{1,1}(X^\sigma,K)^{2n-4}$. Then, $\sum_i \phi'(\alpha_i)\cup\phi'(\beta_i)$ vanishes.

In order to prove the latter, let ω be the hyperplane class on X^σ with

$$[T^\sigma] = \omega^{N+4-n}.$$

With respect to this Kähler class we obtain a decomposition into primitive pieces:

$$\sum_i \phi'(\alpha_i) \cup \phi'(\beta_i) = \delta_0 \cdot \omega^2 + \delta_1 \cup \omega + \delta_2,$$

where $\delta_j \in H^{j,j}(X,\mathbb{C})_{\mathrm{pr}}$. Since ω is an integral class, it follows that δ_j lies in $H^{j,j}(X,K)_{\mathrm{pr}}$. The above identity then shows $\delta_2 \in SH^{1,1}(X,K)$.

At this point, we use the assumption in Proposition 4.4.2 which ensures that K is stable under complex conjugation. Indeed, this assumption allows us to choose for $j=0,1,2$ the following K-rational classes:

$$\eta_j := \overline{\delta_j} \cdot \omega^{n-2-j} \in SH^{1,1}(X^\sigma,K)^{2n-4}.$$

For $j=0,1,2$, we put $\eta=\eta_j$ in (4.32). Then, the Hodge–Riemann bilinear relations yield $\delta_j=0$ for $j=0,1,2$. This finishes the proof of Claim 4.4.8. □

Exploiting the isomorphism of K-algebras ϕ from Claim 4.4.8, the proof of Proposition 4.4.2 is now obtained by changing the notation in the corresponding part of the proof of Proposition 4.4.1. This finishes the proof of Proposition 4.4.2. □

4.5 Some simply connected surfaces with special automorphisms

In this section we construct for any integer $g \geq 1$ a simply connected surface Y_g of geometric genus g and with special automorphisms. In the proof of Theorem 4.1.6 in Section 4.6, we will then apply the construction from Section 4.4 to these surfaces. In Section 4.7, we will use the examples from Section 4.6 in order to prove Theorem 4.1.7. It is only the proof of the latter theorem where it will become important that $b_2(Y_g)$ tends to infinity if g does.

4.5.1 Hyperelliptic curves with special automorphisms

For $g \geq 1$, let C_g denote the hyperelliptic curve with affine equation

$$y^2 = x^{2g+1} - 1,$$

see [87] or Section 2.3.1 in Chapter 2. The complement of this affine piece in C_g is a single point which we denote by ∞. For a primitive $(2g+1)$-th root of unity ζ_{2g+1}, the maps

$$(x,y) \longmapsto (\zeta_{2g+1} \cdot x, y) \quad \text{and} \quad (x,y) \longmapsto (x, -y)$$

induce automorphisms of C_g which we denote by η_g and ι respectively. Then, ι has the $2g+2$ fixed points

$$(1,0), (\zeta_{2g+1}, 0), \ldots, (\zeta_{2g+1}^{2g}, 0) \text{ and } \infty.$$

The automorphism η_g fixes ∞ and performs a cyclic permutation on the remaining fixed points. The corresponding permutation matrix has eigenvalues $1, \zeta_{2g+1}, \ldots, \zeta_{2g+1}^{2g}$.

The holomorphic 1-forms

$$\frac{x^{i-1}}{y} \cdot dx,$$

where $i = 1, \ldots, g$, form a basis of $H^{1,0}(C_g)$. Therefore, η_g^* has eigenvalues $\zeta_{2g+1}, \ldots, \zeta_{2g+1}^{g}$ on $H^{1,0}(C_g)$. Moreover, ι acts on $H^1(C_g, \mathbb{Z})$ by multiplication with -1.

4.5.2 The elliptic curve E_i

Let E_i be the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$, cf. Section 4.2.3. Multiplication by i and -1 induces automorphisms η_i and ι of E_i respectively. The involution ι has four fixed points. The action of η_i fixes two of those fixed points and interchanges the remaining two. On $H^{1,0}(E_i)$, the automorphisms ι and η_i act by multiplication with -1 and i respectively.

4.5.3 Products modulo the diagonal involution

For $g \geq 1$, we consider the product $C_g \times E_i$, where C_g and E_i are defined above. On this product, the involution ι acts via the diagonal. This action has $8g+8$ fixed points. Let $\widetilde{C_g \times E_i}$ be the blow-up of these fixed points. Then,

$$Y_g := \widetilde{C_g \times E_i}/\iota \tag{4.33}$$

is a smooth surface. For instance, $Y_1 = K3(C_1 \times E_i)$ is a Kummer K3 surface, see Section 4.2.4.

Lemma 4.5.1. *The surface Y_g is simply connected.*

Proof. It suffices to prove that the normal surface

$$Y_g' := (C_g \times E_i)/\iota$$

is simply connected. Projection to the second coordinate induces a map

$$\pi : Y_g' \longrightarrow \mathbb{P}^1.$$

Let $U \subseteq \mathbb{P}^1$ be the complement of the 4 branch points of $E_i \to \mathbb{P}^1$. Then, restriction of π to $V := \pi^{-1}(U)$ yields a fiber bundle $\pi|_V : V \to U$ with fiber C_g. Since U is homotopic to a wedge of 3 circles, the long exact homotopy sequence yields a short exact sequence

$$0 \longrightarrow \pi_1(C_g) \longrightarrow \pi_1(V) \longrightarrow \pi_1(U) \longrightarrow 0 \ .$$

Since π has a section, this sequence splits. Since V is the complement of a divisor in Y_g', the natural map $\pi_1(V) \to \pi_1(Y_g')$ is surjective by Proposition 2.10 in [43]. Therefore, the above split exact sequence shows that $\pi_1(Y_g')$ is generated by the fundamental group of a general fiber together with the image of the fundamental group of a section of π. The latter is clearly trivial. Furthermore, the inclusion of a general fiber $C_g \hookrightarrow Y_g'$ is homotopic to the composition of the quotient map $C_g \longrightarrow C_g/\iota$ with the inclusion of a special fiber $C_g/\iota \simeq \mathbb{P}^1$. Since $\mathbb{P}^1$ is simply connected, the inclusion of a general fiber $C_g \hookrightarrow Y_g'$ is trivial on π_1. It follows that the image of $\pi_1(C_g) \to \pi_1(Y_g')$ is trivial. This proves the lemma. □

Definition 4.5.2. *Let Y_g be as in (4.33). Then we define the automorphisms f and f' of Y_g to be induced by $\eta_g \times \mathrm{id}$ and $\mathrm{id} \times \eta_i$ respectively.*

Lemma 4.5.3. *The surface Y_g with automorphisms f and f' as above satisfies (A1)–(A3).*

Proof. In order to describe the second cohomology of Y_g, we denote the exceptional $\mathbb{P}^1$-curves of Y_g by $D_1, \ldots, D_{8g+8}$. Then, for any field K:

$$H^2(Y_g, K) = H^2(C_g \times E_i, K) \oplus \left(\bigoplus_{i=1}^{8g+8} [D_i] \cdot K \right). \tag{4.34}$$

It follows from the discussion in Section 4.5.1 (resp. 4.5.2) that the action of f (resp. f') on $H^2(Y_g,\mathbb{C})$ has eigenvalues $1, \zeta_{2g+1}, \ldots, \zeta_{2g+1}^{2g}$ (resp. $\pm 1, \pm i$). Moreover, the same statement holds for their actions on $H^{1,1}(Y_g,\mathbb{C})$. This proves (A2) and (A3).

By (4.34), nontrivial rational $(1,1)$-classes on C_g and E_i induce classes α and β in $H^{1,1}(Y_g,\mathbb{Q})$ which satisfy (A1). This finishes the prove of the lemma. $\square$

4.6 Multilinear intersection forms on $H^2(-,\mathbb{R})$ and $H^{1,1}(-,\mathbb{C})$

Here we prove Theorem 4.1.6. This will be achieved by Lemma 4.6.1 and Theorem 4.6.2 below, where more precise statements are proven.

Let $n \geq 4$ and $g \geq 1$. Moreover, let Y_g be the simply connected surface with automorphisms f and f' from Definition 4.5.2. We pick an ample divisor on Y_g which is fixed by f and f'. A sufficiently large multiple of this divisor gives an embedding

$$i : Y_g \hookrightarrow \mathbb{P}^N$$

with $n \leq N + 4$ such that the actions of f and f' fix the pullback of the hyperplane class.

Next, let

$$X_g := Bl_{Z_1 \cup \cdots \cup Z_5} \left(Y_g \times Y_g \times \mathbb{P}^N \right)$$

be the blow-up of $Y_g \times Y_g \times \mathbb{P}^N$ along $Z_1 \cup \cdots \cup Z_5$, where Z_i is defined in (4.10). Since $n \leq N + 4$, X_g contains a smooth n-dimensional complete intersection subvariety

$$T_{g,n} \subseteq X_g. \tag{4.35}$$

Since Y_g, f and f' are defined over $\mathbb{Q}[\zeta_{8g+4}] = \mathbb{Q}[\zeta_{2g+1}, i]$, so is X_g and we may assume that the same holds true for $T_{g,n}$.

Lemma 4.6.1. *Let $n \geq 2$, then the variety $T_{g,n}$ from (4.35), as well as each of its conjugates, is simply connected.*

Proof. Since Y_g is simply connected by Lemma 4.5.1, so is X_g. By the Lefschetz hyperplane theorem, $T_{g,n}$ is then simply connected for $n \geq 2$.

Since the curves C_g and E_i in the definition of Y_g are defined over $\mathbb{Z}$, it follows that Y_g is isomorphic to any conjugate Y_g^σ. Thus, Y_g^σ is simply connected and the above reasoning shows that the same holds true for $T_{g,n}^\sigma$, as long as $n \geq 2$. This proves the lemma. □

The next theorem, which implies Theorem 4.1.6 from the introduction, shows that certain automorphisms $\sigma \in \mathrm{Aut}(\mathbb{C})$ which act nontrivially on $\mathbb{Q}[\zeta_{8g+4}]$ change the analytic topology as well as the complex Hodge structure of $T_{g,n}$.

Theorem 4.6.2. *Let $g \geq 1$ and $n \geq 4$ be integers and let $\sigma \in \mathrm{Aut}(\mathbb{C})$ with $\sigma(i) = i$ and $\sigma(\zeta_{2g+1}) \neq \zeta_{2g+1}$ or vice versa. Then, the $\mathbb{R}$-multilinear intersection forms on $H^2(T_{g,n}, \mathbb{R})$ and $H^2(T_{g,n}^\sigma, \mathbb{R})$, as well as the $\mathbb{C}$-multilinear intersection forms on $H^{1,1}(T_{g,n}, \mathbb{C})$ and $H^{1,1}(T_{g,n}^\sigma, \mathbb{C})$, are not weakly isomorphic.*

Proof. For ease of notation, we assume $\sigma(i) = i$ and $\sigma(\zeta_{2g+1}) = \zeta_{2g+1}^{-1}$. The general case is proven similarly.

Since the curves C_g and E_i from Sections 4.5.1 and 4.5.2 are defined over $\mathbb{Z}$, it follows that the isomorphism type of Y_g is invariant under any automorphism of $\mathbb{C}$. Hence, we may identify Y_g with Y_g^σ. Under this identification, $f'^\sigma = f'$ since i is fixed by σ. Moreover, $f^\sigma = f^{-1}$, since it is induced by the automorphism

$$\eta_g^{-1} \times \mathrm{id} \in \mathrm{Aut}(C_g \times E_i).$$

Suppose that the $\mathbb{R}$-multilinear intersection forms on

$$H^2(T_{g,n}, \mathbb{R}) \quad \text{and} \quad H^2(T_{g,n}^\sigma, \mathbb{R})$$

are weakly isomorphic. By Lemma 4.5.3, Proposition 4.4.1 applies and we obtain an $\mathbb{R}$-algebra automorphism of $H^*(Y_g, \mathbb{R})$ with properties (P1) and (P2). By (P1),

$$\psi(i^* h) = b \cdot i^* h$$

for some $b \in \mathbb{R}^\times$. Since the square of $i^* h$ generates $H^4(Y_g, \mathbb{R})$, it follows that in degree 4, the automorphism ψ is given by multiplication with a positive real number.

We extend ψ now $\mathbb{C}$-linearly and obtain an automorphism

$$\psi : H^*(Y_g, \mathbb{C}) \xrightarrow{\sim} H^*(Y_g, \mathbb{C}),$$

which we denote by the same letter and which satisfies

$$\psi \circ f = f^{-1} \circ \psi \quad \text{and} \quad \psi \circ f' = f' \circ \psi. \tag{4.36}$$

Let us now pick nontrivial classes $\omega \in H^{1,0}(C_g)$ and $\omega' \in H^{1,0}(E_i)$ with $\eta_g^*\omega = \zeta_{2g+1} \cdot \omega$ and $\eta_i^*\omega' = i \cdot \omega'$. Then, $\omega \cup \overline{\omega'}$ lies in $H^{1,1}(Y_g)$ and we consider $\psi(\omega \cup \overline{\omega'})$ in $H^2(Y_g, \mathbb{C})$. By (4.36), f^{-1} and f' act on this class by multiplication with ζ_{2g+1} and $-i$ respectively. We claim that the only class in $H^2(Y_g, \mathbb{C})$ with this property is $\overline{\omega} \cup \overline{\omega'}$ and so

$$\psi(\omega \cup \overline{\omega'}) = \lambda \cdot \overline{\omega} \cup \overline{\omega'} \tag{4.37}$$

for some nonzero $\lambda \in \mathbb{C}$. Indeed, since η_i interchanges two of the fixed points of ι on E_i and fixes the remaining two, f'^* has eigenvalues ± 1 on the subspace of exceptional divisors in (4.34). Therefore, $\psi(\omega \cup \overline{\omega'})$ needs to be contained in $H^2(C_g \times E_i, \mathbb{C})$. On this subspace, f^{-1*} and f'^* are given by $(\eta_g^{-1} \times \mathrm{id})^*$ and $(\mathrm{id} \times \eta_i)^*$ respectively. Our claim follows by the explicit description of η_g and η_i in Sections 4.5.1 and 4.5.2.

Together with its complex conjugate, equation (4.37) shows:

$$\psi(\omega \cup \overline{\omega'} \cup \overline{\omega} \cup \omega') = -|\lambda|^2 \cdot \omega \cup \overline{\omega'} \cup \overline{\omega} \cup \omega'.$$

Since the above degree four class generates $H^4(Y_g, \mathbb{C})$, we deduce that ψ is given in degree four by multiplication with $-|\lambda|^2$. As we have seen earlier, this number should be positive, which is a contradiction. This finishes the proof of the first assertion in Theorem 4.6.2.

For the proof of the second assertion, assume that the $\mathbb{C}$-multilinear intersection forms on $H^{1,1}(T_{g,n}, \mathbb{C})$ and $H^{1,1}(T_{g,n}^\sigma, \mathbb{C})$ are weakly isomorphic. By Lemma 4.5.3 and Proposition 4.4.1, this yields an automorphism ψ of $H^{1,1}(Y_g, \mathbb{C})$ which satisfies (4.36). Then, f^{-1} and f' act on $\psi(\omega \cup \overline{\omega'})$ by multiplication with ζ_{2g+1} and $-i$ respectively. This is a contradiction, since $H^{1,1}(Y_g, \mathbb{C})$ does not contain such a class. This finishes the proof of the theorem. $\square$

Recall from (4.35) that $T_{g,n}$ is defined over the cyclotomic number field $\mathbb{Q}[\zeta_{8g+4}]$. This number field contains the totally real subfield

$$K_g := \mathbb{Q}[\zeta_{8g+4} + \zeta_{8g+4}^{-1}].$$

For instance, $K_1 = \mathbb{Q}[\sqrt{3}]$. From Theorem 4.6.2, we deduce the following

Corollary 4.6.3. *Let $K \subseteq \mathbb{C}$ be a subfield with $K_g \subseteq K$, and let $\sigma \in \mathrm{Aut}(\mathbb{C})$ with $\sigma(i) = i$ and $\sigma(\zeta_{2g+1}) \neq \zeta_{2g+1}$ or vice versa. Then the intersection forms on the equidimensional vector spaces $H^{1,1}(T_{g,n}, K)$ and $H^{1,1}(T_{g,n}^\sigma, K)$ are not weakly isomorphic.*

Proof. By Theorem 4.6.2 it suffices to prove that the $(1,1)$-classes on $T_{g,n}$ are spanned by K_g-rational ones. Modulo divisor classes, $H^{1,1}(T_{g,n})$ is given by $H^{1,1}(Y_g) \oplus H^{1,1}(Y_g)$. Furthermore, modulo divisors, $H^{1,1}(Y_g)$ is given by the ι-invariant classes on $E_i \times C_g$. The complex Hodge structure of E_i and C_g is generated by $\mathbb{Q}[i]$- and $\mathbb{Q}[\zeta_{2g+1}]$-rational classes respectively, see [87] for the latter. We may now arrange that the induced generators of $H^{1,1}(Y_g)$ are invariant under complex conjugation and thus lie in the subspace of K_g-rational classes. This concludes the proof of the corollary. □

Remark 4.6.4. *Our types of arguments are consistent with Conjecture 4.1.2 in the sense that they cannot detect conjugate varieties with nonisomorphic algebras of $\mathbb{Q}$-rational (p,p)-classes. This is because the essential ingredient in the proof of Theorem 4.6.2 is a variety Y with an automorphism whose action on $H^{p,p}(Y,K)$ has a set of eigenvalues which is not* $\mathrm{Aut}(\mathbb{C})$*-invariant. (In our arguments, this role is played by the surface Y_g with the automorphism $f \circ f'$.) For $K = \mathbb{Q}$, the characteristic polynomial of such an action has rational coefficients and so the above situation cannot happen.*

Remark 4.6.5. *Using Freedman's classification of simply connected topological* 4*-manifolds, one can prove that simply connected conjugated smooth complex projective surfaces are always homeomorphic. On the other hand, Theorem 4.1.6 shows that in any dimension at least* 4*, there are simply connected conjugate smooth complex projective varieties which are not homeomorphic. The case of dimension three remains open.*

4.7 Nonhomeomorphic conjugate varieties in each birational equivalence class

In this section we prove Theorem 4.1.7. For this purpose, let Z be a given smooth complex projective variety of dimension ≥ 10. Next, let $T_{g,4}$ be the four-dimensional smooth complex projective variety, defined in (4.35). By (4.13) and (4.34), the second Betti number of $T_{g,4}$ equals $24g + 26$. We may therefore choose an integer $g \geq 1$ with

$$b_2(T_{g,4}) > b_4(Z) + 4. \tag{4.38}$$

From some projective space, Z is cut out by finitely many homogeneous polynomials. We denote the field extension of $\mathbb{Q}$ which is generated by the coefficients of these polynomials by L. Since L is finitely generated, and after possibly replacing g by a suitable larger integer, we may pick an automorphism σ of $\mathbb{C}$ which fixes L and i but not ζ_{2g+1}.

Since $T_{g,4}$ has dimension 4, it can be embedded into $\mathbb{P}^9$. The assumption $\dim(Z) \geq 10$ therefore ensures that we may fix an embedding of $T_{g,4}$ into the exceptional divisor of the blow-up $\hat{Z}$ of Z in a point $p \in Z$. We then define the following element in the birational equivalence class of Z:

$$W := Bl_{T_{g,4}}(\hat{Z}). \tag{4.39}$$

Since conjugation commutes with blow-ups, the σ-conjugate of W is given by

$$W^\sigma = Bl_{T^\sigma_{g,4}}(\hat{Z}^\sigma), \tag{4.40}$$

where $\hat{Z}^\sigma$ is the blow-up of Z^σ in a point $p^\sigma \in Z^\sigma$ and $T^\sigma_{g,4}$ is embedded in the exceptional divisor of this blow-up. Since σ fixes L, we have $Z^\sigma \simeq Z$. Therefore, W and W^σ are both birational to Z. Hence, Theorem 4.1.7 follows from the following result.

Theorem 4.7.1. *Let W and σ be as above. Then the graded even-degree real cohomology algebras of W and W^σ are nonisomorphic.*

Proof. For a contradiction, let us assume that there is an isomorphism

$$\gamma : H^{2*}(W,\mathbb{R}) \longrightarrow H^{2*}(W^\sigma,\mathbb{R})$$

of graded $\mathbb{R}$-algebras. Using pullbacks, we regard $H^{2*}(Z,\mathbb{R}) \subseteq H^{2*}(\hat{Z},\mathbb{R})$ and $H^{2*}(Z^\sigma,\mathbb{R}) \subseteq H^{2*}(\hat{Z}^\sigma,\mathbb{R})$ as subalgebras of $H^{2*}(W,\mathbb{R})$ and $H^{2*}(W^\sigma,\mathbb{R})$ respectively. By Theorem 4.2.1,

$$H^2(W,\mathbb{R}) = H^2(Z,\mathbb{R}) \oplus [H]\cdot\mathbb{R} \oplus [D]\cdot\mathbb{R}, \tag{4.41}$$
$$H^2(W^\sigma,\mathbb{R}) = H^2(Z^\sigma,\mathbb{R}) \oplus [H^\sigma]\cdot\mathbb{R} \oplus [D^\sigma]\cdot\mathbb{R}, \tag{4.42}$$

where $H \subset \hat{Z}$ and $H^\sigma \subset \hat{Z}^\sigma$ are the exceptional divisors above the blown-up points, and

$$j : D \hookrightarrow W \quad \text{and} \quad j^\sigma : D^\sigma \hookrightarrow W^\sigma$$

are the exceptional divisors of the blow-ups along $T_{g,4}$ and $T^\sigma_{g,4}$ respectively.

Any cohomology class of positive degree on Z is Poincaré dual to a homology class which does not meet the center of the blow-up $\hat{Z} \to Z$. This shows that for any $\eta \in H^k(Z,\mathbb{R})$, with $k \geq 1$, and for any $\alpha \in H^*(D,\mathbb{R})$,

$$\eta \cup [H] = 0 \quad \text{and} \quad \eta \cup j_*(\alpha) = 0.$$

A similar statement holds on W^σ and we will use these properties tacitly.

The restriction of $-[H]$ to $H \subset \hat{Z}$ is given by $c_1(\mathcal{O}_H(1))$; its restriction to $T_{g,4}$ is therefore ample. By Theorem 4.2.1, we have

$$b_4(W) = b_4(Z) + b_2(T_{g,4}) + 2.$$

It then follows from (4.38) that the second primitive Betti number of $T_{g,4}$ is bigger than $b_4(W)/2$. Since $T_{g,4}$ is four-dimensional, and since $-[H]$ restricts to an ample class on $T_{g,4}$, it follows that $H^2(Z,\mathbb{R}) \oplus [H] \cdot \mathbb{R}$ inside $H^2(W,\mathbb{R})$ is given by those classes whose multiplication on $H^4(W,\mathbb{R})$ has kernel of dimension bigger than $b_4(W)/2$. A similar statement holds for $H^2(Z^\sigma,\mathbb{R}) \oplus [H^\sigma] \cdot \mathbb{R}$ inside $H^2(W^\sigma,\mathbb{R})$ and so γ needs to take $H^2(Z,\mathbb{R}) \oplus [H] \cdot \mathbb{R}$ to $H^2(Z^\sigma,\mathbb{R}) \oplus [H^\sigma] \cdot \mathbb{R}$. Since γ is an isomorphism, it follows that

$$\gamma([D]) = \alpha^\sigma + a \cdot [H^\sigma] + b \cdot [D^\sigma] \tag{4.43}$$

holds for some $\alpha^\sigma \in H^2(Z^\sigma,\mathbb{R})$ and $b \neq 0$.

Cup product with $[D]$ on $H^2(W,\mathbb{R})$ has two-dimensional image, spanned by $[D] \cup [H]$ and $[D]^2$. For any $\beta^\sigma \in H^2(Z^\sigma,\mathbb{R})$, the following classes are therefore linearly dependent:

$$\gamma([D]) \cup \beta^\sigma, \quad \gamma([D]) \cup [H^\sigma] \quad \text{and} \quad \gamma([D]) \cup [D^\sigma].$$

Since $b \neq 0$, this is only possible if $\alpha^\sigma \cup \beta^\sigma = 0$ for all β^σ. Hence, $\alpha^\sigma = 0$.

Since $\alpha^\sigma = 0$, it follows from $[D] \cup [H] \neq 0$ that $\gamma([H]) \in H^2(Z^\sigma,\mathbb{R}) \oplus [H^\sigma] \cdot \mathbb{R}$ cannot be contained in $H^2(Z^\sigma,\mathbb{R})$ and hence

$$\gamma([H]) = \tilde{\alpha}^\sigma + c \cdot [H^\sigma]$$

for some $\tilde{\alpha}^\sigma \in H^2(Z^\sigma,\mathbb{R})$ and $c \neq 0$. As cup product with $[H]$ on $H^2(W,\mathbb{R})$ has two-dimensional image, the above argument which showed $\alpha^\sigma = 0$, also implies $\tilde{\alpha}^\sigma = 0$. Thus, γ takes $[H] \cdot \mathbb{R}$ to $[H^\sigma] \cdot \mathbb{R}$. It follows that γ takes $H^2(Z,\mathbb{R})$ to $H^2(Z^\sigma,\mathbb{R})$, since these are the kernels of cup product with $[H]$ and $[H^\sigma]$ respectively.

Since $T_{g,4}$ is four-dimensional, we have $[H]^5 \cup [D] = 0$. Then application of γ yields:

$$c^5 \cdot [H^\sigma]^5 \cup (a \cdot [H^\sigma] + b \cdot [D^\sigma]) = 0.$$

Since $[H^\sigma]^5 \cup [D^\sigma]$ vanishes, whereas $[H^\sigma]^6$ is nontrivial, it follows from $c \neq 0$ that a vanishes. Thus, γ maps $[D] \cdot \mathbb{R}$ to $[D^\sigma] \cdot \mathbb{R}$ and we conclude that γ respects the decompositions (4.41) and (4.42).

The latter implies that γ induces an $\mathbb{R}$-linear isomorphism between the ideals $([D]) \subseteq H^{2*}(W,\mathbb{R})$ and $([D^\sigma]) \subseteq H^{2*}(W^\sigma,\mathbb{R})$. In order to state the key-property of this isomorphism, we identify cohomology classes on $T_{g,4}$ and $T^\sigma_{g,4}$ with their pullbacks to the exceptional divisors D and D^σ respectively.

Lemma 4.7.2. *For every $\alpha \in H^{2k}(T_{g,4}, \mathbb{R})$, there exists a unique*

$$\alpha^\sigma \in H^{2k}(T^\sigma_{g,4}, \mathbb{R})$$

with

$$\gamma([D] \cup j_*(\alpha)) = [D^\sigma] \cup j^\sigma_*(\alpha^\sigma).$$

Proof. For $0 \le k \le 2$, let us fix some $\alpha \in H^{2k}(T_{g,4}, \mathbb{R})$ and note that

$$H^{2k+2}(W^\sigma, \mathbb{R}) = H^{2k+2}(Z^\sigma, \mathbb{R}) \oplus [H^\sigma]^{k+1} \cdot \mathbb{R} \oplus j^\sigma_*(H^{2k}(D^\sigma, \mathbb{R})).$$

Since γ maps $[D]$ to a multiple of $[D^\sigma]$, and since products of $[D^\sigma]$ with positive-degree classes on Z^σ always vanish, the above identity shows

$$\gamma([D] \cup j_*(\alpha)) = [D^\sigma] \cup j^\sigma_*(\alpha^\sigma) + e \cdot [D^\sigma] \cup [H^\sigma]^{k+1} ,$$

for some $\alpha^\sigma \in H^{2k}(D^\sigma, \mathbb{R})$ and $e \in \mathbb{R}$.

The restrictions of $-[H]$ to $T_{g,4}$ and $-[H^\sigma]$ to $T^\sigma_{g,4}$ are ample classes

$$\omega \in H^2(T_{g,4}, \mathbb{R}) \ \text{ and } \ \omega^\sigma \in H^2(T^\sigma_{g,4}, \mathbb{R})$$

respectively. Now suppose that α in the above formula is primitive with respect to ω. Then the cup product of the above class with $\gamma([H])^{5-2k}$ vanishes. Since $\gamma([H])$ is a multiple of $[H^\sigma]$,

$$[D^\sigma] \cup j^\sigma_*(\alpha^\sigma \cup (\omega^\sigma)^{5-2k}) + e \cdot (-1)^{k+1} j^\sigma_*((\omega^\sigma)^{6-k}) = 0.$$

This implies firstly that $e = 0$ and secondly that $\alpha^\sigma \cup (\omega^\sigma)^{5-2k}$ vanishes as class on D^σ. By the Hard Lefschetz Theorem, the latter already implies that α^σ, which a priori is only a class on D^σ, is in fact a primitive class on $T^\sigma_{g,4}$.

For arbitrary $\alpha \in H^k(T_{g,4}, \mathbb{R})$, the existence of α^σ now follows – since γ takes $[H] \cdot \mathbb{R}$ to $[H^\sigma] \cdot \mathbb{R}$ – from the Lefschetz decompositions with respect to ω and ω^σ; the uniqueness is immediate from Theorem 4.2.1. This concludes Lemma 4.7.2. □

By Lemma 4.7.2, we are now able to define an $\mathbb{R}$-linear map

$$\phi : H^{2*}(T_{g,4}, \mathbb{R}) \longrightarrow H^{2*}(T^\sigma_{g,4}, \mathbb{R}),$$

by requiring

$$\gamma([D] \cup j_*(\alpha)) = b \cdot \gamma([D]) \cup j^\sigma_*(\phi(\alpha))$$

for all $\alpha \in H^*(T_{g,4}, \mathbb{R})$, where b is, as above, the nontrivial constant with $\gamma([D]) = b \cdot [D^\sigma]$. Applying the same argument to γ^{-1}, we obtain an $\mathbb{R}$-linear inverse of ϕ.

By Theorem 4.6.2, ϕ cannot be an isomorphism of algebras and so we will obtain a contradiction as soon as we have seen that ϕ respects the product structures. For this purpose, let α and β denote even-degree cohomology classes on $T_{g,4}$. Then, by Theorem 4.2.1 and Lemma 4.2.2, it suffices to prove

$$b\cdot\gamma([D])^3 \cup j_*^\sigma(\phi(\alpha\cup\beta)) = b\cdot\gamma([D])^3 \cup j_*^\sigma(\phi(\alpha)\cup\phi(\beta)).$$

Using (4.3), the latter is seen as follows:

$$\begin{aligned} b\cdot\gamma([D])^3 \cup j_*^\sigma(\phi(\alpha\cup\beta)) &= \gamma([D])^2 \cup \gamma([D]\cup j_*(\alpha\cup\beta)) \\ &= \gamma([D]^2 \cup j_*(1)\cup j_*(\alpha\cup\beta)) \\ &= \gamma([D]\cup j_*(\alpha)\cup[D]\cup j_*(\beta)) \\ &= b^2\cdot\gamma([D])^2 \cup j_*^\sigma(\phi(\alpha))\cup j_*^\sigma(\phi(\beta)) \\ &= b^2\cdot\gamma([D])^2 \cup j_*^\sigma(1)\cup j_*^\sigma(\phi(\alpha)\cup\phi(\beta)) \\ &= b\cdot\gamma([D])^3 \cup j_*^\sigma(\phi(\alpha)\cup\phi(\beta)). \end{aligned}$$

This concludes the proof of Theorem 4.7.1. □

4.8 Examples with nonisotrivial deformations

In this section we prove that the examples in Theorem 4.1.7 may be chosen to have nonisotrivial deformations. Here, a family $(X_s)_{s\in S}$ of varieties over a connected base S is called nonisotrivial if there are two points $s_0, s_1 \in S$ with $X_{s_0} \ncong X_{s_1}$. The idea of the proof is to vary the blown-up point $p\in Z$ in the construction of Section 4.7. In order to state our result, we write $X\sim Y$ if two varieties X and Y are birationally equivalent.

Theorem 4.8.1. *Let Z be a smooth complex projective variety of dimension ≥ 10. Then there is a nonisotrivial family $(W_p)_{p\in U}$ of smooth complex projective varieties W_p over some smooth affine variety U, and an automorphism $\sigma\in \mathrm{Aut}(\mathbb{C})$ such that for all $p\in U$:*

$$W_p \sim Z \sim W_p^\sigma \quad \textit{and} \quad H^{2*}(W_p,\mathbb{R}) \ncong H^{2*}(W_p^\sigma,\mathbb{R}).$$

Proof. As in Section 4.7, we may pick some $\sigma\in\mathrm{Aut}(\mathbb{C})$ and some $g\geq 1$ such that

$$Z\simeq Z^\sigma,\quad \sigma(i)=i,\quad \sigma(\zeta_{2g+1})\neq\zeta_{2g+1} \quad\text{and}\quad b_2(T_{g,4}) > b_2(Z)+4.$$

Next, let $U\subseteq Z$ be a Zariski open and dense subset with trivial tangent bundle. Let $\Delta\subseteq U\times Z$ be the graph of the inclusion $U\hookrightarrow Z$ and consider the blow-up $Bl_\Delta(U \times Z)$. The normal bundle of Δ in $U \times Z$ is trivial, since U has trivial tangent bundle. Hence, the exceptional divisor of $Bl_\Delta(U \times Z)$ is isomorphic to $\Delta \times \mathbb{P}^{n-1}$. Since $n \geq 10$, we may fix an embedding of $\Delta \times T_{g,4}$ into this exceptional divisor and consider the blow-up

$$Bl_{\Delta \times T_{g,4}}(Bl_\Delta(U \times Z)).$$

Projection to the first coordinate then gives a family

$$(W_p)_{p \in U}$$

of smooth complex projective varieties, birational to Z. Then, for all $p \in U$, the conjugate varieties W_p and W_p^σ are as in (4.39) and (4.40) respectively. Thus, $W_p \sim Z$ and $W_p^\sigma \sim Z^\sigma$. By Theorem 4.7.1 and since $Z \simeq Z^\sigma$, we obtain for all $p \in U$:

$$W_p \sim Z \sim W_p^\sigma \quad \text{and} \quad H^{2*}(W_p, \mathbb{R}) \not\cong H^{2*}(W_p^\sigma, \mathbb{R}).$$

To conclude Theorem 4.8.1, it therefore remains to prove

Claim 4.8.2. *After replacing Z by another representative of its birational equivalence class, and for a suitable choice of U, the family $(W_p)_{p \in U}$ is nonisotrivial.*

Let us prove this claim. By the arguments of Theorem 4.7.1, one sees that any isomorphism $g : W_p \to W_q$ induces an isomorphism g^* on cohomology which respects the decomposition (4.41). This implies that g respects the exceptional divisors and thus induces an isomorphism of Z which takes p to q.

The above argument, applied to $p = q$, shows that W_p admits no automorphism which takes points from the exceptional divisors to $Z - \{p\}$. In particular, W_p contains a Zariski open subset with trivial tangent bundle and with two points that cannot be interchanged by an automorphism of W_p. Since W_p is birational to Z, we may therefore, after possibly replacing Z by another representative of its birational equivalence class, assume that U already contains points p and q which cannot be interchanged by any automorphism of Z. Then, as we have seen, W_p and W_q are not isomorphic. This finishes the proof of Claim 4.8.2 and so concludes Theorem 4.8.1. □

Remark 4.8.3. *In contrast to Theorem 4.8.1, most of the previously known examples of nonhomeomorphic pairs of conjugate varieties tend to be rather rigid and do in general not occur in nonisotrivial families. This was already observed by D. Reed in [67]. However, it is often possible to obtain nonisotrivial families as products of previously known examples with nonrigid varieties, e.g. one could take products of Serre's examples [78] with a smooth hypersurface of degree at least 3 in $\mathbb{P}^3$, since the latter are simply connected and come in nonisotrivial families.*

5 Theta divisors with curve summands and the Schottky Problem

Abstract. We prove the following converse of Riemann's Theorem: let (A, Θ) be an indecomposable principally polarized abelian variety whose theta divisor can be written as a sum of a curve and a codimension two subvariety $\Theta = C + Y$. Then C is smooth, A is the Jacobian of C, and Y is a translate of $W_{g-2}(C)$. As applications, we determine all theta divisors that are dominated by a product of curves and characterize Jacobians by the existence of a d-dimensional subvariety with curve summand whose twisted ideal sheaf is a generic vanishing sheaf.

5.1 Introduction

This chapter provides new geometric characterizations of Jacobians inside the moduli stack of all principally polarized abelian varieties over the complex numbers. For a recent survey on existing solutions and open questions on the Schottky Problem, we refer the reader to [32]. By slight abuse of notation, we will denote a ppav (principally polarized abelian variety) by (A, Θ), where $\Theta \subseteq A$ is a theta divisor that induces the principal polarization on the abelian variety A; the principal polarization determines $\Theta \subseteq A$ uniquely up to translation.

5.1.1 A converse of Riemann's theorem

Let $(J(C), \Theta_C)$ be the Jacobian of a smooth curve C of genus $g \geq 2$. We fix a base point on C and consider the corresponding Abel–Jacobi embedding $C \longrightarrow J(C)$. Addition of points induces morphisms

$$\mathrm{AJ}_k : C^{(k)} \longrightarrow J(C),$$

This chapter is based on [75].

whose image is denoted by $W_k(C)$. Riemann's Theorem [4, p. 27] says

$$\Theta_C = W_{g-1}(C).$$

That is,

$$\Theta_C = W_1(C) + W_{g-2}(C)$$

has a curve summand $W_1(C) \simeq C$. We prove the following converse.

Theorem 5.1.1. *Let (A, Θ) be an indecomposable g-dimensional ppav and suppose that there is a curve C and a codimension two subvariety Y in A such that*

$$\Theta = C + Y.$$

Then C is smooth and there is an isomorphism $(A, \Theta) \simeq (J(C), \Theta_C)$ which identifies C and Y with translates of $W_1(C)$ and $W_{g-2}(C)$ respectively.

Recall that a d-dimensional subvariety $Z \subseteq A$ is called geometrically nondegenerate [65, p. 466] if there is no nonzero decomposable holomorphic d-form on A which restricts to zero on Z, see also Section 5.2 below. For instance, $W_d(C)$ inside the Jacobian of a smooth curve is geometrically nondegenerate.

The intermediate Jacobian of a smooth cubic threefold is an indecomposable ppav which is not isomorphic to the Jacobian of a curve and whose theta divisor can be written as a sum of two geometrically nondegenerate surfaces [16, Sec. 13]. One of Pareschi–Popa's conjectures (Conjecture 5.5.7 below) predicts that apart from Jacobians of curves, intermediate Jacobians of smooth cubic threefolds are the only ppavs whose theta divisors have a geometrically nondegenerate summand of dimension $1 \leq d \leq g-2$. Theorem 5.1.1 proves (a strengthening of) that conjecture if $d = 1$ or $d = g-2$.

5.1.2 Detecting Jacobians via special subvarieties

Recall that a coherent sheaf $\mathcal{F}$ on an abelian variety A is a GV-sheaf if for all i its i-th cohomological support locus

$$S^i(\mathcal{F}) := \left\{ L \in \operatorname{Pic}^0(A) \mid H^i(A, \mathcal{F} \otimes L) \neq 0 \right\}$$

has codimension $\geq i$ in $\operatorname{Pic}^0(A)$, see [60, p. 212].

Using this definition, we characterize $W_d(C) \subseteq J(C)$ among all d-dimensional subvarieties of arbitrary ppavs. Our proof combines Theorem 5.1.1 with the main results in [19] and [60].

Theorem 5.1.2. *Let (A, Θ) be an indecomposable ppav, and let $Z \subsetneq A$ be a geometrically nondegenerate subvariety of dimension d. Suppose that the following holds:*

1. *$Z = Y + C$ has a curve summand $C \subseteq A$,*

2. *the twisted ideal sheaf $\mathcal{I}_Z(\Theta) = \mathcal{I}_Z \otimes \mathcal{O}_A(\Theta)$ is a GV-sheaf.*

Then C is smooth and there is an isomorphism $(A, \Theta) \simeq (J(C), \Theta_C)$ which identifies C, Y and Z with translates of $W_1(C)$, $W_{d-1}(C)$ and $W_d(C)$ respectively.

The sum of geometrically nondegenerate subvarieties $C, Y \subsetneq A$ of dimension 1 and $d-1$ respectively yields a geometrically nondegenerate subvariety of dimension d, see Lemma 5.2.2 below. Therefore, any abelian variety contains lots of geometrically nondegenerate subvarieties Z satisfying the first condition in Theorem 5.1.2.

The point is condition 2 in Theorem 5.1.2. If $d = g-1$, where $g = \dim(A)$, this is known to be equivalent to Z being a translate of Θ, so we recover Theorem 5.1.1 from Theorem 5.1.2. If $1 \le d \le g-2$, condition 2 is more mysterious. It is known to hold for $W_d(C)$ inside the Jacobian $J(C)$, as well as for the Fano surface of lines inside the intermediate Jacobian of a smooth cubic threefold. Pareschi–Popa conjectured (Conjecture 5.5.2 below) that up to isomorphisms these are the only examples; they proved it for subvarieties of dimension one or codimension two.

5.1.3 The DPC Problem for theta divisors

A variety X is DPC (dominated by a product of curves), if there are curves $C_1, \dots, C_n$ together with a dominant rational map

$$C_1 \times \cdots \times C_n \dashrightarrow X.^1$$

For instance, unirational varieties, abelian varieties as well as Fermat hypersurfaces $\{x_0^d + \cdots + x_N^d = 0\} \subseteq \mathbb{P}^N$ of degree $d \ge 1$ are DPC, see [70]. Serre [80] constructed the first example of a variety which is not DPC. Later, Deligne [21, Sec. 7] and Schoen [70] used a Hodge theoretic obstruction to produce many more examples.

On the one hand, the theta divisor of the Jacobian of a smooth curve is DPC by Riemann's Theorem. On the other hand, Schoen found [70, p. 544] that his Hodge theoretic obstruction does not prevent (smooth) theta divisors from

[1] A priori $n \ge \dim(X)$, but by [70, Lem. 6.1], we may actually assume $n = \dim(X)$.

being DPC. This led Schoen [70, Sec. 7.4] to pose the problem of finding theta divisors which are not DPC, if such exist. The following solves that problem completely, which was our initial motivation for this chapter.

Corollary 5.1.3. *Let (A, Θ) be an indecomposable ppav. The theta divisor Θ is DPC if and only if (A, Θ) is isomorphic to the Jacobian of a smooth curve.*

We prove in fact a strengthened version (Corollary 5.6.3) of Corollary 5.1.3, in which the DPC condition is replaced by the existence of a dominant rational map $Z_1 \times Z_2 \dashrightarrow \Theta$, where Z_1 and Z_2 are arbitrary varieties of dimension 1 and $g-2$ respectively. The latter is easily seen to be equivalent to Θ having a curve summand and so Theorem 5.1.1 applies.

We discuss further applications of Theorem 5.1.1 in Sections 5.6.1 and 5.6.2. Firstly, using work of Clemens–Griffiths [16], we prove that the Fano surface of lines on a smooth cubic threefold is not DPC (Corollary 5.6.5). Secondly, for a smooth genus g curve C, we determine in Corollary 5.6.6 all possible ways in which the symmetric product $C^{(k)}$ with $k \leq g-1$ can be dominated by a product of curves. Our result can be seen as a generalization of a theorem of Martens' [59, 64].

5.1.4 Method of proofs

Although Theorem 5.1.1 is a special case of Theorem 5.1.2, it appears to be more natural to prove Theorem 5.1.1 first. Here we use techniques that originated in work of Ran and Welters [63, 65, 95]; they are mostly of cohomological and geometric nature. One essential ingredient is Ein–Lazarsfeld's result [25] on the singularities of theta divisors, which allows us to make Welters' method [95] unconditional. Eventually, Theorem 5.1.1 will be reduced to Matsusaka–Hoyt's criterion [36], asserting that Jacobians of smooth curves are characterized among indecomposable g-dimensional ppavs (A, Θ) by the property that the cohomology class $\frac{1}{(g-1)!}[\Theta]^{g-1}$ can be represented by a curve. Theorem 5.1.2 follows then quickly from Theorem 5.1.1 and work of Debarre [19] and Pareschi–Popa [60].

5.1.5 Conventions

We work over the field of complex numbers. A variety is a separated integral scheme of finite type over $\mathbb{C}$; if not mentioned otherwise, varieties are assumed to be proper over $\mathbb{C}$. A curve is an algebraic variety of dimension one. In particular, varieties (and hence curves) are reduced and irreducible.

If not mentioned otherwise, a point of a variety is always a closed point. A general point of a variety is a closed point in some Zariski open and dense set.

For a codimension one subscheme Z of a normal variety X, we denote by $\mathrm{div}_X(Z)$ the corresponding effective Weil divisor on X; if Z is not pure-dimensional, all components of codimension ≥ 2 are ignored in this definition. Linear equivalence between divisors is denoted by $\sim$.

For subschemes Z and Z' of an abelian variety A, we denote by $Z + Z'$ (resp. $Z - Z'$) the image of the addition (resp. difference) morphism $Z \times Z' \longrightarrow A$, equipped with the natural scheme structure. If Z' is a point $a \in A$, $Z \pm Z'$ is also denoted by $Z_{\pm a}$. Note that for subvarieties Z and Z' of A, the image $Z \pm Z'$ is reduced and irreducible, hence a subvariety of A.

If $Z \subseteq A$ is a subvariety, the tangent space at each point of Z is identified via translation with a subspace of $T_{A,0}$.

5.2 Nondegenerate subvarieties

Following Ran [65, p. 464], a d-dimensional subvariety Z of a g-dimensional abelian variety is called nondegenerate if the image of the Gauß map

$$G_Z : Z \dashrightarrow \mathrm{Gr}(d, g)$$

is via the Plücker embedding not contained in any hyperplane. This condition is stronger then the previously mentioned notion of geometrically nondegenerate subvarieties. We will need the following consequence of [65, Lem. II.1].

Lemma 5.2.1. *Let (A, Θ) be a ppav and let $Z \subseteq A$ be a codimension k subvariety whose cohomology class is a multiple of $\frac{1}{k!}[\Theta]^k$. Then Z is nondegenerate, hence geometrically nondegenerate.*

Ran proved that a d-dimensional subvariety $Z \subseteq A$ is geometrically nondegenerate if and only if for each abelian subvariety $B \subseteq A$, the composition $Z \longrightarrow A/B$ has either d-dimensional image or it is surjective [65, Lem. II.12]. In [18, p. 105], Debarre used Ran's characterization as definition and proved the following.

Lemma 5.2.2. *Let $Z_1, Z_2 \subseteq A$ be subvarieties of dimensions d_1 and d_2 with $d_1 + d_2 \leq \dim(A)$ respectively.*

1. *If Z_1 is geometrically nondegenerate,* $\dim(Z_1 + Z_2) = d_1 + d_2$.
2. *If Z_1 and Z_2 are geometrically nondegenerate, $Z_1 + Z_2 \subseteq A$ is geometrically nondegenerate.*

5.3 A consequence of Ein–Lazarsfeld's Theorem

The purpose of this section is to prove Lemmas 5.3.2 and 5.3.3 below. Under the additional assumption

$$\dim(\mathrm{Sing}(\Theta)) \le \dim(A) - 4, \tag{5.1}$$

these were first proven by Ran [63, Cor. 3.3] and Welters [95, Prop. 2] respectively. The general case is a consequence of the following result of Ein–Lazarsfeld [25].

Theorem 5.3.1 (Ein–Lazarsfeld)**.** *Let (A, Θ) be a ppav. If Θ is irreducible, it is normal and has only rational singularities.*

Let (A, Θ) be an indecomposable ppav of dimension ≥ 2. By the Decomposition Theorem [8, p. 75], Θ is irreducible and we choose a desingularization $f : X \longrightarrow \Theta$. The composition of f with the inclusion $\Theta \subseteq A$ is denoted by $j : X \longrightarrow A$.

Lemma 5.3.2. *Pullback of line bundles induces an isomorphism*

$$j^* : \mathrm{Pic}^0(A) \xrightarrow{\sim} \mathrm{Pic}^0(X).$$

Proof. By Theorem 5.3.1, $f_*\mathcal{O}_X = \mathcal{O}_\Theta$ and $R^i f_* \mathcal{O}_X = 0$ for all $i > 0$. We therefore obtain

$$H^1(X, \mathcal{O}_X) \simeq H^1(\Theta, \mathcal{O}_\Theta) \simeq H^1(A, \mathcal{O}_A),$$

where the first isomorphism follows from the Leray spectral sequence, and the second one from Kodaira vanishing and the short exact sequence

$$0 \longrightarrow \mathcal{O}_A(-\Theta) \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{O}_\Theta = j_*\mathcal{O}_X \longrightarrow 0. \tag{5.2}$$

Hence, $j^* : \mathrm{Pic}^0(A) \longrightarrow \mathrm{Pic}^0(X)$ is an isogeny.

Tensoring (5.2) by a nontrivial $P \in \mathrm{Pic}^0(A)$, we obtain

$$H^0(X, j^*P) \simeq H^0(A, P) = 0,$$

where we applied Kodaira vanishing to $\mathcal{O}_A(-\Theta) \otimes P$. It follows that j^*P is nontrivial. That is, j^* is an injective isogeny and thus an isomorphism. This proves Lemma 5.3.2. □

Lemma 5.3.3. *For any $a \neq 0$ in A, $j : X \longrightarrow A$ induces an isomorphism*

$$j^* : H^0(A, \mathcal{O}_A(\Theta_a)) \xrightarrow{\sim} H^0(X, j^*(\mathcal{O}_A(\Theta_a))).$$

Proof. Following Welters [95, Prop. 2], the assertion follows from (5.2) by tensoring with $\mathcal{O}_A(\Theta_a)$, since $\mathcal{O}_A(\Theta_a - \Theta)$ has no cohomology for $a \neq 0$. □

5.4 Proof of Theorem 5.1.1

Let (A, Θ) be a g-dimensional indecomposable ppav, and suppose that there is a curve $C \subseteq A$ and a $(g-2)$-dimensional subvariety $Y \subseteq A$ such that

$$\Theta = C + Y.$$

After translation, we may assume $\Theta = -\Theta$. We pick a point $c_0 \in C$ and replace C and Y by C_{-c_0} and Y_{c_0}. Hence, we may assume $0 \in C$ and so $Y = 0 + Y$ is contained in Θ.

Since (A, Θ) is indecomposable, Θ is irreducible, hence normal by Theorem 5.3.1. The idea of the proof of Theorem 5.1.1 is to consider the intersection $\Theta \cap \Theta_c$ for nonzero $c \in C$. Since Θ induces a principal polarization, $\Theta \cap \Theta_c$ is a proper subscheme of Θ for all $c \neq 0$. For our purposes it is more convenient to consider the corresponding Weil divisor on Θ, denoted by

$$\mathrm{div}_\Theta(\Theta \cap \Theta_c).$$

Clearly, this divisor is just the pullback of the Cartier divisor Θ_c from A to Θ.

Since $\Theta = -\Theta$, the map $x \longmapsto c - x$ defines an involution of $\Theta \cap \Theta_c$. Since $\Theta = C + Y$, it follows that $\mathrm{div}_\Theta(\Theta \cap \Theta_c)$ contains the effective Weil divisors Y_c and $-Y$. For general c, these divisors are distinct and so we find

$$\mathrm{div}_\Theta(\Theta \cap \Theta_c) = Y_c + Z(c) \tag{5.3}$$

for all $c \neq 0$, where $Z(c)$ is an effective Weil divisor on Θ which contains $-Y$. In the following proposition, we prove that actually $Z(c) = -Y$. As a byproduct of the proof, we will be able to compute the cohomology class of C in terms of the degree of the addition morphism

$$F : C \times Y \longrightarrow \Theta.$$

Our proof uses Welters' method [95].

Proposition 5.4.1. *Let (A, Θ) be a g-dimensional indecomposable ppav with $\Theta = C + Y$, $\Theta = -\Theta$ and $0 \in C$ as above. For any nonzero $c \in C$,*

$$\mathrm{div}_\Theta(\Theta \cap \Theta_c) = Y_c + (-Y). \tag{5.4}$$

Moreover, the cohomology class of C is given by

$$[C] = \frac{\deg(F)}{(g-1)^2 \cdot (g-2)!} \cdot [\Theta]^{g-1}. \tag{5.5}$$

Proof. We fix a resolution of singularities $f: X \longrightarrow \Theta$ and denote the composition of f with the inclusion $\Theta \subseteq A$ by $j: X \longrightarrow A$. Moreover, for each $a \in A$, we fix an effective divisor $\widetilde{\Theta}_a$ in the linear series $|j^*(\Theta_a)|$ on X. For $a \neq 0$, $|j^*(\Theta_a)|$ is zero-dimensional by Lemma 5.3.3. It follows that $\widetilde{\Theta}_a$ is unique if $a \neq 0$; it is explicitly given by

$$\widetilde{\Theta}_a = \mathrm{div}_X(f^{-1}(\Theta_a \cap \Theta)). \tag{5.6}$$

Since Θ is normal, the general point of each component of $\Theta_a \cap \Theta$ lies in the smooth locus of Θ. The above description therefore proves

$$f_*\widetilde{\Theta}_a = \mathrm{div}_\Theta(\Theta_a \cap \Theta), \tag{5.7}$$

for all $a \neq 0$ in A.

Next, we would like to find a divisor $\widetilde{Y}_c$ on X whose pushforward to Θ is Y_c. Since Y_c is in general not Cartier on Θ, we cannot simply take the pullback. Instead, we consider the Weil divisor which corresponds to the scheme theoretic preimage of Y_c,

$$\widetilde{Y}_c := \mathrm{div}_X(f^{-1}(Y_c)). \tag{5.8}$$

Since Θ is normal, Y_c is not contained in the singular locus of Θ. It follows that $f^{-1}(Y_c)$ has a unique component which maps birationally onto Y_c and the remaining components are in the kernel of f_*. Hence,

$$f_*\widetilde{Y}_c = Y_c. \tag{5.9}$$

For all $c \neq 0$ in C, we define

$$\widetilde{Z}(c) := \widetilde{\Theta}_c - \widetilde{Y}_c. \tag{5.10}$$

It follows from (5.3), (5.6) and (5.8) that $\widetilde{Z}(c)$ is effective. Moreover, by (5.3), (5.7) and (5.9),

$$f_*\widetilde{Z}(c) = \mathrm{div}_\Theta(\Theta \cap \Theta_c) - Y_c = Z(c). \tag{5.11}$$

By generic flatness, it follows that there is a Zariski dense and open subset $U \subseteq C$ such that for $c \in U$ the preimages $f^{-1}(Y_c)$ form the fibers of a flat family of schemes over U. By the definition of $\widetilde{Y}_c$ in (5.8), $\widetilde{Y}_c - \widetilde{Y}_{c'}$ is numerically trivial on X for all $c, c' \in U$. Lemma 5.3.2 yields therefore for all $c, c' \in U$ a linear equivalence

$$\widetilde{Y}_c - \widetilde{Y}_{c'} \sim j^*(\Theta_{z(c,c')} - \Theta) \sim \widetilde{\Theta}_{z(c,c')} - \widetilde{\Theta}, \tag{5.12}$$

where $z : U \times U \longrightarrow A$ is the morphism induced by the universal property of

$$\mathrm{Pic}^0(X) \simeq \mathrm{Pic}^0(A).$$

The proof of Proposition 5.4.1 proceeds now in several steps.

Step 1. Let $c' \in U$ and consider the function $x_{c'}(c) := z(c,c') + c'$. For all $c \in U$ with $x_{c'}(c) \neq 0$, we have

$$\mathrm{div}_\Theta(\Theta_{x_{c'}(c)} \cap \Theta) = Y_c + Z(c'). \tag{5.13}$$

Moreover, if $c' \in U$ is general, then $x_{c'}(c)$ is nonconstant in $c \in U$.

Proof. Using the theorem of the square [8, p. 33] on A and pulling back this linear equivalence to X shows $\widetilde{\Theta}_{x_{c'}(c)} \sim \widetilde{\Theta}_{z(c,c')} - \widetilde{\Theta} + \widetilde{\Theta}_{c'}$. By (5.12) and the definition of $\widetilde{Z}(c')$ in (5.10), we therefore obtain:

$$\begin{aligned}\widetilde{\Theta}_{x_{c'}(c)} &\sim \widetilde{\Theta}_{z(c,c')} - \widetilde{\Theta} + \widetilde{\Theta}_{c'}\\ &\sim \widetilde{Y}_c - \widetilde{Y}_{c'} + \widetilde{\Theta}_{c'}\\ &\sim \widetilde{Y}_c + \widetilde{Z}(c').\end{aligned}$$

That is, $\widetilde{Y}_c + \widetilde{Z}(c')$ is an effective divisor linearly equivalent to $\widetilde{\Theta}_{x_{c'}(c)}$. By Lemma 5.3.3, the linear series $|\widetilde{\Theta}_{x_{c'}(c)}|$ is zero-dimensional for all $x_{c'}(c) \neq 0$, and so we actually obtain an equality of Weil divisors:

$$\widetilde{\Theta}_{x_{c'}(c)} = \widetilde{Y}_c + \widetilde{Z}(c').$$

Applying f_* to this equality, (5.13) follows from (5.7), (5.9) and (5.11).

Using again the theorem of the square on A and pulling back the corresponding linear equivalence to X, we obtain $\widetilde{\Theta}_{z(c,c')} - \widetilde{\Theta} \sim \widetilde{\Theta} - \widetilde{\Theta}_{-z(c,c')}$. It therefore follows from (5.12) that

$$\widetilde{\Theta} - \widetilde{\Theta}_{-z(c,c')} \sim Y_c - Y_{c'} = -(Y_{c'} - Y_c) \sim \widetilde{\Theta} - \widetilde{\Theta}_{z(c',c)}.$$

Hence, $-z(c,c') = z(c',c)$ by Lemma 5.3.2.

For a contradiction, suppose that $x_{c'}(c) = z(c,c') + c'$ is constant in c for general (hence for all) $c' \in U$. It follows that $z(c,c')$ is constant in the first variable. Since $z(c,c') = -z(c',c)$, it is also constant in the second variable. Therefore, for general c', $x_{c'}(c) = z(c,c') + c'$ is nonzero and constant in c. This contradicts (5.13), because its right hand side is nonconstant in c as $C + Y = \Theta$. This concludes step 1. □

Let us now fix a general point $c' \in U$. By step 1, the closure of $c \longmapsto x_{c'}(c)$ is a proper irreducible curve $D \subseteq A$.

We say that a subvariety Z of A is translation invariant under D if

$$Z_x = Z_{x'}$$

for all $x, x' \in D$. Equivalently, Z is translation invariant under D if and only if the corresponding cohomology classes on A satisfy $[Z] * [D] = 0$, where $*$ denotes the Pontryagin product. That description shows that the notion of translation invariance depends only on the cohomology classes of Z and D. In particular, Z is translation invariant under D if and only if the same holds for $\pm Z$ or $\pm D$. If Z is not translation invariant under D, we also say that it moves when translated by D.

We will use that $-Y$ moves when translated by D. Indeed, for $x_1, x_2 \in D$ with $Y_{x_1} = Y_{x_2}$, we obtain

$$\Theta_{x_1} = C + Y_{x_1} = C + Y_{x_2} = \Theta_{x_2}.$$

Hence, $x_1 = x_2$ which proves that Y and hence $-Y$ is not translation invariant under D.

For each $c \neq 0$, we decompose the Weil divisor $Z(c)$ on Θ into a sum of effective divisors

$$Z(c) = Z_{\mathrm{mov}}(c) + Z_{\mathrm{inv}}(c), \tag{5.14}$$

where $Z_{\mathrm{inv}}(c)$ contains all the components of $Z(c)$ that are translation invariant under D and the components of $Z_{\mathrm{mov}}(c)$ move when translated by D.

Step 2. We have $x_{c'}(c) = c$ and hence $D = C$. Moreover, for each $c \neq 0$ in U,

$$\mathrm{div}_\Theta(\Theta \cap \Theta_c) = Y_c + (-Y) + Z_{\mathrm{inv}}(c'). \tag{5.15}$$

Proof. Let Z' be a prime divisor in $Z_{\mathrm{mov}}(c')$. It follows from step 1 that $Z'_{-x} \subseteq \Theta$ for general $x \in D$, hence for all $x \in D$. Since Z' moves when translated by $-D$, $Z' - D = \Theta$ and so

$$-Z' + D = -\Theta = \Theta.$$

Since $-Z' \subseteq -\Theta = \Theta$, this equality implies

$$(-Z')_x \subseteq \Theta_x \cap \Theta$$

for all nonzero $x \in D$. Therefore, for each $c \in U$ with $x_{c'}(c) \neq 0$, the prime divisor $(-Z')_{x_{c'}(c)}$ is contained in $\mathrm{div}_\Theta(\Theta_{x_{c'}(c)} \cap \Theta)$. Hence, by (5.13) from step 1:

$$(-Z')_{x_{c'}(c)} \leq Y_c + Z(c'),$$

for all $c \in U$ with $x_{c'}(c) \neq 0$.

Let us now move in the above inequality the point c in C and keep c' fixed and general. By step 1, the point $x_{c'}(c)$ moves. Since Z' is a component of $Z_{\mathrm{mov}}(c')$, the translate $(-Z')_{x_{c'}(c)}$ must also move. The translate Y_c moves because $Y + C = \Theta$. Clearly, $Z(c')$ does not move as we keep c' fixed. The above inequality of effective Weil divisors therefore shows

$$(-Z')_{x_{c'}(c)} = Y_c. \tag{5.16}$$

Recall that the prime divisor $(-Y)$ is contained in $Z(c')$ for all c'. We have explained above step 2 that this prime divisor is not translation invariant under D, hence it is contained in $Z_{\mathrm{mov}}(c')$. Equality (5.16) therefore holds for $Z' = -Y$, which proves $Y_{x_{c'}(c)} = Y_c$. This implies

$$\Theta_{x_{c'}(c)} = Y_{x_{c'}(c)} + C = Y_c + C = \Theta_c.$$

Hence,

$$x_{c'}(c) = c,$$

which implies $D = C$.

It remains to prove (5.15). Since $x_{c'}(c) = c$, equality (5.16) shows that $-Y$ is actually the only prime divisor in $Z_{\mathrm{mov}}(c')$. Hence,

$$Z_{\mathrm{mov}}(c') = \lambda \cdot (-Y)$$

for some positive integer λ. Using $x_{c'}(c) = c$ and (5.14) in the conclusion (5.13) from step 1, we therefore obtain

$$\mathrm{div}_{\Theta}(\Theta \cap \Theta_c) = Y_c + \lambda \cdot (-Y) + Z_{\mathrm{inv}}(c').$$

For (5.15), it now remains to prove $\lambda = 1$. By the above equality of Weil divisors, it suffices to prove that for general points $y \in Y$ and $c \in C$, the intersection $\Theta \cap \Theta_c$ is transverse at the point $-y$. Recall that Θ is normal and so it is smooth at $-y$ for $y \in Y$ general. It thus suffices to see that the tangent space $T_{\Theta,-y}$ meets $T_{\Theta_c,-y} = T_{\Theta,-y-c}$ properly. Since $T_{\Theta,-y}$ and $T_{\Theta,-y-c}$ have codimension one in $T_{A,0}$, it actually suffices to prove

$$T_{\Theta,-y} \neq T_{\Theta,-y-c}$$

for general $c \in C$ and $y \in Y$. In order to see this, it suffices to note that Θ is irreducible and so the Gauß map

$$G_{\Theta} : \Theta \dashrightarrow \mathbb{P}^{g-1}$$

is generically finite [8, Prop. 4.4.2]. Indeed, $T_{\Theta,-y} = T_{\Theta,-y-c}$ for general c and y implies that through the general point of Θ (which is of the form $-y-c$) there is a curve which is contracted by G_{Θ}. This concludes step 2. $\square$

Step 3. We have the following identity in $H^{2g-2}(A,\mathbb{Z})$:

$$[\Theta]^2 * [C] = 2 \cdot \deg(F) \cdot [\Theta], \tag{5.17}$$

where we recall that $F : C \times Y \longrightarrow \Theta$ denotes the addition morphism.

Proof. It follows from the conclusion (5.15) in step 2 that $Z_{\text{inv}}(c')$ is actually independent of the general point $c' \in U$. We therefore write $Z_{\text{inv}} = Z_{\text{inv}}(c')$.

For a contradiction, suppose that there is a prime divisor Z' on Θ with $Z' \le Z_{\text{inv}}$. Let us think of Z' as a codimension two cycle on A. By definition, Z' is translation invariant under D, hence under C by step 2. Therefore, $[Z'] * [C] = 0$ in $H^{2g-2}(A,\mathbb{Z})$. Since this holds for each prime divisor Z' in Z_{inv},

$$[Z_{\text{inv}}] * [C] = 0.$$

For $c \neq 0$, we may consider $\Theta \cap \Theta_c$ as a pure-dimensional codimension two subscheme of A. As such it gives rise to an effective codimension two cycle on A, which is nothing but the pushforward of the cycle $\mathrm{Div}_\Theta(\Theta \cap \Theta_c)$ from Θ to A. Mapping this cycle further to cohomology, we obtain $[\Theta]^2$ in $H^{2g-4}(A,\mathbb{Z})$. Conclusion (5.15) in step 2 therefore implies

$$\begin{aligned} [\Theta]^2 * [C] &= 2 \cdot [Y] * [C] + [Z_{\text{inv}}] * [C] \\ &= 2 \cdot [Y] * [C] \\ &= 2 \cdot \deg(F) \cdot [\Theta], \end{aligned}$$

where we used $[Y] = [Y_c] = [-Y]$ and $[Z_{\text{inv}}] * [C] = 0$. □

Step 4. Assertion (5.5) of Proposition 5.4.1 holds.

Proof. We apply the cohomological Fourier–Mukai functor to the conclusion (5.17) of step 3. Using Lemma 9.23 and Lemma 9.27 in [38], this yields:

$$\frac{2}{(g-2)!} \cdot [\Theta]^{g-2} \cup \mathrm{PD}[C] = \frac{2 \cdot \deg(F)}{(g-1)!} \cdot [\Theta]^{g-1}, \tag{5.18}$$

where PD denotes the Poincaré duality operator. Here we used

$$\mathrm{PD}\left(\frac{1}{k!} \cdot [\Theta]^k\right) = \frac{1}{(g-k)!} \cdot [\Theta]^{g-k}$$

for all $0 \le k \le g$.

By the Hard Lefschetz Theorem, (5.18) implies

$$[C] = \frac{\deg(F)}{(g-1)^2 \cdot (g-2)!} \cdot [\Theta]^{g-1},$$

which is precisely assertion (5.5) of Proposition 5.4.1. □

By Lemma 5.2.1, assertion (5.5) of Proposition 5.4.1 implies that C is geometrically nondegenerate. It follows from Lemma 5.2.2 that no proper subvariety of A is translation invariant under C, hence under D by the second conclusion of step 2. This implies $Z_{inv}(c') = 0$ by its definition in (5.14). Assertion (5.4) of Proposition 5.4.1 follows therefore from assertion (5.15) in step 2. This finishes the proof of Proposition 5.4.1. □

The next step in the proof of Theorem 5.1.1 is the following

Proposition 5.4.2. *In the same notation as above, C is smooth,* $\deg(F) = g-1$ *and* $[C] = \frac{1}{(g-1)!} \cdot [\Theta]^{g-1}$.

Proof. Let us first show that C is smooth. Indeed, (5.4) implies by Lemma 5.2.1 that Y is nondegenerate. Via the Plücker embedding, its Gauß image is therefore not contained in any hyperplane. If $c_0 \in C$ is a singular point, the sum of Zariski tangent spaces $T_{C,c_0} + T_{Y,y}$ has thus for general $y \in Y$ dimension g. It follows that $c_0 + Y$ is contained in the singular locus of Θ, which contradicts its normality (Theorem 5.3.1). Therefore C is smooth.

In order to prove Proposition 5.4.2, it suffices by (5.5) to show $\deg(F) = g-1$. This will be achieved by computing the degree of $i^*\Theta$, where $i : C \longrightarrow A$ denotes the inclusion, in two ways. On the one hand, (5.5) implies

$$\deg(i^*\Theta) = [C] \cup [\Theta] = \frac{\deg(F)}{(g-1)^2 \cdot (g-2)!}[\Theta]^g = \frac{g \cdot \deg(F)}{g-1}. \tag{5.19}$$

On the other hand, we may consider the addition morphism

$$m : C \times C \times Y \longrightarrow A.$$

For $y \in Y$, the restriction of m to $C \times C \times y$ will be denoted by

$$m_y : C \times C \longrightarrow A.$$

Since the degree is constant in flat families, we obtain

$$\deg(i^*\Theta) = \deg(i^*(\Theta_{-c-y})) = \deg\left(\left(m_y^*\Theta\right)|_{C\times c}\right) \tag{5.20}$$

for all $c \in C$ and $y \in Y$.

Let us now fix a general point $y \in Y$. Then the image of m_y is not contained in Θ because $C + C + Y = A$. Therefore, we can pull back the Weil divisor Θ via

$$m_y^*(\Theta) = \mathrm{div}_{C\times C}(m_y^{-1}(\Theta)),$$

where $m_y^{-1}(\Theta)$ denotes the scheme-theoretic preimage, whose closed points are given by

$$\{(c_1, c_2) \in C \times C \mid c_1 + c_2 + y \in \Theta\}.$$

Hence, $m_y^*(\Theta)$ contains the prime divisors $C\times 0$ and $0\times C$. We aim to calculate the right hand side of (5.20) and proceed again in several steps.

Step 1. The multiplicity of $C \times 0$ and $0 \times C$ in $m_y^*(\Theta)$ is one.

Proof. Let λ be the multiplicity of $C \times 0$ in $m_y^*(\Theta)$. For $c \in C$ general, the point $(c, 0)$ has then multiplicity λ in the 0-dimensional scheme

$$m_y^{-1}(\Theta) \cap (c \times C).$$

Since m_y maps $c \times C$ isomorphically to C_{c+y}, the above scheme is isomorphic to

$$\Theta \cap m_y(c \times C) = \Theta \cap (C_{c+y}),$$

and $c + y \in C_{c+y}$ has multiplicity λ in that intersection. If $\lambda \geq 2$, then

$$T_{C,0} = T_{C_{c+y},c+y} \subseteq T_{\Theta,c+y}.$$

Since $c + y$ is a general point of Θ, this inclusion contradicts the previously mentioned fact that the Gauß map G_Θ is generically finite and so the tangent space of Θ at a general point does not contain a fixed line. This proves that $C \times 0$ has multiplicity one in $m_y^*(\Theta)$. A similar argument shows that the same holds for $0 \times C$, which concludes step 1. □

By step 1,

$$m_y^*(\Theta) = \mathrm{div}_{C\times C}(m_y^{-1}(\Theta)) = (C \times 0) + (0 \times C) + \Gamma \tag{5.21}$$

for some effective 1-cycle Γ on $C \times C$ which contains neither $C \times 0$ nor $0 \times C$.

Step 2. Let Γ' be a prime divisor in Γ. Then for each $(c_1, c_2) \in \Gamma'$,

$$-c_1 - c_2 - y \in Y. \tag{5.22}$$

Proof. Condition (5.22) is Zariski closed and so it suffices to prove it for a general point $(c_1, c_2) \in \Gamma'$. Such a point satisfies $c_1 \neq 0 \neq c_2$ and $c_1+c_2+y \in \Theta\cap\Theta_{c_i}$ for $i = 1, 2$. We can therefore apply (5.4) in Proposition 5.4.1 and obtain

$$c_1 + c_2 + y \in \mathrm{supp}(Y_{c_i} + (-Y)),$$

for $i = 1, 2$, where supp$(-)$ denotes the support of the corresponding effective Weil divisor. It follows that $c_1 + c_2 + y$ lies in $Y_{c_1} \cap Y_{c_2}$ or in $(-Y)$.

It suffices to rule out $c_1 + c_2 + y \in Y_{c_1} \cap Y_{c_2}$. But if this is the case, then $c_1 + y$ and $c_2 + y$ are both contained in Y. Since $y \in Y$ is general, the intersection $(C + y)\cap Y$ is proper and so (c_1, c_2) is contained in a finite set of points, which contradicts the assumption that it is a general point of Γ'. This concludes step 2. □

Step 3. The 1-cycle Γ is reduced, i.e. it is a sum of distinct prime divisors.

Proof. We may assume $\Gamma \neq 0$, as otherwise the assertion is trivially true.

In order to see that Γ is reduced, it suffices to prove that the intersections of $m_y^{-1}(\Theta)$ with $c \times C$ and $C \times c$ are both reduced, where $c \in C$ is general. The other assertion being similar, we will only prove that $m_y^{-1}(\Theta) \cap (C \times c_2)$ is reduced, where $c_2 \in C$ is general. Since m_y maps $(C \times c_2)$ isomorphically to C_{c_2+y}, it suffices to see that the intersection

$$C_{c_2+y} \cap \Theta \tag{5.23}$$

is transverse.

Let us consider a point $c_1 \in C$ with $c_1 + c_2 + y \in \Theta$. For $c_1 = 0$, transversality of (5.23) in $c_1 + c_2 + y$ was proven in step 1. For $c_1 \neq 0$, step 2 implies that $y_1 := -(c_1 + c_2 + y)$ is contained in Y. In order to prove that the intersection (5.23) is transverse at $-y_1$, we need to see that

$$T_{C,c_1} = T_{C_{c_2+y},-y_1} \subsetneq T_{\Theta,-y_1}. \tag{5.24}$$

This follows from the fact that c_2 and y are general as follows.

Recall the addition map $m : C \times C \times Y \longrightarrow A$ and consider the scheme theoretic preimage $m^{-1}(-Y)$ together with the projections

$$\mathrm{pr}_{23} : m^{-1}(-Y) \longrightarrow C \times Y \quad \text{and} \quad \mathrm{pr}_3 : m^{-1}(-Y) \longrightarrow Y.$$

Let Γ' be a prime divisor in Γ with $(c_1, c_2) \in \Gamma'$. It follows from step 2 that $\Gamma' \times y$ is contained in some component Z of $m^{-1}(-Y)$. The restriction of pr_{23} to Z is surjective because c_2 and y are general. Hence, $\dim(Z) > \dim(Y)$ and so there is a curve in Z passing through (c_1, c_2, y) which is contracted via m to y_1. This implies that there is some quasi-projective curve T together with a nonconstant morphism $(\tilde{c}_1, \tilde{c}_2, \tilde{y}) : T \longrightarrow C \times C \times Y$, with $\tilde{c}_1(t_0) = c_1$, $\tilde{c}_2(t_0) = c_2$ and $\tilde{y}(t_0) = y$ for some $t_0 \in T$ such that

$$\tilde{c}_1(t) + \tilde{c}_2(t) + \tilde{y}(t) = -y_1,$$

for all $t \in T$. Since $c_2 \in C$ and $y \in Y$ are general, the addition morphism $F : C \times Y \longrightarrow \Theta$ is generically finite in a neighbourhood of (c_2, y). Hence,

$$\tilde{c}_1(t) = -y_1 - \tilde{c}_2(t) - \tilde{y}(t)$$

is nonconstant in t.

For a contradiction, suppose $T_{C,c_1} \subset T_{\Theta,-y_1}$, where we recall $-y_1 = c_1 + c_2 + y$. The image of $(\tilde{c}_2, \tilde{y}) : T \longrightarrow C \times Y$ is a curve through the general point (c_2, y). It follows that $(\tilde{c}_2(t), \tilde{y}(t))$ is a general point of $C \times Y$ for general $t \in T$. Replacing (c_2, y) by $(\tilde{c}_2(t), \tilde{y}(t))$ in the above argument therefore shows

$$T_{C,\tilde{c}_1(t)} \subset T_{\Theta,-y_1}$$

for general (hence all) $t \in T$, since $-y_1 = \tilde{c}_1(t) + \tilde{c}_2(t) + \tilde{y}(t)$. As $\tilde{c}_1(t)$ is non-constant in t, $T_{C,c}$ is contained in the plane $T_{\Theta,-y_1}$ for general $c \in C$. Hence, C is geometrically degenerate, which by Lemma 5.2.1 contradicts (5.5) in Proposition 5.4.1. This contradiction establishes (5.24), which finishes the proof of step 3. □

Step 4. For $c_2 \in C$ general, $\deg(\Gamma|_{C \times c_2}) = \deg(F)$.

Proof. Let $c_2 \in C$ be general. By step 3, Γ is reduced and so its restriction to $C \times c_2$ is a reduced 0-cycle. Since c_2 and y are general, $-c_2 - y$ is a general point of Θ. Therefore, $F^{-1}(-c_2 - y)$ is a disjoint union of $\deg(F)$ reduced points. It thus suffices to construct a bijection between the closed points of the zero-dimensional reduced schemes $\mathrm{supp}(\Gamma) \cap (C \times c_2)$ and $F^{-1}(-c_2 - y)$. This bijection is given by

$$\phi : \mathrm{supp}(\Gamma) \cap (C \times c_2) \longrightarrow F^{-1}(-c_2 - y),$$

where $\phi((c_1, c_2)) = (c_1, -c_1 - c_2 - y)$. The point is here that ϕ is well-defined by step 2; its inverse is given by

$$\phi^{-1}((c_1, y_1)) = (c_1, -c_1 - y_1 - y).$$

This establishes the assertion in step 4. □

By step 4, $\deg(\Gamma|_{C \times c_2}) = \deg(F)$ for a general point $c_2 \in C$. Using (5.20) and (5.21), we obtain therefore

$$\deg(i^*\Theta) = 1 + \deg(\Gamma|_{C \times c_2}) = 1 + \deg(F).$$

Comparing this with (5.19) yields

$$\frac{g \cdot \deg(F)}{g-1} = 1 + \deg(F),$$

hence $\deg(F) = g - 1$, as we want. This finishes the proof of Proposition 5.4.2. □

Proof of Theorem 5.1.1. Let (A, Θ) be an indecomposable ppav with

$$\Theta = C + Y.$$

As explained in the beginning of Section 5.4, we may assume $\Theta = -\Theta$ and $0 \in C$. By Matsusaka–Hoyt's criterion [36, p. 416], Proposition 5.4.2 implies that C is smooth and that there is an isomorphism $\psi : (A, \Theta) \xrightarrow{\sim} (J(C), \Theta_C)$ which maps C to a translate of $W_1(C)$. Since $0 \in C$, it follows that

$$\psi(C) = W_1(C) - x_2$$

for some $x_2 \in W_1(C)$.

For $x_1 \in W_1(C)$ with $x_1 \neq x_2$, Weil [94] proved

$$\operatorname{div}_{W_{g-1}(C)}(W_{g-1}(C) \cap W_{g-1}(C)_{x_1 - x_2}) = W_{g-2}(C)_{x_1} + (-W_{g-2}(C))_{-\kappa - x_2}, \quad (5.25)$$

where $\kappa \in J(C)$ is such that $-W_{g-1}(C) = W_{g-1}(C)_\kappa$. Comparing (5.4) with (5.25), we conclude that $\psi(Y)$ is a translate of $W_{g-2}(C)$. This finishes the proof of Theorem 5.1.1. □

Remark 5.4.3. *Welters [95, p. 440] showed that the conclusion of Proposition 5.4.1 implies the existence of a positive-dimensional family of trisecants of the Kummer variety of (A, Θ). The latter characterizes Jacobians by results of Gunning's [33] and Matsusaka–Hoyt's [36] and could hence be used to circumvent Proposition 5.4.2 in the proof of Theorem 5.1.1. We presented Proposition 5.4.2 here because its proof is elementary and purely algebraic, whereas the use of trisecants involves analytic methods, see [33, 49]. It is hoped that this might be useful in other situations (e.g. in positive characteristics) as well. We also remark that Proposition 5.4.2 can be used to avoid the use of Gunning's results in Welters' work [95].*

Remark 5.4.4. *In [56, p. 254], Little conjectured Theorem 5.1.1 for $g = 4$; a proof is claimed if $\Theta = C + S$ is a sum of a curve C and a surface S, where no translate of C or S is symmetric (hence C is non-hyperelliptic) and some additional nondegeneracy assumptions hold. However, some parts of the proof seem to be flawed and so further assumptions on C and S are necessary in [56], see [55].*

5.5 GV-sheaves, theta duals and Pareschi–Popa's conjectures

The purpose of this section is to prove Theorem 5.1.2 stated in the introduction and to explain two related conjectures of Pareschi and Popa. We need to recall some results of Pareschi–Popa's work [60] first.

Let (A, Θ) be a ppav of dimension g. By [60, Thm. 2.1], a coherent sheaf $\mathcal{F}$ on A is a GV-sheaf if and only if the complex

$$\mathbb{R}\hat{\mathcal{S}}(\mathbb{R}\mathcal{H}om(\mathcal{F}, \mathcal{O}_A)) \tag{5.26}$$

in the derived category of the dual abelian variety $\hat{A}$ has zero cohomology in all degrees $i \neq g$. Here, $\mathbb{R}\hat{\mathcal{S}} : \mathrm{D}^b(A) \longrightarrow \mathrm{D}^b(\hat{A})$ denotes the Fourier–Mukai transform with respect to the Poincaré line bundle [38, p. 201].

For a geometrically nondegenerate subvariety $Z \subseteq A$, Pareschi and Popa consider the twisted ideal sheaf $\mathcal{I}_Z(\Theta) = \mathcal{I}_Z \otimes \mathcal{O}_A(\Theta)$.[2] It follows from their own and Höring's work respectively [60, p. 210] that this is a GV-sheaf if Z is a translate of $W_d(C)$ in the Jacobian of a smooth curve or a translate of the Fano surface of lines in the intermediate Jacobian of a smooth cubic threefold. Both examples are known to have minimal cohomology class $\frac{1}{(g-d)!}[\Theta]^{g-d}$. Pareschi–Popa's Theorem [60, Thm. B] says that this holds in general.

Theorem 5.5.1 (Pareschi–Popa)**.** *Let Z be a d-dimensional geometrically nondegenerate subvariety of a g-dimensional ppav (A, Θ). If $\mathcal{I}_Z(\Theta)$ is a GV-sheaf,*

$$[Z] = \frac{1}{(g-d)!}[\Theta]^{g-d}.$$

Combining Theorem 5.5.1 with Debarre's "minimal class conjecture" in [19], Pareschi and Popa arrive at the following, see [60, p. 210].

Conjecture 5.5.2. *Let (A, Θ) be an indecomposable ppav of dimension g and let Z be a geometrically nondegenerate subvariety of dimension $1 \leq d \leq g-2$. If*

$$\mathcal{I}_Z(\Theta) \text{ is a GV-sheaf,} \tag{5.27}$$

then either (A, Θ) is isomorphic to the Jacobian of a smooth curve C and Z is a translate of $W_d(C)$, or it is isomorphic to the intermediate Jacobian of a smooth cubic threefold and Z is a translate of the Fano surface of lines.

Pareschi and Popa [60, Thm. C] proved Conjecture 5.5.2 for $d = 1$ and $d = g - 2$. Theorem 5.1.2 stated in the introduction proves it for subvarieties with curve summands. Before we can explain the proof of Theorem 5.1.2, we need to recall Pareschi–Popa's notion of theta duals [60, p. 216].

[2] In fact, Pareschi and Popa treat the more general case of an equidimensional closed reduced subscheme $Z \subseteq A$, but for our purposes the case of subvarieties will be sufficient.

Definition 5.5.3. *Let $Z \subseteq A$ be a subvariety. Its theta dual $\mathcal{V}(Z) \subseteq \hat{A}$ is the scheme-theoretic support of the g-th cohomology sheaf of the complex*

$$(-1_{\hat{A}})^* \mathbb{R}\hat{\mathcal{S}}(\mathbb{R}\mathcal{H}om(\mathcal{I}_Z(\Theta), \mathcal{O}_A))$$

in the derived category $\mathrm{D}^b(\hat{A})$.

From now on, we use Θ to identify $\hat{A}$ with A. The theta dual of $Z \subseteq A$ is then a subscheme $\mathcal{V}(Z) \subseteq A$. For $W_d(C)$ inside a Jacobian of dimension $g \geq 2$, Pareschi and Popa proved [60, Sect. 8.1]

$$\mathcal{V}(W_d(C)) = -W_{g-d-1}(C), \tag{5.28}$$

for $1 \leq d \leq g-2$. Apart from this example, it is in general difficult to compute $\mathcal{V}(Z)$. However, the reduced scheme $\mathcal{V}(Z)^{\mathrm{red}}$ can be easily described as follows.

Lemma 5.5.4. *Let $Z \subseteq A$ be a subvariety. The components of the reduced scheme $\mathcal{V}(Z)^{\mathrm{red}}$ are given by the maximal (with respect to inclusion) subvarieties $W \subseteq A$ such that $Z - W \subseteq \Theta$.*

Proof. By [60, p. 216], the set of closed points of $\mathcal{V}(Z)$ is $\{a \in A \mid Z \subseteq \Theta_a\}$. This proves the lemma. □

We will use the following consequence of (5.28) and Lemma 5.5.4.

Lemma 5.5.5. *Let C be a smooth curve of genus $g \geq 2$ and let Z be a $(g-d-1)$-dimensional subvariety of $J(C)$ such that $W_d(C)+Z$ is a translate of the theta divisor Θ_C. Then, Z is a translate of $W_{g-d-1}(C)$.*

Proof. By assumption, there is a point $a \in J(C)$ with $W_d(C)+Z_a = \Theta_C$. Hence, by Lemma 5.5.4 and (5.28),

$$(-Z)_{-a} \subseteq \mathcal{V}(W_d(C)) = -W_{g-d-1}(C).$$

Since $(-Z)_{-a}$ is $(g-d-1)$-dimensional, we deduce $Z = W_{g-d-1}(C)_{-a}$, as claimed. □

For a geometrically nondegenerate subvariety $Z \subseteq A$ of dimension d,

$$\dim(\mathcal{V}(Z)) \leq g - d - 1 \tag{5.29}$$

follows from Lemmas 5.2.2 and 5.5.4. Moreover, if equality is attained in (5.29), then $\Theta = Z - W$ for some component W of $\mathcal{V}(Z)^{\mathrm{red}}$, and so Θ has Z as a d-dimensional summand.

Pareschi and Popa proved the following [60, Thm. 5.2(a)].

Proposition 5.5.6. *Let $Z \subseteq A$ be a geometrically nondegenerate subvariety. If $\mathcal{I}_Z(\Theta)$ is a GV-sheaf, equality holds in (5.29).*

Motivated by Proposition 5.5.6, Pareschi and Popa conjectured [60, p. 222] that Conjecture 5.5.2 holds if one replaces (5.27) by the weaker assumption

$$\dim(\mathcal{V}(Z)) = g - d - 1. \tag{5.30}$$

By the above discussion, their conjecture is equivalent to

Conjecture 5.5.7. *Let (A, Θ) be an indecomposable ppav of dimension g and let Z be a geometrically nondegenerate subvariety of dimension $1 \le d \le g-2$. Suppose that*

$$\Theta = Z + W \tag{5.31}$$

for some subvariety $W \subseteq A$. Then, (A, Θ) is either isomorphic to the Jacobian of a smooth curve C and Z is a translate of $W_d(C)$, or it is isomorphic to the intermediate Jacobian of a smooth cubic threefold and Z is a translate of the Fano surface of lines.

Theorem 5.1.1 proves (a strengthening of) Conjecture 5.5.7 for $d = 1$ and $d = g - 2$. This provides the first known evidence for that conjecture.

Remark 5.5.8. *Conjecture 5.5.2 is implied by Conjecture 5.5.7, as well as by Debarre's "minimal class conjecture" in [19]. Similar implications among the latter two conjectures are not known.*

We end this section with the proof of Theorem 5.1.2.

Proof of Theorem 5.1.2. Let $Z \subsetneq A$ be as in Theorem 5.1.2. Since $\mathcal{I}_Z(\Theta)$ is a GV-sheaf, equality holds in (5.29) by Proposition 5.5.6. The reduced theta dual $\mathcal{V}(Z)^{\mathrm{red}}$ contains thus by Lemmas 5.2.2 and 5.5.4 a $(g-d-1)$-dimensional component W with $Z - W = \Theta$. By assumption 1 in Theorem 5.1.2, $Z = C + Y$ has a curve summand C and so we obtain

$$\Theta = C + Y - W.$$

By Theorem 5.1.1, C is smooth and there is an isomorphism

$$\psi : (A, \Theta) \xrightarrow{\sim} (J(C), \Theta_C)$$

which identifies C and $Y - W$ with translates of $W_1(C)$ and $W_{g-2}(C)$ respectively. Hence,

$$\psi(Z) - \psi(W) = \psi(C) + \psi(Y) - \psi(W) = W_{g-1}(C)_a, \tag{5.32}$$

for some $a \in J(C)$, where $\psi(C)$ and $\psi(Y) - \psi(W)$ are translates of $W_1(C)$ and $W_{g-2}(C)$ respectively. It remains to prove that $\psi(Y)$ is a translate of $W_{d-1}(C)$, the assertion concerning $\psi(Z) = \psi(C) + \psi(Y)$ will then be immediate.

If $d = g-1$, then $\psi(W)$ is a point and $\psi(Y)$ is a translate of $W_{g-2}(C)$, as we want. We may therefore assume $d \leq g-2$ in the following. By Theorem 5.5.1, the GV-condition on $\mathcal{I}_Z(\Theta)$ implies

$$[Z] = \frac{1}{(g-d)!} \cdot [\Theta]^{g-d}.$$

By Debarre's Theorem [19], $\psi(Z)$ is thus a translate of $W_d(C)$ or $-W_d(C)$.

Case 1: $\psi(Z)$ is a translate of $W_d(C)$.

By (5.32), $W_d(C) - \psi(W)$ is a translate of $W_{g-1}(C)$ and so $-\psi(W)$ is a translate of $W_{g-d-1}(C)$ by Lemma 5.5.5. By (5.32), $W_{g-d}(C) + \psi(Y)$ is thus a translate of $W_{g-1}(C)$. Applying Lemma 5.5.5 again shows then that $\psi(Y)$ is a translate of $W_{d-1}(C)$, as we want.

Case 2: $\psi(Z)$ is a translate of $-W_d(C)$.

By (5.32), $W_d(C)+\psi(W)$ is in this case a translate of $-W_{g-1}(C)$ and thus of $W_{g-1}(C)$. By Lemma 5.5.5, $\psi(W)$ is therefore a translate of $W_{g-d-1}(C)$. Since $1 \leq d \leq g-2$, it follows from (5.32) that

$$W_{g-1}(C) = W_1(C) - W_1(C) + W', \tag{5.33}$$

where W' is a translate of $\psi(Y) - W_{g-d-2}(C)$. By Lemma 5.5.5,

$$-W_1(C) + W' = W_{g-2}(C). \tag{5.34}$$

Let $c_0 \in C$ be the preimage of $0 \in J(C)$ under the Abel–Jacobi embedding. Any point on W' is then represented by a divisor $D - g \cdot c_0$ on C, where D is effective of degree g. It follows from (5.34) that $D - c_0 - c$ is effective for all $c \in C$. Thus,

$$D - c_0 \in W^1_{g-1}(C) \subseteq \mathrm{Pic}^{g-1}(C)$$

is a divisor whose linear series is positive-dimensional. By (5.34), we have $\dim(W') \geq g-3$ (in fact equality holds) and so $\dim(W^1_{g-1}(C)) \geq g-3$. A theorem of Martens [4, p. 191] implies that C is hyperelliptic and so case 1 applies. This concludes the proof. □

5.6 Dominations by products

5.6.1 The DPC Problem for theta divisors

We have the following well-known

Lemma 5.6.1. *Let A be an abelian variety and let $F : Z_1 \times Z_2 \dashrightarrow A$ be a rational map from a product of smooth varieties Z_1 and Z_2. Then there are morphisms $f_i : Z_i \longrightarrow A$ for $i = 1, 2$ such that $F = f_1 + f_2$.*

Proof. Since A does not contain rational curves, F is in fact a morphism, which by the universal property of Albanese varieties factors through

$$\mathrm{Alb}(Z_1) \times \mathrm{Alb}(Z_2).$$

We conclude as morphisms between abelian varieties are translates of homomorphisms. □

The following result shows that property 1 in Theorem 5.1.2 is in fact a condition on the birational geometry of Z.

Corollary 5.6.2. *An n-dimensional subvariety Z of an abelian variety A has a d-dimensional summand if and only if there is a dominant rational map $F : Z_1 \times Z_2 \dashrightarrow Z$, where Z_1 and Z_2 are varieties of dimension d and $n - d$ respectively.*

Proof. If Z has a d-dimensional summand Z_1, the decomposition $Z = Z_1 + Z_2$ for a suitable Z_2 gives rise to a dominant rational map $F : Z_1 \times Z_2 \dashrightarrow Z$ as we want. Conversely, if $F : Z_1 \times Z_2 \dashrightarrow Z$ is given, after resolving the singularities of Z_1 and Z_2, the assertion follows from Lemma 5.6.1. This proves Corollary 5.6.2. □

Corollary 5.1.3 stated in the introduction is an immediate consequence of Riemann's Theorem and

Corollary 5.6.3. *Let (A, Θ) be an indecomposable g-dimensional ppav. Suppose there is a dominant rational map*

$$F : Z_1 \times Z_2 \dashrightarrow \Theta,$$

where Z_1 and Z_2 are varieties of dimension 1 and $g - 2$ respectively. Then (A, Θ) is isomorphic to the Jacobian of a smooth curve C. Moreover, if we identify Θ with $W_{g-1}(C)$, there are rational maps

$$f_1 : Z_1 \dashrightarrow W_1(C) \quad \text{and} \quad f_2 : Z_2 \dashrightarrow W_{g-2}(C)$$

with $F = f_1 + f_2$.

Proof. After resolving the singularities of Z_1 and Z_2, we may assume that both varieties are smooth. By Lemma 5.6.1, $F : Z_1 \times Z_2 \dashrightarrow \Theta \subseteq A$ is then a sum of morphisms $f_1 : Z_1 \longrightarrow A$ and $f_2 : Z_2 \longrightarrow A$. Hence,

$$f_1(Z_1) + f_2(Z_2) = \Theta,$$

and so Corollary 5.6.3 follows from Theorem 5.1.1. □

Remark 5.6.4. *For an arbitrary ppav (A,Θ), Corollary 5.1.3 implies that each component of Θ is DPC if and only if (A,Θ) is a product of Jacobians of smooth curves. Indeed, if $(A,\Theta) = (A_1,\Theta_1)\times\cdots\times(A_r,\Theta_r)$ with indecomposable factors (A_i,Θ_i), then Θ has r components which are isomorphic to $\Theta_i\times\prod_{j\neq i}A_j$ where $i=1,\dots,r$. Since abelian varieties are DPC, it follows that the components of Θ are DPC if and only if each Θ_i is DPC, hence the result by Corollary 5.1.3.*

Corollary 5.6.5. *The Fano surface of lines on a smooth cubic threefold $X\subseteq\mathbb{P}^4$ is not dominated by a product of curves.*

Proof. By [16, Thm. 13.4.], the theta divisor of the intermediate Jacobian $(J^3(X),\Theta)$ is dominated by the product $S\times S$, where S is the Fano surface of lines on X. Since $(J^3(X),\Theta)$ is indecomposable and not isomorphic to the Jacobian of a smooth curve [16, p. 350], Corollary 5.6.5 follows from Corollary 5.6.3. □

5.6.2 Dominations of symmetric products of curves

Theorem 5.1.1 is nontrivial even in the case where (A,Θ) is known to be a Jacobian. This allows us to classify all possible ways in which the symmetric product $C^{(k)}$ of a smooth curve C of genus $g\geq k+1$ can be dominated by a product of curves. Before we explain the result, we should note that

$$\mathrm{AJ}_k: C^{(k)}\longrightarrow W_k(C)$$

is a birational morphism for $g\geq k$, and that $-W_{g-1}(C)$ is a translate of $W_{g-1}(C)$. In particular, multiplication by -1 on $J(C)$ induces a nontrivial birational automorphism

$$\iota: C^{(g-1)} \overset{\sim}{\dashrightarrow} C^{(g-1)}.$$

Corollary 5.6.6. *Let C be a smooth curve of genus g. Suppose that for some $k\leq g-1$, there are smooth curves $C_1,\dots,C_k$ together with a dominant rational map*

$$F: C_1\times\cdots\times C_k \dashrightarrow C^{(k)}.$$

Then there are dominant morphisms $f_i: C_i\longrightarrow C$ with the following property:

- *If $k<g-1$, then $F=f_1+\cdots+f_k$.*
- *If $k=g-1$, then $F=f_1+\cdots+f_{g-1}$ or $F=\iota\circ(f_1+\cdots+f_{g-1})$.*

Proof. We use the birational morphism $\mathrm{AJ}_k : C^{(k)} \longrightarrow W_k(C)$ to identify $C^{(k)}$ birationally with its image $W_k(C)$ in $J(C)$. By Lemma 5.6.1, the rational map

$$\mathrm{AJ}_k \circ F : C_1 \times \cdots \times C_k \dashrightarrow W_k(C)$$

is a sum of morphisms $C_i \longrightarrow W_k(C)$. If C_i' denotes the image of C_i in $J(C)$, then

$$\Theta_C = C_1' + \cdots + C_k' + W_{g-k-1}(C) \tag{5.35}$$

by Riemann's Theorem. Proposition 5.4.2 yields therefore $[C_i'] = \frac{1}{(g-1)!}[\Theta_C]^{g-1}$ for all i. It follows for instance from Debarre's Theorem [19] that each C_i' is a translate of C or of $-C$, where $C \subseteq J(C)$ is identified with its Abel–Jacobi image. If C is hyperelliptic, Corollary 5.6.6 follows.

Assume now that C is non-hyperelliptic. Then there is some $0 \le r \le k$, such that C_i is a translate of $-C$ for precisely r many indices $i \in \{1, \dots, k\}$. By (5.35), $W_{g-r-1}(C) - W_r(C)$ is then a translate of Θ_C. However, Lemma 5.5 in [19] yields

$$[W_{g-r-1}(C) - W_r(C)] = \binom{g-1}{r} \cdot [\Theta_C],$$

which coincides with $[\Theta_C]$ if and only if $r = 0$ or $r = g-1$. This proves Corollary 5.6.6. □

Corollary 5.6.6 implies a theorem of Martens [59, 64] asserting that any birational map

$$C_1^{(k)} \overset{\sim}{\dashrightarrow} C_2^{(k)}$$

between the k-th symmetric products of smooth curves C_1 and C_2 of genus $g \ge k+2$ is induced by an isomorphism $C_1 \xrightarrow{\sim} C_2$.

For $k \ge g$, the symmetric product $C^{(k)}$ is birational to $J(C) \times \mathbb{P}^{k-g}$. This shows that Corollary 5.6.6 is sharp as for $k \ge g$, the product $J(C) \times \mathbb{P}^{k-g}$ admits a lot of nontrivial dominations. For instance, it is dominated by $k-g$ arbitrary curves (whose product dominates $\mathbb{P}^{k-g}$) together with any choice of g curves in $J(C)$ whose sum is $J(C)$.

Bibliography

[1] H. Abelson, *Topologically distinct conjugate varieties with finite fundamental group*, Topology **13** (1974), 161–176.

[2] J. Amorós and I. Biswas, *Compact Kähler manifolds with elliptic homotopy type*, Adv. Math. **224** (2010), 1167–1182.

[3] D. Arapura, *Geometric Hodge structures with prescribed Hodge numbers*, Preprint, arXiv:1406.2142. (to appear in Recent Advances in Hodge Theory, Cambridge)

[4] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, *Geometry of algebraic curves I*, Springer-Verlag, New York, 1985.

[5] M. Atiyah and F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology **1** (1962), 25–45.

[6] W.P. Barth, K. Hulek, C.A.M. Peters and A. Van de Ven, *Compact Complex Surfaces*, Springer–Verlag, Berlin, 2004.

[7] I. Bauer, F. Catanese and F. Grunewald, *Faithful actions of the absolute Galois group on connected components of moduli spaces*, Inv. Math. **199** (2015), 859–888.

[8] C. Birkenhake and H. Lange, *Complex abelian varieties*, 2nd edition, Springer–Verlag, 2004.

[9] P. Cascini and L. Tasin, *On the Chern numbers of a smooth threefold*, Preprint, arXiv:1412.1686.

[10] F. Catanese, *Moduli and classification of irregular Kähler manifolds (and algebraic varieties) with Albanese general type fibrations*, Inv. Math. **104** (1991), 263–289.

[11] F. Catanese and M. Schneider, *Bounds for stable bundles and degrees of Weierstrass schemes*, Math. Ann. **293** (1992), 579–594.

[12] F. Charles, *Conjugate varieties with distinct real cohomology algebras*, J. reine angew. Math. **630** (2009), 125–139.

[13] F. Charles and C. Schnell, *Notes on absolute Hodge classes*, in "Hodge Theory", ed. E. Cattani, F. El Zein, P.A. Griffiths and L.D. Trang, Princeton University Press, 2013, 469–530.

[14] M.-C. Chang, *Distributions of Chern numbers of complete intersection three folds*, GAFA **7** (1997), 861–872.

[15] M.-C. Chang and A. Lopez, *A linear bound on the Euler number of threefolds of Calabi-Yau and of general type*, Manu. Math. **105** (2001), 47–67.

[16] C.H. Clemens and P.A. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. **95** (1972), 281–356.

[17] S. Cynk and K. Hulek, *Construction and examples of higher-dimensional modular Calabi–Yau manifolds*, Canad. Math. Bull. **50** (2007), 486–503.

[18] O. Debarre, *Tores et variétès abéliennes complex*, Cours Spécialisés 6, Société Mathématique de France, EDP Sciences, 1999.

[19] O. Debarre, *Minimal cohomology classes and Jacobians*, J. Alg. Geom. **4** (1995), 321–335.

[20] P. Deligne, *La conjecture de Weil pour les surfaces K3*, Inv. Math. **15** (1972), 206–226.

[21] P. Deligne, *Hodge cycles on abelian varieties* (notes by J. S. Milne), in Lecture Notes in Mathematics **900**, Springer–Verlag, 1982, 9–100.

[22] J.P. Demailly, T. Peternell and M. Schneider, *Compact complex manifolds with numerically effective tangent bundles*, J. Alg. Geom. **3** (1994), 295–345.

[23] I. Dolgachev, *Algebraic surfaces with* $q = p_g = 0$, Algebraic surfaces, 97–215, C.I.M.E. Summer Sch., 76, Springer, Heidelberg,

[24] R.W. Easton and R. Vakil, *Absolute Galois acts faithfully on the components of the moduli space of surfaces: a Belyi-type theorem in higher dimension*, Int. Math. Res. Not. **20** (2007), 10pp.

[25] L. Ein and R. Lazarsfeld, *Singularities of theta divisors and the birational geometry of irregular varieties*, J. Amer. Math. Soc. **10** (1997), 243–258.

[26] F.T. Farrell and L.E. Jones, *Rigidity in geometry and topology*, Proc. of the ICM, Vol. **I, II** (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, 653–663.

[27] E. Freitag, *Siegelsche Modulfunktionen*, Grundlehren der mathematischen Wissenschaften 254, Springer–Verlag, Berlin, 1983.

[28] R. Friedman and J.W. Morgan, *Algebraic surfaces and 4-manifolds: some conjectures and speculations*, Bulletin of the AMS **18** January 1988.

[29] R. Friedman and J.W. Morgan, *On the diffeomorphism types of certain algebraic surfaces. I*, J. Diff. Geom. 27 (1988), 297–369.

[30] R. Friedman and J.W. Morgan, *Smooth four-manifolds and complex surfaces*, Springer–Verlag, Berlin, 1994.

[31] B. van Geemen, *Some equations for the universal Kummer variety*, Preprint, arXiv:1307.2463, (2013).

[32] S. Grushevsky, *The Schottky problem*, Current Developments in Algebraic Geometry, MSRI Publications **59**, Cambridge Univ. Press (2012), 129–164.

[33] R.C. Gunning, *Some curves in abelian varieties.* Inv. Math. **66** (1982), 377–389.

[34] F. Hirzebruch, *Some problems on differentiable and complex manifolds*, Ann. of Math. **60** (1954), 213–236; reprinted in [35].

[35] F. Hirzebruch, *Gesammelte Abhandlungen*, Band I, Springer–Verlag, 1987.

[36] W.L. Hoyt, *On products and algebraic families of Jacobian varieties*, Ann. of Math. **77** (1963), 415–423.

[37] B. Hunt, *Complex manifold geography in dimension 2 and 3*, J. Diff. Geo. **30** (1989), 51–153.

[38] D. Huybrechts, *Fourier-Mukai transforms in Algebraic Geometry*, Oxford Mathematical Monographs, Oxford, 2006.

[39] V. Iskovkikh and Y. Prokhorov, *Fano varieties*, Algebraic geometry V, 1–247, ed. A. Parshin and I. Shafarevich, Springer–Verlag, 1999.

[40] M.A. Kervaire, *Le théorèm de Barden–Mazur–Stallings*, Comment. Math. Helv. **40** (1965), 31–42.

[41] J. Kollár, *Lectures on resolution of singularities*, Princeton University Press, Princeton, 2007.

[42] J. Kollár, *Low degree polynomial equations: arithmetic, geometry and topology*, European Congress of Mathematics, Vol. I (Budapest, 1996), 255–288, Progr. Math., **168**, Birkhäuser, Basel, 1998.

[43] J. Kollár, *Shafarevich Maps and Automorphic Forms*, Princeton University Press, Princeton, 1995.

[44] J. Kollár, Y. Miyaoka and S. Mori, *Rational connectedness and boundedness of Fano varieties*, J. Diff. Geom. **36** (1992), 765–779.

[45] D. Kotschick, *Chern numbers and diffeomorphism types of projective varieties*, J. of Topology **1** (2008), 518–526.

[46] D. Kotschick, *Characteristic numbers of algebraic varieties*, Proc. Natl. Acad. Sci. USA **106** (2009), 10114–10115.

[47] D. Kotschick, *Topologically invariant Chern numbers of projective varieties*, Adv. Math. **229** (2012), 1300–1312.

[48] D. Kotschick and S. Schreieder, *The Hodge ring of Kähler manifolds*, Compositio Math. **149** (2013), 637–657.

[49] I. Krichever, *Characterizing Jacobians via trisecants of the Kummer Variety*, Ann. of Math. **172** (2010), 485–516.

[50] R. Lazarsfeld, *Positivity in Algebraic Geometry II*, Springer–Verlag, Berlin, 2004.

[51] R. Lazarsfeld and M. Popa, *Derivative complex, BGG correspondence, and numerical inequalities for compact Kähler manifolds*, Inv. Math. **182** (2010), 605–633.

[52] C. LeBrun, *Topology versus Chern numbers of complex 3-folds*, Pacific J. Math. **191** (1999), 123–131.

[53] S. Lee, *On Masek's criterion on very ampleness*, Comm. Algebra **30** (2002), 4383–4399.

[54] A.S. Libgober and J. W. Wood, *Uniqueness of the complex structure on Kähler manifolds of certain homotopy types*, J. Diff. Geom. **32** (1990), 139–154.

[55] J. Little, *Correction to: On Lie's approach to the study of translation manifolds*, on his personal webpage: http://mathcs.holycross.edu/ little/Corrs.html.

[56] J. Little, *On Lie's approach to the study of translation manifolds*, J. Diff. Geom. **26** (1987), 253–272.

[57] L. Lombardi, *Inequalities for the Hodge numbers of irregular compact Kaehler manifolds*, IMRN **1** (2013), 63–83.

[58] T. Matsusaka, *On a characterization of a Jacobian variety*, Mem. Coll. Sci. Kyoto Ser. A Math. **32** (1959), 1–19.

[59] H.H. Martens, *An extended Torelli Theorem*, Amer. J. Math. **87** (1965), 257–261.

[60] G. Pareschi and M. Popa, *Generic vanishing and minimal cohomology classes on abelian varieties*, Math. Ann. **340** (2008), 209–222.

[61] U. Persson, *Chern invariants of surfaces of general type*, Compositio Math. **43** (1981) 3–58.

[62] C.S. Rajan, *An example of non-homeomorphic conjugate varieties*, Math. Res. Lett. **18**, (2011), 937-943.

[63] Z. Ran, *A characterization of five-dimensional Jacobian varieties*, Inv. Math. **67** (1982), 395–422.

[64] Z. Ran, *On a theorem of Martens*, Rend. Sem. Mat. Univers. Politecn. Torino **44** (1986), 287–291.

[65] Z. Ran, *On subvarieties of abelian varieties*, Inv. Math. **62** (1981), 459–479.

[66] M. Rapoport and T. Zink, *Über die lokale Zetafunktion von Shimuravarietaäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Characteristik*, Invent. Math. **68** (1982), 21–101.

[67] D. Reed, *The topology of conjugate varieties*, Math. Ann. **305**, (1996), 287–309.

[68] Q. Ren, S.V. Sam, G. Schrader and B. Sturmfels, *The universal Kummer threefold*, Experimental Mathematics **22** (2013), 327–362.

[69] X. Roulleau and G. Urzúa, *Chern slopes of simply connected complex surfaces of general type are dense in [2,3]*, Ann. of Math. **182** (2015), 287–306.

[70] C. Schoen, *Varieties dominated by product varieties*, Internat. J. Math. **7** (1996), 541–571.

[71] S. Schreieder, *Chern-Zahlen von Projektivisierungen*, Bachelor Thesis, LMU München, 2011.

[72] S. Schreieder, *Hodge theory*, Part III Essay, Cambridge, 2012.

[73] S. Schreieder, *Dualization invariance and a new complex elliptic genus*, J. reine angew. Math. **692** (2014), 77–108.

[74] S. Schreieder, *Multiplicative sub-Hodge structures of conjugate varieties*, Forum of Mathematics, Sigma **2** (2014), e1, 37pp.

[75] S. Schreieder, *Theta divisors with curve summands and the Schottky problem*, Preprint 2014, arXiv:1409.3134. (to appear in Math. Ann.)

[76] S. Schreieder, *On the construction problem for Hodge numbers*, Geometry & Topology **19** (2015), 395–342.

[77] S. Schreieder and L. Tasin, *Algebraic structures with unbounded Chern numbers*, Preprint 2015, arXiv:1505.03086.

[78] J-P. Serre, *Exemples de variétés projectives conjuguées non homéomorphes*, C. R. Acad. Sci. Paris **258** (1964), 4194–4196.

[79] J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier, Grenoble **6**, (1955-1956), 1–42.

[80] J.-P. Serre, *Letter to Grothendieck, March 31, 1964*, in: *Grothendieck-Serre correspondence*, AMS, Providence R.I. (2004).

[81] J.-P. Serre, *Sur la topologie des variétés algébriques en caractéristique p*, Symposium Internacional de Topologia Algebraica, Mexico (1958), 24–53; in Oeuvres vol. 1, 501–530.

[82] I.R. Shafarevich, *Basic Algebraic Geometry 1*, Springer Verlag, Berlin, 1994.

[83] I. Shimada *Non-homeomorphic conjugate complex varieties*, in "Singularities-Niigata-Toyama 2007", **56**, Adv. Stud. Pure Math., Math. Soc. Japa, Tokyo, (2009), 285–301.

[84] C. Simpson, *The construction problem in Kähler geometry*, in *Different faces of geometry*, ed. S.K. Donaldson, Y. Eliashberg and M. Gromov, p. 365–402, Int. Math. Ser. (N. Y.), 3, Kluwer/Plenum, New York, 2004.

[85] S. Smale, *On the structure of manifolds*, Amer. J. Math. **84** (1962), 387–399.

[86] R.E. Stong, *Notes on cobordism theory*, Math. Notes, Princeton University Press, Cambridge, 1968.

[87] Y. Tashiro, S. Yamazaki, M. Ito and T. Higuchi, *On Riemann's period matrix of* $y^2 = x^{2n+1} - 1$, RIMS Kokyuroku **963** (1996), 124–141.

[88] H. Tsuji, *Stability of tangent bundles of minimal algebraic varieties*, Topology **27** (1988), 429–442.

[89] C. Voisin, *Hodge Theory and Complex Algebraic Geometry, I*, Cambridge University Press, Cambridge, 2002.

[90] C. Voisin, *Hodge Theory and Complex Algebraic Geometry, II*, Cambridge University Press, Cambridge, 2003.

[91] C. Voisin, *On the homotopy types of Kähler compact and complex projective manifolds*, Inv. Math. **157** (2004), 329–343.

[92] C. Voisin, *Some aspects of the Hodge conjecture*, Jpn. J. Math. **2** (2007), 261–296.

[93] C.T.C. Wall, *On simply connected 4-manifolds*, J. London Math. Soc. **39** (1964), 141–149.

[94] A. Weil, *Zum Beweis des Torellischen Satzes*, Nachr. Akad. Wiss. Göttingen (1957), 33–53.

[95] G.E. Welters, *A characterization of non-hyperelliptic Jacobi varieties*, Inv. Math. **74** (1983), 437–440.

[96] S.-T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Natl. Acad. Sci. USA **74** (1977), 1798–1799.

[97] Y. Zhang, *Miyaoka–Yau inequality for minimal projective manifolds of general type*, Proc. Amer. Math. Soc. **137** (2009), 2749–2754.