Seminar on Complex Geometry

The seminar provides an introduction to complex geometry as presented in [5]. We introduce complex manifolds and study Kähler metrics on them. Complex manifolds with Kähler metrics are called Kähler manifolds; an important class of examples of Kähler manifolds is provided by smooth complex projective varieties, that is, zero sets of polynomials in projective space. We will see how Hodge theory leads to surprisingly strong consequences for the topology and the cohomology of Kähler manifolds. In particular, we will see that many topological manifolds can not be equipped with the structure of a complex projective variety. The seminar ends with the Kodaira embedding theorem, which characterizes those Kähler manifolds which are isomorphic to a smooth complex projective variety.

The seminar takes place during the summer term 2015 on Thursdays, 16-18, in Seminar room 0.007. If you are interested in participating, please send an email listing two or three talks you would be willing to give either to huybrech@math.uni-bonn.de or to schreied@math.uni-bonn.de. We will assign the first talks (one talk per participant) by early March. Two weeks before his talk, each participant should briefly discuss his topic with me (Stefan Schreieder). Please contact schreied@math.uni-bonn.de to make an appointment.

Prerequisites: We assume basic concepts of differential geometry, such as differentiable manifolds, differential forms and de Rham cohomology. Some of this material is collected in [5, Appendix A]. If you are not yet familiar with sheaf cohomology, then you should have a look at [5, Appendix B]. You should also know the theory of holomorphic functions in one variable and have a look at Section 1.1 of [5] where some principal facts about the local theory of holomorphic functions in several variables is presented.

Our program follows [5]; other sources are [1, 2] and more advanced references which go deeper into the subject are [3, 4, 6]. All references in the following list of talks go to [5].

1. **Complex manifolds and holomorphic vector bundles (M. Parucha, 9.4.)**

   Assigned reading: Sections 2.1 and 2.2.

   Talk: Define complex manifolds, sheaf of holomorphic functions and holomorphic maps. Give some examples, e.g. projective space, smooth projective hypersurfaces and complex tori. Define holomorphic vector bundles and line bundles, discuss the exponential sequence and first Chern classes. Discuss the normal bundle sequence and the adjunction formula.

2. **Differential forms on complex manifolds (B. Fluhr, 16.4.)**

   Assigned reading: pp. 25–28. Sections 1.3 and 2.6.

   Talk: Define almost complex manifolds. Prove 2.6.2 and 2.6.4. Explain 2.6.8 and prove 2.6.11. Define the Dolbeault complex and Dolbeault cohomology groups. Prove 2.6.21.

3. **Kähler manifolds (M. Talebi, 23.4.)**

   Assigned reading: pp. 28–29, pp. 48–49, pp. 116–120.

   Talk: Prove 1.3.12. Define Kähler manifolds and prove 3.1.8. Explain Examples 3.1.9 (i) and (ii). Prove 3.1.10 and 3.1.11.

4. **Hermitian linear algebra (F. Zickenheiner, 30.4.)**

   Assigned reading: Section 1.2.

   Talk: Recall the definition of the Lefschetz operator (1.2.18) and its dual (1.2.21). Prove that they define an $\mathfrak{sl}_2$-representation on $\Lambda^* V^*$ (1.2.26) and the Lefschetz decomposition theorem (1.2.30).

5. **Kähler identities (S. Floccari, 7.5.)**

   Assigned reading: Section 3.1.

   Talk: Give an overview over the operators occurring in the Kähler identities (3.1.12) and prove the identities.

6. **Hodge decomposition (J. Morrissey, 21.5.)**

   Assigned reading: Section 3.2.

   Talk: Define the various spaces of harmonic forms and prove 3.2.6. State 3.2.8 and prove 3.2.9 and 3.2.12. Use Remark 3.2.7 to deduce the corresponding symmetries for the cohomology groups and explain the diagram on p. 138.

7. **Lefschetz theorems (E. Brakkee, 11.6.)**

   Assigned reading: Section 3.3.

   Talk: Prove the Lefschetz (1,1)-theorem (3.3.1 and 3.3.2). Prove 3.3.10, define primitive cohomology and prove the Hard Lefschetz theorem (3.3.13).

8. **Hodge–Riemann bilinear relations (L. Kühne, 18.6.)**

   Assigned reading: Section 3.3.

   Talk: Prove the Hodge–Riemann bilinear relations (3.3.15) and deduce the Hodge index theorem (3.3.16) and the signature theorem (3.3.18). This talk is slightly shorter and should be finished within one hour.

9. **Connections and Curvature (C. Roschinski, 25.6.)**

   Assigned reading: Section 4.2 and 4.3.

   Talk: Introduce the Chern connection (4.2.14). Discuss Example 4.2.16 (ii). Explain the difference to a holomorphic connection and prove 4.2.19. Explain the relationship between the curvature of the Chern connection and the Atiyah class (4.3.10). Explain Example 4.3.12. This talk is slightly longer and should start already in the previous session.

10. **Chern–Weil theory and Hirzebruch–Riemann–Roch (D. Valloni, 2.7.)**

    Assigned reading: Sections 4.4 and 5.1.

    Talk: Introduce Chern classes, Chern characters and Todd classes (4.4.8). Discuss Example 4.4.11 and prove Exercise 4.4.7. Explain the statement of the Hirzebruch–Riemann–Roch theorem and the examples on p. 233.

11. **Kodaira Vanishing and Weak Lefschetz (T. Q. T. Nguyen, 9.7.)**

    Assigned reading: Section 5.2.

    Talk: Prove Kodaira Vanishing, leaving out the proof of Lemma 5.2.3. Discuss Example 5.2.5. Prove the Weak Lefschetz theorem.

12. **Kodaira Embedding (??)**

    Assigned reading: Sections 2.5 and 5.3.

    Talk: Explain how a complete linear system induces a closed embedding if and only if it separates points and tangent vectors. Prove the Kodaira embedding theorem.

References

[1] W. Ballmann, *Lectures on Kähler manifolds*, ESI Lectures in Mathematics and Physics, EMS, 2006.

[2] M. A. de Cataldo, *The Hodge theory of projective manifolds*, Imperial College Press, 2007.

[3] J.-P. Demailly, *Complex differential geometry*, http://www-fourier.ujf-grenoble.fr/ demailly/manuscripts/agbook.pdf

[4] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & sons, New York, 1978.

[5] D. Huybrechts, *Complex Geometry*, Springer, Berlin, 2005.

[6] C. Voisin, *Hodge Theory and complex algebraic geometry I*, Cambridge University Press, Cambridge, 2002.