ON THE RATIONALITY PROBLEM FOR QUADRIC BUNDLES

STEFAN SCHREIEDER

Abstract. We classify all positive integers n and r such that (stably) non-rational complex r-fold quadric bundles over rational n-folds exist. We show in particular that for any n and r, a wide class of smooth r-fold quadric bundles over $\mathbb{P}^n$ are not stably rational if $r \in [2^{n-1}-1, 2^n-2]$. In our proofs we introduce a generalization of the specialization method of Voisin and Colliot-Thélène–Pirutka which avoids universally CH_0-trivial resolutions of singularities.

1. Introduction

A quadric bundle is a flat morphism of projective varieties $f : X \longrightarrow S$, whose generic fibre is a smooth quadric. We will always assume that the base S is a rational variety. It is then an interesting and old problem, which goes back at least to the work of Artin and Mumford [2], to decide whether X is rational as well. By Springer's theorem [35], X is rational if f admits a rational multisection of odd degree. By a theorem of Lang [26], such a section exists whenever $r > 2^n - 2$, where r denotes the dimension of the fibres of f and $n = \dim(S)$ denotes the dimension of the base.

On the other hand, quite little is known about the non-rationality of such bundles. For instance, unless the fibres of f are conic curves or quadric surfaces, no smooth example is known to be non-rational. Our first result is as follows.

Theorem 1. *Let n and r be positive integers. Smooth (stably) non-rational complex r-fold quadric bundles over rational bases of dimension n exist if and only if $r \leq 2^n - 2$.*

While the rationality problem is solved for many types of conic bundles [1, 2, 6, 7, 8, 16, 41], even for smooth quadric surface bundles, progress has been made only recently by Hassett, Pirutka and Tschinkel. They proved that the very general fibres of three families of quadric surface bundles over $\mathbb{P}^2$, degenerated over plane octic curves, are not stably rational [17, 18, 19]. Each family contains a dense set of smooth rational fourfolds and so they obtained the first examples of non-rational varieties with rational

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deformation types. The next result shows more generally that for any positive integer r, deformation invariance of rationality fails for r-fold quadric bundles over rational bases.

Theorem 2. *Let r be a positive integer. Then there is a smooth complex projective family $\pi : \mathcal{X} \longrightarrow B$ of smooth complex varieties such that each fibre $X_b = \pi^{-1}(b)$ is an r-fold quadric bundle over some complex projective space, satisfying the following:*

(1) for very general $t \in B$, the fibre X_t is not stably rational;

(2) the set $\{b \in B \mid X_b \text{ is rational}\}$ is dense in B for the analytic topology.

The case $r = 1, 2$ is due to [17]; the density result is thereby proven via an argument of Voisin [41]. Even though that argument does not seem to apply to higher-dimensional quadric bundles, we are able to reduce the above density result to one about quadric surface bundles over surfaces. The main difficulty lies then in proving the stable non-rationality assertion.

More explicitly, we discuss now the rationality problem for a natural and interesting class of r-fold quadric bundles over $\mathbb{P}^n$. We start with a generically non-degenerate line bundle valued quadratic form $q : \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(l)$, where $\mathcal{E} := \bigoplus_{i=0}^{r+1} \mathcal{O}_{\mathbb{P}^n}(-l_i)$ is a split vector bundle on $\mathbb{P}^n$. If $q_s \neq 0$ for all $s \in \mathbb{P}^n$, then $X := \{q = 0\} \subset \mathbb{P}(\mathcal{E})$ is an r-fold quadric bundle over $\mathbb{P}^n$. We may identify q with a symmetric matrix $A = (a_{ij})$ of homogeneous polynomials of degrees $|a_{ij}| = l_i + l_j + l$. Locally over $\mathbb{P}^n$, X is given by

$$\sum_{i,j=0}^{r+1} a_{ij} z_i z_j = 0.$$

The deformation type of X depends only on the integers $d_i := 2l_i + l$, which have all the same parity. We call any such bundle of type $(d_i)_{0 \leq i \leq r+1}$, cf. Section 3.5 below. We then have the following; see Theorem 40 and Remark 41 below for a more general statement.

Theorem 3. *Let n, r be positive integers with $2^{n-1} - 1 \leq r \leq 2^n - 2$, and let $d_0, \ldots, d_{r+1}$ be integers of the same parity such that $d_i \geq 2^n + n - 1$ for all i. Then a very general complex r-fold quadric bundle of type $(d_i)_{0 \leq i \leq r+1}$ over $\mathbb{P}^n$ is not stably rational.*

The lower bound $2^n + n - 1$ on the degrees is bounded from above by $n + 2r + 1$. As an example, we thus see that for n, r as above, very general complex hypersurfaces $X \subset \mathbb{P}^n \times \mathbb{P}^{r+1}$ of bidegree $(d, 2)$ with $d \geq n + 2r + 1$ are not stably rational. In contrast, if $r \geq 2$, some smooth hypersurfaces of that kind are rational; in fact, for $r \geq 2$, all examples in Theorem 3 have rational deformation types, see Corollary 21 below.

As another application, we consider singular hypersurfaces $X \subset \mathbb{P}^{N+1}$ of degree d. If X is not a cone and contains a singular point whose multiplicity is roughly as large as the degree, then X tends to be quite close to being rational, no matter how large d is. For instance, a single point $x \in X$ of multiplicity $d - 1$ forces X to be rational. In contrast, Theorem 3 implies that many degree d hypersurfaces with points of multiplicity $d-2$ are not even stably rational.

Corollary 4. *Let n and r be positive integers with $2^{n-1}-1 \leq r \leq 2^n-2$. Set $N := n+r$ and $m := 2^n+n+1$. Then a very general complex hypersurface $X \subset \mathbb{P}^{N+1}$ of degree $d \geq m$ and with multiplicity $d-2$ along an r-plane is not stably rational.*

The above hypersurfaces are birational to r-fold quadric bundles over $\mathbb{P}^n$, cf. Lemma 22. The upper bound $r \leq 2^n-2$ is thus sharp by the aforementioned result of Lang [26]. The lower bound m on the degree satisfies $m \in [N+3, 2N-n+3]$ and it lies on the boundary of that interval if $r = 2^n-2$ or $r = 2^{n-1}-1$, respectively.

Building on work of Kollár [24], Totaro showed [37] that a very general smooth complex hypersurface $X \subset \mathbb{P}^{N+1}$ of degree $d \geq \lceil 2(N+2)/3 \rceil$ is not stably rational. Our lower bounds differ roughly by a factor $\lambda \in [\frac{3}{2}, 3]$ from those bounds.

The proofs of the above results are based on two main ingredients, which we explain in the following two subsections respectively.

1.1. Examples à la Artin–Mumford and Colliot-Thélène–Ojanguren in higher dimensions.

For any complex projective variety Y, there are unramified cohomology groups $H^i_{nr}(Y, \mathbb{Z}/l)$, which are stable birational invariants of Y. These invariants have been introduced by Colliot-Thélène and Ojanguren [10] in their reinterpretation of the famouse Artin–Mumford example [2]. The results in [2] and [10] show (cf. Lemma 12 below) that for $n = 2$ and $r = 1, 2$, or $n = 3$ and $r = 3, 4, 5, 6$, there is a singular unirational r-fold quadric bundle Y over $\mathbb{P}^n$ with $H^n_{nr}(Y, \mathbb{Z}/2) \neq 0$. For $n = r = 2$, different examples with the same property have recently been constructed by Pirutka [31] and Hassett–Pirutka–Tschinkel [17].

Using an algebraic approach of Peyre [29], Asok showed that for arbitrary positive integers n and r with $2^{n-1}-1 \leq r \leq 2^n-2$, there is a collection of singular unirational r-fold quadric bundles $Y_1, \ldots, Y_s$ over $\mathbb{P}^{2n}$, with $s = \binom{2n}{n} - 1$, such that their common fibre product over $\mathbb{P}^{2n}$ has nontrivial unramified cohomology in degree n, see [3, Theorem 4.2] and Lemma 12 below.

For $r \geq 7$, r-fold quadric bundles over rational bases with nontrivial unramified cohomology are not known. Generalizing [2] and [10], the next result provides such examples for any r.

Theorem 5. *Let n and r be positive integers with $2^{n-1}-1 \leq r \leq 2^n-2$. Then there is a unirational complex projective r-fold quadric bundle $Y \longrightarrow \mathbb{P}^n$ with $H^n_{nr}(Y, \mathbb{Z}/2) \neq 0$.*

While the upper bound $r \leq 2^n-2$ is sharp by [26], the lower bound $r \geq 2^{n-1}-1$ is not essential; see Theorem 36 for a more general result which works without that bound. As in [3], the above result relies on Voevodsky's proof of the Milnor conjecture [39].

1.2. **A specialization method without resolutions.** Voisin [41] introduced and Colliot-Thélène–Pirutka [11] developed further a specialization technique which led to numerous applications in the study of (stable) rationality properties of rationally connected varieties, see for instance [1, 6, 7, 8, 12, 16, 17, 18, 19, 20, 28, 30, 37]. Roughly speaking, in order to prove stable non-rationality of a projective variety X, one has to find a degeneration Y of X which admits both, some obstruction for stable rationality (e.g. non-trivial unramified cohomology) and a universally CH_0-trivial resolution of singularities $\tau : \widetilde{Y} \longrightarrow Y$. In order to check this last property in practice, one has to provide explicit local charts for the resolution $\widetilde{Y}$ and show that all scheme-theoretic fibres of τ have universally trivial Chow groups of zero-cycles. This is a quite subtle condition, which occupied main parts in several applications mentioned above, see for instance [12, 17, 18]. In particular, the method applies only to situations where Y has very mild singularities and resolutions can be described explicitly. For instance, it had been impossible to apply the method to several (reasonable) special fibres Y, where obstructions for rationality were known, but the singularities did not seem to allow manageable resolutions.

The main idea of this paper is to replace the existence of a universally CH_0-trivial resolution of Y by a weaker condition, which is easier to check, see Proposition 25 below. This leads to more general specialization theorems which also apply in situations where it seems impossible to compute a resolution of singularities explicitly, letting alone to check that a universally CH_0-trivial one exists.

To state such a result, note that we define in this paper so called CTO type quadrics over rational function fields and produce examples in arbitrary dimensions, see Definition 16 and Proposition 29. These quadrics appear as generic fibres in the examples of Theorem 5. We then prove the following specialization theorem; for what it exactly means that a variety degenerates or specializes to another variety, see Section 2.2 below.

Theorem 6. *Let X be a projective variety which specializes to a complex projective variety Y with a morphism $f : Y \longrightarrow S$ to a rational n-fold S with $n \geq 2$. If the generic fibre Y_η of f is smooth and stably birational to a CTO type quadric Q over $\mathbb{C}(S)$, then X is not stably rational.*

Remarkably, the only condition on Y that we have to impose in the above theorem concerns the generic fibre of $f : Y \longrightarrow S$. For instance, f does not need to be flat and Y does not need to have a universally CH_0-trivial resolution. In fact, there is no assumption whatsoever on the singularities of Y at points which do not dominate S.

This significantly extends the number of possible applications. The main point is that for any smooth quadric Q over $\mathbb{C}(\mathbb{P}^n)$, and for any rational n-fold S, there is a wide range of different models $f : Y \longrightarrow S$ with Q as generic fibre. If Q is of CTO type, then the above theorem applies and so any variety which specializes to Y is not stably rational.

We emphasize that in general one must be quite careful when trying to deduce non-rationality for X from non-rationality properties of some specialization Y of X. For instance, a smooth cubic surface is rational and degenerates to a cone over an elliptic curve, which is non-rational. In fact, the situation is worse: $\mathbb{P}^N$ specializes to the cone over any hypersurface $Z \subset \mathbb{P}^N$ (cf. [37, §4]) and hence to a projective variety Y, stably birational to any given projective variety of dimension $N-1$. For instance, if $N \geq 4$, we may choose a specialization Y of $\mathbb{P}^N$ with a rational map $f : Y \dashrightarrow \mathbb{P}^n$ whose generic fibre is stably birational to a CTO type quadric. In Theorem 6, such degenerations are excluded by the assumption that f is a morphism with smooth generic fibre. It is however possible to weaken those assumptions so that $f : Y \dashrightarrow S$ is only a dominant rational map, but Y must have sufficiently mild singularities locally along the closure of a general fibre of f, see Theorem 38 for the precise statement.

Remark 7. *The quadric surfaces over $\mathbb{C}(\mathbb{P}^2)$, recently constructed by Pirutka* [31] *and Hassett, Pirutka and Tschinkel* [17]*, are not of CTO type. Nonetheless, our specialization method without resolutions works also for those quadrics. This simplifies* [17, 18, 19]*, but it also yields much more general results which seemed inaccessible before. The details appear elsewhere* [34].

2. Preliminaries

2.1. **Conventions and notations.** All schemes are separated. A variety is an integral scheme of finite type over a field. Two varieties X and Y over a field k are stably birational, if $X \times \mathbb{P}^m$ is birational (over k) to $Y \times \mathbb{P}^n$ for some $n, m \geq 0$. A resolution of a variety Y is a proper birational morphism of varieties $\tau : \widetilde{Y} \longrightarrow Y$, with $\widetilde{Y}$ smooth. If $Z \subset Y$ is a closed subscheme of a variety Y, then a log resolution of the pair (Y, Z) is a resolution of singularities $\tau : \widetilde{Y} \longrightarrow Y$ such that the reduced subscheme which underlies $\tau^{-1}Z$ is a simple normal crossing divisor. A very general point of some scheme, is a closed point which is contained in a countable intersection of dense open subsets.

2.2. **What it means that a variety specializes or degenerates to another one.** We say that a variety X over a field L specializes (or degenerates) to a variety Y over a field k, if there is a discrete valuation ring R with residue field k and fraction field F with an injection $F \hookrightarrow L$ of fields, together with a flat proper morphism $\mathcal{X} \longrightarrow \operatorname{Spec} R$ of finite type, such that Y is isomorphic to the special fibre $Y \simeq \mathcal{X} \times k$ and $X \simeq \mathcal{X} \times L$ is isomorphic to a base change of the generic fibre $\mathcal{X} \times F$.

The next lemma shows that this terminology allows quite some flexibility.

Lemma 8. *Let $\pi : \mathcal{X} \longrightarrow B$ be a flat proper morphism of complex varieties with integral fibres, and let $0 \in B$ be a closed point. Then for any very general point $t \in B$, the fibre X_t specializes to X_0.*

Proof. The family π is obtained as base change of some family $\pi' : \mathcal{X}' \longrightarrow B'$ defined over some countable algebraically closed subfield $k \subset \mathbb{C}$. Let $U \subset B(\mathbb{C})$ be the union of all closed points $b \in B$, which do not lie on $Z' \times_k \mathbb{C}$ for some proper subvariety $Z' \subsetneq B'$. Since there are only countably many such subvarieties Z', any very general point of B lies in U. Moreover, for any $t \in U$, there is a field isomorphism $\varphi : \overline{\mathbb{C}(B)} \xrightarrow{\simeq} \mathbb{C}$ which identifies the geometric generic fibre $\mathcal{X} \times \overline{\mathbb{C}(B)}$ with the very general fibre X_t, see for instance [38, Lemma 2.1]. This shows that the fibres X_t with $t \in U$ are all abstractly isomorphic (i.e. differ only by the action of $\mathrm{Aut}(\mathbb{C})$) and so it suffices to find one $t \in U$ such that X_t degenerates to X_0. Hence, we may reduce to the case where B is a curve. Taking normalizations, we may also assume that B is smooth and so the statement is clear because $\mathcal{O}_{B,0}$ is a discrete valuation ring under these assumptions. $\square$

2.3. **Chow groups of zero-cycles.** A morphism $f : X \longrightarrow Y$ of varieties over a field k is universally CH_0-trivial, if $f_* : \mathrm{CH}_0(X \times L) \xrightarrow{\simeq} \mathrm{CH}_0(Y \times L)$ is an isomorphism for all field extensions L of k. If the structure morphism $f : X \longrightarrow \operatorname{Spec} k$ is universally CH_0-trivial, then we say that the Chow group of zero-cycles of X is universally trivial. This is equivalent to the existence of an integral decomposition of the diagonal $\Delta_X \in \mathrm{CH}_{\dim(X)}(X \times X)$ as in (2) below. The Chow group of zero-cycles of a smooth projective variety X over a field is a stable birational invariant, see [11, Lemme 1.5] and [37, Theorem 1.1] and references therein.

2.4. **Galois cohomology of fields.** Let K be a field of characteristic coprime to l. We identify the Galois cohomology group $H^n(K, \mu_l^{\otimes n})$ with the étale cohomology group $H^n_{\text{ét}}(\mathrm{Spec}(K), \mu_l^{\otimes n})$, where $\mu_l \subset \mathbb{G}_m$ denotes the group of l-th roots of unity. We also use the identification $H^1(K, \mu_l) \simeq K^*/(K^*)^l$, induced by the Kummer sequence. For $a_1, \dots, a_n \in K^*$, we denote by $(a_1, \dots, a_n) \in H^n(K, \mu_l^{\otimes n})$ the class obtained by cup product. Classes of this form are called symbols.

If A is a discrete valuation ring with fraction field K and residue field κ whose characteristic is coprime to l, then there are residue maps $\partial^n_A : H^n(K, \mu_l^{\otimes n}) \longrightarrow H^{n-1}(\kappa, \mu_l^{\otimes (n-1)})$. If ν denotes the corresponding valuation on K, we also write $\partial^n_\nu = \partial^n_A$.

The following lemma computes the residue of a symbol explicitly in the case of μ_2-coefficients, where squares can be ignored.

Lemma 9. *Let A be a discrete valuation ring with residue field κ and fraction field K, both of characteristic different from 2. Suppose that -1 is a square in K. Let $\pi \in A$ be a uniformizer, $0 \leq m \leq n$ be integers and let $a_1 \dots, a_n \in A^*$ be units in A. Then the following identity holds in $H^{n-1}(\kappa, \mu_2^{\otimes (n-1)})$:*

$$\partial^n_A(\pi a_1, \dots, \pi a_m, a_{m+1}, \dots, a_n) = \left(\sum_{i=1}^{m} (a_1, \dots, \widehat{a_i}, \dots, a_m) \right) \cup (a_{m+1}, \dots, a_n),$$

where $(a_1, \dots, \widehat{a_i}, \dots, a_m)$ *denotes the symbol where* a_i *is omitted. Here we use the convention that the above sum* $\sum_{i=1}^m$ *is one if* $m = 1$ *and it is zero if* $m = 0$.

Proof. The cases $m = 0, 1$ follow for instance from [10, Proposition 1.3]. In order to prove the lemma, it thus suffices to show the following:

$$(\pi a_1, \dots, \pi a_m, a_{m+1}, \dots, a_n) = \left(\sum_{i=0}^{m} (a_1, \dots, a_{i-1}, \pi, a_{i+1}, \dots, a_m) \right) \cup (a_{m+1}, \dots, a_n).$$

To prove this identity, note that the Steinberg relations $(a, 1-a) = 0$ for $a \in K \setminus \{0, 1\}$ imply $(a, -a) = 0$ for all $a \in K^*$, see for instance [23, Lemma 2.2]. Since -1 is a square in K, $(\pi, \pi) = 0$. Using this, the formula follows immediately. $\square$

We will use the following compatibility of residues, see [10, p. 143]. It can be seen as direct consequence of Lemma 9 and the fact that any class is a sum of symbols by [39].

Lemma 10. *Let* $f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ *be a dominant morphism of schemes, where* A *and* B *are discrete valuation rings with fraction fields* $K = \operatorname{Frac} A$ *and* $L = \operatorname{Frac} B$ *and residue fields* κ_A *and* κ_B*, respectively. Then there is a commutative diagram*

$$\begin{array}{ccc} H^n(L, \mu_2^{\otimes n}) & \xrightarrow{\partial_B^n} & H^{n-1}(\kappa_B, \mu_2^{\otimes (n-1)}) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle e \cdot f^*} \\ H^n(K, \mu_2^{\otimes n}) & \xrightarrow{\partial_A^n} & H^{n-1}(\kappa_A, \mu_2^{\otimes (n-1)}), \end{array}$$

where $e = \nu_B(\pi_A) \in \mathbb{Z}$ *is the valuation with respect to* B *of a uniformizer* π_A *of* A.

Finally, we will use the following basic vanishing result, see [33, II.4.2].

Theorem 11. *Let* K *be the function field of an* n*-dimensional variety over an algebraically closed field of odd characteristic. Then,* $H^i(K, \mu_2^{\otimes i}) = 0$ *for all* $i > n$.

2.5. **Rost cycle modules.** Let k be a field. For any finitely generated field extension L of k, we denote by $\operatorname{Val}(L/k)$ the set of all geometric discrete valuations of rank one on L over k. Such valuations are characterized by the property that the corresponding valuation ring $\mathcal{O}_\nu \subset L$ is the local ring $\mathcal{O}_{X,x}$ at a codimension one point $x \in X^{(1)}$ of some normal variety X over k with $k(X) = L$, see [25, Proposition 1.7].

A Rost cycle module M^* over k is a functor from the category of finitely generated field extensions of k to $\mathbb{Z}$-graded abelian groups with some additional properties, see [32] and [25, Section 2]. An important one for us is the existence of residue maps $\partial_\nu^i : M^i(L) \longrightarrow M^{i-1}(E)$, for all $\nu \in \operatorname{Val}(L/k)$, where L/k is a finitely generated field extension and E is the residue field of ν. The group of unramified elements is

$$M^i_{nr}(L) := \{\alpha \in M^i(L) \mid \partial_\nu^i \alpha = 0 \text{ for all } \nu \in \operatorname{Val}(L/k)\}.$$

A class $\alpha \in M^i_{nr}(L)$ is called nontrivial, if it is not in the image of $M^i(k) \longrightarrow M^i_{nr}(L)$.

If X is a variety over k, then we write $M^i_{nr}(X) := M^i_{nr}(k(X))$. If X and Y are smooth proper varieties over k, then for any cycle $\Gamma \in \mathrm{CH}_{\dim(X)}(X \times Y)$, there is a homomorphism

$$\Gamma^* : M^i_{nr}(Y) \longrightarrow M^i_{nr}(X),$$

which is trivial whenever Γ does not dominate X, see [22, RC-I and proof of RC.9]. Via these actions, unramified cohomology descends to a functor on the category of integral correspondences between smooth and proper k-varieties, see [22, RC.3-4]. If Γ is the graph of a rational map $f : X \dashrightarrow Y$, we obtain pullback maps $\Gamma^* = f^*$.

2.6. **Unramified cohomology.** An important example of a Rost cycle module over a field k is given by Galois cohomology $M^i(L) = H^i(L, \mu_l^{\otimes i})$, with l coprime to char(k). The corresponding unramified cohomology groups are denoted by $H^i_{nr}(L, \mu_l^{\otimes i})$; if we want to emphasize the base field k, we also write $H^i_{nr}(L/k, \mu_l^{\otimes i})$ for this group. If k is algebraically closed and $i \geq 1$, then $H^i(k, \mu_l^{\otimes i}) = 0$ and so any $0 \neq \alpha \in H^i_{nr}(L/k, \mu_l^{\otimes i})$ is a nontrivial unramified cohomology class in the sense of Section 2.5 above. For equivalent definitions of unramified cohomology, see [9, Theorem 4.1.1].

If X is a variety over k, $H^i_{nr}(X, \mu_l^{\otimes i}) := H^i_{nr}(k(X)/k, \mu_l^{\otimes i})$ is a stable birational invariant of X, see [10, Proposition 1.2]. If $k = \mathbb{C}$ and X is smooth and projective, then $H^3_{nr}(X, \mu_l^{\otimes 3})$ and $H^4_{nr}(X, \mu_l^{\otimes 4})$ are related to failure of the integral Hodge conjecture for codimension two cycles on X and to torsion in the third Griffiths group, annihilated by the Abel–Jacobi map, respectively, see [13] and [40].

3. Quadric bundles and quadrics over non-closed fields

3.1. **Quadratic forms and Pfister neighbours.** Let K be a field of characteristic different from 2. Any quadratic form q on an n-dimensional K-vector space can be diagonalized, $q = \langle a_1, \dots, a_n \rangle$ for some $a_i \in K$, and we call n the dimension of q. We associate to q the quadric hypersurface $Q := \{q = 0\} \subset \mathbb{P}^{n-1}_K$, given by $\sum_i a_i z_i^2 = 0$. Two quadratic forms are similar if and only if the corresponding quadric hypersurfaces are isomorphic. The form q is isotropic if and only if Q admits a K-rational point.

The form q is called (n-fold) Pfister form, if it is isomorphic to the tensor product of forms of type $\langle 1, -a_i \rangle$ with nonzero $a_i \in K$, where $i = 1, \dots, n$. We denote this tensor product by $\langle\langle a_1, \dots, a_n \rangle\rangle$; it is a form of dimension 2^n. The sign can be ignored if -1 is a square in K. A non-degenerate quadratic form q_1 is called a Pfister neighbour if it is similar to a subform of a Pfister form q_2 with $2\dim(q_1) > \dim(q_2)$.

3.2. **Birational geometry of quadrics.** Let K be a field of characteristic different from 2. We say that two quadratic forms q_1 and q_2 over K are stably birational, if the associated quadric hypersurfaces are stably birational over K. The following lemma is well-known (cf. [21, Proposition 2]); for more results on the birational geometry of quadrics, we refer to [36] and references therein.

Lemma 12. *Let q_2 be a Pfister form over K. Then any Pfister neighbour q_1 of q_2 is stably birational to q_2.*

Proof. Let Q_i be the quadric associated to q_i. It suffices to prove that the generic fibre of $\mathrm{pr}_i : Q_1 \times Q_2 \longrightarrow Q_i$ is rational for $i = 1, 2$. Since q_1 is a subform of q_2, Q_2 has a $K(Q_1)$-rational point and so this is clear for $i = 1$. Conversely, q_2 is isotropic over $K(Q_2)$ and so Q_1 has a $K(Q_2)$-rational point, because $2\dim(q_1) > \dim(q_2)$ and isotropic Pfister forms are hyperbolic [14, II.9.10]. This proves the lemma. □

Remark 13. *An anisotropic quadratic form q_1 over K is stably birational to an anisotropic Pfister form q_2 if and only if q_1 is a Pfister neighbour of q_2, see* [21, Proposition 2].

The following unirationality criterion goes back to Colliot-Thélène and Ojanguren.

Lemma 14. *Let $n \geq 2$, and let $K = \mathbb{C}(x_1, \ldots, x_n)$ be the function field of $\mathbb{P}^n$. Consider the quadratic form $q = \langle 1, a_1, a_2, \ldots, a_r \rangle$ over K for some $a_i \in K^*$. Suppose that $a_1 = f/g$ with $f, g \in \mathbb{C}[x_1, \ldots, x_n]$, satisfying one of the following:*

(1) f and g are linear;
(2) f and g have degree at most two and the homogenization $q \in L[x_0, \ldots, x_n]$ of $gz^2 - f$, where $L = \mathbb{C}(z)$, is a quadratic form of rank ≥ 3 over L.

Then the quadric hypersurface Q determined by q is unirational over $\mathbb{C}$; more precisely, a degree two extension of $K(Q)$ is purely transcendental over $\mathbb{C}$.

Proof. The proof is similar to the arguments in [10, Propositions 2.1 and 3.1]. If a_1 is a square, then Q is rational over K and so the statement is clear. Otherwise, $K' := K[z]/(z^2 - a_1)$ is a field. Since $Q \times K'$ has a K'-rational point, it is rational over K'. It thus suffices to see that $K' \simeq \mathbb{C}(\mathbb{P}^n)$. To this end, consider $L = \mathbb{C}(z)$ and let $Z \subset \mathbb{P}^n_L$ be the projective closure of $\{gz^2 - f = 0\}$. By construction, $K' = L(Z)$ and so it suffices to prove that Z is rational over L. This is clear if f and g are linear. Otherwise, our assumptions imply that Z is a cone over a smooth quadric Z' over L of dimension at least one. Since $L = \mathbb{C}(z)$ is a C_1-field, Z' has a L-rational point and so Z is rational. This concludes the lemma. □

3.3. **A result of Orlov, Vishik and Voevodsky.** Voevodsky's proof of the Milnor conjecture [39] together with an exact sequence of Orlov, Vishik and Voevodsky [27, Theorem 2.1], implies the following important result.

Theorem 15 (Orlov–Vishik–Voevodsky)**.** *Let K be a field of characteristic zero, and let q be a Pfister neighbour of the Pfister form $\langle\langle a_1,\dots,a_n\rangle\rangle$, with $a_i\in K^*$. Let $f: Q\longrightarrow \operatorname{Spec} K$ be the projective quadric associated to q. Then the kernel of*

$$f^*: H^n(K,\mu_2^{\otimes n})\longrightarrow H^n(K(Q),\mu_2^{\otimes n})$$

is generated by $(a_1,\dots,a_n)$.

Proof. By [27, Theorem 2.1] and [39], the result holds for the Pfister neighbour $q=\langle\langle a_1,\dots,a_{n-1}\rangle\rangle\oplus\langle -a_n\rangle$. The stated result follows therefore from Lemma 12, because $\operatorname{im}(f^*)\subset H^n_{nr}(K(Q)/K,\mu_2^{\otimes n})$ and unramified cohomology is a stable birational invariant [10, Proposition 1.2]. □

3.4. **Quadrics à la Artin–Mumford and Colliot-Thélène–Ojanguren.** The following definition summarizes the conditions in [10, Propositions 2.1 and 3.1] of Colliot-Thélène and Ojanguren's paper, where the cases $n=2$ and 3 are studied.

Definition 16. *Let $n\geq 2$ be an integer and consider the function field $K=\mathbb{C}(\mathbb{P}^n)$. Suppose that there are elements $a_1,\dots,a_{n-1},b_1,b_2\in K^*$ such that for $j=1,2$, the class $\alpha_j:=(a_1,\dots,a_{n-1},b_j)\in H^n(K,\mu_2^{\otimes n})$ is nonzero and satisfies the following:*

($*$) for any $\nu\in\operatorname{Val}(K/\mathbb{C})$, $\partial^n_\nu\alpha_j=0$ for $j=1$ or 2.

Then any projective quadric $Q=\{q=0\}$ over K defined by a Pfister neighbour q of the n-fold Pfister form $\langle\langle a_1,\dots,a_{n-1},b_1b_2\rangle\rangle$ is called a quadric of CTO type.

Since Pfister neighbours are non-degenerate by definition, we note that CTO type quadrics are always smooth.

The results in [10] can be summarized as follows: if $n=2$ or 3, then CTO type quadrics exist and have nontrivial unramified $\mathbb{Z}/2$-cohomology in degree n; the Artin–Mumford example [2] is a CTO type conic over $\mathbb{C}(\mathbb{P}^2)$.

While the existence result in [10] is quite subtle, the argument which proves nontriviality of $H^n_{nr}(K(Q)/\mathbb{C},\mu_2^{\otimes n})$ works in arbitrary dimensions, cf. [10, Assertion 2.1.1].

Proposition 17 (Colliot-Thélène–Ojanguren)**.** *Let $n\geq 2$ and let $f:Q\longrightarrow\operatorname{Spec}K$ be a CTO type quadric over $K=\mathbb{C}(\mathbb{P}^n)$. Then, $0\neq f^*\alpha_1\in H^n_{nr}(K(Q)/\mathbb{C},\mu_2^{\otimes n})$.*

Proof. By Theorem 15, $f^*\alpha_1=f^*\alpha_2$ and we denote this class by α'. Let $\nu\in\operatorname{Val}(K(Q)/\mathbb{C})$ and consider the restriction $\mu:=\nu|_K$. If μ is trivial, then $\partial_\nu\alpha'=0$ by Lemma 9. Otherwise, $\mu\in\operatorname{Val}(K/\mathbb{C})$ by [25, Proposition 1.4]. By Lemma 10, there is some $e\in\mathbb{Z}$ such that $\partial^n_\nu\alpha'=e\cdot f^*(\partial^n_\mu\alpha_j)$ for $j=1,2$. Hence, $\partial^n_\nu\alpha'=0$, because $\partial^n_\mu\alpha_j=0$ for $j=1$ or 2 by assumptions. Therefore, $\alpha'=f^*\alpha_1\in H^n_{nr}(K(Q)/\mathbb{C},\mu_2^{\otimes n})$ is unramified over $\mathbb{C}$. To prove that it is nonzero, it suffices by Theorem 15 to see that $\alpha_1\neq 0$ and $\alpha_1\neq\alpha_1+\alpha_2$. This follows from $\alpha_j\neq 0$ for all $j=1,2$. □

Remark 18. *Proposition 17 implies that CTO type quadrics are always anisotropic.*

3.5. **Quadric bundles.** A quadric bundle is a flat morphism $f : X \longrightarrow S$ of projective varieties over a field k whose generic fibre is a smooth quadric over $k(S)$. For simplicity, we will assume that k is algebraically closed of characteristic zero (usually $k = \mathbb{C}$).

Let $q : \mathcal{E} \longrightarrow L$ be a generically non-degenerate line bundle valued quadratic form on some vector bundle $\mathcal{E}$ on S such that $q_s \neq 0$ for all $s \in S$. Then the hypersurface $X := \{q = 0\} \subset \mathbb{P}(\mathcal{E})$ is a quadric bundle over S; flatness follows because all fibres $X_s = \{q_s = 0\} \subset \mathbb{P}(\mathcal{E}_s)$ have the same Hilbert polynomial. The degeneration locus on S is given by the divisor where q does not have full rank.

We will always assume that $\mathcal{E} = \bigoplus_{i=0}^{r+1} L_i^{-1}$ splits into a sum of line bundles. Under this assumption, q corresponds to a symmetric matrix $A = (a_{ij})$, where a_{ij} is a global section of $L_i \otimes L_j \otimes L$. Locally over the base S, X is given by

$$\sum_{i,j=0}^{r+1} a_{ij} z_i z_j = 0, \tag{1}$$

where z_i denotes a local coordinate which trivializes $L^{-i} \subset \mathcal{E}$. If $a_{ij} = 0$ for $i \neq j$, then we also write $q := \langle a_{00}, \dots, a_{r+1,r+1} \rangle$.

If $L_i^{\otimes 2} \otimes L$ is base point free for all i, then for sufficiently general choice of q, Bertini's theorem implies that $X = \{q = 0\}$ is non-singular; it is a quadric bundle over S if $\binom{r+3}{2} > \dim(S)$ or if $L_i \otimes L_j \otimes L$ is trivial for some i and j.

Lemma 19. *Let S be a smooth complex projective rational variety and let $L_0, \dots, L_{r+1}$ and L be line bundles on S such that $L_i \otimes L_j \otimes L$ is base point free for all i, j. Let X be a smooth r-fold quadric bundle over S, given by a symmetric matrix $A = (a_{ij})$ of global sections $a_{ij} \in H^0(S, L_i \otimes L_j \otimes L)$ as in (1) above. If $r \geq \dim(S)$, then X deforms to a smooth rational variety.*

More precisely, if $r \geq \dim(S)$, $a_{mm} = 0$ for some $0 \leq m \leq r+1$, and the remaining a_{ij} are sufficiently general, then the corresponding quadric bundle X is smooth and rational.

Proof. Since all quadric bundles of the given type are parametrized by some open subset of $H^0(S, \mathrm{Sym}^2(\mathcal{E}^\vee) \otimes L)$, where $\mathcal{E}^\vee = \bigoplus_{i=0}^{r+1} L_i$, we see that they have all the same deformation type. It thus suffices to prove that for general sections $a_{ij} \in H^0(S, L_i \otimes L_j \otimes L)$ with $a_{mm} = 0$, X is smooth; X is then automatically rational because it admits a section. We may for simplicity assume $m = r + 1$. By Bertini's theorem, the only possible singularity occurs at $z_0 = \cdots = z_r = 0$, where we use the local chart (1). Using the Jacobian criterion, we see that a singular point of X must lie on the fibre above a point of S where $a_{r+1,i}$ vanishes for $i = 0, \dots, r$. Since $r \geq \dim(S)$, this locus is empty by our base point freeness assumption. This proves the lemma. ☐

Deforming X as in (1) to a quadric bundle whose symmetric matrix $A = (a_{ij})$ is diagonal with a_{ii} general, shows that the deformation type of X depends only on the line bundles $L_i^{\otimes 2} \otimes L$. In the case of line bundle valued quadratic forms $q : \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(l)$ on $\mathbb{P}^n$, where $\mathcal{E} = \bigoplus_{i=0}^{r+1} \mathcal{O}_{\mathbb{P}^n}(-l_i)$ is split, this observation gives rise to the following definition.

Definition 20. *Let $r, n \geq 1$ and $l_0, \dots, l_{r+1}, l$ be integers. An r-fold quadric bundle X over $\mathbb{P}^n$, which is given by a symmetric matrix $A = (a_{ij})$ of homogeneous polynomials of degrees $|a_{ij}| = l_i + l_j + l$ as in (1), is called of type $(d_i)_{0 \leq i \leq r+1}$ if $d_i = 2l_i + l$.*

We usually assume that $d_i \geq 0$ for all i. This is justified by the observation that X admits a section if $d_i < 0$ for some i and so it is rational in that case.

For any $d_i \geq 0$, a quadric bundle of type $(d_i)_{0 \leq i \leq r+1}$ on $\mathbb{P}^n$ exists if and only if all d_i have the same parity and additionally one of the following holds: $\binom{r+3}{2} > n$ or $d_i = 0$ for some i. The following is an immediate consequence of Lemma 19.

Corollary 21. *Let n and r be positive integers with $r \geq n$ and let $(d_i)_{0 \leq i \leq r+1}$ be a tuple of non-negative integers of the same parity. Then some smooth r-fold quadric bundles of type $(d_i)_{0 \leq i \leq r+1}$ over $\mathbb{P}^n$ are rational.*

The following two examples of quadric bundles are well-known.

Lemma 22. *Let n, r be integers with $\binom{r+3}{2} > n > 0$. Let $P \subset \mathbb{P}^{n+r+1}$ be an r-plane, and let $X \subset \mathbb{P}^{n+r+1}$ be a general hypersurface of degree $d + 2$ with multiplicity d along P. Then, X is birational to a general r-fold quadric bundle of type $(d, \dots, d, d+2)$ over $\mathbb{P}^n$.*

Proof. Choose coordinates $x_0, \dots, x_n, y_0, \dots, y_r$ on $\mathbb{P}^{n+r+1}$ such that $P = \{x_0 = \dots = x_n = 0\}$. If $X = \{f = 0\}$, then

$$f = \sum_{i,j=0}^{r} a_{ij} y_i y_j + \sum_{k=0}^{r} (a_{k,r+1} + a_{r+1,k}) y_k + a_{r+1,r+1},$$

for some homogeneous polynomials $a_{ij} = a_{ji}$, $a_{k,r+1} = a_{r+1,k}$ and $a_{r+1,r+1}$ in $x_0, \dots, x_n$ of degree d, $d + 1$ and $d + 2$, respectively. We introduce an additional variable y_{r+1} and homogenize the above equation with respect to the y_i's. This shows that the symmetric matrix $A = (a_{ij})_{0 \leq i,j \leq r+1}$ corresponds to a general r-fold quadric bundle of type $(d, \dots, d, d + 2)$, which is clearly birational to X. (In fact, it is the blow-up $Bl_P X$.) $\square$

Lemma 23. *Let $n, r \geq 1$ be integers. Let $P \subset \mathbb{P}^{n+r}$ be an $(r-1)$-plane, and let $D \subset \mathbb{P}^{n+r}$ be a general hypersurface of even degree $d+2$ with multiplicity d along P. Then the double covering $X \xrightarrow{2:1} \mathbb{P}^{n+r}$, branched along D, is birational to a general r-fold quadric bundle of type $(0, d, \dots, d, d + 2)$ over $\mathbb{P}^n$.*

Proof. The double cover X is given by $s^2 = f$, where $D = \{f = 0\}$. Choosing coordinates $x_0, \dots, x_n, y_1, \dots, y_r$ of $\mathbb{P}^{n+r}$, similarly as in the proof of Lemma 22 shows that X is birational to a quadric bundle over $\mathbb{P}^n$ of type $(0, d, \dots, d, d+2)$; the coordinate s plays the role of y_0 in the proof of Lemma 22. The corresponding symmetric matrix $A = (a_{ij})_{0 \leq i,j \leq r+1}$ satisfies $a_{00} = 1$ and $a_{i0} = 0$ for $i \geq 1$; the remaining entries of A are general. Conversely, if $A = (a_{ij})_{0 \leq i,j \leq r+1}$ is the symmetric matrix of a general r-fold quadric bundle of type $(0, d, \dots, d, d+2)$, then a_{00} is a nonzero constant and so we can transform A into a symmetric matrix with $a_{00} = 1$ and $a_{i0} = 0$ for all $i \geq 1$. This proves the lemma. $\square$

Proposition 24 (Voisin)**.** *Let $d_0 = 0$, $d_1 = d_2 = 2$ and $d_3 = 4$. Let W be the complex vector space of symmetric 4×4-matrices $A = (a_{ij})_{0 \leq i,j \leq 3}$ such that $a_{ij} \in \mathbb{C}[x_0, x_1, x_2]$ is homogeneous of degree $(d_i + d_j)/2$ with $a_{i0} = 0$ for $i = 1, 2, 3$. Then the set of points in $\mathbb{P}(W)$ which parametrize smooth quadric surface bundles of type $(0, 2, 2, 4)$ over $\mathbb{P}^2$ with a rational section is dense in the analytic topology.*

Proof. There is a Zariski open subset $B \subset \mathbb{P}(W)$ which parametrizes smooth quadric surface bundles of type (d_0, d_1, d_2, d_3) over $\mathbb{P}^2$. There is a universal family $\pi : \mathcal{X} \longrightarrow B$. As we have seen in Lemma 23, this family coincides with the universal family of (blow-ups of) double covers of $\mathbb{P}^4$, branched along a quartic hypersurface which is singular along a fixed line. If the fibre X_b above $b \in B$ admits a rational multisection of odd degree, then X_b admits a rational section by Springer's theorem [35]. Since the integral Hodge conjecture is known for codimension two cycles on quadric surface bundles over surfaces (cf. [13, Corollaire 8.2]), it suffices to show that the set of points $b \in B$ such that X_b admits a Hodge class of type $(2,2)$ which intersects the general fibre of $X_b \longrightarrow \mathbb{P}^2$ in odd degree is dense in B. The latter is proven in [42, Proposition 2.4], which is not affected by the gap; similar arguments have later been used in [17] and [19]. $\square$

4. The specialization method via weak decompositions of the diagonal

Recall from Section 1.2 that we aim to generalize the method of Voisin [41] and Colliot-Thélène–Pirutka [11] to degenerations where an explicit resolution of singularities of the special fibre can be avoided. The first step is the following small but crucial improvement of the original technique in [41] and [11]; the proof is inspired by [41, 11], Totaro's paper [37] and the original arguments of Bloch and Srinivas.

Proposition 25. *Let R be a discrete valuation ring with fraction field K and residue field k, with k algebraically closed. Let $\pi : \mathcal{X} \longrightarrow \operatorname{Spec} R$ be a flat proper scheme of finite type over R with geometrically integral fibres. Let $Y := \mathcal{X} \times k$ be the special fibre and suppose that there is a resolution of singularities $\tau : \widetilde{Y} \longrightarrow Y$ with the following properties:*

(1) for some Rost cycle module M^* over k, there is an unramified class $\alpha \in M^i_{nr}(\widetilde{Y})$ which is nontrivial, i.e. $\alpha \notin M^i(k)$;

(2) there is an open subset $U \subset Y$ such that $\tau^{-1}(U) \longrightarrow U$ is universally CH_0-trivial, and such that each irreducible component E_i of $\widetilde{Y} \setminus \tau^{-1}(U)$ is smooth and the restriction of α to E_i is trivial.

Then, no resolution of singularities of the geometric generic fibre $X := \mathcal{X} \times \overline{K}$ admits an integral decomposition of the diagonal.

The assumptions on the resolution τ in Proposition 25 are weaker and easier to check than those in [41, Theorem 2.1] and [11, Théorème 1.14]. Roughly speaking, instead of a universally CH_0-trivial resolution of Y, we ask for a resolution which is universally CH_0-trivial only over some open subset $U \subset Y$ and such that α restricts to zero on the complement. In this paper we will mostly use the special case where $\tau^{-1}(U) \simeq U$ is an isomorphism and so CH_0-triviality is automatic. The idea is to replace the Chow theoretic condition on the resolution τ from [11] by a cohomological one ($\alpha|_{E_i}$ is trivial), which is typically much more accessible. Since our assumptions are weaker, we will not conclude that $\widetilde{Y}$ admits a decomposition of the diagonal in the usual sense; we only obtain a weak decomposition of the form (4) below.

Proof of Proposition 25. It suffices to prove that there is an algebraically closed field F which contains K and such that some resolution of $X \times F$ does not admit an integral decomposition of the diagonal. Up to replacing R by its completion (which does not change the residue field), we may thus assume that R is a complete discrete valuation ring. For a contradiction, we assume that some resolution of X admits an integral decomposition of the diagonal. Pushing forward to X, we obtain a decomposition

$$\Delta_X = [X \times z_X] + B_X, \tag{2}$$

where $z_X \in \mathrm{CH}_0(X)$ is a zero-cycle of degree one, and where $\mathrm{supp}(B_X) \subset S_X \times X$ for some proper closed subset $S_X \subsetneq X$. Since $k = \overline{k}$, the specialization homomorphism on Chow groups [15, Example 20.3.5] gives a decomposition of the diagonal of Y:

$$\Delta_Y = [Y \times z] + B_Y, \tag{3}$$

where z is a zero-cycle of degree one on Y, and where $\mathrm{supp}(B_Y) \subset S_Y \times Y$ for some proper closed subset $S_Y \subsetneq Y$.

Let $\widetilde{U} := \tau^{-1}(U)$ and $E := \widetilde{Y} \setminus \widetilde{U}$. By assumptions, $\widetilde{U} \longrightarrow U$ is universally CH_0-trivial. Hence, for any field extension L of k, the localization exact sequence [15, Proposition

1.8] gives the following commutative diagram, with exact rows:

$$\begin{tikzcd} \mathrm{CH}_0(E\times L) \arrow[r] & \mathrm{CH}_0(\widetilde{Y}\times L) \arrow[r] \arrow[d, "\tau_*"] & \mathrm{CH}_0(\widetilde{U}\times L) \arrow[r] \arrow[d, "\simeq"] & 0 \\ & \mathrm{CH}_0(Y\times L) \arrow[r] & \mathrm{CH}_0(U\times L) \arrow[r] & 0 \end{tikzcd}$$

We apply this to $L = k(Y)$ and think about $\widetilde{Y} \times L$ and $Y \times L$ as generic fibres of the projections $\mathrm{pr}_1 : \widetilde{Y} \times \widetilde{Y} \longrightarrow \widetilde{Y}$ and $\mathrm{pr}_1 : Y \times Y \longrightarrow Y$ to the first factors, respectively. We claim that this gives rise to a decomposition

$$\Delta_{\widetilde{Y}} = [\widetilde{Y} \times \tilde{z}] + B + C, \tag{4}$$

where $\tilde{z} \in \mathrm{CH}_0(\widetilde{Y})$ has degree one (and maps to z), $\mathrm{supp}(C) \subset \widetilde{Y} \times E$ and $\mathrm{supp}(B) \subset S \times \widetilde{Y}$, for some proper closed subset $S \subsetneq \widetilde{Y}$. Indeed, since $k = \overline{k}$, we may choose a lift $\tilde{z}$ of z and then the above diagram together with (3) shows that the image of $\Delta_{\widetilde{Y}} - [\widetilde{Y} \times \tilde{z}]$ in $\mathrm{CH}_0(\widetilde{Y} \times L)$ restricts to zero on $\widetilde{U} \times L$, where $L = k(Y)$. This yields (4), as claimed.

Pull back from the second factor to the first gives an action

$$\Delta_{\widetilde{Y}}^* = [\widetilde{Y} \times \tilde{z}]^* + B^* + C^* : M^i_{nr}(\widetilde{Y}) \longrightarrow M^i_{nr}(\widetilde{Y}),$$

which is the identity because $\Delta_{\widetilde{Y}}$ is the class of the diagonal. As recalled in Section 2.5, B^* acts trivially because B does not dominate the first factor. Moreover, for each $y \in \widetilde{Y}$, $[\widetilde{Y} \times y]^*$ factors through $M^i_{nr}(y) = M^i_{nr}(k)$ and the induced map $M^i_{nr}(k) \longrightarrow M^i_{nr}(\widetilde{Y})$ is the natural one. The image of $[\widetilde{Y} \times \tilde{z}]^*$ is therefore contained in the subgroup of trivial unramified elements $M^i(k) \subset M^i_{nr}(\widetilde{Y})$. The above decomposition of the diagonal thus shows that, up to trivial unramified elements from $M^i(k)$, we have $\alpha = C^*(\alpha)$.

We may write $C = \sum_i C_i$, where $\mathrm{supp}(C_i) \subset \widetilde{Y} \times E_i$, and where the E_i denote the irreducible components of E. Since E_i is smooth, $C_i^* : M^i_{nr}(\widetilde{Y}) \longrightarrow M^i_{nr}(\widetilde{Y})$ factors through the restriction map $M^i_{nr}(\widetilde{Y}) \longrightarrow M^i_{nr}(E_i)$. Our assumptions therefore imply $C_i^*(\alpha) \in M^i(k) \subset M^i_{nr}(\widetilde{Y})$ for all i. This implies $\alpha \in M^i(k) \subset M^i_{nr}(\widetilde{Y})$, which contradicts our assumption that α is nontrivial. This finishes the proof of the proposition. $\square$

Remark 26. *The unramified cohomology group M^i_{nr} in item (1) of Proposition 25 can be replaced by any other birational invariant on which integral correspondences act similarly. For instance, Proposition 25 remains true if we replace condition (1) by the existence of a nontrivial differential form $\alpha \in H^0(\widetilde{Y}, \Omega^i_{\widetilde{Y}})$ for some $i \geq 1$, cf.* [37].

5. A vanishing result

If the special fibre Y in Proposition 25 is birational to a quadric bundle over $\mathbb{P}^n$ whose generic fibre is a quadric of CTO type, then condition (1) of Proposition 25 is satisfied by Proposition 17. In this section we establish a vanishing result which ensures that under some mild assumptions, also the second condition in Proposition 25 is satisfied.

Recall that for any dominant rational map $f : Y \dashrightarrow S$, there is a generic fibre Y_η over the function field of S, well-defined up to birational equivalence. An explicit representative of Y_η is given by the generic fibre of $f|_U : U \longrightarrow S$, where $U \subset Y$ is some open dense subset on which f is defined.

Proposition 27. *Let Y be a normal complex projective variety and let S be a normal complex projective rational n-fold for some $n \geq 2$. Let $f : Y \dashrightarrow S$ be a dominant rational map whose generic fibre Y_η is stably birational to a CTO type quadric Q over $K = \mathbb{C}(S)$, defined by a neighbour of the Pfister form $\langle\langle a_1, \dots, a_{n-1}, b_1 b_2\rangle\rangle$, for some $a_i, b_j \in K^*$. Set $\alpha_j := (a_1, \dots, a_{n-1}, b_j) \in H^n(K, \mu_2^{\otimes n})$ and let $\alpha' := f^*\alpha_1 \in H^n_{nr}(Y, \mu_2^{\otimes n})$ be the unramified class from Proposition 17. Then the following holds:*

(**) for any prime divisor $E \subset Y$ which does not dominate S, the restriction of α' to E vanishes: $\alpha'|_E = 0 \in H^n(\mathbb{C}(E), \mu_2^{\otimes n})$.

We will use the following lemma, which reformulates [25, Propositions 1.4 and 1.7] in geometric terms.

Lemma 28. *Let $f : Y \dashrightarrow S$ be a dominant rational map between normal complex projective varieties. Let $y \in Y^{(1)}$ be a codimension one point. Then there is a normal projective model S' of S, such that the induced rational map $f' : Y \dashrightarrow S'$ maps y either to the generic point of S' or to the generic point of a divisor on S'.*

Proof. Since Y is normal, f is defined at y. If $f(y)$ is dense in S, then any projective resolution of singularities $S' \longrightarrow S$ works. Otherwise, the valuation $\nu \in \mathrm{Val}(\mathbb{C}(Y)/\mathbb{C})$ induced by $y \in Y^{(1)}$ restricts to a geometric discrete valuation μ of rank one on $f^*(\mathbb{C}(S)) \subset \mathbb{C}(Y)$, see [25, Propositions 1.4]. By [25, Proposition 1.7], $\mathcal{O}_\mu = \mathcal{O}_{S',s'}$ for some normal projective variety S', birational to S, and some codimension one point $s' \in (S')^{(1)}$. The induced dominant rational map $f' : Y \dashrightarrow S'$ sends y to s'. This proves the lemma. ☐

Proof of Proposition 27. Let $y \in Y^{(1)}$ be the generic point of E. Since Y is normal, f is defined at y. By Lemma 28, we may up to replacing S by some different normal projective model assume that $x := f(y)$ is a codimension one point on S. Consider the discrete valuation rings $A := \mathcal{O}_{S,x}$ and $B := \mathcal{O}_{Y,y}$ and note that f induces an injection $A \hookrightarrow B$. By the definition of CTO type quadrics, there is some $j \in \{1, 2\}$ with $\partial^n_A \alpha_j = 0$.

The generic fibre of f is stably birational to a CTO type quadric associated to a neighbour of the Pfister form $\langle\langle a_1, \dots, a_{n-1}, b_1 b_2\rangle\rangle$. Since unramified cohomology is a stable birational invariant [10], we conclude $f^*\alpha_1 = f^*\alpha_2 \in H^n_{nr}(Y, \mu_2^{\otimes n})$ from Theorem 15. It thus suffices to prove that $f^*\alpha_j$ restricts to zero on E, where j is as above. Since

$\partial_A^n \alpha_j = 0$,

$$\alpha_j \in H^n_{\text{ét}}(\operatorname{Spec} A, \mu_2^{\otimes n}) \subset H^n(K, \mu_2^{\otimes n}),$$

see [9, §3.3 and §3.8]. Functoriality of étale cohomology yields a commutative diagram

$$\begin{tikzcd} H^n_{\text{ét}}(\operatorname{Spec} A, \mu_2^{\otimes n}) \arrow[r] \arrow[d, "f^*"] & H^n(\kappa(x), \mu_2^{\otimes n}) \arrow[d, "f^*"] \\ H^n_{\text{ét}}(\operatorname{Spec} B, \mu_2^{\otimes n}) \arrow[r] & H^n(\mathbb{C}(E), \mu_2^{\otimes n}), \end{tikzcd}$$

where the vertical arrows are induced by restriction to the corresponding closed points, respectively. Since $H^n(\kappa(x), \mu_2^{\otimes n}) = 0$ by Theorem 11, the $\alpha'|_E = 0$ follows from the commutativity of the above diagram. This finishes the proof of the proposition. □

6. Existence of CTO type quadrics in arbitrary dimensions

In this section, we aim to prove that CTO type quadrics (see Section 3.4) exist over $\mathbb{C}(\mathbb{P}^n)$ for arbitrary $n \geq 2$.

6.1. **Construction of quadrics over $\mathbb{C}(\mathbb{P}^n)$ via arrangements of quadrics in $\mathbb{P}^n$.** We choose coordinates $x_0, \dots, x_n$ on $\mathbb{P}^n$. For $i = 1, \dots, n-1$ we consider homogeneous polynomials $h_i \in \mathbb{C}[x_0, \dots, x_n]$ of degree two and define

$$a_i := \frac{h_i}{x_0^2} \tag{5}$$

for $i = 1, \dots, n-1$. In order to obtain a candidate CTO type quadric, we need to define two more rational functions b_1 and b_2, which we will do next.

Choose two homogeneous polynomials $g_{10}, g_{20} \in \mathbb{C}[x_0, \dots, x_n]$ of degree two. For any $\epsilon = (\epsilon_1, \dots, \epsilon_{n-1}) \in I := \{0,1\}^{n-1}$ and any $j = 1, 2$, we then consider

$$g_{j\epsilon} := g_{j0} + \sum_{i=1}^{n-1} \epsilon_i h_i.$$

With this definition, we put

$$g_1 := \prod_{\epsilon \in I} g_{1\epsilon} \quad \text{and} \quad g_2 := \prod_{\epsilon \in I} g_{2\epsilon}. \tag{6}$$

Finally, let $N := 2^n$ and define

$$b_1 := \frac{g_1}{x_0^N} \quad \text{and} \quad b_2 := \frac{g_2}{x_0^N}. \tag{7}$$

6.1.1. *Assumptions.* In the above construction, we will always assume that the homogeneous degree two polynomials h_i and g_{j0} satisfy the following assumptions.

For any $1 \leq i_1 < \cdots < i_c \leq n-1$ with $c \geq 0$, the following holds for $j = 1, 2$:

$$\operatorname{codim}_{\mathbb{P}^n}(\{h_{i_1} = \cdots = h_{i_c} = g_j = 0\}) \geq c+1, \tag{8}$$

$$\operatorname{codim}_{\mathbb{P}^n}(\{h_{i_1} = \cdots = h_{i_c} = g_1 = g_2 = 0\}) \geq c+2. \tag{9}$$

Moreover, we will assume that for $j = 1, 2$, the following symbol is nonzero

$$0 \neq (a_1, \ldots, a_{n-1}, b_j) \in H^n(K, \mu_2^{\otimes n}). \tag{10}$$

6.1.2. *Existence.* Let $l_1, \ldots, l_{2n+2} \in \mathbb{C}[x_0, \ldots, x_n]$ be linear homogeneous polynomials which are general subject to the condition[1] that

$$l_1, l_2, l_3, l_4 \in \mathbb{C}[x_0, x_1, x_2]. \tag{11}$$

We put

$$h_i := l_{2i-1} l_{2i} \quad \text{and} \quad g_{j0} := l_{2n-3+2j} l_{2n-2+2j}. \tag{12}$$

Conditions (8) and (9) are then clearly satisfied; in fact, (8) and (9) follow from

$$\{h_1 = \cdots = h_{n-1} = g_1 = g_2\} = \emptyset,$$

which holds by our genericity assumption on the l_i. To see that also (10) holds, it suffices by symmetry to deal with the case $j = 1$. Note first that our genericity assumption on the l_i implies that x_0 does not vanish on $\{l_4 = l_6 = \cdots = l_{2n} = 0\}$. We may also perform a change of variables to assume $l_{2i} = x_i$ for $i = 2, \ldots, n$. To prove our claim, we take now successive residues of $(a_1, \ldots, a_{n-1}, b_1)$ along $\{l_{2i} = 0\}$ for $i = n, n-1, \ldots, 3, 2$. Using Lemma 9 and our genericity assumptions on the l_i's, we end up with the class $\overline{a_1} \in F^*/(F^*)^2$, where $\overline{a_1}$ is the restriction of a_1 to the line $L := \{x_2 = \cdots = x_n = 0\}$ and $F = \mathbb{C}(L)$ is the function field of that line. The claim then reduces to prove that $\overline{l_1 l_2}/x_0^2$ is not a square for two general linear homogeneous polynomials $\overline{l_1}, \overline{l_2} \in \mathbb{C}[x_0, x_1]$, which is clear.

We have thus proven that the choice of h_i and g_{j0} as in (12) and the resulting a_i and b_j given by (5) and (7), satisfy all our assumptions (8), (9) and (10).

6.1.3. *Key Property.* Besides (8)–(10), the most important property of this construction is as follows. Let g_1 and g_2 be as in (6), then, for any $i = 1, \ldots, n-1$,

$$\text{the image of } g_1 \text{ and } g_2 \text{ in } \mathbb{C}[x_0, \ldots, x_n]/(h_i) \text{ becomes a square.} \tag{13}$$

[1]Condition (11) is not essential; it will only be used later in the proof of density of the rational fibres in the family of Theorem 2.

6.1.4. *Remarks.* The above construction is inspired by [10, Exemple 2.4 and 3.3], where examples of CTO type quadrics for $n = 2, 3$ are given. While that construction yields as degeneration divisor a special configuration of hyperplanes (cf. [10, p. 150, Fig. 2]), our construction relies on a configuration of pairs of hyperplanes given by $h_i = 0$ and quadrics given by $g_{j\epsilon} = 0$. Already for $n = 3$, our construction yields smaller bounds on the total degree of the degeneration divisor. This is important in view of applications such as Theorem 3 and Corollary 4, stated in the introduction, where small bounds on the degrees are desirable.

6.2. **Proof of existence – a key result.** In this section we prove that the above construction yields quadrics of CTO type. To this end, we do not follow the original approach of Colliot-Thélène and Ojanguren. In fact, we do not try to generalize [10, Complément 3.2], because we were unable to see how to divide the argument by the dimension of the center $x \in \mathbb{P}^n$ of the valuation $\nu \in \mathrm{Val}(\mathbb{C}(\mathbb{P}^n)/\mathbb{C})$ for arbitrary n; that strategy had however been used in all previous geometric constructions of quadrics with nontrivial unramified cohomology we are aware of, cf. [10, 31, 17].

Proposition 29. *Let $n \geq 2$ and let $a_i, b_j \in K = \mathbb{C}(\mathbb{P}^n)$ be as in (5) and (7). Suppose that the assumptions (8), (9) and (10) hold. Then any Pfister neighbour of $\langle\langle a_1, \dots, a_{n-1}, b_1 b_2\rangle\rangle$ defines a CTO type quadric over K.*

Proof. By (10), the class $\alpha_j := (a_1, \dots, a_{n-1}, b_j) \in H^n(K, \mu_2^{\otimes n})$ is nonzero. It thus suffices to prove that for each $\nu \in \mathrm{Val}(K/\mathbb{C})$, $\partial_\nu^n \alpha_j = 0$ for $j = 1$ or 2.

To prove this, let $\nu \in \mathrm{Val}(K/\mathbb{C})$. We can choose a normal complex projective variety S together with a proper birational morphism $f : S \longrightarrow \mathbb{P}^n$, such that ν corresponds to a codimension one point $s \in S^{(1)}$. Let $x := f(s) \in \mathbb{P}^n$ be its image on $\mathbb{P}^n$.

By construction, x is a point of codimension at least one on $\mathbb{P}^n$. Hence, there is some i with $x_i(x) \neq 0$. Multiplying the first $n-1$ entries of α_j by x_0^2/x_i^2 and the last entry by $x_0^{2^n}/x_i^{2^n}$ (which does not change the cohomology class α_j), we see that we may without loss of generality assume $x_0(x) \neq 0$. In particular, a_i and b_j are regular functions locally at x and we may from now on work on an affine open subset where $x_0 \neq 0$.

We consider the completion $\widehat{\mathcal{O}_{S,s}}$ and let $\widehat{K} := \mathrm{Frac}(\widehat{\mathcal{O}_{S,s}})$ be its field of fractions. By Lemma 10, the residue $\partial_\nu^n = \partial_B^n$ fits into a commutative diagram

$$\begin{tikzcd} H^n(\widehat{K}, \mu_2^{\otimes n}) \arrow[r, "\partial^n_{\widehat{B}}"] & H^{n-1}(\kappa(s), \mu_2^{\otimes(n-1)}) \\ H^n(K, \mu_2^{\otimes n}) \arrow[u] \arrow[r, "\partial^n_\nu"] & H^{n-1}(\kappa(s), \mu_2^{\otimes(n-1)}). \arrow[u, "\mathrm{id}"'] \end{tikzcd} \tag{14}$$

To prove the proposition, we need to show that $\partial^n_\nu \alpha_j = 0$ for $j = 1$ or 2. We divide the argument into three cases.

Case 1. $g_1(x) \neq 0$.

If $h_i(x) \neq 0$ for all i, then $\partial^n_\nu \alpha_1 = 0$ by Lemma 9. On the other hand, if at least one h_i vanishes at x, then (13) implies that g_1 becomes a nontrivial square in the residue field $\kappa(s)$. By Hensel's lemma, g_1 becomes a square in the completion $\widehat{K}$, and so $\partial^n_\nu \alpha_1 = 0$ by the commutative diagram (14). This concludes Case 1.

Case 2. $g_1(x) = 0$ and $h_i(x) \neq 0$ for all $i = 1, \dots, n-1$.

In this case, we consider α_2. If $g_2(x) \neq 0$, then $\partial^n_\nu \alpha_2 = 0$ by Lemma 9. On the other hand, since $g_1(x) = 0$ by assumptions, $g_2(x) = 0$ implies by (9) that x has codimension at least two in $\mathbb{P}^n$. Moreover, $\partial^n_\nu \alpha_2$ is a multiple of $(a_1, \dots, a_{n-1})$ by Lemma 9, where by slight abuse of notation we do not distinguish between a_i and its image in $\kappa(s)$. But this shows that the residue $\partial^n_\nu \alpha_2$ is a pullback of a class from $H^{n-1}(\kappa(x), \mu_2^{\otimes(n-1)})$ and so it must vanish by Theorem 11 because x has dimension at most $n-2$.

Case 3. $g_1(x) = 0$ and $h_i(x) = 0$ for some $i = 1, \dots, n-1$.

Consider $\alpha_2 = (a_1, \dots, a_{n-1}, b_2)$ and suppose that exactly c-many entries of α_2 vanish at x. Since $g_1(x) = 0$, (8) and (9) imply that x has codimension at least $c+1$. By assumptions $c \geq 1$ and so Lemma 9 shows that we can write

$$\partial^n_\nu \alpha_2 = \beta \cup \gamma,$$

where $\beta \in H^{c-1}(\kappa(s), \mu_2^{\otimes(c-1)})$ and γ is a symbol of degree $n-c$ which is given by (the images of) all entries of α_2 which do not vanish at x. In particular, γ comes from a class of $H^{n-c}(\kappa(x), \mu_2^{\otimes(n-c)})$ and so it vanishes because x is a point of dimension at most $n-c-1$. This concludes Case 3.

Cases 1, 2 and 3 above finish the proof of the proposition. $\square$

Remark 30. *The above proof did not use that the h_i have degree two. In fact, we can start with any collection of homogeneous polynomials $h_i, g_{j0} \in \mathbb{C}[x_0, \dots, x_n]$ of the same even degree $2m$. We may then define g_1 and g_2 as in (6) and put $a_i = h_i/x_0^{2m}$ and $b_j = g_j/x_0^{(2m)^n}$. The proof of Proposition 29 shows then that the a_i and b_j define CTO type quadrics as soon as the assumptions (8), (9) and (10) hold.*

6.3. **Quadric bundles of CTO type and some estimates.** Here we construct and analyse some quadric bundles whose generic fibres are quadrics of CTO type.

We will need a suitable bijection between $\{0,1\}^n$ and $\{0, 1, \dots, 2^n - 1\}$. We start with $I = \{0,1\}^{n-1}$ and define the length of an element $\epsilon \in I$ by $|\epsilon| = \sum \epsilon_i$. We then choose any bijection $\phi' : I \xrightarrow{\sim} \{0, \dots, 2^{n-1} - 1\}$ with $\phi'(\epsilon) \leq \phi'(\epsilon')$ if $|\epsilon| \leq |\epsilon'|$. With this in mind, we define

$$\phi : \{0,1\}^n = I \times \{0,1\} \xrightarrow{\sim} \{0, 1, \dots, 2^n - 1\}, \quad (\epsilon, \epsilon_n) \longmapsto \phi'(\epsilon) + \epsilon_n 2^{n-1}.$$

Definition 31. *Let $n \geq 2$, and let $l_1, \dots, l_{2n+2} \in \mathbb{C}[x_0, x_1, \dots, x_n]$ be linear homogeneous polynomials as in 6.1.2. Equations (6) and (12) then give two homogeneous polynomials g_1 and g_2 of weight 2^n. For $\epsilon \in \{0,1\}^n$, let*

$$c_\epsilon := \left(\prod_{i=1}^{n-1} (l_{2i-1} l_{2i})^{\epsilon_i}\right) (g_1 g_2)^{\epsilon_n}.$$

Let $\phi : \{0,1\}^n \longrightarrow \{0, \dots, 2^n - 1\}$ be the bijection from above. Then we define the following homogeneous polynomials for $i = 0, \dots, 2^n - 1$:

(1) $c_i := c_{\phi^{-1}(i)}$;
(2) $c'_i := l_1 c_i$ if l_1 does not divide c_i and $c'_i := l_1^{-1} c_i$ otherwise;

Moreover, we denote the degrees of the above homogeneous polynomials by $m_i := |c_i|$ and $m'_i := |c'_i|$, respectively.

For later use, we will assume that the bijection ϕ' from above is chosen in such a way that the following holds for $n \geq 3$:

$$c_1 = l_1 l_2, \quad c_2 = l_3 l_4 \quad \text{and} \quad c_n = l_1 l_2 l_3 l_4. \tag{15}$$

In the next definition, we consider the polynomials $\tilde{c}_i$ that are obtained by starting with $l_1 l_3 \cdots l_{2n-3} g_1 c_i$ and absorbing all squares which arise. The homogeneous polynomials $\tilde{c}_i$ obtained this way are balanced, in the sense that $|\tilde{c}_i| = 2^n + n - 1$ for all i. The formal definition is as follows.

Definition 32. *In the notation of Definition 31, and for $\epsilon \in \{0,1\}^n$, let*

$$\tilde{c}_\epsilon := \left(\prod_{i=1}^{n-1} l_{2i-1}^{1-\epsilon_i} l_{2i}^{\epsilon_i}\right) g_1^{1-\epsilon_n} g_2^{\epsilon_n}.$$

Let $\phi : \{0,1\}^n \longrightarrow \{0, \dots, 2^n - 1\}$ be the bijection from above. Then we define the following homogeneous polynomials for $i = 0, \dots, 2^n - 1$:

(1) $\tilde{c}_i := \tilde{c}_{\phi^{-1}(i)}$;
(2) $\tilde{c}'_i := l_1 \tilde{c}_i$ if l_1 does not divide $\tilde{c}_i$ and $\tilde{c}'_i := l_1^{-1} \tilde{c}_i$ otherwise.

Moreover, we denote the degrees of the above homogeneous polynomials by $\tilde{m}_i := |\tilde{c}_i|$ and $\tilde{m}'_i := |\tilde{c}'_i|$, respectively.

With the above definitions, we have the following corollary of Proposition 29.

Corollary 33. *Let $n \geq 2$ and r be integers with $2^{n-1} \leq r + 1 < 2^n$. In the notation of Definitions 31 and 32, the following quadratic forms*

$$\langle c_0, \dots, c_{r+1} \rangle, \quad \langle c'_0, \dots, c'_{r+1} \rangle, \quad \langle \tilde{c}_0, \dots, \tilde{c}_{r+1} \rangle \quad \textit{and} \quad \langle \tilde{c}'_0, \dots, \tilde{c}'_{r+1} \rangle,$$

define hypersurfaces $Y \subset \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \bigoplus_{i=0}^{r+1} \mathcal{O}_{\mathbb{P}^n}(-k_i)$ with

$$k_i = \lfloor m_i/2 \rfloor, \quad k_i = \lfloor m_i'/2 \rfloor, \quad k_i = \lfloor \tilde{m}_i/2 \rfloor \quad \text{and} \quad k_i = \lfloor \tilde{m}_i'/2 \rfloor,$$

respectively, such that the generic fibre of $Y \longrightarrow \mathbb{P}^n$ is a quadric of CTO type. Moreover, the hypersurface Y associated to $\langle c_0, \dots, c_{r+1} \rangle$ is flat over $\mathbb{P}^n$, i.e. it is a quadric bundle.

Proof. The given forms are line bundle valued quadratic forms on $\mathcal{E}$ with values in $\mathcal{O}_{\mathbb{P}^n}$ or $\mathcal{O}_{\mathbb{P}^n}(1)$, depending on whether the entries of the given form have even or odd degrees, cf. Section 3.5. Since the entries of the given quadratic forms are nonzero and have no common factor, they define an integral hypersurface $Y \subset \mathbb{P}_{\mathbb{P}^n}(\mathcal{E})$ whose generic fibre over $\mathbb{P}^n$ is a smooth quadric. This quadric is of CTO type by construction and Proposition 29. The fact that $\langle c_0, \dots, c_{r+1} \rangle$ defines a quadric bundle follows from $c_0 = 1$ and so Y is flat over $\mathbb{P}^n$ in this case. This proves the corollary. $\square$

The following lemma gives some useful estimates for the degrees of the polynomials which appeared in the above corollary.

Lemma 34. *Let $n \geq 2$. In the notation of Definitions 31 and 32, the following holds:*

(1) $m_0 = 0$, $m_1 = 2$ and $m_0' = m_1' = 1$;
(2) $\tilde{m}_i = 2^n + n - 1$ for all i;
(3) $\tilde{m}_i' \leq \tilde{m}_i + 1$ for all i;
(4) $\sum_{i=0}^{r+1} m_i = (r+2)(n+r+1)$ if $r = 2^n - 2$;
(5) $\sum_{i=0}^{r+1} m_i \leq 2(r+2)(n+r)$ for all $2^{n-1} \leq r+1 < 2^n$;
(6) $\sum_{i=0}^{r+1} m_i' \leq 2(r+2)(n+r)$ for all $2^{n-1} \leq r+1 < 2^n$.

Proof. The first three items are clear. Item (4) follows from

$$\sum_{i=0}^{2^n-1} m_i = |(l_1 l_2 \cdots l_{2n-2} g_1 g_2)^{2^{n-1}}| = 2^{n-1}(2^{n+1} + 2n - 2) = 2^n(2^n + n - 1).$$

We next aim to prove (5). If $r = 2^n - 2$, then it follows from (4) and so it suffices to treat the case $r \leq 2^n - 3$. We write $M_s := \sum_{i=0}^{s} m_i$ and set $s_0 := 2^{n-1} - 1$ and $s := s_0 + i$ for some $1 \leq i \leq 2^{n-1} - 1$. Then,

$$\begin{aligned} M_s = |(l_1 \dots l_{2n-2})^{2^{n-1}}| + |c_{s_0+1} \cdots c_{s_0+i}| &\leq 2^{n-1}(2n-2) + i2^{n+1} + i(2n-2) \\ &\leq (s+1)(2n-2) + i2^{n+1}. \end{aligned}$$

Using $i \leq s/2$ and $2^{n-1} \leq s$, we obtain

$$M_s \leq (s+1)(2n-2) + s2^n \leq (s+1)(2n-2) + 2s^2.$$

Setting $s = r + 1$, we get

$$M_{r+1} \leq (r+2)(2n-2) + 2(r+1)^2 \leq (r+2)(2n-2+2r+2) = 2(r+2)(n+r).$$

This proves (5). Item (6) follows via the same argument, which concludes the lemma. ☐

Remark 35. *At least for small values of n, one can work out the integers $m_i, m'_i, \tilde{m}_i$ and $\tilde{m}'_i$ from Definitions 31 and 32 explicitly. For instance, for $n = 2$, we have $\langle c_0, \dots, c_3 \rangle = \langle 1, h_1, g_1g_2, h_1g_1g_2 \rangle$ with $|h_1| = 2$ and $|g_j| = 4$. We thus obtain*

$$(m_0, m_1, m_2, m_3) = (0, 2, 8, 10), \quad (m'_0, m'_1, m'_2, m'_3) = (1, 1, 9, 9)$$

$$(\tilde{m}_0, \tilde{m}_1, \tilde{m}_2, \tilde{m}_3) = (5, 5, 5, 5), \quad (\tilde{m}'_0, \tilde{m}'_1, \tilde{m}'_2, \tilde{m}'_3) = (4, 6, 4, 6).$$

7. Proof of the main results

7.1. **Quadric bundles with nontrivial unramified cohomology.** The following theorem implies Theorem 5 stated in the introduction.

Theorem 36. *Let n and r be positive integers with $r \leq 2^n - 2$. Then there is a unirational r-fold quadric bundle $Y \longrightarrow \mathbb{P}^n$ with $H^k_{nr}(Y, \mu_2^{\otimes k}) \neq 0$, where k is the unique integer with $2^{k-1} - 1 \leq r \leq 2^k - 2$.*

Proof. Since $r+2 \leq 2^k$, we may consider the homogeneous polynomials $c_i \in \mathbb{C}[x_0, \dots, x_k]$ for $i = 0, \dots, r+1$ from Definition 31. Since $c_0 = 1$, the quadratic form $q = \langle c_0, \dots, c_{r+1} \rangle$ defines an r-fold quadric bundle $Y' \longrightarrow \mathbb{P}^k$, whose generic fibre is of CTO type. Since $k \leq n$, the quadratic form q defines also an r-fold quadric bundle $Y \longrightarrow \mathbb{P}^n$ which is stably birational to Y'. Since unramified cohomology is a stable birational invariant,

$$H^k_{nr}(Y, \mu_2^{\otimes k}) \simeq H^k_{nr}(Y', \mu_2^{\otimes k}) \neq 0,$$

by Corollary 33 and Proposition 17. Finally, Y and Y' are unirational by Lemma 14, because $c_0 = 1$ and $c_1 = l_1 l_2$, where $l_1, l_2 \in \mathbb{C}[x_0, x_1, x_2]$ are general linear homogeneous polynomials, see (11) and (15). This proves Theorem 36. ☐

Remark 37. *It follows from* [3, Theorem 3.1] *that the quadric bundle $Y \longrightarrow \mathbb{P}^n$ from Theorem 36 satisfies $H^i_{nr}(Y, \mu_2^{\otimes i}) = 0$ for all $1 \leq i \leq k - 1$.*

7.2. **Specialization theorems without resolutions.** Recall from Section 2.2 what it means that a variety specializes to another variety. The following specialization theorem is a generalization of Theorem 6 stated in the introduction.

Theorem 38. *Let X be a proper variety which specializes to a complex projective variety Y. Suppose that there is a dominant rational map $f : Y \dashrightarrow \mathbb{P}^n$ with the following properties:*

(1) some Zariski open and dense subset $U \subset Y$ admits a universally CH_0-trivial resolution of singularities $\widetilde{U} \longrightarrow U$ such that the induced rational map $\widetilde{U} \dashrightarrow \mathbb{P}^n$ is a morphism whose generic fibre is proper over $K = \mathbb{C}(\mathbb{P}^n)$.

(2) the generic fibre Y_η of f is stably birational to a quadric of CTO type over $\mathbb{C}(\mathbb{P}^n)$.

Then, no resolution of singularities of X admits an integral decomposition of the diagonal. In particular, X is not stably rational.

Proof. To begin with, note that it suffices to prove the theorem after any extension of the base field of X. Since X specializes to a complex variety, we may thus assume that X is defined over an algebraically closed field of characteristic zero.

Taking a suitable blow-up of some projective closure of $\widetilde{U}$, we obtain a proper birational morphism $\tau : \widetilde{Y} \longrightarrow Y$ with $\tau^{-1}(U) = \widetilde{U}$. By assumption (1), $\tau^{-1}(U) \longrightarrow U$ is a universally CH_0-trivial resolution of U. Replacing $\widetilde{Y}$ by a log resolution which does not change $\widetilde{U}$, and which turns the complement $E := \widetilde{Y} \setminus \widetilde{U}$ into a simple normal crossing divisor, we may additionally assume that τ is a resolution of singularities of Y and each irreducible component E_i of E is smooth.

By item (1), $\widetilde{U} \dashrightarrow \mathbb{P}^n$ is a morphism which becomes proper when base changed to some open dense subset of $\mathbb{P}^n$. Therefore, no component E_i of E dominates $\mathbb{P}^n$.

By item (2), Y_η is stably birational to a quadric Q of CTO type over $K = \mathbb{C}(\mathbb{P}^n)$. By Definition 16 and Proposition 17, there are nonzero elements $a_i, b_j \in K^*$ such that $\alpha_1 := (a_1, \dots, a_{n-1}, b_1) \in H^n(K, \mu_2^{\otimes n})$ pulls back to a nontrivial unramified class in $H^n_{nr}(K(Q)/\mathbb{C}, \mu_2^{\otimes n})$. Since unramified cohomology is a stable birational invariant,

$$0 \neq \alpha' := f^*\alpha_1 \in H^n_{nr}(K(Y_\eta)/\mathbb{C}, \mu_2^{\otimes n}).$$

Applying Proposition 27 to the dominant rational map $\widetilde{Y} \dashrightarrow \mathbb{P}^n$, we see that $\alpha'|_{E_i} = 0$ for any irreducible component E_i of E. Therefore, the assumptions of Proposition 25 are satisfied and so no resolution of singularities of X admits an integral decomposition of the diagonal. Since resolutions of singularities exist in characteristic zero, and because stably rational varieties admit integral decompositions of the diagonal (see Section 2.3), it follows that X is not stably rational. This concludes Theorem 38. $\square$

Proof of Theorem 6. Let X be a projective (or proper) variety which specializes to a complex projective variety Y with a morphism $f : Y \longrightarrow S$ to a rational n-fold S, whose generic fibre is smooth and stably birational to a CTO type quadric Q over $K = \mathbb{C}(S)$. We may then consider the smooth locus $U := Y^{\mathrm{sm}}$ of Y and apply Theorem 38 to the universally CH_0-trivial resolution $U \longrightarrow U$, given by the identity. As the generic fibre of f is smooth, the generic fibre of $U \longrightarrow \mathbb{P}^n$ coincides with Y_η and so it is proper. This shows that Theorem 38 applies and so X is not stably rational. This proves Theorem 6. $\square$

Theorem 6 has the following consequence.

Corollary 39. *Let n and r be positive integers with $2^{n-1}-1 \leq r \leq 2^n-2$. Let $e_0, \dots, e_{r+1} \in \mathbb{C}[x_0, \dots, x_n]$ be nonzero homogeneous polynomials without common factor, whose degrees $d_i := |e_i|$ are all odd or even. Suppose that after setting $x_0 = 1$ and possibly multiplying each entry by some nonzero square, the quadratic form $\langle e_0, \dots, e_{r+1} \rangle$ becomes similar to one of the quadratic forms $\langle c_0, \dots, c_{r+1} \rangle$, $\langle c'_0, \dots, c'_{r+1} \rangle$, $\langle \tilde{c}_0, \dots, \tilde{c}_{r+1} \rangle$ or $\langle \tilde{c}'_0, \dots, \tilde{c}'_{r+1} \rangle$ from Corollary 33.*

Then any projective variety which specializes to the hypersurface $Y \subset \mathbb{P}(\mathcal{E})$ given by $\sum_i e_i z_i^2 = 0$, where $\mathcal{E} = \bigoplus_{i=0}^{r+1} \mathcal{O}_{\mathbb{P}^n}(\lfloor d_i/2 \rfloor)$, is not stably rational.

Proof. Our assumption on the e_i guarantees that Y is integral, but note that Y is not necessarily flat over $\mathbb{P}^n$, cf. Section 3.5. Nonetheless, Corollary 33 implies that the generic fibre Y_η of $Y \longrightarrow \mathbb{P}^n$ is a quadric of CTO type. This fact (or the assumption that $e_i \neq 0$ for all i) ensures that Y_η is smooth. The corollary follows therefore from Theorem 6. □

7.3. Proof of Theorem 3 and some applications.

Theorem 40. *Let n and r be positive integers with $2^{n-1}-1 \leq r \leq 2^n-2$, and let $(d_i)_{0 \leq i \leq r+1}$ be a tuple of non-negative integers of the same parity. Consider the non-negative integers m_i, m'_i, $\tilde{m}_i$ and $\tilde{m}'_i$ from Definitions 31 and 32. Suppose that one of the following holds:*

1. *d_0 is even and $d_i \geq m_i$ for all i;*
2. *d_0 is odd and $d_i \geq m'_i$ for all i;*
3. *d_0 has the same parity as $\tilde{m}_0$ and $d_i \geq \tilde{m}_i$ for all i;*
4. *d_0 has the same parity as $\tilde{m}'_0$ and $d_i \geq \tilde{m}'_i$ for all i.*

Then a very general complex r-fold quadric bundle of type $(d_i)_{0 \leq i \leq r+1}$ over $\mathbb{P}^n$ (see Definition 20) is not stably rational.

Proof. Choose a general linear homogeneous polynomial $l \in \mathbb{C}[x_0, \dots, x_n]$, and let $c_i, c'_i, \tilde{c}_i$ and $\tilde{c}'_i$ be as in Definition 31 .

If d_0 is even and $d_i \geq m_i$ for all i, then consider the homogeneous polynomials

$$e_0 := l^{d_0 - m_0} \cdot c_0 \quad \text{and} \quad e_i := x_0^{d_i - m_i} \cdot c_i \quad \text{for } i = 1, \dots, r+1.$$

Since l is general and the d_i and m_i are even, Corollary 39 applies and shows that a very general quadric bundle X over $\mathbb{P}^n$ of type $(d_i)_{0 \leq i \leq r+1}$ is not stably rational.

If d_0 is odd and $d_i \geq m'_i$ for all i, then replace c_i and m_i by c'_i and m'_i, respectively. Since m'_i is odd for all i, we may then argue as before. If d_0 has the same parity as $\tilde{m}_i$ and $d_i \geq \tilde{m}_i$ for all i, then replace c_i and m_i by $\tilde{c}_i$ and $\tilde{m}_i$, respectively, and argue as before. If d_0 has the same parity as $\tilde{m}'_i$ and $d_i \geq \tilde{m}'_i$ for all i, then replace c_i and m_i by $\tilde{c}'_i$ and $\tilde{m}'_i$, respectively, and argue as before. This finishes the proof of the theorem. □

Proof of Theorem 3. By Lemma 34, $\tilde{m}_i = 2^n + n - 1$ and $\tilde{m}'_i \leq 2^n + n$ for all i, and so Theorem 3 follows from Theorem 40. $\square$

Remark 41. *If $r \geq 2$, then all examples in Theorems 3 and 40 have rational deformation type by Corollary 21. If $(d_0, d_1) = (m_0, m_1)$ or $(d_0, d_1) = (m'_0, m'_1)$, then the examples in Theorem 40 are unirational by Lemmas 14 and 34. Unirationality of the examples in Theorem 3 is unknown.*

Proof of Corollary 4. The corollary follows from Lemma 22 and Theorem 3. $\square$

Corollary 42. *Let n and r be positive integers with $2^{n-1} - 1 \leq r \leq 2^n - 2$. Then a very general complex hypersurface $X \subset \mathbb{P}^n \times \mathbb{P}^{r+1}$ of bidegree $(d, 2)$ with $d \geq 2^n + n - 1$ is not stably rational.*

Proof. A very general hypersurface of bidegree $(d, 2)$ in $\mathbb{P}^n \times \mathbb{P}^{r+1}$ is nothing but a very general r-fold quadric bundle of type $(d, \ldots, d)$ over $\mathbb{P}^n$. The corollary follows therefore from Theorem 3. $\square$

Corollary 43. *Let n, r be integers with $n \geq 2$ and $2^{n-1} - 1 \leq r \leq 2^n - 2$ and put $N := r + n$. Then a double cover of $\mathbb{P}^N$, branched along a very general complex hypersurface $Y \subset \mathbb{P}^N$ of even degree $d \geq 2^{n+1} + 2n - 2$ and with multiplicity $d - 2$ along an $(r-1)$-plane is not stably rational.*

Proof. By Lemma 23, we need to prove that a very general r-fold quadric bundle of type $(0, d-2, \ldots, d-2, d)$ is not stably rational if $d \geq 2^{n+1} + 2n - 2$ is even. This follows from item (1) in Theorem 40, because $m_0 = 0$ and $m_i \leq |l_1 l_2 \ldots l_{2n-2} g_1 g_2| = 2^{n+1} + 2n - 2$ for all i. $\square$

7.4. **Proof of Theorem 1.**

Theorem 44. *Let n and r be positive integers with $r \leq 2^n - 2$. Then there is a smooth unirational r-fold quadric bundle X over $S = \mathbb{P}^{n-k} \times \mathbb{P}^k$, where k is the unique integer with $2^{k-1} - 1 \leq r \leq 2^k - 2$, such that X is not stably rational.*

Proof. By Theorem 40 and Remark 41, there are many smooth unirational complex r-fold quadric bundles $Y \longrightarrow \mathbb{P}^k$ which are not stably rational. The product $X := Y \times \mathbb{P}^{n-k}$ is then a smooth unirational r-fold quadric bundle over S which is not stably rational. This proves the theorem. $\square$

Proof of Theorem 1. By a theorem of Lang [26], $K = \mathbb{C}(\mathbb{P}^n)$ is a C_n-field, cf. [33, II.4.5]. It follows that any r-fold quadric bundle over a rational base of dimension n with $r > 2^n - 2$ has a rational section and so it is rational. We thus conclude via Theorem 44 that smooth stably non-rational r-fold quadric bundles over rational bases of dimension n exist if and only if $r \leq 2^n - 2$. This proves Theorem 1. $\square$

Remark 45. *In the proof of Theorems 1 and 44, it is essential that Theorem 40 yields smooth r-fold quadric bundles over rational bases which are not stably rational, non-rationality would not be enough.*

Remark 46. *The cases $r = 1, 2$ in Theorems 1 and 44 follow from* [41] *and* [17]*, respectively. If one allows singular bundles, the results follow from* [2, 10] *if $r \leq 6$. However, even without the smoothness assumption, the results are new for any $r \geq 7$ (and follow in that case also from Theorem 36 above).*

7.5. **Proof of Theorem 2.**

Theorem 47. *Let n, r and d be integers, with d even if r is even, and such that $n \geq 2$, $2^{n-1} - 1 \leq r \leq 2^n - 2$ and $d \geq 2(n + r)(r + 2)$.*

There is a smooth complex projective family $\pi : \mathcal{X} \longrightarrow B$ over a complex variety B, such that each fibre $X_b = \pi^{-1}(b)$ is a smooth r-fold quadric bundle over $\mathbb{P}^n$, degenerated over a hypersurface of degree d in $\mathbb{P}^n$, satisfying the following:

(1) for very general $t \in B$, the r-fold quadric bundle X_t over $\mathbb{P}^n$ is not stably rational;
(2) all fibres of π are unirational and, if $r \geq 2$, then some fibres are rational;
(3) if $r \geq 3$ and d is even, the set $\{b \in B \mid X_b \text{ is rational}\}$ is dense in B for the analytic topology.

Proof. We first define some non-negative integers d_i for $i = 0, \dots, r + 1$ of the same parity and use the notation from Definition 31. If d is even, we put $d_i := m_i$ for $i = 0, \dots, r$ and $d_{r+1} := d - \sum_{i=0}^{r} m_i$. If d is odd, we define $d_i := m'_i$ for $i = 0, \dots, r$ and $d_{r+1} := d - \sum_{i=0}^{r} m'_i$. By Lemma 34, $d_{r+1} \geq m_{r+1}$ and $(d_0, d_1) = (0, 2)$ if d is even, and $d_{r+1} \geq m'_{r+1}$ and $(d_0, d_1) = (1, 1)$ if d is odd.

Let $\mathcal{E}^\vee := \bigoplus_{i=0}^{r+1} \mathcal{O}(\lfloor d_i/2 \rfloor)$ and consider $V' := H^0(\mathbb{P}^n, \mathrm{Sym}^2(\mathcal{E}^\vee) \otimes \mathcal{O}_{\mathbb{P}^n}(d_0))$. We identify points in V' with symmetric matrices $A = (a_{ij})_{0 \leq i,j \leq r+1}$. To such a matrix, we associate the minor $M(A) := (a_{ij})_{i,j \in \{0,1,2,n\}}$, which is a symmetric 4×4 matrix. We define $V \subset V'$ as the linear subspace given by all symmetric matrices $A = (a_{ij})$ with $a_{ij} \in \mathbb{C}[x_0, x_1, x_2]$ for all $i, j \in \{0, 1, 2, n\}$, and such that $a_{i0} = 0$ for $i = 1, 2, n$ if d is even and $n \geq 3$. We let $B \subset \mathbb{P}(V)$ be the subset of points $[A] \in \mathbb{P}(V)$ such that A defines a smooth r-fold quadric bundle of type $(d_i)_{0 \leq i \leq r+1}$ over $\mathbb{P}^n$; if $n \geq 3$, then we also assume that $M(A)$ defines a smooth quadric surface bundle of type (d_0, d_1, d_2, d_n) over $\mathbb{P}^2$. By Bertini's theorem, B is an open dense subset of $\mathbb{P}(V)$. There is a universal hypersurface $\mathcal{X} \subset B \times \mathbb{P}(\mathcal{E})$. Projection to the first factor gives a smooth projective morphism $\pi : \mathcal{X} \longrightarrow B$ of complex varieties. The fibre X_b above $b \in B$ is a smooth r-fold quadric bundle over $\mathbb{P}^n$, which degenerates over a hypersurface of degree d in $\mathbb{P}^n$. Let $t \in B$ be very general. Since $c_0, c_1, c_2, c_n, c'_0, c'_1, c'_2, c'_n \in \mathbb{C}[x_0, x_1, x_2]$ by (11) and (15), X_t specializes by Lemma 8 to the hypersurface $Y \subset \mathbb{P}(\mathcal{E})$, given by $\sum_{i=0}^{r+1} e_i z_i^2 = 0$, where $e_i = c_i$ (resp.

$e_i = c_i'$) for $i = 0, \ldots, r$ and $e_{r+1} = x_0^{d_{r+1}-m_{r+1}} c_{r+1}$ (resp. $e_{r+1} = x_0^{d_{r+1}-m'_{r+1}} c'_{r+1}$), if d is even (resp. odd), and where we use the notation from Definition 31. It thus follows from Corollary 39 that X_t is not stably rational. This proves item (1).

Recall $(d_0, d_1) \in \{(1,1), (0,2)\}$. Up to replacing B by some open dense subset which is given by a certain genericity assumption on $(a_{ij})_{0 \leq i,j \leq 1}$, Lemma 14 thus ensures that all fibres of π are unirational. If $r \geq 2$, then our assumptions imply $r \geq n$. As in Lemma 19, Bertini's theorem shows then that we may additionally assume that B contains points which correspond to matrices $A = (a_{ij})$ with $a_{r+1,r+1} = 0$. The corresponding quadric bundles admit sections and so they are rational. This proves item (2).

Let us now assume that $r \geq 3$ and d is even. Then, $n \geq 3$ and $(d_0, d_1, d_2, d_n) = (0, 2, 2, 4)$ (see (15)) and we consider the vector space W of symmetric 4×4 matrices from Proposition 24. There is a dominant morphism

$$M : B \longrightarrow \mathbb{P}(W), \quad [A] \longmapsto [M(A)].$$

If the quadric surface bundle over $\mathbb{P}^2$ which is defined by $M(A)$ admits a rational section, then the r-fold quadric bundle over $\mathbb{P}^n$ defined by A admits a rational section as well. By Proposition 24, the set of points $[M(A)] \in \mathbb{P}(W)$ with that property is dense for the analytic topology. This proves item (3), i.e. $\{b \in B \mid X_b \text{ is rational}\}$ is dense in B for the analytic topology. This concludes Theorem 47. □

Proof of Theorem 2. The case $r = 1, 2$ follows from [17], because the examples treated there (hypersurfaces of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$) are both, quadric surface bundles over $\mathbb{P}^2$, as well as conic bundles over $\mathbb{P}^3$. The case $r \geq 3$ follows from Theorem 47. □

Remark 48. *The restriction on the parity of d if r is even is necessary in Theorem 47. Indeed, the Fano variety of m-planes on a smooth quadric of dimension $2m$ has two connected components, and so any smooth $2m$-fold quadric bundle over $\mathbb{P}^n$ gives rise to a double cover of $\mathbb{P}^n$, branched along the degeneration divisor. This forces the degree of the degeneration divisor to be even.*

Remark 49. *It is conceivable that item (3) in Theorem 47 holds for $r \geq 2$ and without the restriction on the parity of d. To prove this, it would be enough to generalize Proposition 24 to other quadric surface bundles of type $(d_0, \ldots, d_3)$ over $\mathbb{P}^2$. Even though this problem seems tractable with the existing methods, we do not try to pursue this here.*

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Mathematical Institute of the University of Bonn, Endenicher Allee 60, D-53115 Bonn, Germany.

E-mail address: `schreied@math.uni-bonn.de`