In this ARGOS we want to study the Witt vector affine Grassmannian, often also called mixed-characteristic affine Grassmannian, $\text{Gr}_W^{\text{GL}_n}$. For any perfect field $k$, its $k$-valued points are given by $W(k)$-lattices in $W(k)[1/p]^n$, where $W(k)$ are the Witt vectors of $k$. For a long time, the Witt vector affine Grassmannian was only known through this set of points. This is in contrast to the case of equal characteristics, where there is an affine Grassmannian (an ind-scheme) parametrizing $R[[t]]$-lattices in $R((t))^n$.

M. Kreidl, [K], investigated to what extent $\text{Gr}_W^{\text{GL}_n}$, when considered as a functor on all rings of characteristic $p$, was ind-representable. This seems to be not the case, but Kreidl already observed that the functor had better properties when restricted to perfect rings, where it parametrizes $W(R)$-lattices in $W(R)[1/p]^n$. This led to the question whether this functor on perfect rings might be representable by an ind-(perfect scheme). The aim of the seminar is to go through the paper [BS] which proves this result. Importantly, X. Zhu, [Z], had previously proved that the functor is representable by an ind-(perfect algebraic space). The methods used in the two papers are independent.

As the functor $X \mapsto X_{\text{perf}}$ from schemes of characteristic $p$ to perfect schemes preserves the étale topology and underlying topological space, all topological questions such as connected components, dimensions, or étale cohomology, can thus be defined for objects like affine Deligne-Lusztig varieties.

Let us give a brief overview of the construction. The affine Grassmannian $\text{Gr}_W^{\text{GL}_n}$ is the functor taking a perfect $\mathbb{F}_p$-algebra $R$ to the set of finite projective $W(R)$-submodules $M \subset W(R)[1/p]^n$ such that $M[1/p] = W(R)[1/p]^n$. This is covered by the subfunctors where $p^N W(R)^n \subset M \subset p^{-N} W(R)^n$; up to translation, we may replace this by $p^2 N W(R)^n \subset M \subset W(R)^n$.

In the analogous equal characteristic situation, the $R[[t]]$-module $R[[t]]^n/M$ is actually a $R$-module, and one can define a line bundle

$$\mathcal{L} = \text{det}_R(R[[t]]^n/M)$$

This line bundle turns out to be ample, and gives a projective embedding.

However, in mixed characteristic, $W(R)^n/M$ is in general not a $R$-module. Still, one can often (e.g., if $R$ is a field) filter $W(R)^n/M$ such that all associated graded $M_i$ are finite projective $R$-modules. Then, one can define

$$\mathcal{L} = \bigotimes_i \text{det}_R(M_i)$$

One would like to know that this defines a canonical line bundle, independent of the choice of the filtration. We give two proofs of this fact. Both make use of $h$-descent for line bundles on perfect schemes: It turns out that various descent properties hold true over perfect schemes without any flatness assumptions. The first uses $K$-theory. In $K$-theoretic language, we want to construct a map

$$\tilde{\text{det}} : K(W(R)\text{ on } R) \rightarrow \text{Pic}^Z(R)$$

1 Moreover, Zhu proves the geometric Satake equivalence in this setup.

2 This is reminiscent of the theory of perfectoid spaces, but more elementary.
from a $K$-theory spectrum to the groupoid of graded line bundles on $R$.\footnote{The quotient $W(R)^n/M$ defines a point of $K(W(R)_{\text{on}} R)$, and its image is the desired line bundle.} It is well-known that there is a map
\[\det : K(R) \to \text{Pic}^Z(R),\]
and the task is to extend this along the map $\alpha : K(R) \to K(W(R)_{\text{on}} R)$. But $\alpha$ is known to be an equivalence if $R$ is the perfection of a regular $\mathbb{F}_p$-algebra. To deal with the general case, it remains to do a descent from the regular case; this is possible by $h$-descent of line bundles on perfect schemes.

Our second proof is more elementary. Here, we observe that $\mathcal{L}$ is naturally a line bundle on the Demazure resolution of a closed Schubert cell, which parametrizes filtrations of $W(R)^n/M$ whose associated graded are finite projective $R$-modules. The question becomes that of descending a line bundle along a map $f : Y \to X$ of perfect schemes. Here, we prove that if $f$ (is proper perfectly finitely presented and) satisfies $Rf_*\mathcal{O}_Y = \mathcal{O}_X$ (a condition that can be checked on fibres), then a line bundle $\mathcal{L}$ on $Y$ descends to $X$ if and only if it is trivial on all geometric fibres. After the line bundle $\mathcal{L}$ is constructed, we prove that it is ample by using a theorem of Keel on semiample line bundles in characteristic $p$.

1) **Affine Grassmannians in equal characteristics**
Define the affine Grassmannian in equal characteristics, and prove that it is representable by a strict ind-projective ind-scheme, cf. [BL, Proposition 2.3, Proposition 2.4].

2) **The $h$-topology**
Define Voevodsky’s $h$-topology, following [BS, Section 2], and characterize the class of $h$-sheaves, [BS, Proposition 2.7, Theorem 2.8], at least in the case of sheaves of groupoids (i.e., stacks).

3) **Perfect schemes**
Prove various basic properties on the comparison between schemes and perfect schemes as in [BS, Section 3].

4) **The $v$-topology on perfect schemes**
Prove that the $v$-topology on perfect schemes is subcanonical (i.e., representable presheaves are sheaves), and that the $v$-cohomology of the structure sheaf vanishes on affines, cf. [BST, Section 3]. Moreover, prove $v$-descent for vector bundles, [BS, Section 4].

5) **The geometric construction of line bundles**
Prove the geometric criterion [BS, Theorem 6.7, Lemma 6.8] for descent of vector bundles. Moreover, prove [BS, Lemma 6.10].

6) **The $K$-theoretic construction of line bundles: Recollection on $\det$**
Recall the definition of the map $\det : K(R) \to \text{Pic}^Z(R)$ from the $K$-theory spectrum of a commutative ring $R$ to the groupoid of graded line bundles on $R$, cf. [BS, Appendix].

7) **The $K$-theoretic construction of line bundles**
Extend $\det$ to a map $\tilde{\det} : K(W(R)_{\text{on}} R) \to \text{Pic}^Z(R)$ by using $v$-descent of line
bundles, cf. [BS, Section 5].

8) **Families of torsion** $W(k)$-modules

Prove the basic lemmas on torsion $W(R)$-modules, [BS, Section 7]. In particular, prove that the fibres of the Demazure resolution are geometrically connected, [BS, Lemma 7.12].

9) **The Witt vector affine Grassmannian**

Define the closed Schubert cells $Gr_{\leq \lambda}$ in the affine Grassmannian, and the Demazure resolution $\tilde{Gr}_\lambda \to Gr_{\leq \lambda}$, cf. [BS, Section 8]. Prove that the natural line bundle $\tilde{L}$ on $\tilde{Gr}_\lambda$ descends to a line bundle $L$ on $Gr_{\leq \lambda}$, cf. [BS, Theorem 8.8].

10) **Projectivity of the Witt vector affine Grassmannian**

Finish the proof by showing that the line bundle $L$ on $Gr_{\leq \lambda}$ is ample, so that $Gr_{\leq \lambda}$ is the perfection of a projective algebraic variety, cf. [BS, Section 8.4].

**References**


