

# Integral $p$ -adic Hodge theory – announcement

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## 1 INTRODUCTION

Let  $C$  be an algebraically closed, nonarchimedean field of mixed characteristic which is complete under a rank one valuation; let  $\mathcal{O} \subseteq C$  be its ring of integers, with maximal ideal  $\mathfrak{m}$  and residue field  $k$ . The reader may assume that  $C$  is the completion of an algebraic closure of  $\mathbb{Q}_p$ .

The main aim of this note is to outline a proof of the following result, which was first announced by the third author during his series of Fall 2014 lectures at the MSRI. Details, generalisations, and further results will be presented in a forthcoming article. In particular, this will include a “comparison isomorphism”-style result.

**Theorem 1.1.** *Let  $\mathfrak{X}$  be a proper, smooth, formal scheme over  $\mathcal{O}$ , and let  $i \geq 0$ . Assume that the crystalline cohomology  $H_{\text{crys}}^i(\mathfrak{X}_k/W(k))$  of its special fibre  $\mathfrak{X}_k$  is torsion-free. Then the  $p$ -adic étale cohomology  $H_{\text{ét}}^i(X, \mathbb{Z}_p)$  of its generic fibre  $X := \mathfrak{X}_C$  is also torsion-free.*

There are previous results of a similar flavour, at least in the case where  $\mathfrak{X}$  is the  $p$ -adic completion of the base-change to  $\mathcal{O}$  of a proper smooth scheme over a finite extension  $V$  of  $W(k)$ , of ramification index  $e$ . Specifically, under stronger assumptions on torsion-freeness (for Hodge cohomology, and assuming degeneration of the mod  $p$  Hodge-de Rham spectral sequence), G. Faltings proves the result if  $\dim X < p - 1$ , cf. [6, Theorem 6]. On the other hand, X. Caruso [3] proves that if  $ie < p - 1$ , then the torsion in crystalline and in étale cohomology agree, which implies our theorem in that case. In the last section, we give an example showing that there may be torsion in crystalline cohomology while the étale cohomology is torsion-free; thus one cannot hope to extend Caruso’s result to the general case.

As an example of our theorem which falls outside the scope of applicability of previous results, one may take for  $\mathfrak{X}$  an Enriques surface over a 2-adic base. In that case, there is 2-torsion in  $H_{\text{ét}}^2(X, \mathbb{Z}_2)$ , so that our theorem implies the existence of 2-torsion in  $H_{\text{crys}}^2(\mathfrak{X}_k/W(k))$ , and thus non-vanishing of  $H_{\text{dR}}^1(\mathfrak{X}_k/k)$ , which is a known pathology of Enriques surfaces in characteristic 2, cf. [10, Proposition 7.3.8].

We note that the theorem may have applications to the study of torsion in the étale cohomology of Shimura varieties. For example, K.-W. Lan and J. Suh [11] prove first a vanishing result for torsion in crystalline cohomology, and then deduce the same for étale cohomology using existing results in integral  $p$ -adic Hodge theory.

## 2 STRATEGY OF THE PROOF

Let  $\mathcal{O}^b := \varprojlim_{\mathfrak{p}} \mathcal{O}/\mathfrak{p}$  be the tilt (in the terminology of [13, §3]; classically  $\mathcal{O}^b$  was denoted by  $\mathcal{R}_{\mathcal{O}}$  [8, 1.2.2]) of  $\mathcal{O}$ , whose field of fractions  $C^b$  is a perfectoid field of characteristic  $p$ ,

and let  $\mathbb{A}_{\text{inf}} := W(\mathcal{O}^b)$  be the first of Fontaine's period rings, on which we denote by  $\varphi$  the usual Witt vector Frobenius. We will require the elements  $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}^b$  and  $\mu := [\varepsilon] - 1 \in \mathbb{A}_{\text{inf}}$ , where  $\zeta_p, \zeta_{p^2}, \dots \in \mathcal{O}$  is a chosen compatible sequence of  $p$ -power roots of unity and  $[\cdot]$  denotes the Teichmüller lifting. The results of our constructions will be independent of this chosen sequence of  $p$ -power roots of unity.

The theorem is proved by constructing a Zariski sheaf  $\mathbb{A}\Omega_{\mathfrak{X}/\mathcal{O}}$  of complexes of  $\mathbb{A}_{\text{inf}}$ -modules on any smooth, formal  $\mathcal{O}$ -scheme  $\mathfrak{X}$ , whose cohomology, when  $\mathfrak{X}$  is proper, interpolates the  $p$ -adic étale and crystalline cohomologies of the generic and special fibres of  $\mathfrak{X}$  respectively. More precisely, it satisfies the following local properties in general:

- (A1)  $\mathbb{A}\Omega_{\mathfrak{X}/\mathcal{O}} \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} \mathbb{A}_{\text{inf}}[\frac{1}{\mu}] \simeq R\nu_* \mathbb{A}_{\text{inf}, X} \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} \mathbb{A}_{\text{inf}}[\frac{1}{\mu}]$ , where  $\nu$  is the projection from the pro-étale site of  $X$  to the Zariski site of  $\mathfrak{X}$  (see [14, §3] for the necessary theory of the pro-étale site of a rigid analytic space);
- (A2)  $\mathbb{A}\Omega_{\mathfrak{X}/\mathcal{O}} \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} W(k) \simeq W\Omega_{\mathfrak{X}_k/k}^\bullet$  after  $p$ -adic completion of the left-hand side;
- (A3)  $\mathbb{A}\Omega_{\mathfrak{X}/\mathcal{O}}$  is equipped with a  $\varphi$ -semilinear endomorphism  $\varphi_{\mathbb{A}}$  which is compatible with the previous two isomorphisms.

Now assume that the hypotheses of the theorem are satisfied, i.e., that  $\mathfrak{X}$  is moreover proper and that the  $i^{\text{th}}$  crystalline cohomology of its special fibre is torsion-free. Then a Fitting ideal argument using (A2) and (A3) shows that the hypercohomology group  $H_{\mathbb{A}}^i(\mathfrak{X}) := H^i(\mathfrak{X}, \mathbb{A}\Omega_{\mathfrak{X}/\mathcal{O}})$  is a finite free  $\mathbb{A}_{\text{inf}}$ -module equipped with a  $\varphi$ -semilinear endomorphism  $\varphi_{\mathbb{A}}$  which becomes an isomorphism after inverting  $\varphi(\xi)$ , where  $\xi := \mu/\varphi^{-1}(\mu)$ , in the sense that  $\varphi_{\mathbb{A}} : \varphi^* H_{\mathbb{A}}^i(\mathfrak{X}) \otimes_{\mathbb{A}_{\text{inf}}} \mathbb{A}_{\text{inf}}[\frac{1}{\varphi(\xi)}] \xrightarrow{\sim} H_{\mathbb{A}}^i(\mathfrak{X}) \otimes_{\mathbb{A}_{\text{inf}}} \mathbb{A}_{\text{inf}}[\frac{1}{\varphi(\xi)}]$ ; in other words,  $H_{\mathbb{A}}^i(\mathfrak{X})$  is a Breuil–Kisin module in the sense of L. Fargues [7, §4].

Moreover, the Breuil–Kisin module  $H_{\mathbb{A}}^i(\mathfrak{X})$  has the following étale and crystalline specialisations, which follow respectively from (A1) and (A2):

- $(H_{\mathbb{A}}^i(\mathfrak{X}) \otimes_{\mathbb{A}_{\text{inf}}} W(\mathcal{O}^b))^{\varphi_{\mathbb{A}}=1} \cong H_{\text{ét}}^i(X, \mathbb{Z}_p)$ ;
- there is a canonical inclusion  $H_{\mathbb{A}}^i(\mathfrak{X}) \otimes_{\mathbb{A}_{\text{inf}}} W(k) \hookrightarrow H_{\text{crys}}^i(\mathfrak{X}_k/W(k))$  with torsion cokernel, which is an isomorphism if  $H_{\text{crys}}^{i+1}(\mathfrak{X}_k/W(k))$  is also assumed to be torsion-free.

Hence  $H_{\text{ét}}^i(X, \mathbb{Z}_p)$  is a free  $\mathbb{Z}_p$ -module of the same rank as  $H_{\text{crys}}^i(\mathfrak{X}_k/W(k))$ . Assuming the existence of the complex  $\mathbb{A}\Omega_{\mathfrak{X}/\mathcal{O}}$ , this completes the sketched proof of the theorem and defines our new cohomology theory  $H_{\mathbb{A}}^i(\mathfrak{X})$  interpolating the  $p$ -adic étale and crystalline cohomologies of the generic and special fibres of  $\mathfrak{X}$  respectively.

**Remark 2.1.** When  $\mathfrak{X}$  is the  $p$ -adic completion of a proper smooth  $\mathcal{O}$ -scheme, the theorem follows more directly from the existence of  $\mathbb{A}\Omega_{\mathfrak{X}/\mathcal{O}}$ , at least if  $H_{\text{crys}}^{i+1}(\mathfrak{X}_k/W(k))$  is also torsion-free. Indeed,  $R\Gamma_{\mathcal{O}^b}(\mathfrak{X}) := R\Gamma(\mathfrak{X}, \mathbb{A}\Omega_{\mathfrak{X}/\mathcal{O}}) \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_p$  is a perfect complex of  $\mathcal{O}^b$ -modules, and perhaps the simplest new observable of our construction, whose fibres interpolate the étale and crystalline cohomology modulo  $p$ , thanks to (A1) and (A2):

- $R\Gamma_{\mathcal{O}^b}(\mathfrak{X}) \otimes_{\mathcal{O}^b}^{\mathbb{L}} C^b \simeq R\Gamma_{\text{ét}}(X, \mathbb{F}_p) \otimes_{\mathbb{F}_p}^{\mathbb{L}} C^b$ ,
- $R\Gamma_{\mathcal{O}^b}(\mathfrak{X}) \otimes_{\mathcal{O}^b}^{\mathbb{L}} k \simeq R\Gamma(\mathfrak{X}_k, \Omega_{\mathfrak{X}_k/k}^\bullet)$ .

By semicontinuity this gives  $\dim_{\mathbb{F}_p} H_{\text{ét}}^i(X, \mathbb{F}_p) \leq \dim_k H^i(\mathfrak{X}_k, \Omega_{\mathfrak{X}_k/k}^\bullet)$ , which yields the third of the following (in)equalities:

$$\dim_{W(k)[\frac{1}{p}]} H_{\text{crys}}^i(\mathfrak{X}_k/W(k))[\frac{1}{p}] = \dim_{\mathbb{Q}_p} H_{\text{ét}}^i(X, \mathbb{Q}_p) \leq \dim_{\mathbb{F}_p} H_{\text{ét}}^i(X, \mathbb{F}_p) \leq \dim_k H^i(\mathfrak{X}_k, \Omega_{\mathfrak{X}_k/k}^\bullet).$$

Here the first equality is a consequence of the crystalline comparison theorem and thus uses that  $\mathfrak{X}$  is algebraic, while the second inequality is formal. If we now assume further that  $H_{\text{crys}}^*(\mathfrak{X}_k/W(k))$  for  $* = i, i + 1$  is torsion free, then the first and last dimensions in the above chain of (in)equalities are equal, and thus all the dimensions are equal. In particular,

$$\dim_{\mathbb{Q}_p} H_{\text{ét}}^i(X, \mathbb{Q}_p) = \dim_{\mathbb{F}_p} H_{\text{ét}}^i(X, \mathbb{F}_p),$$

which implies that  $H_{\text{ét}}^i(X, \mathbb{Z}_p)$  is torsion free. (Note also that, by GAGA results [9, Thm. 3.7.2],  $H_{\text{ét}}^i(X, \mathbb{Z}_p)$  identifies with the  $p$ -adic étale cohomology of the generic fibre of the proper smooth  $\mathcal{O}$ -scheme of which  $\mathfrak{X}$  is the  $p$ -adic completion.)

**Remark 2.2.** One can lift (A2) above to a slightly larger portion of  $\mathbb{A}_{\text{inf}}$ : the base change  $\mathbb{A}\Omega_{\mathfrak{X}/\mathcal{O}} \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} \mathbb{A}_{\text{crys}}$  can be identified with the (Zariski sheafified) crystalline cohomology of  $\mathfrak{X}$  relative to  $\mathbb{A}_{\text{crys}}$ , where the latter is given the standard pd-structure.

### 3 OUTLINE OF CONSTRUCTION OF $\mathbb{A}\Omega_{\mathfrak{X}/\mathcal{O}}$

It is sufficient to naturally (in an  $\infty$ -categorical sense) define the value of  $\mathbb{A}\Omega_{\mathfrak{X}/\mathcal{O}}$  on arbitrary affine open formal subschemes of  $\mathfrak{X}$ , so we may assume that  $\mathfrak{X} = \text{Spf } R$ , where  $R$  is the  $p$ -adic completion of a smooth  $\mathcal{O}$ -algebra. Hence  $X = \mathfrak{X}_C$  is the rigid analytic space  $\text{Spa}(R[\frac{1}{p}], R)$  over  $C$ , on whose pro-étale site we denote by  $\mathbb{A}_{\text{inf}, X} := W(\varprojlim_{\varphi} \mathcal{O}_X^+/p)$  the first period sheaf from [14, Def. 6.1]; note that  $\mathbb{A}_{\text{inf}, X}$  is a sheaf of  $\mathbb{A}_{\text{inf}}$ -modules.

#### 3.1 Modifying torsion via the décalage functor $L\eta$

We must first describe a general process for modifying torsion in complexes, based on P. Deligne’s [4, 1.3.3] décalé filtration and used previously by P. Berthelot and A. Ogus [1, Def. 8.6]. Let  $A$  be a ring,  $f \in A$  a non-zero divisor, and  $N$  a cochain complex of  $A$ -modules supported in non-negative degrees such that  $H^0(N)$  is  $f$ -torsion-free. Then we may replace  $N$  by a quasi-isomorphic complex – still supported in non-negative degrees – consisting of  $f$ -torsion-free  $A$ -modules, and we then define  $L\eta_f N$  to be the subcomplex of  $N$  given by  $L\eta_f N^n := \{x \in f^n N^n : dx \in f^{n+1} N^{n+1}\}$  for  $n \geq 0$ . Then  $L\eta_f : D_{f\text{-tf}}^{\geq 0}(A) \rightarrow D_{f\text{-tf}}^{\geq 0}(A)$  is a well-defined functor, where  $D_{f\text{-tf}}^{\geq 0}(A)$  is the derived (or, more usefully,  $\infty$ -) category of complexes of  $A$ -modules supported in non-negative degrees with  $f$ -torsion-free  $H^0$ . Evidently there is a natural transformation  $L\eta_f \rightarrow \text{id}$ .

Among various easily established identities (e.g., if  $g \in A$  is another non-zero divisor then  $L\eta_{fg} = L\eta_f L\eta_g$  and  $(L\eta_f -) \otimes_A^{\mathbb{L}} A/gA \simeq L\eta_{f \bmod gA}(- \otimes_A^{\mathbb{L}} A/gA)$ , under mild hypotheses), the most important is that there is a natural quasi-isomorphism

$$L\eta_f N \otimes_A^{\mathbb{L}} A/fA \xrightarrow{\sim} (H^\bullet(N \otimes_A^{\mathbb{L}} A/fA), \text{Bock}),$$

where the right side denotes the complex formed from the cohomology of  $N \otimes_A^{\mathbb{L}} A/fA$  with differential given by the Bockstein operator  $H^n(N \otimes_A^{\mathbb{L}} A/fA) \rightarrow H^{n+1}(N \otimes_A^{\mathbb{L}} A/fA)$  associated to  $f$ .

### 3.2 Constructing pro Witt complexes

We now describe two general processes for building “ $F$ - $V$ -procomplexes over the  $\mathcal{O}$ -algebra  $R$ ” in the sense of Langer–Zink [12], among which their relative de Rham–Witt complex  $\{W_r\Omega_{R/\mathcal{O}}^\bullet\}_r$  is the initial object. Note first that Fontaine’s usual homomorphism of  $p$ -adic Hodge theory  $\theta : \mathbb{A}_{\text{inf}} \rightarrow \mathcal{O}$ , whose kernel is generated by the non-zero divisor  $\xi$ , lifts to a homomorphism  $\theta_r : \mathbb{A}_{\text{inf}} \rightarrow W_r(\mathcal{O})$  whose kernel is generated by  $\xi_r := \mu/\varphi^{-r}(\mu)$ ; the Restriction  $R$ , Frobenius  $F$ , and Verschiebung  $V$  on the Witt vectors  $W_r(\mathcal{O})$  correspond via the homomorphisms  $\theta_r$  respectively to  $\text{id}$ ,  $\varphi$ , and  $\xi\varphi^{-1}$  on  $\mathbb{A}_{\text{inf}}$ .

Let  $D$  be a coconnective, commutative, differential graded (or, more generally,  $\mathbb{E}_\infty$ -) algebra over  $\mathbb{A}_{\text{inf}}$  which is equipped with a  $\varphi$ -semilinear automorphism  $\varphi_D$  and such that  $H^0(D)$  is  $\mu$ -torsion-free. Assume also that there are isomorphisms of  $W_r(\mathcal{O})$ -algebras  $\lambda_r : W_r(R) \xrightarrow{\sim} H^0(D \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} \mathbb{A}_{\text{inf}}/\xi_r)$  for all  $r \geq 1$  in such a way that the Restriction, Frobenius, and Verschiebung on the left correspond respectively to the canonical projection,  $\varphi$ , and  $\xi\varphi^{-1}$  on the right.

First process: Equipping the cohomology groups  $\mathcal{W}_r^n(D)_{\text{pre}} := H^n(D \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} \mathbb{A}_{\text{inf}}/\xi_r)$  with the Bockstein differential  $\mathcal{W}_r^n(D)_{\text{pre}} \rightarrow \mathcal{W}_r^{n+1}(D)_{\text{pre}}$  associated to  $\xi_r$  makes  $\mathcal{W}_r^\bullet(D)_{\text{pre}}$  into a commutative differential graded  $W_r(\mathcal{O})$ -algebra. Equipping them further with a Restriction, Frobenius, and Verschiebung given by “ $\varphi^{-r}(\xi)^n$  times the canonical projection”,  $\varphi$ , and  $\xi\varphi^{-1}$  makes  $\{\mathcal{W}_r^\bullet(D)_{\text{pre}}\}_r$  into an  $F$ - $V$ -procomplex over the  $\mathcal{O}$ -algebra  $R$ . (To be precise, the necessary identity that  $Fd\lambda_r([x]) = \lambda_r([x^{p-1}])d\lambda_r([x])$  for all  $x \in R$  does not appear to be automatic for arbitrary  $D$ , but will be satisfied in our cases of interest.) By the universal property of Langer–Zink’s relative de Rham–Witt complex, there are then induced natural morphisms  $\{W_r\Omega_{R/\mathcal{O}}^\bullet\}_r \rightarrow \{\mathcal{W}_r^\bullet(D)_{\text{pre}}\}_r$  of  $F$ - $V$ -procomplexes over the  $\mathcal{O}$ -algebra  $R$ .

Improved process: It turns out that  $\mathcal{W}_r^\bullet(D)_{\text{pre}}$  must be adjusted by some  $\varphi^{-r}(\mu)$ -torsion. To do this, we modify the first process in order to equip the cohomology groups  $\mathcal{W}_r^\bullet(D) := H^\bullet(L\eta_{\varphi^{-r}(\mu)}D \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} \mathbb{A}_{\text{inf}}/\xi_r)$  with the structure of an  $F$ - $V$ -procomplex over the  $\mathcal{O}$ -algebra  $R$ . There are natural resulting morphisms  $\{W_r\Omega_{R/\mathcal{O}}^\bullet\}_r \rightarrow \{\mathcal{W}_r^\bullet(D)\}_r \rightarrow \{\mathcal{W}_r^\bullet(D)_{\text{pre}}\}_r$  of  $F$ - $V$ -procomplexes over the  $\mathcal{O}$ -algebra  $R$ .

The following is the key technical step in the construction of  $\mathbb{A}\Omega_{R/\mathcal{O}}$ :

**Theorem 3.1.** *Apply the “Improved process” to  $D = R\Gamma_{\text{proét}}(X, \mathbb{A}_{\text{inf}, X})$ . Then the induced map  $W_r\Omega_{R/\mathcal{O}}^\bullet \rightarrow \mathcal{W}_r^\bullet(R\Gamma_{\text{proét}}(X, \mathbb{A}_{\text{inf}, X}))$  descends to the  $p$ -adic completion of  $W_r\Omega_{R/\mathcal{O}}^\bullet$ , inducing an almost (wrt. the ideal  $W_r(\mathfrak{m}) \subseteq W_r(\mathcal{O})$ ) quasi-isomorphism of complexes of  $W_r(\mathcal{O})$ -modules for each  $r \geq 1$ :*

$$\widehat{W_r\Omega_{R/\mathcal{O}}^\bullet} \xrightarrow{\text{al.}} \mathcal{W}_r^\bullet(R\Gamma_{\text{proét}}(X, \mathbb{A}_{\text{inf}, X}))$$

*Proof.* By a localisation and étale base change argument, the assertions reduce to the case that  $R = \mathcal{O}\langle \underline{T}^{\pm 1} \rangle := \mathcal{O}\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$ . Let  $\tilde{R} := \mathcal{O}\langle \underline{T}^{\pm 1}/p^\infty \rangle$ , which is equipped with a continuous action of  $\mathbb{Z}_p^d$  via  $\mathcal{O}$ -algebra homomorphisms, where the generator in the  $i^{\text{th}}$ -coordinate of  $\mathbb{Z}_p^d$  acts via  $T_i^{j/p^k} \mapsto \zeta_{p^k}^j T_i^{j/p^k}$ , for  $j \in \mathbb{Z}$  and  $k \geq 0$ . In fact,  $\tilde{R}[\frac{1}{p}]$  is a perfectoid  $C$ -algebra occurring as the global sections of a pro-étale cover  $\tilde{X} := \varprojlim_k \text{Spa}(K\langle \underline{T}^{\pm 1}/p^k \rangle, \mathcal{O}\langle \underline{T}^{\pm 1}/p^k \rangle)$  of  $X$ , and there is a corresponding Hochschild–Serre map  $R\Gamma_{\text{cont}}(\mathbb{Z}_p^d, H_{\text{proét}}^0(\tilde{X}, \mathbb{A}_{\text{inf}, X})) \rightarrow R\Gamma_{\text{proét}}(X, \mathbb{A}_{\text{inf}, X})$ . After tensoring by  $\mathbb{A}_{\text{inf}}/\xi_r \cong$

$W_r(\mathcal{O})$ , this is an almost quasi-isomorphism (wrt. the ideal  $W_r(\mathfrak{m}) \subseteq W_r(\mathcal{O})$ ) by Faltings' almost purity theorem [5, Thm. 3.1] [14, Thm. 6.5].

Applying the “Improved process” also to  $D = R\Gamma_{\text{cont}}(\mathbb{Z}_p^d, H_{\text{proét}}^0(\tilde{X}, \mathbb{A}_{\text{inf}, X}))$  we arrive at a commutative diagram of commutative differential graded  $W_r(\mathcal{O})$ -algebras

$$\begin{array}{ccc} W_r\Omega_{R/\mathcal{O}}^\bullet & \xrightarrow{\hspace{10em}} & \mathcal{W}_r^\bullet(R\Gamma_{\text{proét}}(X, \mathbb{A}_{\text{inf}, X})) \\ & \searrow & \nearrow \\ & \mathcal{W}_r^\bullet(R\Gamma_{\text{cont}}(\mathbb{Z}_p^d, H_{\text{proét}}^0(\tilde{X}, \mathbb{A}_{\text{inf}, X}))) & \end{array}$$

where the right diagonal arrow is an almost quasi-isomorphism by the final assertion of the previous paragraph. Up to explicit  $\varphi^{-r}(\mu)$ -torsion, the terms appearing in the complex at the bottom of the diagram are  $H_{\text{cont}}^\bullet(\mathbb{Z}_p^d, W_r(\tilde{R}))$ ; the top left of the diagram is even more explicit, using Langer–Zink's description of their relative de Rham–Witt complex in the case of a polynomial algebra [12, §2]. Using such explicit descriptions we verify directly that the diagonal left arrow becomes an *isomorphism* after  $p$ -adically completing  $W_r\Omega_{R/\mathcal{O}}^\bullet$ , and this completes the proof.  $\square$

### 3.3 The final step: glueing

From the definition of the improved  $\mathcal{W}_r^\bullet(-)$ -process and the stated properties of the  $L\eta$  functor, there is a chain of quasi-isomorphisms

$$\begin{aligned} \mathcal{W}_r^\bullet(R\Gamma_{\text{proét}}(X, \mathbb{A}_{\text{inf}, X})) &= \left( H^\bullet(L\eta_{\varphi^{-r}(\mu)} R\Gamma_{\text{proét}}(X, \mathbb{A}_{\text{inf}, X}) \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} \mathbb{A}_{\text{inf}}/\xi_r), \text{Bock} \right) \\ &\simeq L\eta_{\xi_r} L\eta_{\varphi^{-r}(\mu)} R\Gamma_{\text{proét}}(X, \mathbb{A}_{\text{inf}, X}) \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} \mathbb{A}_{\text{inf}}/\xi_r \\ &\simeq L\eta_\mu R\Gamma_{\text{proét}}(X, \mathbb{A}_{\text{inf}, X}) \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} \mathbb{A}_{\text{inf}}/\xi_r, \end{aligned}$$

and hence the previous theorem may be rewritten as an almost quasi-isomorphism  $\widehat{W_r\Omega_{R/\mathcal{O}}^\bullet} \simeq L\eta_\mu R\Gamma(X, \mathbb{A}_{\text{inf}, X}) \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} \mathbb{A}_{\text{inf}}/\xi_r$ .

Let  $R_k = R \otimes_{\mathcal{O}} k$ . For each  $r \geq 1$ , we can reduce the preceding map from  $\mathbb{A}_{\text{inf}}/\xi_r \cong W_r(\mathcal{O})$  to  $W_r(k)$  to obtain the lower horizontal map in the diagram

$$\begin{array}{ccc} \mathbb{A}_r\Omega_{R/\mathcal{O}} & \longrightarrow & L\eta_\mu R\Gamma_{\text{proét}}(X, \mathbb{A}_{\text{inf}, X}) \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} \mathbb{A}_{\text{inf}}/p^r \\ \downarrow & & \downarrow \\ W_r\Omega_{R_k/k}^\bullet & \longrightarrow & L\eta_\mu R\Gamma_{\text{proét}}(X, \mathbb{A}_{\text{inf}, X}) \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} W_r(k), \end{array}$$

defining the top left complex  $\mathbb{A}_r\Omega_{R/\mathcal{O}}$  of  $\mathbb{A}_{\text{inf}}/p^r$ -modules via (homotopy) pullback. As the lower two terms are almost zero, the upper horizontal arrow is an almost isomorphism. Moreover, the right vertical map is a quasi-isomorphism after applying  $-\otimes_{\mathbb{A}_{\text{inf}}/p^r}^{\mathbb{L}} W_r(k)$ , as  $W_r(k) \otimes_{\mathbb{A}_{\text{inf}}/p^r}^{\mathbb{L}} W_r(k) \simeq W_r(k)$ . This implies that there is a quasi-isomorphism

$$\mathbb{A}_r\Omega_{R/\mathcal{O}} \otimes_{\mathbb{A}_{\text{inf}}/p^r}^{\mathbb{L}} W_r(k) \simeq W_r\Omega_{R_k/k}^\bullet.$$

We may now finally define

$$\mathbb{A}\Omega_{R/\mathcal{O}} := \text{holim}_r \mathbb{A}_r\Omega_{R/\mathcal{O}},$$

which is quasi-isomorphic, up to  $W(\mathfrak{m}^b)$ -torsion (we avoid the word “almost” here since  $W(\mathfrak{m}^b) \neq W(\mathfrak{m}^b)^2$ ), to  $L\eta_\mu R\Gamma(X, \mathbb{A}_{\text{inf}, X})$  and satisfies  $\mathbb{A}\Omega_{R/\mathcal{O}} \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} W(k) \simeq W\Omega_{R_k/k}^\bullet$  after  $p$ -adic completion of the left-hand side. The complex  $\mathbb{A}\Omega_{R/\mathcal{O}}$  is equipped with a  $\varphi$ -semilinear operator  $\varphi_{\mathbb{A}}$  obtained by glueing those on  $W\Omega_{R_k/k}^\bullet$  and  $L\eta_\mu R\Gamma_{\text{proét}}(X, \mathbb{A}_{\text{inf}, X})$ .

Since  $\mu \in W(\mathfrak{m}^b)$ , we have

$$\mathbb{A}\Omega_{R/\mathcal{O}} \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} \mathbb{A}_{\text{inf}}[\frac{1}{\mu}] \simeq L\eta_\mu R\Gamma_{\text{proét}}(X, \mathbb{A}_{\text{inf}, X}) \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} \mathbb{A}_{\text{inf}}[\frac{1}{\mu}] \simeq R\Gamma_{\text{proét}}(X, \mathbb{A}_{\text{inf}, X}) \otimes_{\mathbb{A}_{\text{inf}}}^{\mathbb{L}} \mathbb{A}_{\text{inf}}[\frac{1}{\mu}],$$

verifying (A1). We have already verified (A2). This completes the outline of the construction of  $\mathbb{A}\Omega_{\mathfrak{X}/\mathcal{O}}$ .

### 3.4 $q$ -de Rham complexes

An alternative perspective on  $\mathbb{A}\Omega_{\mathfrak{X}/\mathcal{O}}$  and its construction is offered by the idea of  $q$ -de Rham complexes.

If  $A$  is a ring,  $q \in A^\times$ , and  $U$  is a formal variable, then the “ $q$ -de Rham complex”  $q\text{-}\Omega_{A[U^{\pm 1}]/A}^\bullet$  of the Laurent polynomial algebra  $A[U^{\pm 1}]$  is defined to be

$$A[U^{\pm 1}] \longrightarrow A[U^{\pm 1}] \text{dlog } U, \quad U^j \mapsto [j]_q U^j \text{dlog } U,$$

where  $\text{dlog } U$  is a formal symbol and  $[j]_q := \frac{q^j - 1}{q - 1}$  is the “ $q$ -analogue of the integer  $j$ ”. In the more general case of a Laurent polynomial algebra in several variables  $U_1, \dots, U_d$ , set

$$q\text{-}\Omega_{A[U_1^{\pm 1}, \dots, U_d^{\pm 1}]/A}^\bullet := \bigotimes_{i=1}^m q\text{-}\Omega_{A[U_i^{\pm 1}]/A}^\bullet.$$

Note that if  $q = 1$  then the  $q$ -de Rham complex equals the usual de Rham complex; more generally, it is a deformation of the de Rham complex over  $\overline{A} := A/(q - 1)$ , in the sense that  $q\text{-}\Omega_{A[U_1^{\pm 1}, \dots, U_d^{\pm 1}]/A}^\bullet \otimes_A \overline{A} = \Omega_{\overline{A}[U_1^{\pm 1}, \dots, U_d^{\pm 1}]/\overline{A}}^\bullet$ .

If  $A$  is a topological ring, whose topology is  $I$ -adic for some ideal  $I$ , then define  $q\text{-}\Omega_{A\langle U_1^{\pm 1}, \dots, U_d^{\pm 1} \rangle/A}^\bullet$  to be the  $I$ -adic completion of  $q\text{-}\Omega_{A[U_1^{\pm 1}, \dots, U_d^{\pm 1}]/A}^\bullet$ .

**Proposition 3.2.** *If  $R = \mathcal{O}\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$  then  $\mathbb{A}\Omega_{R/\mathcal{O}}$  is quasi-isomorphic to the  $q$ -de Rham complex  $q\text{-}\Omega_{\mathbb{A}_{\text{inf}}\langle U_1^{\pm 1}, \dots, U_d^{\pm 1} \rangle/\mathbb{A}_{\text{inf}}}^\bullet$  associated to  $q = [\varepsilon] \in \mathbb{A}_{\text{inf}}^\times$ .*

*Proof.* It follows from the proof of Theorem 3.1, and the general definition of  $\mathbb{A}\Omega_{R/\mathcal{O}}$ , that in this case  $\mathbb{A}\Omega_{R/\mathcal{O}}$  is quasi-isomorphic (and not just almost so) to

$$L\eta_\mu R\Gamma_{\text{cont}}(\mathbb{Z}_p^d, H_{\text{proét}}^0(\tilde{X}, \mathbb{A}_{\text{inf}, X})).$$

This complex can be explicitly calculated and is quasi-isomorphic to  $q\text{-}\Omega_{\mathbb{A}_{\text{inf}}\langle U_1^{\pm 1}, \dots, U_d^{\pm 1} \rangle/\mathbb{A}_{\text{inf}}}^\bullet$  ( $U_i$  corresponds to the Teichmüller lift of  $(T_i, T_i^{1/p}, T_i^{1/p^2}, \dots) \in \tilde{R}^b$ ).  $\square$

Our complex  $\mathbb{A}\Omega_{\mathfrak{X}/\mathcal{O}}$  may therefore be interpreted as a natural extension of the  $q$ -de Rham complex  $q\text{-}\Omega_{\mathbb{A}_{\text{inf}}\langle U_1^{\pm 1}, \dots, U_d^{\pm 1} \rangle/\mathbb{A}_{\text{inf}}}^\bullet$ , defined initially only for tori  $\text{Spf } \mathcal{O}\langle T_1^{\pm 1}, \dots, T_d^{\pm 1} \rangle$  with a fixed choice of coordinates, to arbitrary smooth, formal  $\mathcal{O}$ -schemes  $\mathfrak{X}$ .

## 4 SOME EXAMPLES

We give an example illustrating the sharpness of our result, namely that it is possible to have more torsion in crystalline cohomology than in étale cohomology.

Although the following does not fall within the scope of our result, as it involves non-smooth stacks, it heuristically explains what can happen:

**Example 4.1.** Let  $G = \mathbb{Z}/p\mathbb{Z}$ , and  $H = \mu_p$ , both viewed as finite flat group schemes over  $\mathcal{O}_C$ . Choose a map  $\eta : G \rightarrow H$  that is an isomorphism on the generic fibre and trivial on the special fibre by using a primitive  $p^{\text{th}}$  root of unity in  $C$ . Let  $T \rightarrow BH$  be the universal  $H$ -torsor and  $S = T \times_{BH} BG$  its pullback along  $\eta$ . Then  $S_C$  is a point as it is the universal  $G$ -torsor, while  $S_k \cong BG_k \times \mu_p$  is non-reduced. It is then easy to see that the crystalline cohomology of  $S_k$  has a lot of torsion, while the étale cohomology of  $S_C$  is trivial.

One can easily push the preceding example into the world of proper smooth schemes:

**Example 4.2.** Let  $p = 2$ , and fix an Enriques surface  $S/\mathcal{O}$  whose special fibre is “singular” in the sense that it has fundamental group  $\mathbb{Z}/2\mathbb{Z}$  [2, §3]. Fix also an auxiliary elliptic curve  $E/\mathcal{O}$ , and choose a map  $\mathbb{Z}/2\mathbb{Z} \rightarrow E$  that is non-trivial on the generic fibre and trivial on the special fibre. Pushing out the universal cover of  $S$  along this map gives an  $E$ -torsor  $D \rightarrow S$  whose generic fibre  $D_C \rightarrow S_C$  is non-split, i.e., does not admit a section, but whose special fibre  $D_k \rightarrow S_k$  is split.

Thus  $D$  is a proper smooth  $\mathcal{O}$ -scheme of relative dimension 3 and one can show, using the Leray spectral sequence for  $D \rightarrow X$ , that  $H_{\text{ét}}^2(D_C, \mathbb{Z}_2)$  is torsion free while  $H_{\text{crys}}^2(D_k/W(k))$  has non-trivial 2-torsion.

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