Dr. Stefan Schreieder Sommersemester 2017

S4A1 - Graduate Seminar on Algebraic Geometry: Complex Geometry

The seminar provides an introduction to complex geometry. We introduce complex manifolds and study Kähler metrics on them. Complex manifolds with Kähler metrics are called Kähler manifolds; an important class of examples is provided by smooth complex projective varieties, that is, submanifolds of complex projective space that are given by the zero set of polynomial equations. We will see how Hodge theory provides a particularly useful tool to study this class of manifolds. For instance, this can be used to prove many topological constraints a Kähler manifold must satisfy. In particular, we will see that many topological spaces cannot be realized as a smooth complex projective variety.

The seminar takes place during the summer term 2017 on Thursdays, 16 (c.t.) -18, in seminar room 1.008 (Endenicher Allee 60). Two weeks before his talk, each participant should briefly discuss his topic with me. Please contact schreied@math.uni-bonn.de to make an appointment.

Prerequisites: Basic knowledge of differential geometry (differentiable manifold, real and complex vector bundles, differential forms, etc.) and holomorphic function theory in one variable (definition of a holomorphic function and basic properties, such as the maximum principle) are assumed. Some knowledge of sheaf theory is also required, but, depending on the audience, parts of this material will be briefly sketched/recalled during the seminar. Knowledge of algebraic geometry is helpful but not required.

Our program follows mostly [5] and [6]; alternative sources are [1, 2, 3, 4]. If not mentioned otherwise, all references in the following list of talks go to [5].

1. **Complex manifolds (J. Gruner, 20.4.2017)**

   Assigned reading: Sections 1.1 and 2.1

   Talk: Define complex manifolds, submanifolds, sheaf of holomorphic functions and holomorphic maps. At this point you could briefly recall the definition of a sheaf. Prove Proposition 2.1.5 and discuss Exercise 2.1.2. Give some examples, e.g. projective space, smooth projective hypersurfaces, elliptic curves, complex tori and Hopf manifolds. Define complex projective manifolds. Discuss Exercise 2.1.4.

2. **Holomorphic vector bundles and a review of sheaf cohomology (P. Magni, 27.4.2017)**

   Assigned reading: [6, Section 4] and Section 2.2

   Talk: Recall briefly the definition of sheaf cohomology, using for instance flasque resolutions as in Appendix B. The most important property is Proposition B.0.35.

   Recall briefly the concept of Čech cohomology. State Proposition B.043 and mention that frequently, Čech cohomology coincides with sheaf cohomology in all degrees.

   Define holomorphic vector bundles and mention some natural operations on them: e.g. direct sum, tensor product, duals and pullbacks.

   Define the Picard group and prove Corollary 2.2.10. Introduce the exponential sequence and use it to describe the Picard group; define in particular the first Chern class of a line bundle.

3. **The holomorphic tangent bundle and the example of projective space (U. Meha, 27.4.2017)**

   Assigned reading: Section 2.2 and 2.4

   Talk: Define the holomorphic tangent bundle of a complex manifold. Define the bundle of holomorphic p-forms and define the canonical bundle. Prove Lemma 2.2.15 and Proposition 2.2.17. Compute the tangent bundle of a complex torus.

   Define the sheaf of holomorphic sections of a holomorphic vector bundle and prove Proposition 2.2.19. Define the cohomology groups of holomorphic vector bundles and define the Hodge numbers of a compact complex manifold.

   Explain Proposition 2.2.6 and use it to define $\mathcal{O}(k)$ on $\mathbb{P}^n$. Define the natural map $\mathbb{C}[z_0, \ldots, z_n]_k \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}(k))$ and prove Proposition 2.4.1. Explain all maps in the Euler sequence (2.4.4) and sketch its proof if time permits. Deduce $K_{\mathbb{P}^n} = \mathcal{O}(-n-1)$.

4. **Differential forms on complex manifolds (T. Beckmann und T. Bülles, 4.5.2017)**

   Assigned reading: pp. 25–28; Sections 1.3 and 2.6; [6, Section 4].

   Talk: Define almost complex manifolds (2.6.1) and note that this is equivalent to a complex vector bundle structure on the real tangent bundle T_X. Prove 2.6.2 and 2.6.4. Explain 2.6.8 and prove 2.6.11. Define the Dolbeault complex and Dolbeault cohomology groups (2.6.20).

   Recall that sheaf cohomology can be computed with any Γ-acyclic resolution, see for instance [6, Proposition 4.32]. An important example of Γ-acyclic sheaves are modules over the sheaf of differentiable functions: indeed, such a sheaf is soft, hence acyclic (B.0.39). As an application, use the Poincaré Lemma (see for instance Proposition A.0.3) to prove the de Rham theorem [6, Theorem 4.1]. Use the same line of argument to prove 2.6.21.

   Sketch the proof that for locally contractible spaces, sheaf cohomology with constant coefficients is isomorphic to the corresponding singular cohomology groups, see [6, Theorem 4.47].

5. **Kähler manifolds (P.R. Hoefgeest, 11.5.2017)**

   Assigned reading: pp. 28–29, pp. 48–49, pp. 116–120.

   Talk: Explain the set-up of Proposition 1.3.12 and sketch the proof of this proposition. Define hermitian structures (3.1.1) and explain that any complex manifold admits such a structure (in fact many) because of partitions of unity, see Exercise 3.1.1.

   Define Kähler metrics (3.1.6) and prove Corollary 3.1.8. Explain Examples 3.1.9 (i) and (ii). Prove 3.1.10 and 3.1.11.

6. **Hermitian linear algebra (X.L. Flamm, 18.5.2017)**

   Assigned reading: Section 1.2.

   Talk: In this talk we discuss some linear algebra results on a given Hermitian vector space; you should think of this vector space as the tangent space of a Kähler manifold X at a given point x. In order to explain this point of view, you should recall the definition of a Kähler manifold, i.e. a complex manifold with Kähler metric, and explain how this structure gives point-wise rise to the situation discussed in the following.

   Introduce the Lefschetz operator (1.2.18) and its dual (1.2.21). Prove that they define an $\mathfrak{sl}_2$-representation on $\Lambda^* V^*$ (1.2.26) and the Lefschetz decomposition theorem (1.2.30).

   Discuss Exercises 1.2.9 and 3.1.8, and conclude that the n-sphere S^n admits a Kähler structure if and only if $n = 2$.

7. **Kähler identities (P. Reichenbach, 1.6.2017)**

   Assigned reading: Section 3.1.

   Talk: Introduce the operators occurring in the Kähler identities (3.1.12) and prove the identities. Explain Remark 3.1.14.

8. **Hodge decomposition (N. Tsakanikas, 22.6.2017)**

   Assigned reading: Section 3.2

   Talk: Define the various spaces of harmonic forms and prove 3.2.6. State 3.2.8 and prove 3.2.9 and 3.2.12.

   Use Remark 3.2.7 to deduce the Hodge symmetries $H^{p,q}(X) = H^{q,p}(X) = H^{n-p,n-q}(X)$, where X is compact Kähler of complex dimension n. Explain the diagram on p. 138.

   Deduce that odd Betti numbers of Kähler manifolds are even: $b_{2k+1}(X) \equiv 0 \mod 2$, see Exercise 3.2.6. Use this to give an example of a complex manifold which is not Kähler, hence not projective.

9. **Lefschetz Theorems (L. Hendrian, 29.6.2017)**

   Assigned reading: Section 3.3.

   Talk: Prove the Lefschetz (1,1)-theorem (3.3.1 and 3.3.2). Define the Neron–Severi group (Remark 3.3.3); convince yourself that there is a complex torus $X = \mathbb{C}^2/\Gamma$ with $\mathrm{NS}(X) = 0$ and explain that such a torus is not projective, see Exercise 3.3.6.

   Prove 3.3.10, define primitive cohomology and prove the Hard Lefschetz theorem (3.3.13). If time permits, use the Hodge diamond (see p. 138) to illustrate the inequalities on the Hodge numbers which follow from Hard Lefschetz.

10. **Hodge–Riemann bilinear relations (P. Koulakidou, 6.7.2017)**

    Assigned reading: Section 3.3.

    Talk: Prove the Hodge–Riemann bilinear relations (3.3.15) and deduce the Hodge index theorem (3.3.16) and the signature theorem (3.3.18). (Note that the signature is an invariant of the underlying smooth manifold which is independent of the complex structure, whereas the individual Hodge numbers may very well depend on the complex structure.) Prove Exercise 3.3.3.

    If time permits, you could also discuss Exercise 3.3.7, which is a nice application of the Lefschetz (1,1) theorem from last talk. (Hint: Use a suitable version of the exponential sequence to compute the group $H^1(X, \mathcal{C}^*_{X,\mathbb{C}})$ of isomorphism classes of complex line bundles on X.)

11. **Connections and Curvature (G. Vardosanidze, 13.7.2017)**

    Assigned reading: Section 4.2 and 4.3.

    Talk: Define a connection on a complex vector bundle E (4.2.1), sketch Proposition 4.2.3 and explain Remark 4.2.5. Explain how additional structure, i.e. a hermitian metric on E or a complex structure on M, give rise to the notion of an hermitian connection (4.2.9) and a connection compatible with the holomorphic structure (4.2.12). Introduce the Chern connection (4.2.14); explain Example 4.2.16 (i) and maybe also Example 4.2.16 (ii).

    Introduce the curvature of a connection (4.3.1). Prove Lemma 4.3.2 and conclude that $F_\nabla \in \mathcal{A}^2(M, \mathrm{End}(E))$ is an $\mathrm{End}(E)$-valued differentiable 2-form on M. Prove the Bianchi identity (4.3.5).

    If time permits, introduce the Atiyah class of a holomorphic vector bundle (4.2.18). Explain Proposition 4.2.19 and emphasize the difference between a holomorphic connection

(4.3.7) and a usual connection. Explain the relationship between the curvature of the Chern connection and the Atiyah class (4.3.10).

12. **Chern–Weil theory and Hirzebruch–Riemann–Roch (J. Lourenço, 20.7.2017)**

    Assigned reading: Sections 4.4 and 5.1.

    Talk: The idea of Chern–Weil theory is as follows: If E is a complex vector bundle on a differentiable manifold M, then we can choose a connection ∇ on E (partition of unity!). We have seen in the last talk that this gives rise to the curvature form $F_\nabla \in \mathcal{A}^2(M, \mathrm{End}(E))$, which satisfies the Bianchi identity (4.3.5). The idea is then to apply pointwise certain linear algebra constructions to F_∇ to cook up for each k a $2k$-form on M; as a consequence of the Bianchi identity, these forms will be closed and so we obtain cohomology classes $c_k(E) \in H^{2k}(M, \mathbb{C})$ on M which turn out to be important topological invariants of the complex vector bundle E.

    If you want, you can start your talk with a brief sketch of the above idea. Introduce then k-multilinear symmetric maps on a vector space and define what it means that such a map is invariant (4.4.1). Prove Lemma 4.4.4 and Corollary 4.4.5. Mention Lemma 4.4.6 and Remark 4.4.7. Use this to introduce Chern classes, Chern characters and Todd classes (4.4.8). These classes are defined for arbitrary complex vector bundles; if we talk about Chern classes of complex manifolds, then we mean the Chern class of its tangent bundle, see 4.4.10. These classes are topological invariants of the complex vector bundle which underlies the holomorphic tangent bundle; they are in general not topological invariants of the complex manifold itself. Discuss Example 4.4.11. Mention Proposition 4.4.12 and sketch its proof if time permits.

    Explain the statement of the Hirzebruch–Riemann–Roch theorem (5.1.1) and the examples on p. 233.

13. **Kodaira Vanishing and Weak Lefschetz (S. Düzlü, 27.7.2017)**

    Assigned reading: Section 5.2.

    Talk: Prove Kodaira Vanishing, leaving out the proof of Lemma 5.2.3. Discuss Example 5.2.5. Prove the Weak Lefschetz theorem.

    You should explain some applications of the Weak Lefschetz theorem. For instance, you could discuss Exercise 5.2.6. You could also explain how to compute the Betti numbers $b_i(X) := \dim(H^i(X, \mathbb{C}))$ of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d and illustrate this in one or two explicit examples. (Hint: the computation of $b_i(X)$ for $i \neq n$ follows directly from the Weak Lefschetz theorem and Poincaré duality. In order to compute the middle degree Betti number $b_n(X)$, it then suffices to compute the Euler number, which is nothing but the top degree Chern number $c_n(X) = \sum_i (-1)^i b_i(X)$. This last number is computed in Example 4.4.11.)

Literatur

[1] W. Ballmann, *Lectures on Kähler manifolds*, ESI Lectures in Mathematics and Physics, EMS, 2006.

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[5] D. Huybrechts, *Complex Geometry*, Springer, Berlin, 2005.

[6] C. Voisin, *Hodge Theory and complex algebraic geometry I*, Cambridge University Press, Cambridge, 2002.