SEPARATION IN CLASS FORCING EXTENSIONS

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ABSTRACT. We investigate the validity of instances of the Separation scheme in generic extensions for class forcing.

CONTENTS

1. BASIC DEFINITIONS AND NOTATION

In this paper, we will work with transitive second-order models of set theory, that is models of the form $\mathbb{M} = \langle M, \mathcal{C} \rangle$, where M is transitive and denotes the collection of *sets* of $\mathbb{M}$ and $\mathcal{C}$ denotes the collection of *classes* of $\mathbb{M}$.[1] We require that $M \subseteq \mathcal{C}$ and that elements of $\mathcal{C}$ are subsets of M, and we call elements of $\mathcal{C} \setminus M$ *proper classes* (of $\mathbb{M}$). Classical transitive first-order models of set theory are covered by our approach when we let $\mathcal{C}$ be the collection of classes definable over $\langle M, \in \rangle$. The theories that we will be working in will be fragments of *Gödel-Bernays set theory* GB. This is the same basic setup as in [HKL+16].

Notation. (1) We denote by GB^- the theory in the two-sorted language with variables for sets and classes, with the set axioms given by ZF^- with class parameters allowed in the schemata of Separation and Collection, and the class axioms of extensionality, foundation and first-order class comprehension (i.e. involving only set quantifiers). GB^- enhanced with the power set axiom is the common collection of axioms of GB. GBC is GB together with the axiom of global choice.

(2) By a countable transitive model of GB^-, GB or GBC, we mean a transitive second-order model $\mathbb{M} = \langle M, \mathcal{C} \rangle$ of GB^-, GB or GBC respectively, such that both M and $\mathcal{C}$ are countable in V.

(3) Given a transitive second-order model of the form $\mathbb{M} = \langle M, \mathcal{C} \rangle$, we let $\mathrm{Def}(\mathbb{M})$ denote the collection of subsets of M that are first-order definable over M using class parameters from $\mathcal{C}$ as predicates. Note that the axiom of first-order class comprehension implies that if $\mathbb{M} = \langle M, \mathcal{C} \rangle \models \mathsf{GB}^-$, then $\mathcal{C}$ is closed under first-order definability (over $\mathbb{M}$), that is $\mathrm{Def}(\mathbb{M}) = \mathcal{C}$.

Fix a countable transitive model $\mathbb{M} = \langle M, \mathcal{C} \rangle$ of GB^-. By a *notion of class forcing* (for $\mathbb{M}$) we mean a partial order $\mathbb{P} = \langle P, \leq_{\mathbb{P}} \rangle$ such that $P, \leq_{\mathbb{P}} \in \mathcal{C}$. We will frequently identify $\mathbb{P}$ with its domain P. In the following, we also fix a notion of class forcing $\mathbb{P} = \langle P, \leq_{\mathbb{P}} \rangle$ for $\mathbb{M}$.

We call σ a $\mathbb{P}$-*name* if all elements of σ are of the form $\langle \tau, p \rangle$, where τ is a $\mathbb{P}$-name and $p \in \mathbb{P}$. Define $M^{\mathbb{P}}$ to be the set of all $\mathbb{P}$-names that are elements of M and define $\mathcal{C}^{\mathbb{P}}$ to be the set of all

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[1] Arguing in the ambient universe V, we will sometimes refer to classes of such a model $\mathbb{M}$ as sets, without meaning to indicate that they are sets of $\mathbb{M}$. In particular this will be the case when we talk about subsets of M.

$\mathbb{P}$-names that are elements of $\mathcal{C}$. In the following, we will usually call the elements of $M^{\mathbb{P}}$ simply $\mathbb{P}$-*names* and we will call the elements of $\mathcal{C}^{\mathbb{P}}$ *class* $\mathbb{P}$-*names*. If $\sigma \in M^{\mathbb{P}}$ is a $\mathbb{P}$-name, we define

$$\operatorname{rank}\sigma = \sup\{\operatorname{rank}\tau + 1 \mid \exists p \in \mathbb{P}\,[\langle\tau, p\rangle \in \sigma]\}$$

to be its *name rank*. We will sometimes also need to use the usual set theoretic rank of some $\sigma \in M$, which we will denote as $\mathsf{rnk}(\sigma)$.

We say that a filter G on $\mathbb{P}$ is $\mathbb{P}$-*generic over* $\mathbb{M}$ if G meets every dense subset of $\mathbb{P}$ that is an element of $\mathcal{C}$. Given such a filter G and a $\mathbb{P}$-name σ, we recursively define the G-*evaluation* of σ as

$$\sigma^G = \{\tau^G \mid \exists p \in G\,[\langle\tau, p\rangle \in \sigma]\},$$

and similarly we define Γ^G for $\Gamma \in \mathcal{C}^{\mathbb{P}}$. Moreover, if G is $\mathbb{P}$-generic over $\mathbb{M}$, then we set $M[G] = \{\sigma^G \mid \sigma \in M^{\mathbb{P}}\}$ and $\mathcal{C}[G] = \{\Gamma^G \mid \Gamma \in \mathcal{C}^{\mathbb{P}}\}$.

Given an $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{m-1}, \vec{\Gamma})$, where $\vec{\Gamma} \in (\mathcal{C}^{\mathbb{P}})^n$ are class name parameters, $p \in \mathbb{P}$ and $\vec{\sigma} \in (M^{\mathbb{P}})^m$, we write $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\vec{\sigma}, \vec{\Gamma})$ if for every $\mathbb{P}$-generic filter G over $\mathbb{M}$ with $p \in G$, $\langle M[G], \Gamma_0^G, \ldots, \Gamma_{n-1}^G\rangle \models \varphi(\sigma_0^G, \ldots, \sigma_{m-1}^G, \Gamma_0^G, \ldots, \Gamma_{n-1}^G)$.

A fundamental result in the context of set forcing is the *forcing theorem*. It consists of two parts, the first one of which, the so-called *definability lemma*, states that the forcing relations are definable in the ground model, and the second part, denoted as the *truth lemma*, says that every formula which is true in a generic extension $M[G]$ is forced by some condition in the generic filter G. In the context of second-order models of set theory, this has the following natural generalization:

Definition 1.1. Let $\varphi \equiv \varphi(v_0, \ldots, v_{m-1}, \vec{\Gamma})$ be an $\mathcal{L}_\in$-formula with class name parameters $\vec{\Gamma} \in (\mathcal{C}^{\mathbb{P}})^n$.

(1) We say that $\mathbb{P}$ *satisfies the definability lemma for* φ *over* $\mathbb{M}$ if

$$\{\langle p, \sigma_0, \ldots, \sigma_{m-1}\rangle \in P \times (M^{\mathbb{P}})^m \mid p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma_0, \ldots, \sigma_{m-1}, \vec{\Gamma})\} \in \mathcal{C}.$$

(2) We say that $\mathbb{P}$ *satisfies the truth lemma for* φ *over* $\mathbb{M}$ if for all $\sigma_0, \ldots, \sigma_{m-1} \in M^{\mathbb{P}}$, and every filter G which is $\mathbb{P}$-generic over $\mathbb{M}$ with

$$\langle M[G], \Gamma_0^G, \ldots, \Gamma_{n-1}^G\rangle \models \varphi(\sigma_0^G, \ldots, \sigma_{m-1}^G, \Gamma_0^G, \ldots, \Gamma_{n-1}^G),$$

there is $p \in G$ with $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma_0, \ldots, \sigma_{m-1}, \vec{\Gamma})$.

(3) We say that $\mathbb{P}$ *satisfies the forcing theorem for* φ *over* $\mathbb{M}$ if $\mathbb{P}$ satisfies both the definability lemma and the truth lemma for φ over $\mathbb{M}$.

A particular notion of class forcing that we will make use of at several points in this paper is the following. Given $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$, let $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$ denote the notion of forcing with conditions of the form $p\colon n \to \mathrm{Ord}^M$ for $n \in \omega$, ordered by reverse inclusion. Any generic for this forcing clearly gives rise to a cofinal sequence from ω to Ord^M, this forcing does not add any new sets (i.e. $M[G] = M$ whenever G is $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$-generic over M) and it satisfies the forcing theorem. The latter (easy) facts were verified in [Fri00, Proposition 2.25] and follow from more general results in [HKL+16, Lemma 2.2, Lemma 6.3 and Theorem 6.4] or from the results of [HKS17, Section 3].

2. What is a Generic Extension for Class Forcing?

As in set forcing, where one starts with a first-order model of set theory and defines what a generic extension of such a model is for a given filter that is generic over that model, we want to define generic extensions of second-order models of set theory by class forcing. If the ground model is of the form $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$ and G is an $\mathbb{M}$-generic filter for some notion of class forcing for $\mathbb{M}$, as in set forcing one may simply consider $M[G]$ to be the generic extension (we will call such a first-order extension a *generic set-extension* below). But it seems more natural to require the generic extension to again be a second-order model, that in particular includes a predicate for the generic filter G and (after all, it should be an *extension*) that includes all the predicates that were available to us already in the ground model, that is it seems natural to require all elements of $\mathcal{C}$ to be classes in the generic extension. Note that $\mathcal{C}$ in particular includes a predicate for the ground model M. Moreover, considering that we would like our extensions to potentially be models of GB^- (at least if G is generic for a sufficiently well-behaved notion of class forcing), in the light of the axiom of first-order class comprehension we also make the natural requirement that the classes of our generic class extensions be closed under definability.

Definition 2.1. We say that $\mathcal{N}$ is a *generic class extension* of $\mathbb{M} = \langle M, \mathcal{C}\rangle$ for the notion of class forcing $\mathbb{P} \in \mathcal{C}$ and the $\mathbb{M}$-generic filter $G \subseteq \mathbb{P}$ if $\mathcal{N} = \langle N, \mathcal{E}\rangle$ with $N = M[G]$ and $\mathcal{C} \cup \{G\} \subseteq \mathcal{E} \subseteq \mathcal{C}[G]$, and $\mathcal{E}$ is closed under definability (over $\mathcal{N}$).

We will show that assuming the forcing theorem holds for $\mathbb{P}$, there is a unique generic class extension (given a fixed $\mathbb{P}$-generic filter G over $\mathbb{M}$, of course). We start with a simple observation which shows our above requirements to be somewhat reasonable.

Observation 2.2. $\mathcal{C} \cup \{G\} \subseteq \mathcal{C}[G]$.

Proof. We have to check that for every $E \in \mathcal{C} \cup \{G\}$, there is $\Gamma \in \mathcal{C}^{\mathbb{P}}$ with $\Gamma^G = E$. If $E \in \mathcal{C}$, this is witnessed by $\check{E} = \{\langle \check{x}, \mathbb{1}_{\mathbb{P}}\rangle \mid x \in E\}$. For $E = G$, this is witnessed by $\dot{G} = \{\langle \check{p}, p\rangle \mid p \in \mathbb{P}\}$. $\square$

Let $\mathbb{M}[G]$ denote $\langle M[G], \mathcal{C}[G]\rangle$. The next observation shows that if $\mathbb{P}$ satisfies the forcing theorem, then $\mathbb{M}[G]$ is actually a generic class extension of $\mathbb{M}$. We will then continue to show that it is the unique such.

Observation 2.3. *If $\mathbb{P}$ is a notion of class forcing that satisfies the forcing theorem, then*

$$\mathrm{Def}(\langle M[G], \mathcal{C}[G]\rangle) \subseteq \mathcal{C}[G].$$

Proof. Let $\varphi(v, x, \vec{C})$ be a first-order formula with parameter $x \in M[G]$ and class parameters $\vec{C} \in (\mathcal{C}[G])^n$. Let σ in $M^{\mathbb{P}}$ be a name for x, let $\vec{\Gamma} \in (\mathcal{C}^{\mathbb{P}})^n$ be class names for the elements of $\vec{C}$. Let $\Gamma = \{\langle \mu, p\rangle \mid p \Vdash_{\mathbb{P}} \varphi(\mu, \sigma, \vec{\Gamma})\} \in \mathcal{C}$. The claim follows, as in $\mathbb{M}[G]$, $\Gamma^G = \{y \mid \varphi(y, x, \vec{C})\}$. $\square$

For $\mu, \nu \in M^{\mathbb{P}}$, we let $\mathsf{op}(\mu, \nu)$ denote the canonical name in $M^{\mathbb{P}}$ for the ordered pair $\langle \mu^G, \nu^G\rangle$, that is

$$\mathsf{op}(\mu, \nu) = \{\langle\{\langle \mu, \mathbb{1}_{\mathbb{P}}\rangle\}, \mathbb{1}_{\mathbb{P}}\rangle, \langle\{\langle \mu, \mathbb{1}_{\mathbb{P}}\rangle, \langle \nu, \mathbb{1}_{\mathbb{P}}\rangle\}, \mathbb{1}_{\mathbb{P}}\rangle\}.$$

Let $E = \{\langle \sigma, \sigma^G\rangle \mid \sigma \in M^{\mathbb{P}}\}$ be the *G-evaluation function.* Let $\dot{E} = \{\mathsf{op}(\check{\sigma}, \sigma) \mid \sigma \in M^{\mathbb{P}}\} \in \mathcal{C}$, and note that $\dot{E}^G = E$, hence $E \in \mathcal{C}[G]$. The following observation shows that every element of $\mathcal{C}[G]$ is definable over $M[G]$ using predicates in $\mathcal{C}$, the predicate for the generic filter G and the G-evaluation function E.

Observation 2.4. $\mathrm{Def}(\langle M[G], \mathcal{C} \cup \{G, E\}\rangle) \supseteq \mathcal{C}[G]$.

Proof. Given $C \in \mathcal{C}[G]$, let $\Gamma \in \mathcal{C}^{\mathbb{P}}$ be such that $\Gamma^G = C$. Thus

$$\begin{aligned} C &= \{\tau^G \mid \exists p \in G\, [\langle p, \tau\rangle \in \Gamma]\} = \{x \mid \exists \tau \in M\, \exists p \in G\, [\langle p, \tau\rangle \in \Gamma \wedge \tau^G = x]\} \\ &= \{x \mid \exists \tau \in M\, \exists p \in G\, [\langle p, \tau\rangle \in \Gamma \wedge \langle \tau, x\rangle \in E]\} \in \mathrm{Def}(\langle M[G], \mathcal{C} \cup \{G, E\}\rangle). \end{aligned}$$

Lemma 2.5. *E is first-order definable over $\langle M[G], M, P, G\rangle$.* $\square$

Proof. Given a $\mathbb{P}$-name $\sigma \in M^{\mathbb{P}}$, we recursively define

$$\bar{\sigma} = \left[\{\langle \mathsf{op}(\check{\sigma}, \sigma), \mathbb{1}_{\mathbb{P}}\rangle\} \cup \bigcup_{\langle \tau, p\rangle \in \sigma} \bar{\tau}\right] \in M^{\mathbb{P}}.$$

Note that using $\bar{\sigma}$, we may (uniformly) carry out the usual recursion for defining σ^G. We can now define E by a Σ_1-formula which states that $\sigma \in M$ and that there is a recursion which defines σ^G according to the usual recursive definition of σ^G.[2] $\square$

Corollary 2.6. *Assume $\mathbb{P}$ satisfies the forcing theorem. Then* $\mathrm{Def}(\langle M[G], \mathcal{C} \cup \{G\}\rangle) = \mathcal{C}[G]$ *and hence there is a unique generic class extension of $\mathbb{M}$ for the forcing $\mathbb{P}$ and the generic filter G, namely* $\mathbb{M}[G] = \langle M[G], \mathcal{C}[G]\rangle$. $\square$

One may want to weaken the assumption that $\mathcal{C} \cup \{G\} \subseteq \mathcal{E}$ at times, therefore we also introduce the following notions.

Definition 2.7. Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a model of GB^-.

[2]Similary, we can define E by a Π_1-formula which states that for every recursion which defines σ^G according to the usual recursive definition of σ^G, σ^G assumes the correct value.

(1) We say that $\mathcal{N}$ is a *generic class pseudo-extension* of $\mathbb{M} = \langle M, \mathcal{C}\rangle$ for the forcing $\mathbb{P} \in \mathcal{C}$ and the generic $G \subseteq \mathbb{P}$ if $\mathcal{N} = \langle N, \mathcal{E}\rangle$ with $N = M[G]$ and $\{M, G\} \subseteq \mathcal{E} \subseteq \mathcal{C}[G]$, and $\mathcal{E}$ is closed under definability.
(2) We say that $\mathcal{N}$ is the *generic set-extension* for the forcing $\mathbb{P} \in \mathcal{C}$ and the generic $G \subseteq \mathbb{P}$ if $\mathcal{N} = \langle M[G], \in\rangle$, or (with the usual identification) we also say so about N in case $N = M[G]$. We also identify $\mathcal{N}$ with $\langle M[G], \mathrm{Def}(\mathcal{N})\rangle$.
(3) We call any of the above models (generic class extensions, generic class pseudo-extensions or generic set-extensions) a *generic extension* (of $\mathbb{M}$).

Note that if we consider $\mathcal{N} = \langle M[G], G\rangle$, then M may not be an element of $\mathrm{Def}(\mathcal{N})$, see [Ant15]. Similarly, if G is $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$-generic over $\mathbb{M}$, then $G \notin \mathrm{Def}(\langle M[G], \mathcal{C}\rangle) = \mathcal{C}$, since forcing with $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$ does not add any new sets.

3. Failures of Separation

Recall that $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$ is the notion of forcing with conditions given by $\{p : n \to \mathrm{Ord}^M \mid n \in \omega\}$, ordered by reverse inclusion. This forcing adds a predicate $F = \bigcup G$ which is a cofinal function from ω to Ord^M. Hence Replacement and Collection fail in $\mathbb{M}[G]$, and indeed in every generic class pseudo-extension of $\mathbb{M}$.

It is also easy to see that Separation fails in each such extension. We show that there is no name for the set $\{n \in \omega \mid F(n) \text{ is even}\}$. Let $\dot{F}$ be a class name for the generic function F. Assuming that Separation holds and using the forcing theorem, there are $p \in \mathrm{Col}_*(\omega, \mathrm{Ord})^M$ and a $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$-name σ so that $p \Vdash \sigma = \{n \in \check{\omega} \mid \dot{F}(n) \text{ is even}\}$. Let $\alpha = \mathsf{rnk}(\sigma)$. Now, using an easy density argument, we may extend p to some condition q so that $q(n) = \beta > \alpha$ for some $n \in \omega$. Let π be the automorphism of $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$ that for any condition r swaps the values β and $\beta + 1$ of $r(n)$. Moreover, for $\tau \in M^{\mathbb{P}}$ we recursively define

$$\pi^*(\tau) = \{\langle \pi^*(\mu), \pi(r)\rangle \mid \langle \mu, r\rangle \in \tau\}.$$

Then $\pi^*(\sigma) = \sigma$. Consider $q' = \pi(q)$ and pick a $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$-generic filter G with $q \in G$, let $G' = \pi''G$ and note that $q' \in G'$ and that $\sigma^G = \sigma^{G'}$. But this equation clearly contradicts that $q \Vdash \dot{F}(n)$ is even if and only if $q' \Vdash \dot{F}(n)$ is odd.

In the remainder of this section, we consider generic set-extensions. We want to show that a failure of Separation (and in fact also of Replacement and of Collection) in such an extension is possible as well. The notion of forcing that will witness this will be an adaption of $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$, which does not only add a predicate that is a cofinal function from ω to Ord^M, but also codes this predicate into the values of the continuum function of $M[G]$.

Theorem 3.1. *Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a countable transitive model of $\mathsf{GBC} + \mathsf{GCH}$. There is a cofinality-preserving notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ that satisfies the forcing theorem, such that Separation fails in any $\mathbb{P}$-generic set-extension $M[G]$.*

Proof. Let $\mathbb{P}$ be the forcing notion with conditions of the form $p = \langle p(i) \mid i < n(p)\rangle$ for some $n(p) \in \omega$, with each $p(i)$ of the form $p(i) = \langle \alpha_i(p), C_i(p)\rangle$ where $\alpha_i(p)$ is a regular uncountable cardinal and $C_i(p)$ is a condition in $\mathrm{Add}(\alpha_i(p), \alpha_i(p)^{++})$, the forcing that adds $\alpha_i(p)^{++}$ Cohen subsets to $\alpha_i(p)$, and $\langle \alpha_i(p) \mid i < n\rangle$ is strictly increasing. Given $p \in \mathbb{P}$, we let $\alpha(p) = \langle \alpha_i(p) \mid i < n(p)\rangle$. The ordering on $\mathbb{P}$ is given by stipulating that q is stronger than p iff $\alpha(q)$ end-extends $\alpha(p)$ and for every $i < n(p)$, $C_i(q)$ extends $C_i(p)$ in the usual ordering of $\mathrm{Add}(\alpha_i(p), \alpha_i(p)^{++})$.

Claim 1. *$\mathbb{P}$ satisfies the forcing theorem.*

Proof. We will show that $\mathbb{P}$ is approachable by projections (see [HKL+16, Definition 6.1]). By [HKL+16, Theorem 6.4], this implies that $\mathbb{P}$ satisfies the forcing theorem. For this purpose, for each ordinal α, let

$$\mathbb{P}_\alpha = \begin{cases} \{p \in \mathbb{P} \mid \alpha_{n(p)-1}(p) < \aleph_\alpha\}, & \alpha \in \mathrm{Lim}, \\ \{p \in \mathbb{P} \mid \alpha_{n(p)-1}(p) \leq \aleph_\alpha \wedge [\alpha_{n(p)-1}(p) = \aleph_\alpha \to C_{n(p)-1}(p) = \emptyset]\}, & \text{otherwise.} \end{cases}$$

For successor ordinals α, we define projections $\pi_\alpha : \mathbb{P} \to \mathbb{P}_\alpha$ as follows. If $\alpha_{n(p)-1}(p) < \aleph_\alpha$ then $\pi_\alpha(p) = p$. Otherwise, let $\pi_\alpha(p) = \langle p(i) \mid i < k\rangle^\frown\langle \aleph_\alpha, \emptyset\rangle$, where k is maximal such that $\alpha_{k-1}(p) < \aleph_\alpha$. It is easy to check that $\langle \mathbb{P}_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ and $\langle \pi_{\alpha+1} \mid \alpha \in \mathrm{Ord}\rangle$ witness that $\mathbb{P}$ is approachable by projections. □

If G is $\mathbb{P}$-generic over $\mathbb{M}$ and $p \in G$, we denote by G_p the $\prod_{i<n(p)} \mathrm{Add}(\alpha_i(p), \alpha_i(p)^{++})$-generic filter induced by G.

Claim 2. *For every $p \in \mathbb{P}$ and every $\sigma \in M^{\mathbb{P}}$, there is $q \leq_{\mathbb{P}} p$ and a $\prod_{i<n(q)} \mathrm{Add}(\alpha_i(q), \alpha_i(q)^{++})$-name $\bar{\sigma}$ such that $\sigma^G = \bar{\sigma}^{G_q}$ whenever $q \in G$.*

Proof. Suppose that $\sigma \in (\mathsf{V}_\gamma)^M$ and $p \in \mathbb{P}$. Now choose $q \leq_{\mathbb{P}} p$ such that $\alpha_{n(q)-1}(q) \geq \gamma$. By recursion on the name rank, we define for $\tau \in M^{\mathbb{P}}$,

$$\tau^q = \{\langle \pi^q, \bar{r}\rangle \mid \langle \pi, r\rangle \in \tau \wedge \alpha(r) \subseteq \alpha(q)\},$$

where for $r \in \mathbb{P}$, $\bar{r} = \langle C_i(r) \mid i < n(q)\rangle \in \prod_{i<n(q)} \mathrm{Add}(\alpha_i(q), \alpha_i(q)^{++})$ with $C_i(r) = \emptyset$ for $n(r) \leq i < n(q)$. Then q and $\bar{\sigma} = \sigma^q$ are as desired, since whenever $\langle \tau, r\rangle \in \sigma$ such that $\alpha(r) \nsubseteq \alpha(q)$ then q and r are incompatible by construction of q. $\square$

Claim 3. *Assume that G is $\mathbb{P}$-generic over $\mathbb{M}$ and α is an ordinal of M. Then there is $r \in G$ such that whenever σ is a $\mathbb{P}$-name with $\sigma^G \subseteq \alpha$, then there is a $\prod_{i<n(r)} \mathrm{Add}(\alpha_i(r), \alpha_i(r)^{++})$-name $\tilde{\sigma}$ so that $\tilde{\sigma}^{G_r} = \sigma^G$.*

Proof. Let G and $\alpha \in \mathrm{Ord}^M$ be given and choose $r \in G$ such that $\alpha_{n(r)-1}(r) \geq \alpha$. Assume that $\sigma \in M^{\mathbb{P}}$, and extend r to p in G to obtain a $\prod_{i<n(p)} \mathrm{Add}(\alpha_i(p), \alpha_i(p)^{++})$-name $\bar{\sigma}$ so that $\sigma^G = \bar{\sigma}^{G_p}$, using the previous claim. Let $\mathbb{Q} = \prod_{i<n(p)} \mathrm{Add}(\alpha_i(p), \alpha_i(p)^{++})$ and observe that $\mathbb{Q} \cong \mathbb{Q}_0 \times \mathbb{Q}_1$, where $\mathbb{Q}_0 = \prod_{i<n(r)} \mathrm{Add}(\alpha_i(r), \alpha_i(r)^{++})$ and $\mathbb{Q}_1 = \prod_{n(r)\leq i<n(p)} \mathrm{Add}(\alpha_i(p), \alpha_i(p)^{++})$. Moreover, $\mathbb{Q}_0$ satisfies the κ^+-cc and $\mathbb{Q}_1$ is $<\kappa^+$-closed for $\kappa = \alpha_{n(r)-1}(r)$. Let $p \cong \langle p_0, p_1\rangle$, where $p_0 \in \mathbb{Q}_0$ and $p_1 \in \mathbb{Q}_1$. For each $\beta < \alpha$, we consider the set

$$D_\beta = \{q \leq_{\mathbb{Q}_1} p_1 \mid \text{there is a maximal antichain } A \subseteq \mathbb{Q}_0 \text{ such that } \forall a \in A\,(\langle a, q\rangle \text{ decides } \check{\beta} \in \bar{\sigma})\}.$$

We show that each D_β is open dense below p_1 in $\mathbb{Q}_1$. It is obvious that D_β is open. In order to check density, pick some $q \leq_{\mathbb{Q}_1} p_1$. Inductively, we construct a decreasing sequence $\langle q_i \mid i < \gamma\rangle$ of conditions in $\mathbb{Q}_1$ below q and a sequence $\langle a_i \mid i < \gamma\rangle$ in $\mathbb{Q}_0$ which enumerates an antichain so that each pair $\langle a_i, q_i\rangle$ decides $\check{\beta} \in \bar{\sigma}$, for some $\gamma < \kappa^+$. Suppose that q_i, a_i are given for all $i < \xi$. If $\{a_i \mid i < \xi\}$ is not a maximal antichain, then we can extend both sequences, using that $\mathbb{Q}_1$ is $<\kappa^+$-closed. Since $\mathbb{Q}_0$ satisfies the κ^+-cc, there must be some $\gamma < \kappa^+$ such that $A = \{a_i \mid i < \gamma\}$ is maximal. Invoking the closure of $\mathbb{Q}_1$ once again, we can find $q_\gamma \in \mathbb{Q}_1$ which extends each $q_i, i < \gamma$. Then $q_\gamma \leq q$ and $q_\gamma \in D_\beta$, as desired.

Since $\mathbb{Q}_1$ is $<\kappa^+$-closed, $D = \bigcap_{\beta<\alpha} D_\beta$ is also open dense below p_1. Pick $q \in D \cap H$, where H is the $\mathbb{Q}_1$-generic filter induced by G, and for each $\beta < \alpha$, pick a maximal antichain $A_\beta \subseteq \mathbb{Q}_0$ witnessing that $q \in D_\beta$. It follows that

$$\tilde{\sigma} = \{\langle \check{\beta}, a\rangle \mid a \in A_\beta \wedge \langle a, q\rangle \Vdash_{\mathbb{Q}_0 \times \mathbb{Q}_1} \check{\beta} \in \bar{\sigma}\} \in M^{\mathbb{Q}_0}$$

is as desired. $\square$

Claim 4. *$\mathbb{P}$ is cofinality-preserving and hence preserves all cardinals.*

Proof. Assume it is not. Let $\sigma \in M^{\mathbb{P}}$ name a witness, i.e. a function f from κ to λ that is cofinal, where $\kappa < \lambda$ are regular cardinals in M. By Claim 3, f has a name in some finite product of Cohen forcings. However, this forcing notion is cofinality-preserving using the GCH, a contradiction. $\square$

Claim 5. *$M[G]$ satisfies the power set axiom, and whenever α is an infinite cardinal of M, $M[G] \models 2^\alpha = \alpha^{++}$ if and only if there are $p \in G$ and $i < n(p)$ such that $\alpha = \alpha_i(p)$.*

Proof. Let α be an infinite M-cardinal. Using Claim 3, choose $p \in G$ such that for every $\sigma \in M^{\mathbb{P}}$ such that $\sigma^G \subseteq \alpha$ there is a $\prod_{i<n(p)} \mathrm{Add}(\alpha_i(p), \alpha_i(p)^{++})$-name $\tilde{\sigma}$ with $\tilde{\sigma}^{G_p} = \sigma^G$. Then $\mathcal{P}(\alpha)^{M[G]} = \mathcal{P}(\alpha)^{M[G_p]}$. Since M is a model of $\mathsf{ZFC} + \mathsf{GCH}$, together with Claim 4 this proves both statements of the claim. $\square$

Claim 6. *$M[G]$ does not satisfy Separation.*

Proof. Suppose the contrary and consider

$$x = \{\langle n, \alpha\rangle \in \omega \times \omega_1 \mid \exists \beta \in \mathrm{Ord}^M (f(n) = \aleph_{\omega_1 \cdot \beta + \alpha})\},$$

where $f(n)$ denotes the n-th cardinal at which the GCH fails. Separation implies that $x \in M[G]$. Since for $\alpha < \omega_1$ the set

$$D_\alpha = \{p \in \mathbb{P} \mid \exists \beta \in \mathrm{Ord}^M \exists i < n(p) \ (\alpha_i(p) = \aleph_{\omega_1 \cdot \beta} + \alpha)\}$$

is a dense subet of $\mathbb{P}$ that is definable over M, it follows that x defines a surjection from ω onto ω_1, contradicting that $\mathbb{P}$ is cofinality-preserving. □

This finishes the proof of Theorem 3.1. □

4. Valid Instances of Separation

In this last section, we want to investigate the question as to whether some simple instances of the axiom of Separation might provably hold in all generic extensions for class forcing. A very canonical and well-known set of such instances is provided by the rudimentary functions.

Definition 4.1. X is *rudimentarily closed* if it is closed under *rudimentary functions*, which are defined as follows:

- $f_0(x_1, \ldots, x_k) = x_i$,
- $f_1(x_1, \ldots, x_k) = x_i \setminus x_j$ and
- $f_2(x_1, \ldots, x_k) = \{x_i, x_j\}$ are rudimentary.
- If h and g_i for $1 \leq i \leq l$ are rudimentary, then so is
$$f_3(x_1, \ldots, x_k) = h(g_1(x_1, \ldots, x_k), \ldots, g_l(x_1, \ldots, x_k)).$$
- If g is rudimentary, then so is $f_4(x_1, \ldots, x_k) = \bigcup_{y \in x_1} g(y, x_2, \ldots, x_k)$.

Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a countable transitive model of GB^- throughout this section. The following definitions appear in [HKL+16, Section 5].

Definition 4.2. If $\mathbb{B}$ is a Boolean algebra, then we say that $\mathbb{B}$ is *M-complete* if the supremum $\sup_{\mathbb{B}} A$ of all elements in A exists in $\mathbb{B}$ for every $A \in M$ with $A \subseteq \mathbb{B}$.

We say that $\mathbb{P}$ *has a Boolean completion in* $\mathbb{M} = \langle M, \mathcal{C}\rangle$ if there is an M-complete Boolean algebra $\mathbb{B} = \langle B, 0_{\mathbb{B}}, 1_{\mathbb{B}} \neg, \wedge, \vee\rangle$ such that B, all Boolean operations of $\mathbb{B}$ and an injective dense embedding from $\mathbb{P}$ into $\mathbb{B} \setminus \{0_{\mathbb{B}}\}$ are elements of $\mathcal{C}$.

In [HKL+16, Theorem 5.5] it is shown that if $\mathbb{M}$ has a hierarchy (or rather, satisfies the weaker notion of *representatives choice* – see [HKL+16, Definition 3.2]), then the existence of a Boolean completion for a separative notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ is equivalent to the forcing theorem for $\mathbb{P}$ over $\mathbb{M}$. The next lemma observes that M-complete Boolean algebras produce rudimentarily closed forcing extensions.

Lemma 4.3. *If $\mathbb{P}$ is an M-complete Boolean algebra and G is $\mathbb{M}$-generic, then $M[G]$ is rudimentarily closed.*

Proof. Closure under projections and compositions of rudimentary functions is obvious. Assume that $\sigma, \tau \in M^{\mathbb{P}}$. Clearly, $\{\langle \rho, [\![\rho \notin \tau]\!] \wedge p\rangle \mid \langle \rho, p\rangle \in \sigma\}$ is a name for $\sigma^G \setminus \tau^G$ and $\{\langle \sigma, \mathbb{1}_{\mathbb{P}}\rangle, \langle \tau, \mathbb{1}_{\mathbb{P}}\rangle\}$ is a name for the unordered pair $\{\sigma^G, \tau^G\}$. Next, suppose that $g(v_0, v_1)$ is a rudimentary function. We have to find a name for $\bigcup_{x \in \tau^G} g(x, \sigma^G)$. Since g is rudimentary, for every $\rho \in \mathrm{dom}(\tau)$ there is a $\mathbb{P}$-name $\pi_{\rho,\sigma}$ for $g(\rho^G, \sigma^G)$. Now put

$$\theta = \{\langle \eta, p \wedge q\rangle \mid \exists \rho (\langle \rho, p\rangle \in \tau \wedge \langle \eta, q\rangle \in \pi_{\rho,\sigma})\}.$$

Clearly, $\theta^G \subseteq \bigcup_{x \in \tau^G} g(x, \sigma^G)$. For the converse, consider $\langle \rho, p\rangle \in \tau$ with $p \in G$ and $y \in g(\rho^G, \sigma^G)$. Hence there must be $\langle \eta, q\rangle \in \pi_{\rho,\sigma}$ such that $q \in G$ and $y = \eta^G$. Then also $p \wedge q \in G$, so $\eta^G \in \theta^G$. □

What happens when a notion of class forcing $\mathbb{P}$ is not an M-complete Boolean algebra? Let us additionally assume that $\mathbb{M}$ has a hierarchy and that $\mathbb{P}$ is separative and satisfies the forcing theorem. Then by [HKL+16, Theorem 5.5], $\mathbb{P}$ has a Boolean completion in $\mathbb{M}$ and in fact the proof of [HKL+16, Theorem 5.5] yields that the completion $\mathbb{B}(\mathbb{P})$ that is constructed in this proof is *minimal*, in the sense that it injectively embeds into every Boolean completion of $\mathbb{P}$, by an injection in $\mathcal{C}$.

We would like to know whether $\mathbb{P}$-generic extensions are rudimentarily closed. One way towards a positive answer may be to compare $\mathbb{P}$-generic and $\mathbb{B}(\mathbb{P})$-generic extensions of $\mathbb{M}$ and hope that they are the same and thus rudimentarily closed by Lemma 4.3.[3] However we do not know whether this is the case.

A related (and perhaps easier) question is whether the $\mathbb{B}(\mathbb{P})$-generic extensions of $\mathbb{M}$ are just the rudimentary closures of the corresponding $\mathbb{P}$-generic extensions of $\mathbb{M}$. If this were the case, then a positive answer to the question whether every $\mathbb{P}$-generic extension is rudimentarily closed would also yield a positive answer to the question whether $\mathbb{P}$ and $\mathbb{B}(\mathbb{P})$ produce the same generic extensions.

In the following, we will present some results that yield weak instances of rudimentary closure in arbitrary forcing extensions. Note that in particular we do not assume the forcing theorem for $\mathbb{P}$ anywhere throughout the remainder of this section. We also do not assume that $\mathbb{M}$ has a hierarchy for the following results. However all of our results are restricted to what we call *almost nice names*, a slight generalization of nice names.

Definition 4.4. Let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$ and let $\alpha \in \mathrm{Ord}^M$. We say that $\sigma \in M$ is an *almost nice* $\mathbb{P}$*-name* for a subset of α if $\sigma = \bigcup_{\beta<\alpha}\{\check{\beta}\} \times X_\beta$ with each $X_\beta \subseteq P$.

Lemma 4.5. *Let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. Let $\alpha \in \mathrm{Ord}^M$ be an ordinal and let $\sigma, \tau \in M^{\mathbb{P}}$ be almost nice names for subsets of α. If G is $\mathbb{P}$-generic over $\mathbb{M}$, then there is a $\mathbb{P}$-name $\mu \in M^{\mathbb{P}}$ such that $\mu^G = \sigma^G \cap \tau^G$.*

Proof. Since σ and τ are almost nice names, they are of the form

$$\sigma = \bigcup_{\beta<\alpha}\{\check{\beta}\} \times X_\beta \text{ and } \tau = \bigcup_{\beta<\alpha}\{\check{\beta}\} \times Y_\beta,$$

where $\langle X_\beta \mid \beta < \alpha\rangle, \langle Y_\beta \mid \beta < \alpha\rangle \in M$. Let G be $\mathbb{P}$-generic over $\mathbb{M}$ and let $\beta_0 \in \alpha$ be minimal such that $\beta_0 \in \sigma^G \cap \tau^G$. Put $\mu = \{\langle\check{\beta}_0, \mathbb{1}_{\mathbb{P}}\rangle\} \cup \{\langle\mu_\beta, p\rangle \mid \beta \in (\beta_0, \alpha), p \in X_\beta\}$, where $\mu_\beta = \check{\beta}_0 \cup \bigcup_{\gamma<\beta}\{\check{\gamma}\} \times Y_\beta$ for every $\beta \in (\beta_0, \alpha)$. Then

$$(\mu_\beta)^G = \begin{cases} \beta, & \beta \in \tau^G \\ \beta_0, & \text{otherwise.} \end{cases}$$

Clearly, $\mu^G = \sigma^G \cap \tau^G$ is as desired. $\square$

Remark 4.6. Note that it is in general not possible to find a $\mathbb{P}$-name μ as in Lemma 4.5 such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \mu = \sigma \cap \tau$. For example, consider a notion of class forcing $\mathbb{P}$ which contains compatible conditions $p, q \in \mathbb{P}$ such that the class $\{r \in \mathbb{P} \mid r \leq_{\mathbb{P}} p, q\}$ contains no predense subset that is an element of M. Then there is no such $\mathbb{P}$-name for the intersection of $\sigma = \{\langle\check{0}, p\rangle\}$ and $\tau = \{\langle\check{0}, q\rangle\}$.

In the remainder of this section, we consider partial results towards the existence of relative complements in class generic extensions.

Lemma 4.7. *Let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. If σ is an almost nice $\mathbb{P}$-name for a subset of ω and G is $\mathbb{P}$-generic over $\mathbb{M}$, then there is a name τ such that $\tau^G = \omega \setminus \sigma^G$.*

Proof. Let $\sigma = \bigcup_{n\in\omega}\{\check{n}\} \times X_n$ with $\langle X_n \mid n \in \omega\rangle \in M$. If σ^G or $\omega \setminus \sigma^G$ are finite, then it is easy to define a name τ as required. Suppose that both σ^G and $\omega \setminus \sigma^G$ are infinite.

We construct a name for the relative complement of σ in ω as follows. For each $n \geq 1$, we define a name τ_n which searches for the next $m \notin \sigma^G$ above n, i.e. such that $\tau_n^G = \min\{m \geq n \mid m \notin \sigma^G\}$. Let $\tau' = \{\langle\tau_n, \mathbb{1}_{\mathbb{P}}\rangle \mid 1 \leq n < \omega\}$. Then either $\tau = \tau'$ or $\tau = \{\langle\check{0}, \mathbb{1}_{\mathbb{P}}\rangle\} \cup \tau'$ has the required property.

It remains to define the τ_n. Each name τ_n is built by testing successively for each $m \geq n$ whether $m \in \sigma^G$ and adding 1 in this case. More precisely, we will define a sequence of names σ_n^m for $m \geq n$ such that $(\sigma_n^m)^G$ increases by 1 until just before we reach the next element of the

[3]By [HKL$^+$16, Theorem 9.4], if $\mathbb{P}$ does not satisfy the Ord-cc over $\mathbb{M}$, then $\mathbb{B}(\mathbb{P})$ is not the unique Boolean completion of $\mathbb{P}$ in $\mathbb{M}$. It is not hard to find examples of a forcing $\mathbb{P}$ and a Boolean completion $\mathbb{B}$ of $\mathbb{P}$ in $\mathbb{M}$ that produce different generic extensions. In fact, we shall show in the forthcoming [HKS16] that such a Boolean completion of $\mathbb{P}$ exists whenever $\mathbb{P}$ is not pretame.

complement of σ^G and is constant from then on, for all $\mathbb{P}$-generic filters G over V. We then define $\tau_n = \check{n} \cup \{\langle \sigma_n^m, \mathbb{1}_\mathbb{P} \rangle \mid m \geq n\}$.

To define σ_n^m for $m \geq n$, we first construct a sequence of auxiliary names as follows. For $k \in \omega$, let $\sigma_n^{n,k} = (n \check{-} 1) \cup [\{n \check{-} 1\} \times X_{n+k}]$ and let $\sigma_n^{m+1,k} = \sigma_n^{m,k} \cup \left[\{\sigma_n^{m,k+1}\} \times X_{n+k}\right]$ for $m \geq n$. Moreover let $\sigma_n^m = \sigma_n^{m,0}$. Then

$$(\sigma_n^{n,k})^G = \begin{cases} n-1 & n+k \notin \sigma^G \\ n & n+k \in \sigma^G \end{cases}$$

and hence

$$(\sigma_n^n)^G = \begin{cases} n-1 & n \notin \sigma^G \\ n & n \in \sigma^G. \end{cases}$$

Therefore the statement $(*)_n^{m,k}$

$$(\sigma_n^{m,k})^G = \min\{m, \min\{i \mid i \geq n-1,\ i+k+1 \notin \sigma^G\}\}$$

holds for $m = n$ and all $k \in \omega$. It is straightforward to check via the definition of $\sigma_n^{m+1,k}$ that $(*)_n^{m,k}$ for all $k \in \omega$ implies $(*)_n^{m+1,k}$ for all $k \in \omega$, by considering the cases $(\sigma_n^{m,k})^G = n-1$ and $(\sigma_n^{m,k})^G \geq n$. □

Remark 4.8. Note that it is in general not possible to find a $\mathbb{P}$-name τ as in Lemma 4.7 such that $\mathbb{1}_\mathbb{P} \Vdash_\mathbb{P} \tau = \omega \setminus \sigma$. For example, consider a notion of class forcing $\mathbb{P}$ which contains a condition $p \in \mathbb{P}$ such that the class $\{q \in \mathbb{P} \mid p \perp q\}$ contains no predense subset that is an element of M. Then there is no such $\mathbb{P}$-name for the relative complement of $\sigma = \{\langle \check{0}, p \rangle\}$ in ω.

Definition 4.9. If $\mathbb{P}$ is a partial order that is closed under meets, given $\langle X_i \mid i < n < \omega \rangle$ with each $X_i \subseteq \mathbb{P}$, let

$$\bigwedge_{i<n} X_i = X_0 \wedge \ldots \wedge X_{n-1} = \{p_0 \wedge \ldots \wedge p_{n-1} \mid \forall i < n\ p_i \in X_i\}.$$

Let the value of an empty intersection be $\{\mathbb{1}_\mathbb{P}\}$. Given $p \in \mathbb{P}$ and $X \subseteq \mathbb{P}$, we let

$$p \wedge X = \{p \wedge q \mid q \in X\}.$$

If σ is a $\mathbb{P}$-name and $X \subseteq \mathbb{P}$, we let

$$\sigma \wedge X = \bigcup_{\langle \tau, p \rangle \in \sigma} \{\tau\} \times (p \wedge X).$$

Given an ordinal β we let $[\beta]$ denote the largest limit ordinal $\leq \beta$.

The following is a sample result on the existence of relative complements in ω^2, that illustrates the increasing difficulties when one aims for a general positive result in this direction. It should however not be too difficult to push these results a little further, to obtain for example relative complements in the ordinal ω^ω.

Lemma 4.10. *Let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$ that is closed under meets. If σ is an almost nice $\mathbb{P}$-name for a subset of some ordinal $\xi \leq \omega^2$ and G is $\mathbb{P}$-generic over $\mathbb{M}$, then there is a $\mathbb{P}$-name τ with $\tau^G = \xi \setminus \sigma^G$.*

Proof. Let $\sigma = \bigcup_{\alpha<\xi} \{\check{\alpha}\} \times X_\alpha$. We may assume (by possibly shrinking ξ) that the complement of σ^G in ξ is unbounded in ξ, and that $\xi \geq \omega$ is a limit ordinal. If $\sigma^G \subseteq \omega$, then the claim follows from Lemma 4.7. Hence we suppose that $\sigma^G \nsubseteq \omega$. Let ν be the least infinite element of σ^G. It is sufficient to define a name τ with $\tau^G = [\nu, \xi) \setminus \sigma^G$, since we can obtain the required name from τ by invoking Lemma 4.7.

We will first define auxiliary names $\sigma_\alpha^{\beta,k}$ for $\nu \leq \alpha \leq \beta < \xi$ and $k < \omega$, similar to as we did in the proof of Lemma 4.7. Let $\mu = \min([\alpha, \xi) \setminus \sigma^G)$. Here the names $\sigma_\alpha^{\beta,k}$ will have the following weaker property $(*)_\alpha^{\beta,k}$

$(*_1)$ $(\sigma_\alpha^{\beta,k})^G + k < \mu$ and
$(*_2)$ $(\sigma_\alpha^{\beta,k})^G = \beta$ if $\beta + k < \mu$.

We will need the additional freedom for the values of the names (in comparison to the ones appearing in the proof of Lemma 4.7) in order to be able to continue our construction through limit levels. We then define $\tau_\alpha = \check{\alpha} \cup \{\langle \sigma_\alpha^{\beta,0}, \mathbb{1}_\mathbb{P}\rangle \mid \alpha \leq \beta < \xi\}$ for α with $\nu \leq \alpha < \xi$.

Claim. *Suppose that $\nu \leq \alpha < \xi$ and $(*)_\alpha^{\beta,0}$ holds for all β with $\alpha \leq \beta < \xi$. Then $\tau_\alpha^G = \min([\alpha,\xi) \setminus \sigma^G)$.*

Proof. Let $\mu = \min([\alpha,\xi) \setminus \sigma^G)$. Since $(\sigma_\alpha^{\beta,0})^G < \mu$ for all $\alpha \leq \beta < \xi$ by $(*)_\alpha^{\beta,0}$, it follows that $\tau^G \subseteq \mu$. Since $(\sigma_\alpha^{\beta,0})^G = \beta$ for all $\alpha \leq \beta < \xi$ by $(*)_\alpha^{\beta,0}$, it follows that $\tau^G = \mu$. □

Let $\tau = \{\langle \tau_\alpha, \mathbb{1}_\mathbb{P}\rangle \mid \nu \leq \alpha < \xi\}$. It follows from the previous claim that $\tau^G = [\nu,\xi) \setminus \sigma^G$. This implies the statement of the theorem.

It remains to define $\sigma_\alpha^{\beta,k}$ and prove $(*)_\alpha^{\beta,k}$ for all $\nu \leq \alpha \leq \beta < \xi$. For every limit ordinal $\beta < \xi$, let $f_\beta \colon \omega \to \beta$ be a bijection. We inductively define the following names for $\nu \leq \alpha \leq \beta < \xi$, limit ordinals γ with $\nu \leq \alpha \leq \gamma < \xi$ and $k < \omega$.

$$\sigma_\alpha^{\alpha,k} = \bigcup_{\beta<\alpha} \{\check{\beta}\} \times [X_\alpha \wedge \ldots \wedge X_{\alpha+k}]$$

$$\sigma_\alpha^{\beta+1,k} = \sigma_\alpha^{\beta,k} \cup \{\sigma_\alpha^{\beta,k+1}\} \times \left[\bigwedge_{i\in(f''_{[\beta]}(\beta-[\beta]))\setminus\alpha} X_i \;\wedge \bigwedge_{i\in[[\beta],\beta+k+1]\setminus\alpha} X_i\right]$$

$$\sigma_\alpha^{\gamma,k} = \bigcup_{\alpha\leq\beta<\gamma} \sigma_\alpha^{\beta,k} \wedge [X_\gamma \wedge \ldots \wedge X_{\gamma+k}]$$

It is easy to check the properties

(a) $(\sigma_\alpha^{\beta,k})^G$ is an ordinal,
(b) $(\sigma_\alpha^{\beta,k+1})^G \leq (\sigma_\alpha^{\beta,k})^G$,
(c) $(\sigma_\alpha^{\beta+1,k})^G = (\sigma_\alpha^{\beta,k})^G$ or $(\sigma_\alpha^{\beta+1,k})^G = (\sigma_\alpha^{\beta,k})^G + 1$,
(d) $(\sigma_\alpha^{\gamma,k})^G \leq \sup_{\alpha\leq\beta<\gamma} (\sigma_\alpha^{\beta,k})^G$

for all $\nu \leq \alpha \leq \beta < \xi$, all limit ordinals γ with $\nu \leq \alpha < \gamma < \xi$ and all $k < \omega$. The next claim completes the proof.

Claim. *$(*)_\alpha^{\beta,k}$ holds for all $\nu \leq \alpha \leq \beta < \xi$, i.e.*

(1) $[\alpha, (\sigma_\alpha^{\beta,k})^G + k] \subseteq \sigma^G$ and
(2) if $[\alpha, \beta + k] \subseteq \sigma^G$, then $(\sigma_\alpha^{\beta,k})^G = \beta$.

Proof. Suppose that $\nu \leq \alpha < \xi$. We prove the claim by induction for all β with $\alpha \leq \beta < \xi$, simultaneously for all $k < \omega$. If $\beta = \alpha$, then $(\sigma_\alpha^{\alpha,k})^G = 0$ or $(\sigma_\alpha^{\alpha,k})^G = \alpha$ and the statements of the claim are easy to check.

In the successor step, we show that $\sigma_\alpha^{\beta+1,k}$ has the required properties if $\alpha \leq \beta < \xi$. We first show (1). Let $\mu = \min([\alpha,\xi) \setminus \sigma^G)$. If $[\beta] \leq \mu \leq \beta + k + 1$, then $(\sigma_\alpha^{\beta+1,k})^G = (\sigma_\alpha^{\beta,k})^G$, and hence condition (1) follows from the inductive assumption for $\sigma_\alpha^{\beta,k}$. If $\mu > \beta + k + 1$, then $(\sigma_\alpha^{\beta,k})^G = (\sigma_\alpha^{\beta,k+1})^G = \beta$ by the inductive assumption. Therefore $(\sigma_\alpha^{\beta+1,k})^G + k = \beta + k + 1 \in \sigma^G$, and hence condition (1) follows. Suppose that $\mu < [\beta]$. If $(\sigma_\alpha^{\beta,k})^G + k + 1 < \mu$, then (1) follows from the inductive assumption and from the condition (c) preceding the claim. Suppose that $\mu = (\sigma_\alpha^{\beta,k})^G + k + 1$. Since (1) is immediate from the definition of $\sigma_\alpha^{\beta+1,k}$ if $(\sigma_\alpha^{\beta,k+1})^G < (\sigma_\alpha^{\beta,k})^G$, we can assume that $(\sigma_\alpha^{\beta,k+1})^G = (\sigma_\alpha^{\beta,k})^G$ by condition (b) preceding the claim. Then $\mu = (\sigma_\alpha^{\beta,k})^G + k + 1 = (\sigma_\alpha^{\beta,k+1})^G + k + 1$. This contradicts the inductive assumption (1) for $\sigma_\alpha^{\beta,k+1}$. We now show (2). If $[\alpha, \beta + k + 1] \subseteq \sigma^G$, it follows from the definition of $\sigma_\alpha^{\beta+1,k}$ and the inductive assumption for $\sigma_\alpha^{\beta,k+1}$ that $(\sigma_\alpha^{\beta+1,k})^G = \beta + 1$.

In the limit step, we show that $\sigma_\alpha^{\gamma,k}$ has the required properties if γ is a limit ordinal with $\alpha < \gamma < \xi$. Suppose that $\gamma = \omega \cdot (n + 1)$. We first show (1). If some $\delta \in [\gamma, \gamma + k]$ is not in σ^G, then $(\sigma_\alpha^{\gamma,k})^G = 0$ by the definition of $\sigma_\alpha^{\gamma,k}$. Then (1) and (2) are trivially valid. Thus we can assume that $[\gamma, \gamma + k] \subseteq \sigma^G$. Suppose that some $\delta \in [\alpha, \gamma)$ is not in σ^G. By the definition in the successor case,[4] there is some $\bar{\gamma} < \gamma$ such that for every $\delta \in [\bar{\gamma}, \gamma)$, $(\sigma_\alpha^{\delta,k})^G$ has the same

[4]Here is now the only point in the proof where we make use of the bijections $\langle f_\delta \mid \delta < \xi\rangle$, and also the reason why the above argument does not work for relative complements of subsets of $\xi > \omega^2$, as the following would not remain valid.

value, i.e. $\langle(\sigma_\alpha^{\delta,k})^G \mid \delta < \gamma\rangle$ is eventually constant. This yields (1) via the inductive assumption for $\sigma_\alpha^{\bar{\gamma},k}$. We now show (2). If $[\alpha, \gamma+k] \subseteq \sigma^G$, then the definition of $\sigma_\alpha^{\gamma,k}$ together with our inductive assumptions on $\sigma_\alpha^{\beta,k}$ for $\alpha \leq \beta < \gamma$ immediately yield that $(\sigma_\alpha^{\gamma,k})^G = \gamma$. □

This completes the proof of Lemma 4.10. □

The previous results cannot be directly generalized to ω_1. The following forcing $\mathbb{P}$ is a candidate for a forcing such that the generic extension is not rudimentarily closed. Let $\mathbb{P}$ be the forcing $\mathrm{Col}(\omega_1, \mathrm{Ord})$ together with conditions p_α for all $\alpha < \omega_1$ that correspond to the Boolean value of the statement that $\dot{f}(\alpha)$ is even, where $\dot{f}$ is a name for the surjection $f\colon \omega_1 \to \mathrm{Ord}$ added by the forcing. We do not know whether the set $\{\alpha < \omega_1 \mid f(\alpha) \text{ is odd}\}$ has a $\mathbb{P}$-name.

5. Open Questions

One of the questions left open in Section 4 is the following.

Question 5.1. *Are class forcing extensions provably rudimentarily closed, and in particular closed under relative complements?*

If this is not the case, one might try to prove the following.

Question 5.2. *Assume $\mathbb{P}$ is separative and antisymmetric and satisfies the forcing theorem over $\mathbb{M}$, a countable transitive model of GB^- with a hierarchy, and thus $\mathbb{P}$ has a minimal Boolean completion $\mathbb{B}(\mathbb{P})$. Are the $\mathbb{B}(\mathbb{P})$-generic extensions the rudimentary closures of the corresponding $\mathbb{P}$-generic extensions?*

A much more basic sample question is the following.

Question 5.3. *If A and B are subsets of ω in some $\mathbb{P}$-generic extension $\mathbb{M}[G]$, does $A \setminus B$ have a $\mathbb{P}$-name, so that $A \setminus B \in M[G]$?*

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