Perfectoid Spaces and their Applications

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Abstract. We survey the theory of perfectoid spaces and its applications.

Mathematics Subject Classification (2010). Primary: 14G22, 11F80 Secondary: 14G20, 14C30, 14L05, 14G35, 11F03

Keywords. Perfectoid spaces, rigid-analytic geometry, almost mathematics, p-adic Hodge theory, Shimura varieties, Langlands program.

1. Introduction

In algebraic geometry, one of the most important dichotomies is the one between characteristic 0 and positive characteristic p. Our intuition is formed from the study of complex manifolds, which are manifestly of characteristic 0, but in number theory, the most important questions are in positive or mixed characteristic. Algebraic geometry gives a framework to transport intuition from characteristic 0 to positive characteristics. However, there are also several new phenomena in characteristic p, such as the presence of the Frobenius map, which acts naturally on all spaces of characteristic p. Using the Frobenius, one can formulate the Weil conjectures, and more generally the theory of weights. This makes many results accessible over fields such as $\mathbb{F}_p((t))$, which are wide open over fields of arithmetic interest such as $\mathbb{Q}_p$. The theory of perfectoid spaces was initially designed as a means of transporting information available over $\mathbb{F}_p((t))$ to $\mathbb{Q}_p$, but has since found a number of independent applications. The purpose of this report is to give an overview of the developments in the field since perfectoid spaces were introduced in early 2011.

2. $\mathbb{Q}_p$ vs. $\mathbb{F}_p((t))$

To study the transition between characteristic 0 and characteristic p, it is useful to look at the corresponding local fields. In characteristic 0, we have the field of p-adic numbers $\mathbb{Q}_p$:

$$\mathbb{Q}_p = \{ \sum_{n \gg -\infty} a_n p^n \mid a_n \in \{0, 1, \ldots, p-1\}\} \ ,$$

*This work was done while the author was a Clay Research Fellow.

which can be regarded as 'power series in the variable p'. On the other hand, one has the actual field of power series in the variable t:

$$\mathbb{F}_p((t)) = \{ \sum_{n \gg -\infty} a_n t^n \mid a_n \in \{0, 1, \ldots, p-1\} = \mathbb{F}_p\} \ .$$

Although these two fields have formally 'the same' elements, the basic addition and multiplication operations are different: In $\mathbb{Q}_p$, one computes with carry, but in $\mathbb{F}_p((t))$ without carry. Also, $t \mapsto t^p$ defines the Frobenius map of $\mathbb{F}_p((t))$, but there is no map $\mathbb{Q}_p \to \mathbb{Q}_p$ that sends p to p^p.

There are several strategies to pass from one field to the other. Let us recall the most important ones.

Letting $p \to \infty$. In model theory, one can formalize the idea that $\mathbb{Q}_p$ becomes isomorphic to $\mathbb{F}_p((t))$ as $p \to \infty$. This has the following implication: A first-order statement is true for almost all fields $\mathbb{Q}_p$ (for varying p) if and only if it is true for almost all fields $\mathbb{F}_p((t))$. The first application of this was the Ax-Kochen theorem, [4], that a homogeneous polynomial of degree d in more than d^2 variables admits a solution over $\mathbb{Q}_p$, except for a finite list of primes p (which depends only on d). In fact, the same result is true over $\mathbb{F}_p((t))$ for all p. However, there are counterexamples to the general statement over $\mathbb{Q}_p$, such as a quartic form in 18 variables over $\mathbb{Q}_2$ without a solution. More strikingly, this transfer principle is used in the proof of the fundamental lemma: Ngô, [52], has proved the fundamental lemma over $\mathbb{F}_p((t))$ (for sufficiently large p), which could then be transferred to $\mathbb{Q}_p$, if p is sufficiently large.[1]

However, this strategy cannot be used to get information about any fixed prime number p. One of the ways in which one wants to compare two fields is to compare the categories of finite extension fields. This is encapsulated by the absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ of a field K, where $\overline{K}$ is some separable closure of K. If K is a local field such as $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, it comes with a decreasing ramification filtration

$$G_K \supset G_K^{(0)} = I_K \supset G_K^{(1)} = P_K \supset G_K^{(2)} \supset \ldots \ ;$$

here, $P_K \subset I_K \subset G_K$ are the wild inertia, resp. inertia subgroups. The 'tame quotient' $G_K^{\mathrm{tame}} = G_K/G_K^{(1)} = G_K/P_K$ admits an explicit description, and P_K is a (not very explicit) pro-p-group.

Restricting ramification. From the explicit description of G_K^{tame} in the case of local fields, one knows that $G_{\mathbb{Q}_p}^{\mathrm{tame}} \cong G_{\mathbb{F}_p((t))}^{\mathrm{tame}}$ canonically. In other words, there is a canonical procedure to associate to a tame extension of $\mathbb{Q}_p$ a tame extension of $\mathbb{F}_p((t))$. This result can be strengthened if one passes to extension fields. More precisely, for any $n \geq 1$,

$$G_{\mathbb{Q}_p(p^{1/n})}/G^{(n)}_{\mathbb{Q}_p(p^{1/n})} \cong G_{\mathbb{F}_p((t))(t^{1/n})}/G^{(n)}_{\mathbb{F}_p((t))(t^{1/n})} \ .$$

This is a result of Deligne, [21], relying on ideas of Krasner, [49], which formalizes the idea that $\mathbb{Q}_p(p^{1/n})$ and $\mathbb{F}_p((t))(t^{1/n})$ are 'close local fields' (which get 'closer' as

[1] One does not need model theory to do this, as Waldspurger, [66], had earlier shown this transfer principle for large p directly.

$n \to \infty$). Note that, again, this result plays a crucial role in the Langlands program, namely it is used (through Henniart's numerical local Langlands correspondence, [37]) in the original proof of the local Langlands correspondence for GL_n over $\mathbb{Q}_p$ by Harris-Taylor, [34].[2]

There is yet another approach, which gives a comparison of the whole Galois group.

Making things perfect(oid). Let $K = \mathbb{Q}_p(p^{1/p^\infty}) = \bigcup_m \mathbb{Q}_p(p^{1/p^m})$, which we will momentarily confuse with its completion, which has the same absolute Galois group G_K. Then a theorem of Fontaine-Wintenberger, [32], states that the absolute Galois groups of G_K and $G_{\mathbb{F}_p((t))}$ are isomorphic. This can be considered as a limit case of Deligne's theorem as one lets $n = p^m$, $m \to \infty$. Indeed, note that as $\mathbb{F}_p((t))(t^{1/p^m})$ is a purely inseparable extension of $\mathbb{F}_p((t))$, $G_{\mathbb{F}_p((t))(t^{1/p^m})} \cong G_{\mathbb{F}_p((t))}$. In fact, if one lets $K^\flat$ be the completion of

$$\mathbb{F}_p((t))(t^{1/p^\infty}) = \bigcup_m \mathbb{F}_p((t))(t^{1/p^m}) ,$$

then the theorem of Fontaine-Wintenberger states equivalently that $G_K \cong G_{K^\flat}$. This theorem is one of the foundational cornerstones of p-adic Hodge theory. Moreover, it is true in a wide variety of cases: Any 'deeply ramified' extension of $\mathbb{Q}_p$ can be used in place of $\mathbb{Q}_p(p^{1/p^\infty})$.

Note that the last approach gives the cleanest result: It works for any fixed p, and produces an isomorphism of the whole Galois groups. However, it comes at the expense of passing to infinite extensions. The theory of perfectoid spaces is a generalization of this procedure to higher-dimensional objects.

3. The (generalized) Fontaine-Wintenberger isomorphism

To start, let us explain the general statement of the Fontaine-Wintenberger isomorphism.[3]

Definition 3.1. A perfectoid field is a complete topological field K, whose topology comes from a nonarchimedean norm $|\cdot| : K \to \mathbb{R}_{\geq 0}$ with dense image, such that $|p| < 1$ and, letting $\mathcal{O}_K = \{x \in K \mid |x| \leq 1\}$ be the ring of integers, the Frobenius map $\Phi : \mathcal{O}_K/p \to \mathcal{O}_K/p$ is surjective.

Examples include the completions of $\mathbb{Q}_p(p^{1/p^\infty})$, $\mathbb{Q}_p(\mu_{p^\infty})$, $\overline{\mathbb{Q}}_p$ and $\mathbb{F}_p((t))(t^{1/p^\infty})$, $\overline{\mathbb{F}_p((t))}$. Note that perfectoid fields can be of characteristic 0 or p. In the first case,

[2] The alternative proof given in [57] avoids this argument, and gives a proof of the local Langlands correspondence for GL_n over $\mathbb{Q}_p$ which is purely in characteristic 0.

[3] It should be noted that the original result of Fontaine-Wintenberger is quite different, at least in emphasis. The theorem as stated was only proved recently, and was noticed independently (at least) by Kedlaya-Liu and the author.

they contain $\mathbb{Q}_p$ naturally, as $|p| < 1$. Note that $\mathbb{Q}_p$ is not a perfectoid field (although $\mathbb{Z}_p/p = \mathbb{F}_p$ has a surjective Frobenius map), because $|\cdot| : \mathbb{Q}_p \to \mathbb{R}_{\geq 0}$ has discrete image $0 \cup p^{\mathbb{Z}} \subset \mathbb{R}_{\geq 0}$. In characteristic p, perfectoid fields are the same thing as perfect complete nonarchimedean fields.

By a construction of Fontaine, one can take any perfectoid field K, and produce a perfectoid field $K^\flat$ of characteristic p, called the tilt of K. First, one defines $\mathcal{O}_{K^\flat} = \varprojlim_\Phi \mathcal{O}_K/p$, and then defines $K^\flat$ as the fraction field of $\mathcal{O}_{K^\flat}$. It comes with a natural norm, with respect to which $\mathcal{O}_{K^\flat} \subset K^\flat$ is the ring of integers. In fact, one has the following alternative description of $K^\flat$.

Lemma 3.2. *There is a natural identification of multiplicative monoids*

$$\mathcal{O}_{K^\flat} = \varprojlim_{x\mapsto x^p} \mathcal{O}_K = \{(x^{(0)}, x^{(1)}, \ldots) \mid x^{(i)} \in \mathcal{O}_K, (x^{(i+1)})^p = x^{(i)}\} \ , K^\flat = \varprojlim_{x\mapsto x^p} K \ .$$

In particular, $x \mapsto x^\sharp := x^{(0)}$ defines a multiplicative map $K^\flat \to K$, and the norm $|x|_{K^\flat} = |x^\sharp|_K$ on $K^\flat$.

Proof. Let us only check the first identification. For this, one has to verify that the projection map

$$\varprojlim_{x\mapsto x^p} \mathcal{O}_K \to \varprojlim_{x\mapsto x^p} \mathcal{O}_K/p = \mathcal{O}_{K^\flat}$$

is bijective. Indeed, any sequence $(\bar{x}_0, \bar{x}_1, \ldots) \in \varprojlim_{x\mapsto x^p} \mathcal{O}_K/p$ lifts uniquely to $(x^{(0)}, x^{(1)}, \ldots)$, where

$$x^{(i)} = \lim_{n\to\infty} \widetilde{\bar{x}_{i+n}}^{p^n} \ ,$$

where $\widetilde{\bar{x}}_j \in \mathcal{O}_K$ denotes an arbitrary lift of $\bar{x}_j \in \mathcal{O}_K/p$. The existence of this p-adic limit follows from the basic fact that if $a \cong b \mod p$, then $a^{p^n} \cong b^{p^n} \mod p^{n+1}$. ∎

As a basic example of the tilting equivalence, the perfectoid field K which is the completion of $\mathbb{Q}_p(p^{1/p^\infty})$ tilts to the perfect nonarchimedean field $K^\flat$ which is the completion of $\mathbb{F}_p((t))(t^{1/p^\infty})$. Under the identification

$$K^\flat = \varprojlim_{x\mapsto x^p} K \ ,$$

the element t corresponds to the sequence $(p, p^{1/p}, p^{1/p^2}, \ldots)$. In particular, $t^\sharp = p$, so in a vague sense, the map $x \mapsto x^\sharp$ is the map 'replace t by p'. However, calculating it in general involves a p-adic limit, so e.g.

$$(1+t)^\sharp = \lim_{n\to\infty} (1 + p^{1/p^n})^{p^n} \ .$$

This already shows that any general theory of perfectoid objects has to be of an analytic nature.

Theorem 3.3 ([47, Theorem 3.5.6],[58, Theorem 3.7]). *Let K be a perfectoid field.*

(i) *For any finite extension L/K, L is a perfectoid field.*

(ii) *The association $L \mapsto L^\flat$ defines an equivalence between the category of finite extensions of K and the category of finite extensions of $K^\flat$.*

It is formal to deduce from part (ii) that the absolute Galois groups $G_K \cong G_{K^\flat}$ are isomorphic.

4. Untilting: Work of Fargues-Fontaine

The following question arises naturally: For a given perfectoid field L of characteristic p, in how many ways can it be untilted to a perfectoid field K of characteristic 0, $K^\flat \cong L$? The answer to this question leads naturally to the Fargues-Fontaine curve, [30], [31]. In particular, they prove the following theorem.

Theorem 4.1. *Fix a perfectoid field L of characteristic p. There is a regular noetherian scheme X_L of Krull dimension 1 (locally the spectrum of a principal ideal domain) over $\mathbb{Q}_p$ whose closed points x are in bijection with equivalence classes of pairs (K,ι), where K is a perfectoid field of characteristic 0 and $\iota : L \hookrightarrow K^\flat$ is an injection which makes $K^\flat$ a finite extension of L; here, the pairs (K,ι) and $(K,\iota\circ\Phi^n)$ are regarded as equivalent for any $n\in\mathbb{Z}$. The degree $[K^\flat : L]$ is called the degree of x. Moreover, there are (infinitely many) points of degree 1.*

In particular, one can always untilt a perfectoid field L to characteristic 0, and the ways of doing so are parametrized by a 1-dimensional object. Note that if, e.g., L is algebraically closed, then all points are of degree 1 and have algebraically closed residue field. However, the curve lives only over $\mathbb{Q}_p$, and thus is not of finite type over $\mathbb{Q}_p$. Concretely,

$$X_L = \operatorname{Proj} \bigoplus_{n\geq 0} B^+(L)^{\varphi = p^n} ,$$

where $B^+(L)$ is one of Fontaine's period rings, a certain completion of $W(\mathcal{O}_L)[\frac{1}{p}]$. A point of X_L gives rise to an ideal $I \subset W(\mathcal{O}_L)[\frac{1}{p}]$ (well-defined up to the action of Frobenius), and the corresponding perfectoid field of characteristic 0 is given by $K = W(\mathcal{O}_L)[\frac{1}{p}]/I$. This gives an explicit description of untilting in terms of Witt vectors.

The work of Fargues-Fontaine has further connections with the theory of perfectoid spaces that we cannot explain in detail here, for lack of space, cf. [29]. For the rest of this article, we will usually fix a perfectoid field K in characteristic 0, which amounts to fixing a point $\infty \in X_{K^\flat}$ of degree 1.

5. Perfectoid algebras

Definition 5.1. A perfectoid K-algebra is a Banach K-algebra R for which the subring $R^\circ \subset R$ of powerbounded elements is a bounded subring, and such that the Frobenius map $\Phi : R^\circ/p \to R^\circ/p$ is surjective.

The simplest example is $R = K\langle T^{1/p^\infty}\rangle$ for which $R^\circ = \mathcal{O}_K\langle T^{1/p^\infty}\rangle$ is the completion of $\mathcal{O}_K[T^{1/p^\infty}] = \bigcup_m \mathcal{O}_K[T^{1/p^m}]$. In other words, perfectoid K-algebras are algebras with 'lots of (approximate) p-power roots'. Note that perfectoid K-algebras are always quite big, e.g. nonnoetherian; also, no 'smallness' hypothesis is imposed. The mixture of completeness and nonnoetherianity might cause big trouble (as, e.g., completions of nonnoetherian algebras are not in general flat)! However, it turns out that the 'bigness' condition of surjective Frobenius forces good behaviour.

One can apply Fontaine's construction to any perfectoid K-algebra. This defines the tilting functor: Let R be a perfectoid K-algebra. Set

$$R^{\flat\circ} = \varprojlim_{\Phi} R^\circ/p = \varprojlim_{x\mapsto x^p} R^\circ$$

which is a $\mathcal{O}_{K^\flat}$-algebra, and

$$R^\flat = R^{\flat\circ} \otimes_{\mathcal{O}_{K^\flat}} K^\flat = \varprojlim_{x\mapsto x^p} R\ .$$

Proposition 5.2 ([58, Theorem 5.2]). *Fix a perfectoid field K with tilt $K^\flat$.*

(i) *For any perfectoid K-algebra R, the tilt $R^\flat$ is a perfectoid $K^\flat$-algebra with subring of powerbounded elements $R^{\flat\circ} \subset R^\flat$.*

(ii) *The functor $R \mapsto R^\flat$ defines an equivalence between the category of perfectoid K-algebras and the category of perfectoid $K^\flat$-algebras.*

Note also that for any perfectoid K-algebra R, one has a continuous multiplicative map $R^\flat \to R$, $f \mapsto f^\sharp$.

6. Perfectoid Spaces

As remarked earlier, any theory of perfectoid objects has to be of an analytic nature. This reflects itself algebraically in the fact that perfectoid algebras are Banach algebras. On the level of spaces, it means that we have to work in some category of nonarchimedean analytic spaces. The classical such category is Tate's category of rigid-analytic spaces, [64], but strong finiteness assumptions are built into the foundations of this theory. There are (at least) two more recent approaches to nonarchimedean analytic spaces: Berkovich's analytic spaces, [8], and Huber's adic spaces, [40]. We choose to work with Huber's adic spaces, because we feel that it is the most natural framework; e.g., it interacts well with the theory of formal models. Moreover, one glues spaces along open subsets, which is at least technically convenient.[4]

Following Huber, we make the following definition in the perfectoid world:

[4] Somewhat more importantly, most examples of perfectoid spaces arise as 'inverse limits' of classical finite type spaces. The general topos-theoretic notion of inverse limit developed in SGA4

Definition 6.1. A perfectoid affinoid K-algebra is a pair (R, R^+), where R is a perfectoid K-algebra, and $R^+ \subset R^\circ$ is an open and integrally closed subring.

The role of the integral subalgebra R^+ is certainly secondary, and one may safely assume that $R^+ = R^\circ$ on first reading.

Proposition 6.2. *The association* $(R, R^+) \mapsto (R^\flat, R^{\flat +})$ *with*

$$R^\flat = \varprojlim_{x \mapsto x^p} R \ , \ R^{\flat +} = \varprojlim_{x \mapsto x^p} R^+$$

defines an equivalence between perfectoid affinoid K-algebras and perfectoid affinoid $K^\flat$-algebras.

To a pair (R, R^+), Huber associates a space of continuous valuations.

Definition 6.3. A valuation on R is a map $|\cdot| : R \to \Gamma \cup \{0\}$, where Γ is a totally ordered abelian group (e.g., $\Gamma = \mathbb{R}_{>0}$, but higher-rank valuations are allowed), such that $|0| = 0$, $|1| = 1$, $|xy| = |x||y|$ and $|x+y| \leq \max(|x|, |y|)$. The valuation $|\cdot|$ is continuous if for all $\gamma \in \Gamma$, the subset $\{x \in R \mid |x| < \gamma\} \subset R$ is open.

There is an obvious notion of equivalence of valuations, and one defines $\mathrm{Spa}(R, R^+)$ as the set of equivalence classes of continuous valuations $|\cdot|$ on R such that $|R^+| \leq 1$. For a point $x \in \mathrm{Spa}(R, R^+)$, we denote by $f \mapsto |f(x)|$ the associated valuation. One may find back R^+ as

$$R^+ = \{f \in R \mid |f(x)| \leq 1 \ \forall x \in \mathrm{Spa}(R, R^+)\} \ .$$

One equips $\mathrm{Spa}(R, R^+)$ with the topology generated by rational subsets: For $f_1, \ldots, f_n, g \in R$ which generate R as an ideal, the subset

$$U(f_1, \ldots, f_n; g) = \{x \in \mathrm{Spa}(R, R^+) \mid |f_i(x)| \leq |g(x)|\} \subset \mathrm{Spa}(R, R^+)$$

is a rational subset.

Proposition 6.4 ([38, Theorem 3.5]). *The space* $\mathrm{Spa}(R, R^+)$ *is a spectral space. In particular, it is quasicompact, quasiseparated, and the rational subsets form a basis for the topology consisting of quasicompact open subsets, stable under finite intersections.*

Again, one finds an interesting relation under tilting.[5]

Theorem 6.5 ([58, Theorem 6.3 (i)]). *For any $x \in \mathrm{Spa}(R, R^+)$, one may define a point $x^\flat \in \mathrm{Spa}(R^\flat, R^{\flat +})$ by setting $|f(x^\flat)| := |f^\sharp(x)|$ for $f \in R^\flat$. This defines a homeomorphism* $\mathrm{Spa}(R, R^+) \cong \mathrm{Spa}(R^\flat, R^{\flat +})$ *preserving rational subsets.*

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requires coherent topoi, and adic spaces have an underlying coherent topological space, as well as a coherent étale topos. This makes the machinery of SGA4 available, which gets crucially used in many applications. Berkovich spaces have compact Hausdorff underlying spaces, which are not coherent.

[5] A closely related result was proved earlier by Kedlaya, [46].

It is a priori not clear that $f \mapsto |f^\sharp(x)|$ actually defines a valuation, as the strong triangle inequality might fail because of nonadditivity of $f \mapsto f^\sharp$. Moreover, injectivity of $\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(R^\flat, R^{\flat +})$ is not clear as one only remembers the valuation on the image of $R^\flat \to R$, which is far from dense. However, there is the following crucial approximation lemma.[6]

Lemma 6.6 ([58, Corollary 6.7 (i)]). *Assume K is of characteristic 0. Let $f \in R$ be any element, and fix any $\epsilon > 0$. Then there exists $g \in R^\flat$ such that for all $x \in \mathrm{Spa}(R, R^+)$,*

$$|(f - g^\sharp)(x)| \leq |p|^{1-\epsilon} \max(|f(x)|, \epsilon) \ .$$

This means in particular that $|f(x)| = |g^\sharp(x)|$ except if both are very small. However, $f - g^\sharp$ may be quite large if f is large.

One wants to equip the topological space $X = \mathrm{Spa}(R, R^+)$ with a structure sheaf $\mathcal{O}_X$. For this, let $U = U(f_1, \ldots, f_n; g) \subset X$ be a rational subset. Equip $R[g^{-1}]$ with the topology for which the image of $R^+[\frac{f_1}{g}, \ldots, \frac{f_n}{g}] \to R[g^{-1}]$ is open and bounded. Let $R\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g} \rangle$ be the completion of $R[g^{-1}]$ with respect to this topology; it comes equipped with a natural subring

$$R\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g} \rangle^+ \subset R\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g} \rangle \ .$$

Proposition 6.7 ([39, Proposition 1.3]). *The pair*

$$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = \left(R\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g} \rangle, R\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g} \rangle^+ \right)$$

depends only on the rational subset $U \subset X$ (and not on the choice of $f_1, \ldots, f_n, g \in R$). The map

$$\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to \mathrm{Spa}(R, R^+)$$

is a homeomorphism onto U, preserving rational subsets. Moreover, $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is initial with respect to this property.

The propositions of Huber so far have not used the assumption that R is perfectoid. This assumption is needed, however, to prove that $\mathcal{O}_X$ is actually a sheaf. Huber proved this when R is strongly noetherian, so e.g. if R is topologically of finite type over K. Perfectoid K-algebras are virtually never (strongly) noetherian, so this result does not help.

Theorem 6.8 ([58, Theorem 6.3]). *Let (R, R^+) be any perfectoid affinoid K-algebra with tilt $(R^\flat, R^{\flat +})$. Let $X = \mathrm{Spa}(R, R^+)$, $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat +})$. For any rational subset $U \subset X$, let $U^\flat \subset X^\flat$ be its image under the homeomorphism $X \cong X^\flat$.*

(i) *The presheaves $\mathcal{O}_X$, $\mathcal{O}_{X^\flat}$ are sheaves.*

[6] A slightly stronger version (replacing $1 - \epsilon$ by $p/(p-1) - \epsilon$) appears in [47].

(ii) *For any rational subset* $U \subset X$, *the pair* $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ *is a perfectoid affinoid* K*-algebra, which tilts to* $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$.

(iii) *For any* $i > 0$, *the cohomology group* $H^i(X, \mathcal{O}_X) = 0$ *vanishes. In fact,* $H^i(X, \mathcal{O}_X^+)$ *is almost zero, i.e. killed by the maximal ideal of* $\mathcal{O}_K$.

The resulting spaces $\mathrm{Spa}(R, R^+)$ (equipped with the two sheaves of topological rings $\mathcal{O}_X$, $\mathcal{O}_X^+$) are called affinoid perfectoid spaces (over K). Objects obtained by gluing such spaces are called perfectoid spaces over K.

Corollary 6.9. *The categories of perfectoid spaces over* K *and over* $K^\flat$ *are equivalent. Here, if* X *tilts to* $X^\flat$, *then the underlying topological spaces of* X *and* $X^\flat$ *are canonically homeomorphic. Moreover, a subset* $U \subset X$ *is affinoid perfectoid if and only if* $U^\flat \subset X^\flat$ *is affinoid perfectoid. For any such* U, $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ *is a perfectoid affinoid* K*-algebra with tilt* $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$.

For any perfectoid space X, one may define its étale site $X_{\text{ét}}$.

Theorem 6.10 ([58, Theorem 7.12, Proposition 7.13]). *Under tilting,* $X_{\text{ét}} \cong X^\flat_{\text{ét}}$. *Moreover, if* $X = \mathrm{Spa}(R, R^+)$ *is affinoid perfectoid, then* $H^0(X_{\text{ét}}, \mathcal{O}_X^+) = R^+$ *while* $H^i(X_{\text{ét}}, \mathcal{O}_X^+)$ *is almost zero for* $i > 0$. *In particular,* $H^0(X_{\text{ét}}, \mathcal{O}_X) = R$ *while* $H^i(X_{\text{ét}}, \mathcal{O}_X) = 0$ *for* $i > 0$.

The assertion $X_{\text{ét}} \cong X^\flat_{\text{ét}}$ is a far-reaching generalization of the Fontaine-Wintenberger isomorphism. Indeed, if we put $X = \mathrm{Spa}(K, \mathcal{O}_K)$, which tilts to $X^\flat = \mathrm{Spa}(K^\flat, \mathcal{O}_{K^\flat})$, the assertion is precisely the Fontaine-Wintenberger isomorphism. The assertion about $H^i(X_{\text{ét}}, \mathcal{O}_X^+)$ is a strengthening of Faltings's almost purity theorem, which is essentially the version of it for the finite étale site. Let us state it in our setup.

Theorem 6.11 ([58, Theorem 7.9 (iii)]). *Let* R *be a perfectoid* K*-algebra, and let* S/R *be finite étale. Then* S *is a perfectoid* K*-algebra, and* S° *is a uniformly almost finite étale* R°*-algebra.*

The following is an easy corollary, which gives a higher-dimensional variant of the Fontaine-Wintenberger isomorphism (for the finite étale case).

Corollary 6.12. *Let* R *be a perfectoid* K*-algebra with tilt* $R^\flat$. *Then tilting defines an equivalence between the categories of finite étale* R*-algebras and finite étale* $R^\flat$*-algebras.*

The almost purity theorem is interesting only in characteristic 0; in characteristic p, it is easy. Originally, Faltings proved such statements in situations of good reduction, [23], and then more generally in certain situations of semistable (or more generally toric) reduction, [25]. We note that in Faltings' situation, R was the completion of an inductive limit of regular algebras. Then, by Zariski-Nagata purity, the ramification locus of S° over R° is purely of codimension 1. We know by assumption that there is no ramification in characteristic 0, as S/R is finite étale. At the generic points of R°/p, it follows from (the proof of) the

Fontaine-Wintenberger result that there is *almost* no ramification. If there were none, one would get that S°/R° is finite étale. Faltings made the same argument work in the almost world. It came as a surprise that no regularity assumption is needed for the theorem.

Faltings' almost purity theorem was the technical cornerstone for most of the deep work in p-adic Hodge theory, as it provided a higher-dimensional variant of the Fontaine-Wintenberger isomorphism. The given generalization made many new applications possible, of which some are explained below.

7. Example: Projective spaces

Let us start with an explicit example of the tilting equivalence. Let K be a perfectoid field with tilt $K^\flat$. Let us consider the case of projective space. In all applications of perfectoid spaces, the hard part is to find a way to pass from objects of finite type over K to perfectoid objects. This is not possible in a canonical way, and one has to make a choice.

On $\mathbb{P}^n$, one has the map $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ sending $(x_0 : \ldots : x_n)$ to $(x_0^p : \ldots : x_n^p)$. Consider $\mathbb{P}^n_K$ as an adic space over K. Then there is a perfectoid space $(\mathbb{P}^n_K)^{\mathrm{perf}}$ over K such that

$$(\mathbb{P}^n_K)^{\mathrm{perf}} \sim \varprojlim_{\varphi} \mathbb{P}^n_K \ .$$

Here, $\sim \varprojlim$, read 'being similar to the inverse limit', is a technical notion that accounts for the non-existence of inverse limits in the category of adic spaces, cf. [62, Definition 2.4.1]. Explicitly, $(\mathbb{P}^n_K)^{\mathrm{perf}}$ is glued out of $n+1$ copies of

$$\mathrm{Spa}(K\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle, \mathcal{O}_K\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle)$$

in the usual way. One can make the same construction over $K^\flat$ to get $(\mathbb{P}^n_{K^\flat})^{\mathrm{perf}}$.

Theorem 7.1 ([58, Theorem 8.5]). *The perfectoid space $(\mathbb{P}^n_K)^{\mathrm{perf}}$ tilts to $(\mathbb{P}^n_{K^\flat})^{\mathrm{perf}}$. In particular, there are homeomorphisms of topological spaces underlying the adic spaces*

$$|\mathbb{P}^n_{K^\flat}| \cong |(\mathbb{P}^n_{K^\flat})^{\mathrm{perf}}| \cong |(\mathbb{P}^n_K)^{\mathrm{perf}}| \cong \varprojlim_{\varphi} |\mathbb{P}^n_K| \ .$$

Similarly, there are isomorphisms of étale sites

$$(\mathbb{P}^n_{K^\flat})_{\text{ét}} \cong (\mathbb{P}^n_{K^\flat})^{\mathrm{perf}}_{\text{ét}} \cong (\mathbb{P}^n_K)^{\mathrm{perf}}_{\text{ét}} \cong \varprojlim_{\varphi} (\mathbb{P}^n_K)_{\text{ét}} \ .$$

These constructions give a 'projection map'

$$\pi : \mathbb{P}^n_{K^\flat} \to \mathbb{P}^n_K$$

defined on topological spaces and étale topoi, and given by $(x_0 : \ldots : x_n) \mapsto (x_0^\sharp : \ldots : x_n^\sharp)$ in coordinates.

There are many variants to this theorem. All one needs is a 'dynamical system' (X, φ) over K such that (with respect to a suitable integral model of X) φ is a lift of Frobenius. For example, one might take $X = \mathbb{P}^n$ with

$$\varphi : (x_0 : \ldots : x_n) \mapsto (x_0^p + pP_0(x_0, \ldots, x_n) : \ldots : x_n^p + pP_n(x_0, \ldots, x_n))$$

for arbitrary homogeneous polynomial $P_0, \ldots, P_n \in \mathbb{Z}_p[x_0, \ldots, x_n]$ of degree p. In that case, the tilt will still be $(\mathbb{P}^n_{K^\flat})^{\mathrm{perf}}$. One might also take a canonical lift of an ordinary abelian variety, with its canonical lift of Frobenius. In that case, the tilt will be the perfection of the ordinary abelian variety in characteristic p. However, nothing of this sort works of curves of genus ≥ 2. Currently, there are very few explicit examples of tilting for varieties besides the cases of toric varieties and (semi-)abelian varieties. An interesting case might be the one of flag varieties.

8. Weight-monodromy conjecture

One application of the theory of perfectoid spaces is to a class of cases of the weight-monodromy conjecture. Let us briefly recall the statement, cf. [18].

Let X over $\mathbb{Q}_p$ (or a finite extension thereof) be a proper smooth variety. Fix a prime $\ell \neq p$. On the étale cohomology group $V = H^i(X_{\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_\ell)$, the absolute Galois group $G_{\mathbb{Q}_p}$ acts. Fix a Frobenius element $\mathrm{Frob} \in G_{\mathbb{Q}_p}$. From the Weil conjectures, [19], the Rapoport-Zink spectral sequence, [54], and de Jong's alterations, [42], the following is known about the structure of V:

(i) There is a direct sum decomposition $V = \bigoplus_{j=0}^{2i} V_j$, where all eigenvalues of Frob on V_j are Weil numbers of weight j.

(ii) There is a nilpotent operator $N : V \to V$ mapping $V_j \to V_{j-2}$, coming from the action of the pro-ℓ-inertia.

Conjecture 8.1 ([18]). *For any $j = 0, \ldots, i$, the map $N^j : V_{i+j} \to V_{i-j}$ is an isomorphism.*

This is somewhat reminiscent of the Lefschetz decomposition, and is sometimes said to be 'Mirror dual' to it. There is a similar result for projective smooth families of complex manifolds over a punctured complex disc, which is known to be true by work of Schmid, [56], and Steenbrink, [63].

Deligne proved the analogue for X over $\mathbb{F}_p((t))$ in [20].[7] Our result deduces the conjecture over $\mathbb{Q}_p$ in many cases by reduction to equal characteristic, via tilting.

Theorem 8.2 ([58]). *Let X be a geometrically connected proper smooth variety over a finite extension of $\mathbb{Q}_p$ which is a set-theoretic complete intersection in a projective smooth toric variety. Then the weight-monodromy conjecture holds true for X.*

[7] Actually, he assumed that X is already defined over a curve, but this assumption can be removed.

Note that this result is new even for a smooth hypersurface in projective space. Let us note that strictly speaking, the author is not aware of any (geometrically connected projective smooth) X which does provably not satisfy this assumption. However, we can also not prove it in any reasonable generality.

9. Close local fields: Work of Hattori

Recall that the theory of perfectoid spaces developed as a generalization of the Fontaine-Wintenberger result which worked with infinite extensions of $\mathbb{Q}_p$. Hattori shows that one can, however, use this theory to prove generalizations of Deligne's results on close local fields. Let us state here one of his results.

For a complete discrete valuation field K with valuation v (normalized with image $\mathbb{Z} \cup \{\infty\}$) and residue characteristic p, the absolute ramification index e_K is defined as $e_K = v(p)$. In particular, $e_K = \infty$ if K is of characteristic p.

Theorem 9.1 ([35, Theorem 1.2 (ii)]). *Let K_1 and K_2 be two complete discrete valuation fields of residue characteristic p, such that the residue fields $k_1 \cong k_2$ are isomorphic. Let $j \leq \min(e_{K_1}, e_{K_2})$. Then there is an isomorphism*

$$G_{K_1}/G_{K_1}^{(j)} \cong G_{K_2}/G_{K_2}^{(j)} .$$

The main novelty is that the residue fields k_i are not assumed to be perfect. Thus, Hattori has to use the Abbes-Saito ramification filtration for complete discrete valuation fields with imperfect residue fields, [1]. This is defined in terms of geometrically connected components of certain rigid-analytic varieties. Hattori's approach is to use perfectoid spaces to compare these rigid-analytic varieties in different characteristics. For this, one has to check that connected components do not change when passing to the perfectoid world, i.e. extracting a lot of p-power roots; this uses the bound on the ramification degree and an explicit computation. Then the result follows from the homeomorphism $X \cong X^\flat$ of underlying topological spaces.

In particular, this shows that the theory of perfectoid spaces gives new information on the other approaches to changing the characteristic. We note that in the representation theory of local groups, there are Hecke algebra isomorphisms for not-too-ramified types of close local fields, mirroring the Galois story on the automorphic side, cf. [43]. It would be interesting to see if perfectoid spaces can shed new light on these Hecke algebra isomorphisms as well.

10. Rigid Motives: Work of Vezzani

Another way in which perfectoid spaces have been used to study phenomena of changing the characteristic is in relation to Ayoub's category of rigid motives, cf. [5]. Rigid motives are defined by formally repeating some constructions from $\mathbb{A}^1$-homotopy theory, working with the category of smooth rigid-analytic varieties, and replacing $\mathbb{A}^1$ by the closed unit ball. For any nonarchimedean field K and any ring Λ, one gets the resulting category of rigid motives $\mathrm{RigMot}(K,\Lambda)$ with coefficients in Λ.

The following theorem is due to Vezzani:

Theorem 10.1 ([65]). *Let K be a perfectoid field with tilt $K^\flat$. For any $\mathbb{Q}$-algebra Λ, the categories $\mathrm{RigMot}(K,\Lambda) \cong \mathrm{RigMot}(K^\flat,\Lambda)$ are canonically equivalent.*

This can be regarded as a version of the Fontaine-Wintenberger isomorphism for 'rigid motivic Galois groups'. Vezzani's strategy is to compare both categories to categories of 'perfectoid motives' which one gets from (suitable) perfectoid spaces. It is rather formal that these perfectoid motives are equivalent over K and $K^\flat$, and the task becomes to relate these to classical finite-type objects.

11. p-adic Hodge theory

The subject of p-adic Hodge theory can be regarded as a parallel to Deligne's formulation of complex Hodge theory as the interrelationship between the various cohomology theories associated with compact Kähler manifolds. Let us recall the most important results in the complex setting. Fix a compact Kähler manifold X. One has the singular cohomology $H^i(X,\mathbb{Z})$, the de Rham cohomology $H^i_{\mathrm{dR}}(X)$ defined as hypercohomology of the holomorphic de Rham complex

$$\Omega^\bullet_X = \mathcal{O}_X \to \Omega^1_X \to \ldots$$

and the Hodge cohomology groups $H^i(X,\Omega^j_X)$.

Theorem 11.1 (Poincaré lemma). *The inclusion $\mathbb{C} \to \Omega^\bullet_X$ is a quasi-isomorphism of sheaves of complexes. In particular,*

$$H^i(X,\mathbb{Z}) \otimes \mathbb{C} = H^i(X,\mathbb{C}) \cong H^i_{\mathrm{dR}}(X)\ .$$

Theorem 11.2 (Hodge). *The Hodge-to-de Rham spectral sequence*

$$E_1^{ij} = H^j(X,\Omega^i_X) \Rightarrow H^{i+j}_{\mathrm{dR}}(X)$$

degenerates at E_1. In particular, $H^i_{\mathrm{dR}}(X)$ admits a decreasing de Rham filtration $\mathrm{Fil}^j\, H^i_{\mathrm{dR}}(X)$ with associated graded pieces $H^j(X,\Omega^{i-j}_X)$.

Theorem 11.3 (Hodge). *There is a canonical Hodge decomposition*

$$H^i(X,\mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{j=0}^{i} H^j(X,\Omega^{i-j}_X)\ .$$

Here, $H^j(X,\Omega^{i-j}_X)$ is identified with the intersection

$$\mathrm{Fil}^j\, H^i_{\mathrm{dR}}(X) \cap \overline{\mathrm{Fil}^{i-j}\, H^i_{\mathrm{dR}}(X)}\ ,$$

using transversality of $\mathrm{Fil}^\bullet$ and the complex conjugate filtration $\overline{\mathrm{Fil}^\bullet}$.

Now let C be a complete and algebraically closed extension of $\mathbb{Q}_p$. For example, $C = \mathbb{C}_p$, the completion of $\overline{\mathbb{Q}}_p$. Note that C is perfectoid. Let X be a proper smooth rigid-analytic variety over C. (In particular, X is of finite type, and certainly not perfectoid.) This should be regarded as the analogue of a compact complex manifold, which is not necessarily Kähler. Prior to the author's work on the subject, all work concentrated on the case of algebraic X, but it is shown in [59] that this restriction is not necessary.

Again, one has de Rham and Hodge cohomology groups $H^i_{\mathrm{dR}}(X)$, $H^i(X, \Omega^j_X)$, defined in the same way. What replaces singular cohomology is étale cohomology $H^i_{\text{ét}}(X, \mathbb{Z}_p)$. The following result generalizes a fact well-known for algebraic varieties.

Theorem 11.4 ([59, Theorem 1.1], [60, Theorem 3.17]). *Let X be a proper rigid-analytic variety over C. Then $H^i_{\text{ét}}(X, \mathbb{Z}_p)$ is a finitely generated $\mathbb{Z}_p$-module, which vanishes for $i > 2 \dim X$.*

Properness is crucial here. In fact, already for a closed unit disc, the $\mathbb{F}_p$-cohomology is infinite-dimensional, due to the presence of Artin-Schreier covers. This is in stark contrast with the ℓ-adic case ($\ell \neq p$), where strong finiteness statements are known by work of Berkovich and Huber, [9], [40].

Before explaining the proof of the theorem, let us recall another result from [59].

Theorem 11.5 ([59, Theorem 1.2]). *Let U be a connected affinoid rigid-analytic variety over C. Then U is a $K(\pi, 1)$ for p-torsion coefficients. In other words, for every p-torsion local system $\mathbb{L}$ on U, the natural map*

$$H^i(X_{\text{ét}}, \mathbb{L}) \to H^i(\pi_1(X, \bar{x}), \mathbb{L}_{\bar{x}})$$

is a bijection, where $\bar{x} \in X(C)$ is a base point, and $\pi_1(X, \bar{x})$ is the profinite étale fundamental group.

There is Artin's theorem on good neighborhoods which states that a smooth algebraic variety in characteristic 0 is locally a $K(\pi, 1)$. It is interesting to note that no smallness or smoothness assumption is necessary for this result in the p-adic world. Let us briefly sketch its proof as this gives a good impression on how perfectoid spaces are used in applications to p-adic Hodge theory. Let $\tilde{U} \to U$ be 'the universal cover of U', which is the inverse limit of all finite étale covers. It is not hard to see that $\tilde{U}$ is an affinoid perfectoid space. Essentially, the existence of enough p-th roots is assured as taking p-th roots is finite étale in characteristic 0. By formal nonsense, it is enough to prove that $H^i(\tilde{U}_{\text{ét}}, \mathbb{F}_p) = 0$ for $i > 0$; we already know that $H^1(\tilde{U}_{\text{ét}}, \mathbb{F}_p) = 0$ as this parametrizes finite étale $\mathbb{F}_p$-torsors, of which there are no more. Thus, we need to prove that $H^i(\tilde{U}_{\text{ét}}, \mathbb{F}_p) = 0$ for $i > 1$.

Lemma 11.6. *Let Y be an affinoid perfectoid space. Then $H^i(Y_{\text{ét}}, \mathbb{F}_p) = 0$ for $i > 1$.*

Proof. By tilting, we may assume that Y is of characteristic p. Then we have the Artin-Schreier sequence

$$0 \to \mathbb{F}_p \to \mathcal{O}_Y \to \mathcal{O}_Y \to 0 \ ,$$

and the result follows from vanishing of coherent cohomology: $H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$. □

Thus, the general idea is to cover X locally by pro-étale maps from perfectoid spaces, and then use qualitative properties of perfectoid spaces, which are verified in characteristic p. For this purpose, one introduces the pro-étale site $X_{\text{proét}}$, in which X is locally perfectoid in a suitable sense.[8]

By resolution of singularities for rigid-analytic varieties, the proof of the finiteness theorem reduces to the proper smooth case; moreover, it is enough to handle the case of $\mathbb{F}_p$-coefficients. In that case, the argument is involved and makes heavy use of the full machinery of perfectoid spaces, cf. [59]. Roughly, it proceeds in two steps. First, one shows that $H^i_{\text{ét}}(X, \mathcal{O}_X^+/p)$ is almost finitely generated. This makes use of the Cartan-Serre technique of shrinking covers, and the almost vanishing of $H^i(Y_{\text{ét}}, \mathcal{O}_Y^+)$ on affinoid perfectoid spaces Y. As stated above, this vanishing is a strengthening of Faltings's almost purity theorem. One applies it by locally covering X by perfectoid spaces. Then one uses a variant of the Artin-Schreier sequence

$$0 \to \mathbb{F}_p \to \mathcal{O}_X^+/p \to \mathcal{O}_X^+/p \to 0$$

to deduce finiteness of $\mathbb{F}_p$-cohomology. In fact, one gets the following basic comparison result at the same time.

Theorem 11.7 ([59, Theorem 1.3], [60, Theorem 3.17]). *Let X be a proper rigid-analytic variety over C. Then the natural map*

$$H^i(X_{\text{ét}}, \mathbb{F}_p) \otimes \mathcal{O}_C/p \to H^i(X_{\text{ét}}, \mathcal{O}_X^+/p)$$

is an almost isomorphism, i.e. both the kernel and the cokernel are killed by the maximal ideal of $\mathcal{O}_C$.

This is a variant on a result of Faltings, [25, Theorem §3.8]. It forms the basic result which allows one to pass from étale cohomology to coherent cohomology (including here de Rham and Hodge cohomology). Note that the result implies the following remarkable behaviour of $M = R\Gamma(X_{\text{ét}}, \mathcal{O}_X^+)$. After inverting p, $M[p^{-1}] = R\Gamma(X_{\text{ét}}, \mathcal{O}_X) = R\Gamma(X, \mathcal{O}_X)$ is usual coherent cohomology. However, after (derived) modding out p,

$$M/p = R\Gamma(X_{\text{ét}}, \mathcal{O}_X^+/p) \cong_a R\Gamma(X_{\text{ét}}, \mathbb{F}_p) \otimes \mathcal{O}_C/p$$

is almost isomorphic to étale cohomology. In particular, $M[p^{-1}]$ lives only in degrees 0 through $\dim X$, while M itself has torsion going up until degree $2 \dim X$.

[8] The idea of the pro-étale site has turned out to be quite powerful for foundational questions, even in the case of schemes. For new foundations for ℓ-adic cohomology of schemes, see [10].

It also shows the full strength of the result that $H^i(Y_{\text{ét}}, \mathcal{O}_Y^+)$ is almost zero for $i > 0$, if Y is an affinoid perfectoid space: Certainly, nothing similar is true for a finite type space. It means that all the torsion in the cohomology of $\mathcal{O}_X^+$ gets killed after passing to perfectoid covers. This will be at the heart of the applications to torsion in the cohomology of locally symmetric spaces, cf. Section 17.

Let us now mention the analogues of the theorems in the complex world, stated earlier. For definiteness, we assume here that $X = X_0 \times_k C$ is the base-change of some X_0 defined over a completed discretely valued extension k of $\mathbb{Q}_p$ with perfect residue field. Moreover, we assume that X_0 is proper and smooth.

Theorem 11.8 ([59, Corollary 1.8]). *The G_k-representation $H^i_{\text{ét}}(X, \mathbb{Q}_p)$ is de Rham in the sense of Fontaine, and one has the comparison between étale and de Rham cohomology*

$$H^i_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \cong H^i_{\text{dR}}(X_0) \otimes_k B_{\text{dR}} \ .$$

In particular, $H^i_{\text{dR}}(X_0)$ is the filtered k-vector space associated with the de Rham G_k-representation $H^i_{\text{ét}}(X, \mathbb{Q}_p)$.

This is a known phenomenon in p-adic Hodge theory: To get the comparison theorems, one has to extend scalars to Fontaine's big period rings. Here, we use B_{dR}, which is a complete discrete valuation field with residue field C.

Theorem 11.9 ([59, Corollary 1.8]). *The Hodge-to-de Rham spectral sequence*

$$E_1^{ij} = H^j(X, \Omega_X^i) \Rightarrow H^{i+j}_{\text{dR}}(X)$$

degenerates at E_1. In particular, $H^i_{\text{dR}}(X)$ admits a decreasing de Rham filtration $\text{Fil}^j H^i_{\text{dR}}(X)$ *with associated graded pieces $H^j(X, \Omega_X^{i-j})$.*

Note that no Kähler assumption is necessary here. It is interesting to note that some non-Kähler complex manifolds have p-adic analogues, such as the Hopf surface: Divide $\mathbb{A}^2 \setminus \{(0,0)\}$ by the diagonal action of multiplication by q for some $q \in k$ with $|q| < 1$ to get a proper smooth rigid-analytic variety X. This has Hodge numbers $h^{01} = \dim H^0(X, \Omega_X^1) = 0$ while $h^{10} = \dim H^1(X, \mathcal{O}_X) = 1$, so Hodge symmetry fails. However, Hodge-to-de Rham degeneration holds true for the Hopf surface. Fortunately, Iwasawa manifolds for which the Hodge-to-de Rham degeneration fails, do not have p-adic analogues.

The next result does not need a Kähler assumption either:

Theorem 11.10 ([59, Corollary 1.8], [60, Theorem 3.20]). *There is a Hodge-Tate decomposition*

$$H^i_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \cong \bigoplus_{j=0}^{i} H^{i-j}(X, \Omega_X^j)(-j) \ .$$

Here, $(-j)$ denotes a Tate twist. More generally, if X is only defined over C, there is a Hodge-Tate spectral sequence

$$E_2^{ij} = H^i(X, \Omega_X^j)(-j) \Rightarrow H^{i+j}_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \ .$$

It degenerates at E_2 if X is algebraic or defined over k (and probably does in general), giving a Hodge-Tate filtration on $H^i_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$ with associated graded pieces $H^{i-j}(X, \Omega^j_X)(-j)$.

Note the interesting differences between the Hodge-Tate spectral sequence and the Hodge-de Rham spectral sequence: It starts at E_2, and i and j are interchanged. Moreover, a Tate twist appears.

12. Relative φ-modules: Work of Kedlaya-Liu

At around the same time that the author wrote [58], Kedlaya-Liu, [47], [48], worked out closely related results[9] with the goal of constructing $\mathbb{Q}_p$-local systems on period domains, as were conjectured by Rapoport-Zink, [55]. Let us briefly recall the conjecture of Rapoport-Zink, in the case of the group GL_n.

Fix a perfect field k of characteristic p, and let V be a k-isocrystal, i.e. a $W(k)[p^{-1}]$-vector space V of finite dimension n equipped with a σ-linear isomorphism $\phi : V \to V$. Moreover, fix a 'filtration type', i.e. for each integer $i \in \mathbb{Z}$ a multiplicity $n_i \geq 0$ such that $n = \sum n_i$. The space of decreasing filtrations $\mathrm{Fil}^\bullet V \subset V$ for which $\mathrm{gr}^i V$ has dimension n_i forms naturally an algebraic variety $\mathcal{F}$ over $W(k)[p^{-1}]$; we consider $\mathcal{F}$ as an adic space over $W(k)[p^{-1}]$.

If $x \in \mathcal{F}(K)$ is a point defined over a finite extension K of $W(k)[p^{-1}]$, then, by a theorem of Colmez-Fontaine, [17], the triple $(V, \phi, \mathrm{Fil}^\bullet)$ comes from a crystalline representation $\mathbb{L}(x)$ of G_K if and only if it is weakly admissible. Weak admissibility is an analogue of a semistability condition, comparing Hodge and Newton slopes. There is a maximal open subspace $\mathcal{F}^{wa} \subset \mathcal{F}$ whose classical points are the weakly admissible points, cf. [55].

Conjecture 12.1. *For any smooth subspace $X \subset \mathcal{F}$ such that the universal filtration restricted to X satisfies Griffiths transversality, there exists a natural open subset $X^a \subset X^{wa} := X \cap \mathcal{F}^{wa}$ with the same classical points, and a $\mathbb{Q}_p$-local system $\mathbb{L}(X)$ on X^a, which gives the G_K-representation $\mathbb{L}(x)$ when passing to the fibre over any $x \in X^a(K)$.*

The original conjecture of Rapoport-Zink was more optimistic in that it conjectured the existence of $\mathbb{L}(X)$ for $X = \mathcal{F}$, and not only on subspaces where Griffiths transversality is satisfied. However, this does not fit with the p-adic Hodge theory formalism. Note that if the filtration is of 'minuscule type', meaning that $n_i \neq 0$ for at most two consecutive i, then Griffiths transversality is satisfied on all of $\mathcal{F}$. This assumption is satisfied in all cases investigated in [55], which are related to p-divisible groups.

Kedlaya announced a proof of this conjecture in [45]. Very roughly, the strategy of Kedlaya-Liu is to construct the local system locally in the pro-étale site and then glue. This reduces the problem to the perfectoid case. Moreover, now one has to construct a $\mathbb{Q}_p$-local system on the perfectoid space, or equivalently its tilt. But

[9] In particular, they proved Corollary 6.12 independently.

in characteristic p, $\mathbb{Q}_p$-local systems can be constructed from φ-modules by Artin-Schreier-Witt theory. Thus, first they build a φ-module over a relative Robba ring. Then they need to show that the locus where this φ-module is pure of slope 0 is open, and that locally on this locus, an integral structure exists. These theorems are proved in [47]; they generalize previous results of Kedlaya on slope filtrations and the existence of integral structures in the absolute setting, [44].

13. Universal covers of p-divisible groups and abelian varieties

The following definition of a universal cover arose repeatedly in recent years, cf. e.g. [27], [30]. We identify a formal scheme S with the functor it represents on (discrete) rings, so e.g.

$$(\mathrm{Spf}\, \varprojlim A/I^n)(R) = \varinjlim \mathrm{Hom}(A/I^n, R)\ .$$

By a commutative group G over S, we mean an fqpc sheaf of commutative groups on the category of discrete rings living over S. In other words, for any discrete ring R with an R-valued point $S(R)$, one has a commutative group $G(R)$, satisfying fpqc descent. We are particularly interested in the cases where G is an abelian variety or a p-divisible group.

Definition 13.1. Let S be a formal scheme over $\mathrm{Spf}\, \mathbb{Z}_p$, and let G/S be a commutative group. The universal cover $\tilde{G}$ of G is defined as $\tilde{G} = \varprojlim_{\times p} G$.

Let us record several examples.

(i) If $G = \mathbb{G}_m$ is the multiplicative group over $\mathbb{Z}_p$, then $\tilde{G} = \mathrm{Spf}\, \mathbb{Z}_p\langle T^{\pm 1/p^\infty}\rangle$.

(ii) If $G = \mathrm{Spf}\, R[[T_1, \ldots, T_d]]$ is a formal p-divisible group over a ring R, then $\tilde{G} \cong \mathrm{Spf}\, R[[T_1^{1/p^\infty}, \ldots, T_d^{1/p^\infty}]]$. Indeed, if R is of characteristic p, then this follows from the fact that $p = FV$, and, since G is formal, a power of F is divisible by p, so that $\varprojlim_p G = \varprojlim_F G^{(p^n)}$. In general, it follows from rigidity, cf. below. In particular, if $R = \mathcal{O}_K$ is the ring of integers in a perfectoid field K, then the generic fibre $\tilde{G}_\eta$ of $\tilde{G}$ is canonically a perfectoid space over K.

(iii) If $G = \mathbb{G}_a$ is the additive group, then $\tilde{G} = 0$. Indeed, if R is a discrete ring over $\mathrm{Spf}\, \mathbb{Z}_p$, then p is nilpotent in R. This implies that $\tilde{G}(R) = \varprojlim_{\times p} R$ is Mittag-Leffler zero.

For any formal scheme S over $\mathrm{Spf}\, \mathbb{Z}_p$, we may consider the categories of universal covers of abelian varieties, resp. universal covers of p-divisible groups, over S, as full subcategories of the category of commutative groups over S.

Proposition 13.2. *Let $S' \subset S$ be a closed immersion of formal schemes defined by a topologically nilpotent ideal. Then the categories of universal covers of abelian varieties (resp. p-divisible groups) over S and S' are equivalent.*

Proof. Using that abelian varieties and p-divisible groups deform, one gets essential surjectivity. To prove full faithfulnes, one reduces formally to the case of a nilpotent immersion of schemes $S' \subset S$. In that case, one knows that the categories of abelian varieties (resp. p-divisible groups) up to p-power isogeny are equivalent, cf. e.g. [41]. As passage to universal covers turns p-power isogenies into isomorphisms, the result follows. □

Thus, the universal cover may be considered as a crystal on the infinitesimal site. In particular, let us fix an abelian variety or a p-divisible group G_0 over a perfect field k of characteristic p, of height h. It has a universal deformation space $S \cong \operatorname{Spf} W(k)[[T_1, \ldots, T_k]]$ (cf. [41]), and a universal deformation G/S. However, the universal cover $\tilde{G}$ is constant, equal to the evaluation of the crystal $\tilde{G}_0$ on the thickening $S \to \operatorname{Spec} k$.

Note that inside $\tilde{G}$ one has the Tate module $T_pG = \ker(\tilde{G} \to G) = \varprojlim_{\times p} G[p^n]$. If one fixes a C-valued point of the generic fibre of S, where C is an algebraically closed complete extension of $W(k)[p^{-1}]$, then $\Lambda = (T_pG)(\mathcal{O}_C) \cong \mathbb{Z}_p^h \subset \tilde{G}(\mathcal{O}_C)$ is a $\mathbb{Z}_p$-lattice. Informally, one gets back $G = \tilde{G}/T_pG$ by quotienting $\tilde{G}$ by this $\mathbb{Z}_p$-lattice. Here, $\tilde{G}$ is independent of the chosen point, but the $\mathbb{Z}_p$-lattice varies. This is reminiscent of the complex uniformization of abelian varieties: Their universal cover is constant, and different abelian varieties correspond to different $\mathbb{Z}$-lattices in the universal cover. Riemann's theorem gives a condition on when the quotient exists as an algebraic variety in terms of the existence of a polarization.

The following theorem is proved in joint work with Weinstein.[10]

Theorem 13.3 ([62, Theorem D]). *Fix a p-divisible group G_0 over a perfect field k of height h and dimension d, as well as a complete and algebraically closed extension C of $W(k)[p^{-1}]$. Consider the category of lifts (G, ρ) of G_0 to $\mathcal{O}_C$ up to quasi-isogeny: Here, $G/\mathcal{O}_C$ is a p-divisible group, and $\rho : G_0 \times_k \mathcal{O}_C/p \to G \times_{\mathcal{O}_C} \mathcal{O}_C/p$ is a quasi-isogeny. Then the category of lifts (G, ρ) is equivalent to the category of $\mathbb{Z}_p$-lattices $\Lambda \cong \mathbb{Z}_p^h \subset \tilde{G}_0(\mathcal{O}_C)$ for which there exists a (necessarily unique) $h - d$-dimensional subspace $W \subset M(G_0)(\mathcal{O}_C)[p^{-1}] \cong C^h$ such that the image of Λ under the quasi-logarithm map*

$$\mathrm{qlog} : \tilde{G}_0(\mathcal{O}_C) \to M(G_0)(\mathcal{O}_C)[p^{-1}]$$

lies in W, and

$$0 \to \Lambda[p^{-1}] \to \tilde{G}_0(\mathcal{O}_C) \to C^h/W \to 0$$

is exact.

[10] One may deduce a similar result for abelian varieties by using Serre-Tate theory if one incorporates a polarization to guarantee algebraization.

To understand the last condition, consider the universal vector extension EG_0 of G_0, which is a crystal on the crystalline site of k. Then $M(G_0) = \mathrm{Lie}\, EG_0$ is the covariant Dieudonné module of G_0. Now one has the logarithm map

$$\log_{EG_0} : EG_0(\mathcal{O}_C) \to \mathrm{Lie}\, EG_0(\mathcal{O}_C)[p^{-1}] = M(G_0)(\mathcal{O}_C)[p^{-1}] \ .$$

On the other hand, $\tilde{G}_0 = \widetilde{EG_0}$, as the universal cover of vector groups vanishes. Thus, one gets the quasi-logarithm map

$$\mathrm{qlog} : \tilde{G}_0(\mathcal{O}_C) = \widetilde{EG_0}(\mathcal{O}_C) \to EG_0(\mathcal{O}_C) \to M(G_0)(\mathcal{O}_C)[p^{-1}] \ .$$

After fixing a lift G and composing with the Hodge filtration $M(G_0)(\mathcal{O}_C)[p^{-1}] \to (\mathrm{Lie}\, G)[p^{-1}]$ corresponding to G, one gets the logarithm map $\tilde{G}_0(\mathcal{O}_C) = \tilde{G}(\mathcal{O}_C) \to G(\mathcal{O}_C) \to \mathrm{Lie}\, G[p^{-1}]$. Thus, the quasi-logarithm map is a 'universal logarithm for all possible lifts of G_0'. The condition comes from the fact that the image of Λ lies in the kernel of the Hodge filtration $M(G_0)(\mathcal{O}_C)[p^{-1}] \to (\mathrm{Lie}\, G)[p^{-1}]$, which is of dimension $h - d$. Moreover, the sequence

$$0 \to \Lambda[p^{-1}] \to \tilde{G}(\mathcal{O}_C) \to (\mathrm{Lie}\, G)[p^{-1}] \to 0$$

is exact.

This gives one analogue of Riemann's theorem on the classification of complex abelian varieties. The following theorem, again proved in joint work with Weinstein, [62], and closely related to the previous theorem, gives a different such analogue. For this, we use that any p-divisible group G over $\mathcal{O}_C$ has a Hodge-Tate filtration

$$0 \to (\mathrm{Lie}\, G) \otimes_{\mathcal{O}_C} C(1) \to T_pG \otimes_{\mathbb{Z}_p} C \to (\mathrm{Lie}\, G^*)^* \otimes_{\mathcal{O}_C} C \to 0 \ ,$$

which is an analogue of the Hodge-Tate filtration defined above for proper smooth varieties over C, cf. Theorem 11.10. This Hodge-Tate filtration for p-divisible groups was known previously, and is due to Faltings, [24], cf. also Fargues, [28].

Theorem 13.4 ([62, Theorem B]). *The category of p-divisible groups over $\mathcal{O}_C$ is equivalent to the category of pairs (Λ, W) where Λ is a finite free $\mathbb{Z}_p$-module, and $W \subset \Lambda \otimes_{\mathbb{Z}_p} C$ is a C-subvectorspace.*

The functor is given by $G \mapsto (T_pG, \mathrm{Lie}\, G \otimes_{\mathcal{O}_C} C(1))$. This is analogous to the classification of complex abelian varieties by their first singular homology, together with the Hodge filtration.

14. Lubin-Tate spaces: Work of Weinstein

Weinstein has observed that the Lubin-Tate tower at infinite level carries a natural structure as a perfectoid space. For this, fix an integer $n \geq 1$ and a p-divisible group G_0 of dimension 1 and height n. The Lubin-Tate tower at infinite level

$\mathcal{M}_{G_0,\infty}$ parametrizes triples (G,ρ,α) where (G,ρ) is a deformation of G_0 up to quasi-isogeny as before, and $\alpha : \mathbb{Z}_p^n \to T_pG$ is an infinite level structure.

One may define a p-divisible group $\bigwedge G_0$ of G_0 of dimension 1 and height 1 by taking the highest exterior power of the Dieudonné module $M(G_0)$, and passing back to p-divisible groups. This uses crucially that G_0 is of dimension 1. One may construct an alternating map

$$\det : \tilde{G}_0 \otimes \ldots \otimes \tilde{G}_0 \to \widetilde{\bigwedge G_0} \ .$$

This follows from the work of Hedayatzadeh, [36], or from a result in Dieudonné theory in the joint work with Weinstein, [62]. Fix a perfectoid field K; then this gives a similar map on the generic fibre, base-changed to K:

$$\det : \tilde{G}_{0,K} \otimes \ldots \otimes \tilde{G}_{0,K} \to \widetilde{\bigwedge G_0}_K \ .$$

Inside $\widetilde{\bigwedge G_0}_K$, one has the rational Tate module $V_p(\bigwedge G_0) \subset \widetilde{\bigwedge G_0}_K$ and an exact sequence

$$0 \to V_p(\bigwedge G_0) \to \widetilde{\bigwedge G_0}_K \xrightarrow{\log} \mathbb{G}_{a,K} \to 0 \ .$$

The following theorem is easy to deduce from Theorem 13.3, but was proved earlier directly by Weinstein.

Theorem 14.1 (Weinstein). *The following diagram is cartesian:*

$$\begin{tikzcd}
\mathcal{M}_{G_0,\infty} \arrow[r, hook] \arrow[d] & \left(\tilde{G}_{0,K}\right)^n \arrow[d, "\det"] \\
V_p(\bigwedge G_0) \setminus \{0\} \arrow[r, hook] & \widetilde{\bigwedge G_0}_K
\end{tikzcd}$$

All intervening objects are perfectoid spaces over K, and the inclusions are locally closed (i.e., open subsets of Zariski closed subsets).

All objects in this diagram can be made completely explicit. Weinstein has used this to find explicit affinoid perfectoid subsets of $\mathcal{M}_{G_0,\infty}$ whose cohomology realizes the local Langlands correspondence for specific supercuspidal representations, cf. [12]. Recall that it is known (by the work of Harris-Taylor, [34]) that the cohomology of $\mathcal{M}_{G_0,\infty}$ realizes the local Langlands correspondence for all supercuspidal representations of $\mathrm{GL}_n(\mathbb{Q}_p)$. It is remarkable that while at any finite level, one cannot give an explicit description of the Lubin-Tate tower, it is possible to describe $\mathcal{M}_{G_0,\infty}$, together with all group actions, explicitly.

In [62], it is proved that more general Rapoport-Zink spaces become perfectoid at infinite level, and a description purely in terms of p-adic Hodge theory is given. This made it possible to prove the duality isomorphism for basic Rapoport-Zink spaces. In particular, one gets that Drinfeld and Lubin-Tate tower are isomorphic at infinite level as perfectoid spaces. This improves on earlier results of Faltings, [26], and Fargues, [28], who proved such isomorphisms, but had to struggle with formalizing them, as no category was known in which both infinite level spaces lived a priori. Their method is to work with suitable formal models; for this, new formal models have to be constructed first, which is at least technically challenging.

It was recently suggested by Rapoport-Viehmann, [53], that there should exist a theory of 'local Shimura varieties', which should relate to Rapoport-Zink spaces in the same way that general Shimura varieties relate to Shimura varieties of PEL type. The new perspective on Rapoport-Zink spaces mentioned above should make it possible to prove (parts of) their conjectures.

15. *p*-adic cohomology of the Lubin-Tate tower

The Lubin-Tate tower plays an important role in the Langlands program because its ℓ-adic cohomology for $\ell \neq p$ realizes the local Langlands correspondence, cf. [34]. In the emerging p-adic local Langlands program, which has taken a definitive form only for $\mathrm{GL}_2(\mathbb{Q}_p)$, cf. [13], one hopes for a similar realization of the p-adic local Langlands correspondence. However, the $\mathbb{F}_p$-cohomology of the Lubin-Tate tower is too infinite due to the presence of many Artin-Schreier covers. Still, a variant of Theorem 11.4 holds true in this context; for simplicity, we state only the version with $\mathbb{F}_p$-coefficients; a similar result holds true with $\mathbb{Z}_p$-coefficients.

Let F be a finite extension of $\mathbb{Q}_p$. Fix an admissible $\mathbb{F}_p$-representation π of $\mathrm{GL}_n(F)$. Using the Lubin-Tate tower at infinite level, which is a $\mathrm{GL}_n(F)$-torsor over $\mathbb{P}^{n-1}_{\breve{F}}$, where $\breve{F}$ denotes the completion of the maximal unramified extension of F, one gets an étale sheaf $\mathcal{F}_\pi$ on $\mathbb{P}^{n-1}_{\breve{F}}$. It is naturally $D^\times$-equivariant, and equipped with a Weil descent datum. Here, D is the division algebra of invariant $1/n$ over F. The following theorem is work in progress of the author, and relies on the techniques of the proof of Theorem 11.4 along with the duality between Lubin-Tate and Drinfeld tower.

Theorem 15.1. *Let $C/\breve{F}$ be complete and algebraically closed. Then $H^i(\mathbb{P}^{n-1}_C, \mathcal{F}_\pi)$ is an admissible $D^\times$-representation, which vanishes for $i > 2(n-1)$, and is independent of C. The resulting functor from admissible $\mathrm{GL}_n(F)$-representations to admissible $D^\times \times G_F$-representations is compatible with some global correspondences.*

This makes it possible to pass from $\mathrm{GL}_n(F)$-representations to Galois representations in a purely local way. In the global setup, it proves that the $\mathrm{GL}_n(F)$-representation determines the local Galois group representation.

16. Shimura varieties

Fix a reductive group G over $\mathbb{Q}$ with a Shimura datum of Hodge type, giving rise to a Shimura variety S_K, $K \subset G(\mathbb{A}_f)$, over the reflex field E. There is a

Hecke-equivariant compactification $S^{\ast}_{K}$, finite under the minimal compactification $S^{*}_{K} \to S^{\ast}_{K}$, and a flag variety $\mathcal{F}$ with G-action, such that the following are true.[11]

Theorem 16.1. *Fix a tame level $K^p \subset G(\mathbb{A}_f^p)$ and a map $E \to C$ to a complete and algebraically closed extension C of $\mathbb{Q}_p$. Let $(S^{\ast}_K)^{\mathrm{ad}}$ denote the adic space associated with $S^{\ast}_K \otimes_E C$. Then there is a perfectoid space $S^{\ast}_{K^p}$ over C such that*

$$S^{\ast}_{K^p} \sim \varprojlim_{K_p} (S^{\ast}_{K_pK^p})^{\mathrm{ad}} \ .$$

Moreover, there is a $G(\mathbb{Q}_p)$-action on $S^{\ast}_{K^p}$ and a $G(\mathbb{Q}_p)$-equivariant Hodge-Tate period map

$$\pi_{\mathrm{HT}} : S^{\ast}_{K^p} \to \mathcal{F} \ .$$

The map π_{HT} is equivariant for the Hecke operators prime to p with respect to the trivial action on $\mathcal{F}$; in particular, π_{HT} contracts $G(\mathbb{A}_f^p)$-orbits. There is a cover of $\mathcal{F}$ by affinoid subsets $U \subset \mathcal{F}$ for which $\pi_{\mathrm{HT}}^{-1}(U) \subset S^{\ast}_{K^p}$ is an affinoid perfectoid subset.

The geometry of π_{HT} is very interesting. Consider the case of the modular curve. Here, $\mathcal{F} = \mathbb{P}^1$, and π_{HT} is a p-adic analogue of the embedding of the complex upper half-plane (which is a path-connected component of the inverse limit over all levels $\varprojlim_K S_K(\mathbb{C})$) into $\mathbb{P}^1(\mathbb{C})$. In both cases, the map is given by the Hodge filtration.

In the case of the modular curve, $S^{\ast}_{K^p} = S^{*}_{K^p}$ has a stratification into the ordinary and the supersingular locus, $S^{\mathrm{ord}}_{K^p}$ and $S^{\mathrm{ss}}_{K^p}$.[12] The flag variety is $\mathcal{F} = \mathbb{P}^1$. Then, under π_{HT}, all of $S^{\mathrm{ord}}_{K^p}$ maps into $\mathbb{P}^1(\mathbb{Q}_p)$, while the supersingular locus $S^{\mathrm{ss}}_{K^p}$ maps into Ω^2. Here, $\Omega^2 = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{Q}_p)$ is Drinfeld's upper half-plane, which is reminiscent of the complex upper and lower half-plane, which can be written as $\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{R})$. It follows that π_{HT} contracts connected components of the ordinary locus to points, whereas it does something interesting on the supersingular locus.

On the ordinary locus, the map is given by the position of the canonical subgroup. On the supersingular locus, $S^{\mathrm{ss}}_{K^p}$ is a finite disjoint union of Lubin-Tate towers at infinite level (for $n = 2$); these are isomorphic to the Drinfeld tower at infinite level, which is a pro-finite étale cover of Ω^2. The composite is π_{HT}. In particular, the isomorphism between Lubin-Tate and Drinfeld tower is built into the geometry of π_{HT}.

Let us note another perspective on what the Hodge-Tate period map does. Namely, by Theorem 13.4, giving the Hodge filtration is equivalent to giving the p-divisible group. This means that the Hodge-Tate period map, on geometric points of the good reduction locus, is the map sending an abelian variety to its p-divisible group (equipped with all extra structure).

[11] It should be possible to use the minimal compactification itself, and make $\mathcal{F}$ more explicit, but so far this has not been worked out.

[12] We regard some points of the adic space corresponding to rank-2-valuations as part of the ordinary locus which would usually be considered as part of the supersingular locus. We do so by replacing the ordinary part by its closure.

17. Torsion in the cohomology of locally symmetric varieties

As the final topic, we summarize the application of these ideas to the study of torsion in the cohomology of locally symmetric spaces.

Fix a reductive group G over $\mathbb{Q}$. For any (sufficiently small) compact open subgroup $K \subset G(\mathbb{A}_f)$, one has the locally symmetric space

$$Y_K = G(\mathbb{Q})\backslash (G(\mathbb{R})/K_\infty A_\infty^\circ \times G(\mathbb{A}_f)/K)\ ,$$

where $K_\infty \subset G(\mathbb{R})$ is a maximal compact subgroup, and $A_\infty \subset G(\mathbb{R})$ are the $\mathbb{R}$-valued points of a maximal $\mathbb{Q}$-split central torus, with identity component A_∞°. Fixing a tame level $K^p \subset G(\mathbb{A}_f^p)$, one defines the completed cohomology groups

$$\tilde{H}^i(K^p) = \varprojlim_n \varinjlim_{K_p} H^i(Y_{K_pK^p}, \mathbb{Z}/p^n\mathbb{Z})\ ,\ \tilde{H}^i_c(K^p) = \varprojlim_n \varinjlim_{K_p} H^i_c(Y_{K_pK^p}, \mathbb{Z}/p^n\mathbb{Z})\ .$$

Also recall the cohomological degree q_0, which is 'the first interesting cohomological degree' (namely, the first one to which tempered automorphic representations of G contribute). The following conjecture was proposed by Calegari and Emerton, [14].

Conjecture 17.1. *The completed cohomology groups $\tilde{H}^i(K^p)$, $\tilde{H}^i_c(K^p)$ vanish for $i > q_0$.*

Concretely, this means that all cohomology classes in higher degree become infinitely p-divisible as one goes up along all levels at p. If G is a torus, the conjecture is equivalent to Leopoldt's conjecture. On the other hand, we proved the following theorem.

Theorem 17.2 ([61, Theorem I.7]). *Assume that G gives rise to a Shimura variety, so that q_0 is the (complex) dimension of the associated Shimura variety. Then Conjecture 17.1 holds true for compactly supported cohomology.*

If one establishes that also toroidal compactifications become perfectoid at infinite level, then one gets the same result for usual cohomology. Unfortunately, for all tori which give rise to Shimura varieties, the Leopoldt conjecture is trivially satisfied, as the group of units is finite.

The key to the proof is to translate everything into the setting of Shimura varieties at infinite level as perfectoid spaces. In that case, one can use the basic comparison theorem to pass to the cohomology of $\mathcal{O}^+/p$. But at infinite level, one has almost vanishing of higher cohomology of $\mathcal{O}^+/p$ on affinoids as the space is perfectoid. This shows vanishing above the middle dimension, which is exactly the desired statement.

In fact, the same argument proves the following theorem over $\mathbb{C}$, which the author does not know how to prove directly.

Theorem 17.3. *Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be a closed subvariety of dimension d. For any $m \geq 0$, let $X_m \subset \mathbb{P}^n_{\mathbb{C}}$ be the pullback of X under the map $\mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$ sending $(x_0 : \ldots : x_n)$ to $(x_0^{p^m} : \ldots : x_n^{p^m})$. Then, for any $i > d$,*

$$\varinjlim_m H^i(X_m, \mathbb{F}_p) = 0 \ .$$

For classes in the image of cup product with $c_1(\mathcal{O}(1))$, this follows from the fact that $c_1(\mathcal{O}(1))$ becomes infinitely p-divisible. By hard Lefschetz, this accounts for everything rationally, but it does not say anything about possible p-torsion in the cohomology.

18. Galois representations

It was conjectured since the 1970's by Grunewald that torsion in the cohomology of locally symmetric spaces gives rise to Galois representations. This conjecture was made precise by Ash, [2], and is a 'mod p analogue' of (one direction of) the global Langlands conjectures. Since then, it was numerically verified in many cases: what happens is that a Hecke eigenvalue system matches Frobenius eigenvalues of a Galois representations for the first few hundred primes. However, even in these examples, one could not prove that this happens for *all* primes.

Theorem 18.1 ([61, Theorem I.3]). *Let G be the restriction of scalars of GL_n from a totally real or CM field F. Fix any compact open subgroup $K \subset G(\mathbb{A}_f)$. Then, for any system of Hecke eigenvalues ψ appearing in $H^i(Y_K, \overline{\mathbb{F}}_p)$, there exists a (unique) continuous semisimple Galois representation*

$$\rho_\psi : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$$

such that for all but an explicit finite set of 'ramified' places v of F, the characteristic polynomial of $\rho_\psi(\mathrm{Frob}_v)$ is described by the Hecke eigenvalues.

Moreover, there is a version of this theorem for $\mathbb{Z}/p^n\mathbb{Z}$-cohomology, which in the inverse limit over n gives results for classical automorphic representations. The following result was proved earlier by Harris-Lan-Taylor-Thorne, [33], by a different method.

Theorem 18.2 ([61, Theorem I.4]). *Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, where F is totally real or CM. Fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$. Then there exists a unique continuous semisimple Galois representation*

$$\rho_{\pi,p} : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$$

such that for all but an explicit finite set of 'ramified' places v of F, the characteristic polynomial of $\rho_{\pi,p}(\mathrm{Frob}_v)$ is described by the Satake parameters.

It should be noted that in general, the cohomology of the spaces Y_K has a lot of torsion. The simplest example is the case of GL_2 over an imaginary-quadratic field in which the relevant Y_K are hyperbolic 3-manifolds. In that case, computations, as well as theoretical results, show a huge amount of torsion, cf. e.g. [7]. Therefore, the thrust of the above theorem lies in the p-torsion part of the cohomology. Moreover, recent work of Calegari-Geraghty, [15], explains how one may use sufficiently fine information about existence of Galois representations for torsion classes to prove automorphy lifting theorems for GL_n over F. These properties seem to be within reach. Together with the strong potential automorphy machinery as in the work of Barnet-Lamb–Gee–Geraghty–Taylor, [6], this gives some hope that one can establish potential converse results to Theorem 18.2.

Let us now briefly sketch the proof of Theorem 18.1 in the case $F = \mathbb{Q}$. In that case, one considers the Siegel moduli space S_K, $K \subset \mathrm{GSp}_{2n}(\mathbb{A}_f)$, i.e. the moduli space of principally polarized abelian varieties of dimension n. From the Borel-Serre compactification, [11], it follows that the cohomology of the locally symmetric space for GL_n contributes to the cohomology of the Siegel moduli space. Note that the Borel-Serre compactification is a compactification as a real manifold with corners; this makes it possible that a purely real manifold appears in the boundary of the algebraic variety S_K. Thus, the task becomes to understand torsion in the cohomology of S_K. The theorem is the following.

Theorem 18.3 ([61, Theorem I.5]). *Let S_K, $K \subset G(\mathbb{A}_f)$, be any Shimura variety of Hodge type. Then, for any system of Hecke eigenvalues ψ appearing in $H^i_c(S_{K,\mathbb{C}}, \overline{\mathbb{F}}_p)$, there exists a cuspidal eigenform f (possibly of larger level at p, and undetermined weight) such that the Hecke eigenvalues of f are congruent to ψ modulo p.*

This produces congruences between torsion classes and classical cusp forms in large generality. Note that the classes in which we are interested start life as classes coming from the boundary; still, the theorem produces congruences to cusp forms. In particular, for non-torsion classes, it is interesting as it produces congruences between Eisenstein series and cusp forms. However, in the complementary case where S_K is proper, the theorem is also interesting as it controls all possible torsion classes. For example, it proves the existence of Galois representations for all torsion classes in $U(1, n-1)$-Shimura varieties, which is required in recent work of Emerton and Gee, [22]. The point is that one knows how to attach Galois representations to cusp forms in great generality, through the work on automorphic forms on classical groups by Arthur [3] (cf. also [51] for unitary groups) and the work of Clozel, Kottwitz and Harris-Taylor among others on the cohomology of Shimura varieties, [16], [50], [34].

To prove the theorem, one starts by using the basic comparison theorem

$$H^i_{c,\mathrm{\acute{e}t}}(S_{K,C}, \mathbb{F}_p) \otimes \mathcal{O}_C/p \cong_a H^i_{\mathrm{\acute{e}t}}(S^*_{K,C}, I^+/p) ,$$

where $I^+ \subset \mathcal{O}^+$ is the ideal sheaf of functions vanishing at the boundary. This variant of Theorem 11.7 is proved in [60, Theorem 3.13]. This provides a first bridge to the sheaf of cusp forms I^+, but one still has to compute cohomology on the étale site. Next, one passes to infinite level at p, and reduces to controlling

$$H^i_{\text{ét}}(S^*_{K^p}, I^+/p) \ .$$

Here, $S^*_{K^p}$ is perfectoid, so we know that $H^i_{\text{ét}}(U, I^+/p)$ is almost zero for $i > 0$ and affinoid perfectoid subsets $U \subset S^*_{K^p}$; this is a slight variant on Theorem 6.10. This means that $H^i_{\text{ét}}(S^*_{K^p}, I^+/p)$ can (almost) be computed by a Cech complex whose terms are the sections of I^+/p on affinoid subsets. The remaining task is to approximate these forms on U by globally defined forms (of finite level), without messing up the Hecke eigenvalues. Usually, the strategy is to multiply by a multiple of the Hasse invariant. This kills all poles away from the ordinary locus, and works if U is the ordinary locus. However, in our case we need to do the same for a covering of all of $S^*_{K^p}$.

The crucial property of the Hasse invariant is that it commutes with all Hecke operators prime to p. In our setup, we can use the following construction: As

$$\pi_{\mathrm{HT}} : S^*_{K^p} \to \mathcal{F}$$

is equivariant with respect to the trivial action of the Hecke operators prime to p on $\mathcal{F}$, any function that gets pulled back from $\mathcal{F}$ will commute with all Hecke operators prime to p. The same stays true for sections of automorphic vector bundles; automorphic vector bundles come via pullback from $\mathcal{F}$.[13] In this way, one gets enough 'fake-Hasse invariants' to proceed, and prove the result.

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