Structures without Scattered-Automatic Presentation

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Abstract. Bruyère and Carton lifted the notion of finite automata reading infinite words to finite automata reading words with shape an arbitrary linear order $\mathfrak{L}$. Automata on finite words can be used to represent infinite structures, the so-called word-automatic structures. Analogously, for a linear order $\mathfrak{L}$ there is the class of $\mathfrak{L}$-automatic structures. In this paper we prove the following limitations on the class of $\mathfrak{L}$-automatic structures for a fixed $\mathfrak{L}$ of finite condensation rank α. Firstly, no scattered linear order with finite condensation rank above $\omega^{\alpha+1}$ is $\mathfrak{L}$-automatic. In particular, every $\mathfrak{L}$-automatic ordinal is below ω^{ω^α}. Secondly, we provide bounds on the (ordinal) height of well-founded order trees that are $\mathfrak{L}$-automatic. If α is finite or $\mathfrak{L}$ is an ordinal, the height of such a tree is bounded by $\omega^{\alpha+1}$. Finally, we separate the class of tree-automatic structures from that of $\mathfrak{L}$-automatic structures for any ordinal $\mathfrak{L}$: the countable atomless boolean algebra is known to be tree-automatic, but we show that it is not $\mathfrak{L}$-automatic.

1 Introduction

Finite automata play a crucial role in many areas of computer science. In particular, finite automata have been used to represent certain classes of possibly infinite structures. The basic notion of this branch of research is the class of automatic structures (cf. [11]): a structure is automatic if its domain as well as its relations are recognised by (synchronous multi-tape) finite automata processing finite words. This class has the remarkable property that the first-order theory of any automatic structure is decidable. One goal in the theory of automatic structures is a classification of those structures that are automatic (cf. [5,13,12,10,14]). Besides finite automata reading *finite* or *infinite words* there are also finite automata reading finite or infinite *trees*. Using such automata as representation of structures leads to the notion of tree-automatic structures [3]. The classification of tree-automatic structures is less advanced but some results have been obtained in the last years (cf. [5,7,9]). Bruyère and Carton [4] adapted the notion of finite automata such that they can process words that have the shape of some fixed linear order. If the linear order is countable and scattered, the corresponding class of languages possesses the good closure properties of the class of languages of finite automata for finite words (i.e., closure under intersection, union, complement, and projection) and emptiness of a given language is decidable. Thus, these automata are also well-suited for representing structures. Given a fixed countable scattered linear order $\mathfrak{L}$ this leads to the notion of $\mathfrak{L}$-automatic structures. In case that $\mathfrak{L}$ is an ordinal Schlicht and Stephan

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[17] as well as Finkel and Todorcevic [6] studied the classes of $\mathfrak{L}$-automatic ordinals and $\mathfrak{L}$-automatic linear orders. Here we study $\mathfrak{L}$-automatic linear orders for any countable scattered linear order $\mathfrak{L}$ and we study $\mathfrak{L}$-automatic well-founded order forests (i.e., forests (seen as partial orders) without infinite branches):

1. If a linear order is $\mathfrak{L}$-automatic and $\mathfrak{L}$ has finite condensation rank below α, then it is a finite sum of linear orders of condensation rank below $\omega^{\alpha+1}$. As already shown in [17], this bound is optimal.
2. If a well-founded order forest is $\mathfrak{L}$-automatic for some ordinal $\mathfrak{L}$, then its ordinal height is bounded by $\mathfrak{L} \cdot \omega$.
   If a well-founded order forest is $\mathfrak{L}$-automatic for $\mathfrak{L}$ some linear order of condensation rank $n \in \mathbb{N}$, then its ordinal height is bounded by ω^{n+1}.
   These two bounds are optimal.
3. A well-founded $\mathfrak{L}$-automatic order forest has ordinal height bounded by $\omega^{\omega \cdot (\alpha+1)}$ where α is the finite condensation rank of $\mathfrak{L}$.

In order to prove 1. and 3. we observe that the notion of *finite-type products* from [17] and the notion of *sum-augmentations of tamely colourable box-augmentations* from [9,7], even though defined in completely different terms, have a common underlying idea. We introduce a new notion of tamely colourable sum-of-box augmentations that refines both notions and allows to prove a variant of Delhommé's decomposition method (cf. [5]) for the case of $\mathfrak{L}$-automatic structures. The main results then follow as corollaries using results from [7] and [9]. For the other two results, we provide an $\mathfrak{L}$-automatic scattered linear ordering of all $\mathfrak{L}$-shaped words if $\mathfrak{L}$ has finite condensation rank $n \in \mathbb{N}$ or if $\mathfrak{L}$ is an ordinal. Extending work from [14], we provide a connection between the height of a tree and the finite condensation rank of its Kleene-Brouwer ordering (with respect to this $\mathfrak{L}$-automatic ordering) that allows to derive the better bounds stated in 2.

As a very sketchy summary of these results, one could say that we adapt techniques previously used on trees to use them on linear orders. This raises the question whether there is a deeper connection between $\mathfrak{L}$-automatic structures and tree-automatic structures. It is known that all ω^n-automatic structures are tree-automatic (cf. [6]). Moreover, from [17] and [5] it follows that ω^{ω^ω} is ω^ω-automatic but not tree-automatic. It is open so far whether every tree-automatic structure is $\mathfrak{L}$-automatic for some linear order $\mathfrak{L}$. We make a first step towards a solution by showing that the countable atomless boolean algebra is not $\mathfrak{L}$-automatic for any ordinal $\mathfrak{L}$ (while it is tree-automatic, cf. [1]).

2 Preliminaries

2.1 Scattered Linear Orders

In this section, we recall basic notions concerning scattered linear orders. For a detailed introduction, we refer the reader to [16]. A linear order $(L, \leq)$ is *scattered* if there is no embedding of the rational numbers into $(L, \leq)$.

Given a scattered linear order $\mathfrak{L} = (L, \leq)$, an equivalence relation $\sim$ is called a *condensation* if each $\sim$ class is an interval of $\mathfrak{L}$. We then write $\mathfrak{L}/{\sim} := (L/{\sim}, \leq')$ for the linear order of the $\sim$ classes induced by $\leq$ (i.e., for $\sim$-classes x, y, $x \leq' y$ iff there are $k \in x, l \in y$ such that $k \leq l$). As usual, for $\mathfrak{L}$ a scattered linear order and l, l' elements of $\mathfrak{L}$, we write $[l, l']$ for the closed interval between l and l'. For each ordinal α we define the *α-th condensation* $\sim_\alpha$ by $x \sim_0 y$ iff $x = y$, $x \sim_{\alpha+1} y$ if the closed interval $[x, y]$ in $\mathfrak{L}/{\sim_\alpha}$ is finite and for a limit ordinal β, $x \sim_\beta y$ if there is an $\alpha < \beta$ such that $x \sim_\alpha y$. The *finite condensation rank* $\mathsf{FC}(\mathfrak{L})$ is the minimal ordinal α such that $\mathfrak{L}/{\sim_\alpha}$ is a one-element order. We also let $\mathsf{FC}_*(\mathfrak{L})$ be the minimal ordinal α such that $\mathfrak{L}/{\sim_\alpha}$ is a finite order. There is such an ordinal α if and only if $\mathfrak{L}$ is scattered. It is obvious from these definitions that $\mathsf{FC}_*(\mathfrak{L}) \leq \mathsf{FC}(\mathfrak{L}) \leq \mathsf{FC}_*(\mathfrak{L}) + 1$.

As usual, for a linear order $\mathfrak{L} = (L, \leq)$ and a sequence of linear orders $(\mathfrak{L}_i)_{i \in \mathfrak{L}}$ we denote by $\sum_{i \in L} \mathfrak{L}_i$ the $\mathfrak{L}$-sum of the $(\mathfrak{L}_i)_{i \in \mathfrak{L}}$.

We conclude this section by recalling the notion of Dedekind cuts of a linear order.

Let $\mathfrak{L} = (L, \leq)$ be a linear order. A *cut* of $\mathfrak{L}$ is a pair $c = (C, D)$ where C is a downward closed subset $C \subseteq L$ and $D = L \setminus C$. We write $\mathsf{Cuts}(\mathfrak{L})$ for the *set of all cuts of* $\mathfrak{L}$. For cuts c, d, we say that c and d are the consecutive cuts around some $l \in L$ if $c = (C, D)$ and $d = (C', D')$ such that $C = \{x \in L \mid x < l\}$ and $C' = \{x \in L \mid x \leq l\}$. $\mathsf{Cuts}(\mathfrak{L})$ can be naturally equipped with an order (also denoted by $\leq$) via $c = (C, D) \leq d = (C', D')$ if $C \subseteq C'$. We say a cut $c = (C, D)$ has no direct predecessor (or direct successor), if it has no direct predecessor (or direct successor, respectively) with respect to $\leq$. Let us finally introduce a notation for values appearing arbitrarily close to some cut (from below or from above, respectively).

Definition 1. *Let $\mathfrak{L} = (L, \leq)$ be a linear order, and $w : L \to A$ be a mapping. Let $c = (C, D) \in \mathsf{Cuts}(\mathfrak{L})$. We define*

$$\lim_{c^-} w := \{a \in A \mid \forall l \in C \exists l' \in C \quad l \leq l' \text{ and } w(l') = a\} \text{ and}$$
$$\lim_{c^+} w := \{a \in A \mid \forall l \in D \exists l' \in D \quad l' \leq l \text{ and } w(l') = a\}.$$

2.2 Automata for Scattered Words and Scattered-Automatic Structures

For this section, we fix an arbitrary linear order $\mathfrak{L} = (L, \leq)$.

Definition 2. *Let $\Sigma_\diamond$ be some finite alphabet with $\diamond \in \Sigma_\diamond$. An $\mathfrak{L}$-word (over Σ) is a map $L \to \Sigma_\diamond$. An $\mathfrak{L}$-word w is* finite *if the support $\mathsf{supp}(w) := \{l \in L \mid w(l) \neq \diamond\}$ of w is finite. $W(\mathfrak{L})$ denotes the set of $\mathfrak{L}$-words.*

Definition 3. *Let w_1, w_2 be $\mathfrak{L}$-words over alphabets Σ_1 and Σ_2, respectively. We define the convolution of w_1 and w_2, denoted by $w_1 \otimes w_2$, to be the $\mathfrak{L}$-word over alphabet $\Sigma_1 \times \Sigma_2$ given by $[w \otimes v](l) := (w(l), v(l))$.*

We recall Bruyère and Carton's definition of automata for $\mathfrak{L}$-words [4]. Then we introduce the notion of (finite word) $\mathfrak{L}$-automatic structures generalising the notion of ordinal-automatic structures from [17].

Definition 4. *An $\mathfrak{L}$-*automaton *is a tuple $\mathcal{A} = (Q, \Sigma, I, F, \Delta)$ where Q is a finite set of states, Σ a finite alphabet, $I \subseteq Q$ the initial and $F \subseteq Q$ the final states and Δ is a subset of $(Q \times \Sigma \times Q) \cup \left(2^Q \times Q\right) \cup \left(Q \times 2^Q\right)$ called the transition relation.*

Transitions in $Q \times \Sigma \times Q$ are called *successor transitions*, transitions in $2^Q \times Q$ are called *right limit transitions*, and transitions in $Q \times 2^Q$ are called *left limit transitions*.

Definition 5. *A* run *of $\mathcal{A}$ on the $\mathfrak{L}$-word w is a map $r : \mathsf{Cuts}(\mathfrak{L}) \to Q$ such that*

- $(r(c), w(l), r(d)) \in \Delta$ *for all $l \in L$ and all consecutive cuts c, d around l,*
- $(\lim_{c^-} r, r(c)) \in \Delta$ *for all cuts $c \in \mathsf{Cuts}(\mathfrak{L}) \setminus \{(\emptyset, L)\}$ without direct predecessor,*
- $(r(c), \lim_{c^+} r) \in \Delta$ *for all cuts $c \in \mathsf{Cuts}(\mathfrak{L}) \setminus \{(L, \emptyset)\}$ without direct successor.*

Here the operators $\lim_{c^-}$ and $\lim_{c^+}$ are applied to the order on $\mathsf{Cuts}(\mathfrak{L})$ as opposed to $\mathfrak{L}$. The run r is accepting *if $r((\emptyset, L)) \in I$ and $r((L, \emptyset)) \in F$. The* language *of $\mathcal{A}$ consists of all $\mathfrak{L}$-words w such that there is an accepting run of $\mathcal{A}$ on w.*

For some $\mathfrak{L}$-word w and states q, q' of $\mathcal{A}$ we write $q \xrightarrow[\mathcal{A}]{w} q'$ if there is a run r of $\mathcal{A}$ on w such that $r((\emptyset, L)) = q$ and $r((L, \emptyset)) = q'$.

Example 6. The following $\mathfrak{L}$-automaton accepts the set of finite $\mathfrak{L}$-words over the alphabet Σ. Let $\mathcal{A} = (Q, \Sigma, I, F, \Delta)$ with $Q = \{e_l, e_r, n, p\}$, $I = \{n\}, F = \{n, p\}$, and

$$\begin{aligned}\Delta = &\{(n, \diamond, n), (p, \diamond, n)\} \cup \{(n, \sigma, p), (p, \sigma, p) \mid \sigma \in \Sigma \setminus \{\diamond\}\} \\ &\cup \{(\{n\}, n), (n, \{n\}), (p, \{n\}), (\{p\}, e_l), (e_r, \{p\}), (\{n, p\}, e_l), (e_r, \{n, p\})\}.\end{aligned}$$

For each $w \in W(\mathfrak{L})$, $r((C, D)) = \begin{cases} p & \text{if } \max(C) \text{ exists and } \max(C) \in \mathsf{supp}(w) \\ n & \text{otherwise,} \end{cases}$

defines an accepting run if w is a finite $\mathfrak{L}$-word. On an $\mathfrak{L}$-word w with infinite support, the successor transitions require infinitely many occurrences of state p. But then some limit position is marked with an error state e_l or e_r (where l means 'from left' and r 'from right') and the run cannot be continued (see Appendix B for details).

Automata on words (or infinite words or trees or infinite trees) have been applied fruitfully for representing structures. This can be lifted to the setting of $\mathfrak{L}$-words and leads to the notion of (oracle)-$\mathfrak{L}$-automatic structures.

Definition 7. *Fix an $\mathfrak{L}$-word o (called an oracle). A structure $\mathfrak{A} = (A, R_1, R_2, \ldots R_m)$ is $\mathfrak{L}$-o-*automatic *if there are $\mathfrak{L}$-automata $\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_m$ such that*

- *$\mathcal{A}$ represents the domain of $\mathfrak{A}$ in the sense that $A = \{w \mid w \otimes o \in L(\mathcal{A})\}$, and*
- *for each $i \leq m$, $\mathcal{A}_i$ represents R_i in the sense that $R_i = \{(w_1, w_2 \ldots, w_{r_i}) \mid w_1 \otimes w_2 \otimes \cdots \otimes w_{r_i} \otimes o \in L(\mathcal{A}_i)\}$, where r_i is the arity of relation R_i.*

We say that an $\mathfrak{L}$-o-automatic structure is finite word *$\mathfrak{L}$-o-automatic if its domain consists only of finite $\mathfrak{L}$-words. Let $\mathcal{F}_{\mathfrak{L}}$ denote the class of all finite word $\mathfrak{L}$-oracle-automatic graphs.*

For the constantly $\diamond$-valued oracle o ($\forall x \in \mathfrak{L}\ o(x) = \diamond$), we call an $\mathfrak{L}$-o-automatic structure $\mathfrak{L}$-automatic. We call some structure $\mathfrak{A}$ scattered-automatic (scattered-oracle-automatic, respectively) if there is some countable scattered linear order $\mathfrak{L}'$ (and some oracle o) such that $\mathfrak{A}$ is finite word $\mathfrak{L}'$-automatic ($\mathfrak{L}'$-o-automatic, respectively).

Rispal and Carton [15] showed that $\mathfrak{L}$-oracle-automata are closed under complementation if $\mathfrak{L}$ is countable and scattered which implies the following Proposition.

Proposition 8. *If $\mathfrak{L}$ is a countable scattered linear order, the set of finite word $\mathfrak{L}$-o-automatic structures is closed under first-order definable relations.*

2.3 Order Forests

Definition 9. *An (order)* forest *is a partial order $\mathfrak{A} = (A, \leq)$ such that for each $a \in A$, the set $\{a' \in A \mid a \leq a'\}$ is a finite linear order.*

Later we study the rank (also called ordinal height) of $\mathfrak{L}$-automatic well-founded forests. For this purpose we recall the definition of rank. Let $\mathfrak{A} = (A, \leq)$ be a well-founded partial order. Setting $\sup(\emptyset) = 0$ we define the *rank* of $\mathfrak{A}$ by $\mathsf{rank}(a, \mathfrak{A}) = \sup\{\mathsf{rank}(a', \mathfrak{A}) + 1 \mid a' < a \in A\}$ and $\mathsf{rank}(\mathfrak{A}) = \sup\{\mathsf{rank}(a, \mathfrak{A}) + 1 \mid a \in A\}$.

3 Sum- and Box-Augmentation Technique

Delhommé [5] characterised the set of ordinals that can be represented by finite tree-automata. His results relies on a decomposition of definable substructures into *sum-* and *box-augmentations*. Huschenbett [7] and Kartzow et al. [9] introduced a refined notion of *tamely colourable* box-augmentations in order to bound the ranks of tree-automatic linear orders and well-founded order trees, respectively. We first recall the definitions and then show that the decomposition technique also applies to finite word scattered-oracle-automatic structures.

Before we go into details, let us sketch the ideas underlying the sum- and box-augmentation technique. Given an $\mathfrak{L}$-o-automatic structure $\mathfrak{A}$ with domain A and some automaton $\mathcal{A}$ (called *parameter automaton*) that recognises a subset of $A \times W(\mathfrak{L})$, let us denote by $\mathfrak{A}_p$ the substructure of $\mathfrak{A}$ induced by $\mathcal{A}$ and p, i.e., with domain $\{a \in A \mid a \otimes p \in L(\mathcal{A})\}$. The main proposition of this section says that there is a certain class $\mathcal{C}$ of structures (independent of p) such that each $\mathfrak{A}_p$ is a tamely colourable sum-of-box augmentation of structures from $\mathcal{C}$. $\mathcal{C}$ consists of finitely many $\mathfrak{L}$-oracle-automatic structures and scattered-oracle-automatic structures where the underlying scattered linear order has finite condensation rank strictly below that of $\mathfrak{L}$. This allows to compute bounds on structural parameters (like finite condensation rank of linear orders or ordinal height of well-founded partial orders) by induction on the rank of $\mathfrak{L}$. We say a structural parameter φ is compatible with sum-of-box augmentations if for $\mathfrak{A}$ a sum-of-box augmentation of $\mathfrak{A}_1, \dots, \mathfrak{A}_n$, there is a bound on $\varphi(\mathfrak{A})$ in terms of $\varphi(\mathfrak{A}_1), \dots, \varphi(\mathfrak{A}_n)$. The decomposition result tells us that some $\mathfrak{L}$-automatic structure $\mathfrak{A}$ is (mainly) a sum of boxes of scattered-automatic structures where the underlying orders have lower ranks. Thus, by induction hypothesis φ is bounded on these building blocks of $\mathfrak{A}$. Thus, $\varphi(\mathfrak{A})$ is also bounded if φ is compatible with sum- and box-augmentations.

3.1 Sums and Boxes

The next definition recalls the notion of sum- and box-augmentations. We restrict the presentation to structures with one binary relation (but the general case is analogous).

Definition 10. – *A structure* $\mathfrak{A}$ *is a* sum-augmentation *of structures* $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ *if the domain of* $\mathfrak{A}$ *can be partitioned into* n *pairwise disjoint sets such that the substructure induced by the* i*-th set is isomorphic to* $\mathfrak{A}_i$*.*

– *A structure* $\mathfrak{A} = (A, \leq^A)$ *is a* box-augmentation *of structures* $\mathfrak{B}_1 = (B_1, \leq^{B_1})$, $\dots, \mathfrak{B}_n = (B_n, \leq^{B_n})$ *if there is a bijection* $\eta : \prod_{i=1}^n B_i \to A$ *such that for all* $1 \leq j \leq n$ *and all* $\bar{b} = (b_1, \dots, b_n) \in B_1 \times \dots \times B_n$

$$\mathfrak{B}_j \simeq \mathfrak{A}{\restriction}_{\eta(\{b_1\} \times \dots \times \{b_{j-1}\} \times B_j \times \{b_{j+1}\} \times \dots \times \{b_n\})}.$$

– *Let* $\mathcal{C}_1, \dots, \mathcal{C}_n$ *be classes of structures. A structure* $\mathfrak{A}$ *is a* sum-of-box augmentation *of* $(\mathcal{C}_1, \dots, \mathcal{C}_n)$ *if* $\mathfrak{A}$ *is a sum-augmentation of structures* $\mathfrak{B}_1, \dots, \mathfrak{B}_k$ *such that each* $\mathfrak{B}_j$ *is a box-augmentation of structures* $\mathfrak{C}_{j,1}, \dots, \mathfrak{C}_{j,n}$ *with* $\mathfrak{C}_{j,i} \in \mathcal{C}_i$*.*

Definition 11. *Let* $\mathfrak{A} = (A, \leq)$ *be a sum-of-box augmentation of structures* $\mathfrak{B}_{i,j} = (B_{i,j}, \leq_{i,j})$ *via the map* $\eta : \bigsqcup_{i=1}^n \prod_{j=1}^k B_{i,j} \to A$*. This sum-of-box augmentation is called* tamely colourable *if for each* $1 \leq j \leq k$ *there is a function* $\varphi_j : (\bigsqcup_{i=1}^n B_{i,j})^2 \to C_j$ *with a finite range* C_j *such that the* $(\varphi_j)_{1 \leq j \leq k}$ *determine the edges of* $\mathfrak{A}$ *in the sense that there is a set* $M \subseteq \prod_{j=1}^k C_j$ *such that* $\eta(b_1, \dots, b_k) \leq \eta(b'_1, \dots, b'_k)$ *iff* $(\varphi_1(b_1, b'_1), \dots, \varphi_k(b_k, b'_k)) \in M$*.*

3.2 Decomposition of Scattered-Automatic-Structures

In this section, we prove that the sum- and box-augmentation technique applies to finite word scattered-oracle-automatic structures. Fix an arbitrary scattered order $\mathfrak{L}$ with $\mathsf{FC}(\mathfrak{L}) = \alpha \geq 1$. Assume that $\mathfrak{L} = \sum_{z \in \mathbb{Z}} \mathfrak{L}_z$ where each $\mathfrak{L}_z$ is a (possibly empty) suborder with $\mathsf{FC}(\mathfrak{L}_z) < \alpha$. We first introduce notation concerning definable subgraphs.

Definition 12. *Let* $o \in W(\mathfrak{L})$ *be some oracle. Let* $\mathfrak{G} = (V, E)$ *be a finite word* $\mathfrak{L}$*-o-automatic graph. For each parameter automaton* $\mathcal{A}$ *and parameter* $p \in W(\mathfrak{L})$*, we write* $\mathfrak{G}_p^{\mathcal{A}}$ *for the induced subgraph of* $\mathfrak{G}$ *with domain* $V_p^{\mathcal{A}} := \{w \in V \mid w \otimes p \in L(\mathcal{A})\}$*.*

We write $\mathfrak{G}_p$ and V_p for $\mathfrak{G}_p^{\mathcal{A}}$ and $V_p^{\mathcal{A}}$ if $\mathcal{A}$ is clear from the context.

Definition 13. *Let* $c_0 = (C_0, D_0)$ *and* $c_1 = (C_1, D_1)$ *be cuts of* $\mathfrak{L}$*. For a finite* $\mathfrak{L}$*-word* w *we say* w *is a* (c_0, c_1)*-parameter if* $\mathsf{supp}(w) \subseteq D_0 \cap C_1$*, i.e., the support of* w *is completely between* c_0 *and* c_1*.*

For the rest of this section, we fix two numbers $z_0 < z_1 \in \mathbb{Z}$ *and define the cuts* $c_0 := (\sum_{z < z_0} \mathfrak{L}_z, \sum_{z \geq z_0} \mathfrak{L}_z)$ *and* $c_1 := (\sum_{z \leq z_1} \mathfrak{L}_z, \sum_{z > z_1} \mathfrak{L}_z)$*. We also define the scattered orders* $\mathfrak{L}_{\mathbf{L}} := \sum_{z < z_0} \mathfrak{L}_z$ *and* $\mathfrak{L}_{\mathbf{R}} := \sum_{z > z_1} \mathfrak{L}_z$*.* The main result of this section is a uniform sum-of-box decomposition of all substructures defined by a given parameter automaton.

Theorem 14. *Let* $\mathfrak{G}$ *be some finite word* $\mathfrak{L}$*-oracle-automatic graph* (V, E) *where* E *is recognised by some automaton* $\mathcal{A}_E$ *with state set* Q_E *and let* $\mathcal{A}$ *be a parameter automaton with state set* Q*. There are*

– *a set* $\mathcal{C}_{\mathbf{L}}$ *of* $\exp(|Q|^2 + 2|Q_E|^2)$ *many* $\mathfrak{L}_{\mathbf{L}}$*-oracle-automatic graphs, and*

– *a set* $\mathcal{C}_{\mathbf{R}}$ *of* $\exp(|Q|^2 + 2|Q_E|^2)$ *many* $\mathfrak{L}_{\mathbf{R}}$*-oracle-automatic graphs,*

such that for each (c_0, c_1)*-parameter* p *the subgraph* $\mathfrak{G}_p^{\mathcal{A}}$ *is a tamely-colourable sum-augmentation of box-augmentations of* $(\mathcal{C}_{\mathbf{L}}, \mathcal{F}_{\mathfrak{L}_{z_0}}, \mathcal{F}_{\mathfrak{L}_{z_0+1}}, \dots, \mathcal{F}_{\mathfrak{L}_{z_1}}, \mathcal{C}_{\mathbf{R}})$. [3]

Proof. Let o be the oracle such that $\mathfrak{G}$ is finite word $\mathfrak{L}$-o-automatic. By definition, we can write $\mathfrak{L}$ as the sum $\mathfrak{L}_{\mathbf{L}} + \mathfrak{L}_{z_0} + \mathfrak{L}_{z_0+1} + \dots + \mathfrak{L}_{z_1} + \mathfrak{L}_{\mathbf{R}}$. Induced by this decomposition there is a decomposition of any $\mathfrak{L}$-word w as $w = w_{\mathbf{L}} w_{z_0} w_{z_0+1} \dots w_{z_1} w_{\mathbf{R}}$ such that w_j is an $\mathfrak{L}_j$-word. In particular, our parameter and oracle decompose as

$$p = p_{\mathbf{L}} p_{z_0} p_{z_0+1} \dots p_{z_1} p_{\mathbf{R}} \quad \text{and} \quad o = o_{\mathbf{L}} o_{z_0} o_{z_0+1} \dots o_{z_1} o_{\mathbf{R}}.$$

Independently of the choice of the (c_0, c_1)-parameter p, $p_{\mathbf{L}}$ and $p_{\mathbf{R}}$ are constant functions (with value $\diamond$).

In order to construct a sum-of-box decomposition of $\mathfrak{G}_p$, we first define the building blocks of this decomposition. For this purpose, we define equivalence relations $\sim^i_{p\otimes o}$ for each $i \in \{\mathbf{L}, \mathbf{R}, z_0, z_0+1, \dots, z_1\}$ on $\mathfrak{L}_i$-words as follows. For $\mathfrak{L}_i$-words w, w' set $w \sim^i_{p\otimes o} w'$ if and only if

1. for all $q, q' \in Q \quad q \xrightarrow[\mathcal{A}]{w\otimes p_i \otimes o_i} q' \iff q \xrightarrow[\mathcal{A}]{w'\otimes p_i \otimes o_i} q'$ and
2. for all $q, q' \in Q_E \quad q \xrightarrow[\mathcal{A}_E]{w\otimes w \otimes o_i} q' \iff q \xrightarrow[\mathcal{A}_E]{w'\otimes w' \otimes o_i} q'$.

Note that for fixed i, p, o there are at most $\exp(|Q \times Q| + |Q_E \times Q_E|)$ many $\sim^i_{p\otimes o}$ equivalence classes. As domains of the α_i-oracle-automatic building blocks of our decomposition we use the sets $K(i, w, p, o) := \{x \mid x \sim^i_{p\otimes o} w\}$ for each $\mathfrak{L}_i$-word w. We augment this notation by writing $K(i, v, p, o) := K(i, w, p, o)$ for $\mathfrak{L}$-words v, where w is the restriction of v to $\mathfrak{L}_i$. Now for each $M \subseteq Q_E \times Q_E$ we define a structure $\mathfrak{K}^M(i, w, p, o) = (K(i, w, p, o), E^M)$ where $(w_1, w_2) \in E^M$ if $w_1, w_2 \in K(i, w, p, o)$ and there is a $(q, q') \in M$ such that $q \xrightarrow[\mathcal{A}_E]{w_1 \otimes w_2 \otimes o} q'$. Recall that $p_{\mathbf{L}}$ and $p_{\mathbf{R}}$ are independent of the concrete choice of the (c_0, c_1)-parameter p whence (for fixed o) the sets

$$\mathcal{C}_{\mathbf{L}} := \left\{\mathfrak{K}^M(\mathbf{L}, w, p, o) \mid M \subseteq Q_E \times Q_E, p \text{ a } (c_0, c_1)\text{-parameter}\right\}$$
$$\mathcal{C}_{\mathbf{R}} := \left\{\mathfrak{K}^M(\mathbf{R}, w, p, o) \mid M \subseteq Q_E \times Q_E, p \text{ a } (c_0, c_1)\text{-parameter}\right\}$$

have each at most $\exp(|Q|^2 + 2|Q_E|^2)$ many elements (up to isomorphisms).

Our next goal is the definition of the function η that witnesses the decomposition claimed in this theorem. For this purpose, let $\sim_{p\otimes o}$ denote the equivalence on $\mathfrak{L}$-words that is the product of the $\sim^i_{p\otimes o}$.[4] Let

$$\eta : \bigsqcup_{[w] \in V_p/\sim_{p\otimes o}} K(\mathbf{L}, w, p, o) \times \left(\prod_{i=z_0}^{z_1} K(i, w, p, o)\right) \times K(\mathbf{R}, w, p, o) \longrightarrow V_p$$
$$(x_{\mathbf{L}}, x_{z_0}, x_{z_0+1}, \dots, x_{z_1}, x_{\mathbf{R}}) \mapsto x := x_{\mathbf{L}} x_{z_0} x_{z_0+1} \dots x_{z_1} x_{\mathbf{R}}.$$

[3] Recall that $\mathcal{F}_{\mathfrak{L}}$ is the class of all finite word $\mathfrak{L}$-oracle-automatic graphs, see Definition 7.

[4] Thus, for $w = w_{\mathbf{L}} w_{z_0} w_{z_0+1} \dots w_{z_1} w_{\mathbf{R}}$ and $v = v_{\mathbf{L}} v_{z_0} v_{z_0+1} \dots v_{z_1} v_{\mathbf{R}}$ we have $w \sim_{p\otimes o} v$ iff $w_i \sim^i_{p\otimes o} v_i$ for all $i \in \{\mathbf{L}, \mathbf{R}, z_0, z_0+1, \dots, z_1\}$.

It follows from the definitions that η is a well-defined bijection (using the fact that some $\mathfrak{L}$-word x belongs to V_p iff there is a run

$$q_I \xrightarrow[\mathcal{A}]{x_{\mathbf{L}}\otimes p_{\mathbf{L}}\otimes o_{\mathbf{L}}} q_{z_0} \xrightarrow[\mathcal{A}]{x_{z_0}\otimes p_{z_0}\otimes o_{z_0}} q_{z_0+1}\dots q_{z_1} \xrightarrow[\mathcal{A}]{x_{\mathbf{R}}\otimes p_{\mathbf{R}}\otimes o_{\mathbf{R}}} q_F$$

for some initial state q_I and a final state q_F).

We finally prove that η witnesses that $\mathfrak{G}_p$ is a tamely-colourable sum-augmentation of box-augmentations of $(\mathcal{C}_{\mathbf{L}}, \mathcal{F}_{\mathfrak{L}_{z_0}}, \mathcal{F}_{\mathfrak{L}_{z_0+1}}, \dots, \mathcal{F}_{\mathfrak{L}_{z_1}}, \mathcal{C}_{\mathbf{R}})$. For any $w \in V_p$, let $\mathfrak{F}_w$ be the restriction of $\mathfrak{G}_p$ to $\eta\left(K(\mathbf{L},w,p,o) \times \left(\prod_{i=z_0}^{z_1} K(i,w,p,o)\right) \times K(\mathbf{R},w,p,o)\right)$. It is clear that $\mathfrak{G}_p$ is a sum augmentation of $(\mathfrak{F}_{w_1}, \mathfrak{F}_{w_2}\dots, \mathfrak{F}_{w_k})$ for w_i representatives of the $\sim_{p\otimes o}$-classes. From now on let $I_E(F_E)$ denote the initial (final) states of $\mathcal{A}_E$.

1. Fix $w = w_{\mathbf{L}} w_{z_0} w_{z_0+1}\dots w_{z_1} w_{\mathbf{R}} \in V_p$. We show that $\mathfrak{F}_w$ is a box-augmentation of $(\mathcal{C}_{\mathbf{L}}, \mathcal{F}_{\mathfrak{L}_{z_0}}, \mathcal{F}_{\mathfrak{L}_{z_0+1}}, \dots, \mathcal{F}_{\mathfrak{L}_{z_1}}, \mathcal{C}_{\mathbf{R}})$. For this purpose, fix $i \in \{\mathbf{L}, \mathbf{R}, z_0, z_0 + 1, \dots, z_1\}$ and let $\overleftarrow{w} := w_{\mathbf{L}}\dots w_{i-1}$, $\overleftarrow{o} := o_{\mathbf{L}}\dots o_{i-1}$, $\overrightarrow{w} := w_{i+1}\dots w_{\mathbf{R}}$, and $\overrightarrow{o} := o_{i+1}\dots o_{\mathbf{R}}$. Let M_i be the set defined by

   $$(q_1,q_2) \in M_i \iff \exists q_I \in I_E, q_F \in F_E \quad q_I \xrightarrow[\mathcal{A}_E]{\overleftarrow{w}\otimes\overleftarrow{w}\otimes\overleftarrow{o}} q_1 \text{ and } q_2 \xrightarrow[\mathcal{A}_E]{\overrightarrow{w}\otimes\overrightarrow{w}\otimes\overrightarrow{o}} q_F. \quad (1)$$

   The function

   $$\eta_i^w : K(i,w,p,o) \to V_p, \quad x_i \mapsto w_{\mathbf{L}} w_{z_0} w_{z_0+1}\dots w_{i-1} x_i w_{i+1}\dots w_{z_1} w_{\mathbf{R}}$$

   embeds $\mathfrak{K}^{M_i}(i,w,p,o)$ into $\mathfrak{G}_p$ because

   $$\begin{aligned} &\forall x_i, y_i \in K(i,w,p,o) \quad (x_i,y_i) \in E^{M_i} \\ \Leftrightarrow &\exists (q_1,q_2) \in M_i \quad q_1 \xrightarrow[\mathcal{A}_E]{x_i\otimes y_i\otimes o_i} q_2 \\ \overset{(1)}{\Leftrightarrow} &\exists q_I \in I_E, q_F \in F_E \quad q_I \xrightarrow[\mathcal{A}_E]{\overleftarrow{w}\otimes\overleftarrow{w}\otimes\overleftarrow{o}} q_1 \xrightarrow[\mathcal{A}_E]{x_i\otimes y_i\otimes o_i} q_2 \xrightarrow[\mathcal{A}_E]{\overrightarrow{w}\otimes\overrightarrow{w}\otimes\overrightarrow{o}} q_F \\ \Leftrightarrow &(\eta_i^w(x_i), \eta_i^w(y_i)) \in E. \end{aligned}$$

2. We show that the decomposition is tamely colourable. For all $j \in \{\mathbf{L}, \mathbf{R}, z_0, z_0 + 1, \dots, z_1\}$, let $c_j : (\bigsqcup_{[w]\in V_p/\sim_{p\otimes o}} K(j,w,p,o))^2 \to Q_E^2$ be the colouring function satisfying $c_j(x_j,y_j) := \{(q,q') \in A_E \mid q \xrightarrow[\mathcal{A}_E]{x_j\otimes y_j\otimes o_j} q'\}$. The colour functions $(c_j)_{j\in\{\mathbf{L},\mathbf{R},z_0,z_0+1,\dots z_1)}$ determine E because for $w = w_{\mathbf{L}} w_{z_0} w_{z_0+1}\dots w_{z_1} w_{\mathbf{R}}$ and $v = v_{\mathbf{L}} v_{z_0} v_{z_0+1}\dots v_{z_1} v_{\mathbf{R}}$,

   $$\begin{aligned} &(w_{\mathbf{L}} w_{z_0} w_{z_0+1}\dots w_{z_1} w_{\mathbf{R}}, v_{\mathbf{L}} v_{z_0} v_{z_0+1}\dots v_{z_1} v_{\mathbf{R}}) \in E \\ \iff &\exists q_0, \dots q_k \in Q_E \quad \begin{pmatrix} q_0 \in I_E, q_k \in F_E, \text{ and} \\ q_0 \xrightarrow[\mathcal{A}_E]{w_{\mathbf{L}}\otimes v_{\mathbf{L}}\otimes o_{\mathbf{L}}} q_1 \xrightarrow[\mathcal{A}_E]{w_{z_0}\otimes v_{z_0}\otimes o_{z_0}} q_2 \dots q_{k-1} \xrightarrow[\mathcal{A}_E]{w_{\mathbf{R}}\otimes v_{\mathbf{R}}\otimes o_{\mathbf{R}}} q_k \end{pmatrix} \\ \iff &\exists q_0, \dots q_k \in Q_E \quad \begin{pmatrix} q_0 \in I_E, q_k \in F_E, \text{ and} \\ (q_{i-1},q_i) \in c_j(w_j,v_j) \text{ with } j = \begin{cases} \mathbf{L} & \text{if } i=1, \\ \mathbf{R} & \text{if } i=k, \\ z_0+m & \text{if } i=m \end{cases} \end{pmatrix}. \end{aligned}$$

4 Bounds on Scattered-Oracle-Automatic Structures

4.1 FC-Ranks of Linear-Orders

In this section, we first study the question which scattered linear orders are $\mathfrak{L}$-oracle-automatic for a fixed order $\mathfrak{L}$. We provide a sharp bound on the FC-rank. For the upper bound we lift Schlicht and Stephan's result [17] using our new sum- and box-decomposition from the case where $\mathfrak{L}$ is an ordinal (detailed proof in Appendix C):

Theorem 15. *Let $\mathfrak{L}$ be a scattered order of FC_* rank $1+\alpha$ (0, respectively) for some ordinal α. Then every finite word $\mathfrak{L}$-oracle-automatic scattered linear order $\mathfrak{A}$ satisfies $\mathsf{FC}_*(\mathfrak{A}) < \omega^{\alpha+1}$ ($\mathsf{FC}_*(\mathfrak{A}) < \omega^0 = 1$, respectively).*

If $\mathfrak{L}$ is an ordinal of the form $\omega^{1+\alpha}$, Schlicht and Stephan [17] showed that the supremum of the $\mathfrak{L}$-automatic ordinals is exactly $\omega^{\omega^{\alpha+1}}$ whence Theorem 15 is optimal. From our theorem we can also derive the following characterisation of finite FC-rank presentable ordinals (cf. Appendix E).

Corollary 16. *Let $\mathfrak{L}$ be a countable scattered linear order with $\mathsf{FC}(\mathfrak{L}) < \omega$. The finite word $\mathfrak{L}$-oracle-automatic ordinals are exactly those below $\omega^{\omega^{\mathsf{FC}(\mathfrak{L})+1}}$.*

The oracle in this claim cannot be removed. In fact, 0 and 1 are the only finite word $\mathbb{Z}^n$-automatic ordinals if $n \geq 1$ (any $\mathbb{Z}^n$-automatic linear order with 2 elements contains a copy of $\mathbb{Z}$).

4.2 Ranks of Well-Founded Automatic Order Forests

We next study scattered-oracle-automatic well-founded order forests. Kartzow et al. [9] proved compatibility of the height function with sum- and box-augmentations. Together with our decomposition theorem, this yields a bound on the height of an $\mathfrak{L}$-oracle-automatic well-founded order forest in terms of $\mathsf{FC}(\mathfrak{L})$. Unfortunately, in important cases these bounds are not optimal. For scattered orders $\mathfrak{L}$ where the set of finite $\mathfrak{L}$-words allow an $\mathfrak{L}$-oracle-automatic order which is scattered, we can obtain better bounds. If $\mathfrak{L}$ is an ordinal or has finite FC-rank, the set of $\mathfrak{L}$-words allows such a scattered ordering. If the finite $\mathfrak{L}$-words admit an $\mathfrak{L}$-automatic scattered order $\leq$, the Kleene-Brouwer ordering of an $\mathfrak{L}$-oracle-automatic well-founded order forest with respect to $\leq$ is $\mathfrak{L}$-oracle-automatic again. Thus, its FC-rank is bounded by our previous result. Adapting a result of Kuske et al.[14] relating the FC-rank of the Kleene-Brouwer ordering with the height of the forest, we derive a bound on the height (cf. Appendix D). Our main result on forests is as follows.

Theorem 17.
- *Let $\mathfrak{L}$ be an ordinal or a scattered linear order with $\mathsf{FC}(\mathfrak{L}) < \omega$. Each $\mathfrak{L}$-oracle-automatic forest $\mathfrak{F} = (F, \leq)$ has rank strictly below $\omega^{\mathsf{FC}(\mathfrak{L})+1}$.*
- *Let $\mathfrak{L}$ be some scattered linear order. Each $\mathfrak{L}$-oracle-automatic forest $\mathfrak{F} = (F, \leq)$ has rank strictly below $\omega^{\omega \cdot (\mathsf{FC}(\mathfrak{L})+1)}$.*

Remark 18. The bounds in the first part are optimal: for each ordinal $\mathfrak{L}$ and each $c \in \mathbb{N}$, we can construct an $\mathfrak{L}$-automatic tree of height $\omega^{\mathsf{FC}(\mathfrak{L})} \cdot c$ (cf. Appendix D.5).

5 Separation of Tree- and Ordinal-Automatic Structures

Theorem 19. *The countable atomless Boolean algebra is not finite word $\mathfrak{L}$-automatic for any ordinal $\mathfrak{L}$.*

This theorem is proved by first showing that, if the atomless Boolean algebra is finite word $\mathfrak{L}$-automatic for some ordinal $\mathfrak{L}$, then it already is ω^n-automatic for some $n \in \mathbb{N}$. This follows because any finite word $\mathfrak{L}$-automatic structure for $\mathfrak{L}$ an ordinal above ω^ω has a sufficiently elementary substructure that has a ω^n-automatic presentation for some $n \in \mathbb{N}$. In the case of the countable atomless Boolean algebra any Σ_3-elementary substructure is isomorphic to the whole algebra. Extending Khoussainov et al.'s monoid growth rate argument for automatic structures (cf. [12]) to the ω^n-setting, we can reject this assumption (cf. Appendix F). This answers a question of Frank Stephan.

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A Basics on Scattered Linear Orders

Recall the following basic (folklore) results.

Lemma 20. *Let $\mathfrak{L} = (L, \leq)$ be a scattered linear order with $\mathsf{FC}(\mathfrak{L}) = \alpha$. For all $l, l' \in L$, there are some $n \in \mathbb{N}$ and scattered linear orders $\mathfrak{L}_1, \mathfrak{L}_2, \dots, \mathfrak{L}_n$ of condensation rank strictly below α such that $[l, l'] \cong \mathfrak{L}_1 + \mathfrak{L}_2 + \cdots + \mathfrak{L}_n$.*

Proof. $\mathfrak{L}$ can be written as $\sum_{i \in \mathbb{Z}} \mathfrak{L}_i$ for $\mathfrak{L}_i$ scattered linear orders with $\mathsf{FC}(\mathfrak{L}) < \alpha$. If l comes from the j-th factor of this sum and l' form the j'-th, then $[l, l']$ is isomorphic to $\mathfrak{L}'_j + \sum_{i=j+1}^{j'-1} \mathfrak{L}_i + \mathfrak{L}'_{j'}$ where $\mathfrak{L}'_j$ and $\mathfrak{L}'_{j'}$ are suborders of $\mathfrak{L}_j$ and $\mathfrak{L}_{j'}$ whence they have rank below α. ☐

Lemma 21. *Let $\gamma \in \{\omega, \omega^*, \zeta\}$, $\mathfrak{L}_i$ be a scattered order of FC_* rank α. The order $\mathfrak{L} := \sum_{i \in \gamma} \mathfrak{L}_i$ is of rank $\mathsf{FC}_*(\mathfrak{L}) = \alpha + 1$.*

Proof. Since $\mathsf{FC}_*(\mathfrak{L}_i) = \alpha$, for all $\beta < \alpha$ the β-th condensation of $\mathfrak{L}_i$ contains infinitely many nodes. Thus, also the β-th condensation of $\mathfrak{L}$ contains infinitely many equivalence classes containing elements in $\mathfrak{L}_i$. Thus, for each $i \in \gamma$ such that $i + 2 \in \gamma$ and for every $x_i \in \mathfrak{L}_i$, $x_{i+2} \in \mathfrak{L}_{i+2}$ the β condensation of x_i and the β-condensation of x_{i+2} are separated by infinitely many nodes (the β condensations of $\mathfrak{L}_{i+1}$). Thus, the α condensation of $\mathfrak{L}$ does not identify nodes of $\mathfrak{L}_i$ and $\mathfrak{L}_{i+2}$. Thus, it contains a suborder isomorphic to γ, whence $\mathsf{FC}_*(\mathfrak{L}) \geq \alpha + 1$. On the other hand, since each $\mathfrak{L}_i$ has rank α the α-condensation of $\mathfrak{L}$ is a γ-sum over finite linear orders. Hence its $\alpha + 1$-condensation is finite and $\mathsf{FC}_*(\mathfrak{L}) \leq \alpha + 1$. ☐

Lemma 22. *[Lemma 4.16 of [8]] Let $\mathfrak{L}$ be a linear order and $\alpha < \mathsf{FC}(\mathfrak{L})$. There is a closed interval I of $\mathfrak{L}$ such that I is a scattered linear suborder of $\mathfrak{L}$, $\mathsf{FC}(I) = \alpha + 1$, and $\mathsf{FC}_*(I) = \alpha$.*

B Correctness of the Automaton in Example 6

States e_l and e_r report errors from left and from right, respectively, i.e., a cut is forced to be visited in state e_l if it is a right limit step such that left of this limit infinitely many positive positions appear.

On input w, the successor transitions mark the support of w by state p and all other successor positions in w by n. Let $P(w) \subseteq \mathsf{Cuts}(w)$ be defined by $(C, D) \in P(w)$ if $\exists x \in \mathsf{supp}(w)$ such that $x = \max(C)$. If w has finite support, then

$$r(c) := \begin{cases} p & \text{if } c \in P(w) \\ n & \text{otherwise} \end{cases}$$

defines an accepting run on w.

We now prove that there is no accepting run if w is not a finite word. Heading for a contradiction assume that r is an accepting run of $\mathcal{A}$ on w and w has infinite support. Then $r((\emptyset, L)) = n = r((L, \emptyset))$. We want to show that there is a cut c such that $r(c) = e_l$ or $r(c) = e_r$.

If we are able to show this, we arrive at a contradiction: if $r(c) = e_l$ then c is not the maximal cut. But there is no successor transition and no left limit transition from state e_l. Thus, r cannot assign states to the cuts to the right of c, which is a contradiction. If state e_r occurs, the argument is the same using the cuts to the left of c.

We show that there is a cut that is assigned an error state e_l or e_r. Assume that there is an infinite ascending chain $l_1 < l_2 < l_3 < \dots$ in $\mathfrak{L}$ such that $\{l_i \mid i \in \mathbb{N}\} \subseteq \mathsf{supp}(w)$. For $C := \{x \in L \mid \exists i \in \mathbb{N} \quad x \leq l_i\}$ and $D := L \setminus C$ the cut $c := (C, D)$ has no direct predecessor. Moreover, $p \in \lim_{c^-} r$ because state p occurs at each cut associated to one of the l_i. Thus, if there is a right limit transition applicable at c, it assigns state e_l to c. If there is no infinite ascending chain in $\mathsf{supp}(w)$, then there is an infinite descending chain. The analogous argument shows that then state e_r occurs.

C Proof of Theorem 15

Huschenbett [7] used the sum-of-box decomposition technique in order to prove a strict bound on the finite condensation rank of tree-automatic scattered linear orders. His result relies on the fact that the finite condensation rank behaves well with box-decompositions in the following sense. Let $\alpha_0 \oplus \dots \oplus \alpha_n$ denote the commutative sum of $\alpha_0, ..., \alpha_n$.

Lemma 23. *[Proposition 4.11 in [8]] For each scattered linear order $\mathfrak{A}$ that is a tamely-colourable box-augmentation of $\mathfrak{B}_1, \dots, \mathfrak{B}_n$, its rank is bounded by*

$$\mathsf{FC}_*(\mathfrak{A}) \leq \mathsf{FC}_*(\mathfrak{B}_1) \oplus \mathsf{FC}_*(\mathfrak{B}_2) \oplus \dots \oplus \mathsf{FC}_*(\mathfrak{B}_n).$$

Moreover, Khoussainov et al. have already shown that FC_* rank behaves well with sum-augmentations.

Lemma 24. *[Proposition 4.4 in [13]] For each scattered linear order $\mathfrak{A}$ that is a sum-augmentation of $\mathfrak{B}_1, \dots, \mathfrak{B}_n$, its rank is determined by*

$$\mathsf{FC}_*(\mathfrak{A}) = \max\{\mathsf{FC}_*(\mathfrak{B}_1), \mathsf{FC}_*(\mathfrak{B}_2), \dots, \mathsf{FC}_*(\mathfrak{B}_n)\}.$$

Proposition 25. *Let α be a scattered order of FC rank $1 + \gamma$ (0, respectively) for some ordinal γ. Every α-oracle-automatic scattered linear order has FC_* rank strictly below $\omega^{\gamma+1}$ ($\omega^0 = 1$, respectively).*

Proof. In the case $\mathsf{FC}(\alpha) = 0$ the domain of an α-automatic structure has at most $|\Sigma|$ many elements. The theorem follows because every finite linear order has FC_* rank 0.

Now let $\mathsf{FC}(\alpha) = 1 + \gamma$. As induction hypothesis assume that the theorem holds for all orders β with $\mathsf{FC}(\beta) < 1 + \gamma$. Heading for a contradiction assume that $\mathfrak{L} = (L, \leq)$ is an α-oracle-automatic scattered linear order such that $\mathsf{FC}_*(\mathfrak{L}) \geq \omega^{\gamma+1}$. Let $\leq$ be recognised by some automaton with state set $Q_{\leq}$. Due to Lemma 22 the automaton $\mathcal{A}$ corresponding to the formula $\varphi(x, y_1, y_2) := y_1 \leq x \leq y_2$ is a parameter automaton such that for each $n \in \mathbb{N}$ there is a parameter p_n such that $\mathfrak{L}_{p_n}$ is a scattered linear order with $\mathsf{FC}_*(\mathfrak{L}_{p_n}) = \omega^{\gamma} \cdot n$. Assume that $\mathcal{A}$ has state set Q.

Now, fix some $n_0 \in \mathbb{N}$ such that $n_0 > 4^{2+2\cdot\exp(|Q|^2+2|Q_\le|^2)}$. Due to Theorem 14, there are sets $\mathcal{C}_0, \mathcal{C}_1$ of size $\exp(|Q|^2 + 2|Q_\le|^2)$ such that for each $n \le n_0$, $\mathfrak{L}_{p_n}$ is a tamely-colourable sum-augmentation of box-augmentations of $\mathcal{C}_0, \mathcal{C}_1$ and sets of β_i-oracle-automatic structures where $\mathsf{FC}(\beta_i) < \mathsf{FC}(\alpha)$ (cf. Lemma 20). By choice of n_0, there is some $1 \le m < \frac{n_0}{4}$ such that for all structures $\mathfrak{A} \in \mathcal{C}_0 \cup \mathcal{C}_1$

$$\begin{aligned}&\mathsf{FC}_*(\mathfrak{A}) \le \omega^\gamma \cdot m \text{ or}\\&\mathsf{FC}_*(\mathfrak{A}) > \omega^\gamma \cdot 4m.\end{aligned} \tag{2}$$

Now consider the decomposition of $\mathfrak{L}_{p_{4m}}$. Due to Lemma 24 there is a suborder $\mathfrak{L}'$ of $\mathfrak{L}_{p_{4m}}$ with $\mathsf{FC}_*(\mathfrak{L}') = \omega^\gamma \cdot 4m$ that is tamely-colourable box-augmentation of structures $(\mathfrak{C}_0, \mathfrak{C}_1, \mathfrak{B}_1, \dots, \mathfrak{B}_k)$ where $\mathfrak{C}_0 \in \mathcal{C}_0, \mathfrak{C}_1 \in \mathcal{C}_1$, and $\mathfrak{B}_i$ a β_i-oracle-automatic structure for each $1 \le i \le k$. Note that for each $1 \le i \le k$, by induction hypothesis $\mathsf{FC}_*(\mathfrak{B}_i) < \omega^{\gamma_i+1}$ for some $\gamma_i < \gamma$. Thus,

$$\mathsf{FC}_*(\mathfrak{B}_1) \oplus \dots \oplus \mathsf{FC}_*(\mathfrak{B}_k) < \omega^{\max\{\gamma_i | 1 \le i \le k\}+1} \le \omega^\gamma.$$

Moreover, since $\mathfrak{C}_0$ and $\mathfrak{C}_1$ are substructures of $\mathfrak{L}'$, we have $\mathsf{FC}_*(\mathfrak{C}_i) \le \omega^\gamma \cdot 4m$ whence (2) implies that $\mathsf{FC}_*(\mathfrak{C}_i) \le \omega^\gamma \cdot m$ for $i \in \{0, 1\}$. Due to the properties of $\oplus$ and Lemma 23 we arrive at the contradiction

$$\begin{aligned}\mathsf{FC}_*(\mathfrak{L}') = \omega^\gamma \cdot 4m &\le \mathsf{FC}_*(\mathfrak{C}_0) \oplus \mathsf{FC}_*(\mathfrak{C}_1) \oplus \mathsf{FC}_*(\mathfrak{B}_1) \oplus \dots \oplus \mathsf{FC}_*(\mathfrak{B}_k)\\&\le \omega^\gamma \cdot m \oplus \omega^\gamma \oplus \omega^\gamma \cdot m\\&< \omega^\gamma \cdot 4m.\end{aligned}$$

□

Theorem 15 now follows as a corollary of this Proposition.

Corollary 26. *Let α be a scattered order of FC_* rank $1 + \gamma$ (0, respectively) for some ordinal γ. Every finite word α-oracle-automatic scattered linear order has FC_* rank strictly below $\omega^{\gamma+1}$ ($\omega^0 = 1$, respectively).*

Proof. If α is a scattered linear order such that $\mathsf{FC}_*(\alpha) = 1 + \gamma$, then there are linear orders α_i with $\mathsf{FC}(\alpha_i) \le 1 + \gamma$ for $1 \le i \le k$ such that $\alpha = \sum_{i=1}^k \alpha_i$.

Theorem 14 implies that each finite word α-oracle-automatic scattered linear order $\mathfrak{L}$ is a tamely colourable sum-of-box augmentations of $(\mathcal{F}_{\alpha_1}, \dots, \mathcal{F}_{\alpha_k})$, the classes of finite word α_i-oracle-automatic structures. Due to Lemmas 23 and 24 there are α_i-oracle-automatic scattered linear orders $\mathfrak{L}_i$ (for $1 \le i \le k$) such that $\mathsf{FC}_*(\mathfrak{L}) \le \mathsf{FC}_*(\mathfrak{L}_1) \oplus \dots \oplus \mathsf{FC}_*(\mathfrak{L}_k)$. Since $\mathsf{FC}_*(\mathfrak{L}_i) < \omega^{\gamma+1}$ for each $1 \le i \le k$, we immediately conclude that $\mathsf{FC}_*(\mathfrak{L}) < \omega^{\gamma+1}$. □

D Ranks of Forests

We now introduce a variant of the height of a well-founded partial order called *infinity rank* and denoted by ∞-rank.

Definition 27. *Let $\mathfrak{P} = (P, \leq)$ be a well-founded partial order. We define the ordinal valued* ∞-rank *of a node $p \in P$ inductively by*

$$\infty\text{-}\mathsf{rank}(p, \mathfrak{P}) = \sup\{\alpha + 1 \mid \exists^\infty p'(p' < p \textit{ and } \infty\text{-}\mathsf{rank}(p', \mathfrak{P}) \geq \alpha)\}.$$

The ∞-rank *of $\mathfrak{P}$ is then*

$$\infty\text{-}\mathsf{rank}(\mathfrak{P}) = \sup\{\alpha + 1 \mid \exists^\infty p \in P \quad \infty\text{-}\mathsf{rank}(p, \mathfrak{P}) \geq \alpha\}.$$

Lemma 28. *[9] For $\mathfrak{P}$ a well-founded partial order, we have*

$$\infty\text{-}\mathsf{rank}(\mathfrak{P}) \leq \mathsf{rank}(\mathfrak{P}) < \omega \cdot (\infty\text{-}\mathsf{rank}(\mathfrak{P}) + 1).$$

In this section, we prove the following bound on the ranks of α-automatic order forests.

Theorem 29. *Let α be some scattered linear order.*

1. *Every α-oracle-automatic order forest $\mathfrak{F} = (F, \leq)$ such that*
   - *F is also the domain of some α-oracle-automatic scattered linear order, and*
   - $\mathsf{FC}(\alpha) = 1 + \gamma$

   has rank strictly below $\omega^{1+\gamma+1}$ and ∞-rank *strictly below $\omega^{\gamma+1}$.*
2. - *If $\mathsf{FC}(\alpha) < \omega$, then every α-oracle-automatic order forest has rank strictly below $\omega^{\mathsf{FC}(\alpha)+1}$.*
   - *If $\mathsf{FC}(\alpha) = \omega + c_0$ for some $c_0 < \omega$, then every α-oracle-automatic order forest has rank strictly below $\omega^{\omega \cdot (c_0+1)}$.*
   - *If $\mathsf{FC}(\alpha) = \omega \cdot c_1 + c_0$ for $c_0, c_1 < \omega$ and $c_1 \geq 2$, then every α-oracle-automatic order forest has rank strictly below $\omega^{\omega^2 \cdot (c_1 - 1) + \omega \cdot (c_0+1)}$.*
   - *If $\mathsf{FC}(\alpha) \geq \omega^2$, then every α-oracle-automatic order forest has rank strictly below $\omega^{\omega \cdot \mathsf{FC}(\alpha) + \omega}$.[5]*

Remark 30. If α is an ordinal or $\mathsf{FC}(\alpha) < \omega$, we show in the next section that every α-oracle-automatic set F of finite α-words allows a scattered linear order. Thus, if α satisfies one of these conditions, then the better bounds hold.

D.1 A Scattered Order of Scattered Words

We first show that scattered orders α of finite rank allow a scattered order of all finite α-words that is α-automatic. Afterwards, we show that the analogous result holds in case that α is an ordinal. Our first claim is proved by induction on the FC-rank and the FC_*-rank of α. We prepare our result by defining an automaton that determines at every cut the left and the right rank of this cut. Given a cut $c = (C, D)$ without direct predecessor, the left rank is the minimal rank of the induced suborders of nonempty upwards closed subsets of C. Analogously, the right rank is the minimal rank of the induced suborders of nonempty downwards closed subsets of D.

[5] In particular, if $\mathsf{FC}(\alpha) = \omega^n \cdot c_n + \omega^{n-1} \cdot c_{n-1} + \cdots + \omega \cdot c_1 + c_0$ such that $n \geq 2$, $c_1, c_2, \ldots, c_n < \omega$, and $c_n \neq 0$, then every α-oracle-automatic order forest has rank strictly below $\omega^{\omega^{n+1} \cdot c_n + \omega^{n-1+1} \cdot c_{n-1} \cdots \omega^2 \cdot c_1 + \omega \cdot (c_0+1)}$.

Definition 31. *For Σ arbitrary, let $\mathcal{C}_n = (Q_n, \Sigma, I_n, F_n, \Delta_n)$ be an automaton with state set $Q_n := \{0, 1, \dots, n\} \times \{0, 1, \dots, n\}$, initial states $I_n = \{0\} \times \{0, 1, \dots, n\}$ and final state $F_n = \{0, 1, \dots, n\} \times \{0\}$. In order to define its transition relation, we use the following notation for $i \leq n$, let $\mathcal{P}_i$ be defined by*

$$\{S \in 2^{Q_n} \mid \forall j > i \; \forall k \quad (j,k),(k,j) \notin S \text{ and } \exists k \leq i \quad (i,k) \in S \text{ or } (k,i) \in S\}.$$

The transition relation of $\mathcal{C}_n$ is

$$\begin{aligned} \Delta_n = &\{((i,0), \sigma, (0,j)) \mid \sigma \in \Sigma \text{ and } i,j \in \{0,1,\dots n\}\} \\ &\cup \{((i,j), X) \mid X \in \mathcal{P}_j\} \\ &\cup \{(X, (i,j)) \mid X \in \mathcal{P}_i\} \end{aligned}$$

Lemma 32. *Let α be some scattered linear order and w an arbitrary α-word. Interpreting $\mathcal{C}_n$ as an α-automaton, there is an accepting run r of $\mathcal{C}_n$ on w if and only if $\mathsf{FC}_*(\alpha) \leq n$. In this case, r is the unique accepting run and for every cut $c = (C, D)$ the state at c is*

- *in $\{0\} \times \{0, 1, \dots, n\}$ if c has a direct predecessor,*
- *in $\{0, 1, \dots, n\} \times \{0\}$ if c has a direct successor,*
- *in $\{k\} \times \{0, 1, \dots, n\}$ (with $k \geq 1$) if c has no direct predecessor, and for each cut $c' < c$ there is a cut c'' such that $c' < c'' < c$ and $\mathsf{FC}(\alpha\restriction_{(c'',c)}) = k$, and*
- *in $\{0, 1, \dots, n\} \times \{k\}$ (with $k \geq 1$) if c has no direct successor, and for each cut $c' > c$ there is a cut c'' such that $c' > c'' > c$ and $\mathsf{FC}(\alpha\restriction_{(c,c'')}) = k$.*

Proof. First, let $n \geq \mathsf{FC}_*(\alpha)$. This implies, that for all cuts c'' and c, the suborder induced by (c'', c) has FC-rank at most $\mathsf{FC}_*(\alpha) \leq n$. Moreover, if c is a cut without direct predecessor, and if $c_1 < c_2 < c_3 < \cdots < c$ is an infinite chain of cuts whose limit is c, then $\mathsf{FC}(\alpha\restriction_{(c_i,c)})$ stabilises at some i_0. Thus, the following function r is well-defined. It is a function $r : \mathsf{Cuts}(\alpha) \to Q_n$ where for each cut $c = (C, D)$ we have $r(C, D) = (i, j)$ such that

1. $i = 0$ if c has a direct predecessor or $C = \emptyset$,
2. otherwise, $i = \min\{\mathsf{FC}((c', c)) \mid c' < c\}$,
3. $j = 0$ if c has a direct successor or $D = \emptyset$,
4. otherwise, $j = \min\{\mathsf{FC}((c, c')) \mid c' > c\}$.

A straightforward induction on the left and right rank of each cut in α shows that r is consistent with the transition relation, i.e., r is an accepting run of $\mathcal{C}_n$ on each α-word.

We next show that r is the unique run of $\mathcal{C}_n$ on α-words. Heading for a contradiction assume that r' is another accepting run on some α-word and that $c = (C, D)$ satisfies $r(c) = (i, j) \neq r'(c) = (i', j')$. Without loss of generality (the other case is symmetric), we may assume that $i \neq i'$ and c has been chosen such that i is minimal with this property. We distinguish the following cases:

- Assume that $i = 0$. Since r' is accepting, c cannot be the minimal cut. Thus, c has a direct predecessor c'. But independent of the successor transition used between c' and c, $r'(c) \in \{0\} \times \{0, 1, \dots, n\}$ whence $i = i' = 0$ contradicting the assumption $i \neq i'$.

- Assume that $i \geq 1$. The right limit transition applied by r at c shows that there is a cut $c' < c$ such that for all $c'' \in (c', c)$, $r(c'') \in \{0, 1, \ldots, i-1\}^2$. By minimality of i, r and r' agree on this interval. But then again the applicable right limit transitions always imply that $i' = i$ contradicting $i' \neq i$.

Finally, we have to show that there are no accepting runs of $\mathcal{C}_n$ on α-words if $\mathsf{FC}_*(\alpha) > n$. Assume that $\mathsf{FC}_*(\alpha) > n$. Due to Lemma 22, α contains an interval α' with $\mathsf{FC}_*(\alpha') = n + 1$. We show that there is no function $r : \alpha' \to Q_n$ which is consistent with the transition relation Δ_n. Up to symmetry, α' contains an upwards closed interval of the form $\sum_\omega \beta_i$ with $\mathsf{FC}(\beta_i) = n$. As shown in the first part, there is an accepting run r' of $\mathcal{C}_{n+1}$ on this sum. For the maximal cut $c_{\max}$ of α', we have $r'(c_{\max}) = (n + 1, 0)$. In fact, one easily sees that the previous arguments apply to any (possibly non-accepting run) on α' in the sense that any run on α' satisfies $r'(c_{\max}) \in \{n+1\} \times \{0, 1, \ldots, n+1\}$. Since $\Delta_n \subsetneq \Delta_{n+1}$, any run of $\mathcal{C}_n$ on α is also a run of $\mathcal{C}_{n+1}$ that does not use states from $\{n + 1\} \times \{0, 1, \ldots, n + 1\}$. But we have seen that any run of $\mathcal{C}_{n+1}$ on α' would label $c_{\max}$ with such a state. Thus, there is no run of $\mathcal{C}_n$ on α' whence there can neither be a run of $\mathcal{C}_n$ on α. □

The automaton $\mathcal{C}_n$ will be useful to decompose an order α with $\mathsf{FC}_*(\alpha) = n$ into finitely many pieces $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ of FC-rank at most n.

Lemma 33. *Let α be an order with $\mathsf{FC}_*(\alpha) = n$ and r the accepting run of $\mathcal{C}_n$ on α-words. Let c, d be consecutive cuts of maximal rank in the sense that*

- *c is minimal or $r(c) = (i, j)$ with $\max(i, j) = n$,*
- *d is maximal or $r(d) = (k, l)$ with $\max(k, l) = n$, and*
- *for all $e \in (c, d)$, $r(e) = (x, y)$ we have $\max(x, y) < n$.*

Then the interval (c, d) of α has FC-rank at most n.

Remark 34. In particular, this lemma implies that in an order α with $\mathsf{FC}_*(\alpha) = n$ there are only finitely many cuts of left or right rank n.

Proof. By induction on i, we prove that for arbitrary cuts $c \leq d$ the following holds. If for all cuts e strictly between c and d we have $r(e) \in \{0, 1, \ldots, i - 1\}^2$ then $\mathsf{FC}((c, d)) \leq i$.

For $i = 0$, the condition implies that $c = d$ whence $\mathsf{FC}((c, d)) = \mathsf{FC}(\emptyset) = 0$. Now assume that this claim holds for $i - 1$ and that for all cuts $e \in (c, d)$ we have $r(e) \in \{0, 1, \ldots, i - 1\}^2$. By definition of the limit transitions, we know that $r(c) \in \{0, 1, \ldots, n\} \times \{0, 1, \ldots, i\}$ and that $r(d) \in \{0, 1, \ldots, i\} \times \{0, 1, \ldots, n\}$. From our construction of the accepting run r (compare the previous proof), we conclude that there are cuts $c < c_1 \leq d_1 < d$ such that $\mathsf{FC}((c, c_1)) \leq i - 1$ and $\mathsf{FC}((d_1, d)) \leq i - 1$. Next, we claim that there are only finitely many cuts $c_1 < e < d_1$ such that $r(e) \in M_{i-1} := (\{i-1\} \times \{0, 1, \ldots, \{i-1\}) \cup (\{0, 1, \ldots, \{i-1\}) \times \{i-1\})$. Otherwise there would be an infinite ascending or descending chain of cuts in M_{i-1} whose limit e would satisfy $c_1 \leq e \leq d_1$ and $r(e) \notin \{0, 1, \ldots, i-1\}^2$ contradicting our assumptions on the interval (c, d). Thus, let $c_1 = e_1 < e_2 < \cdots < e_{n-1} < e_n = d_1$ be a finite sequence of cuts such that for all $c_1 \leq e \leq d_1$ we have $r(e) \in M_{i-1}$ only if there is a $1 \leq j \leq n$ with

$e = e_j$. Thus, $(c, d) = (c, c_1) + \sum_{i=1}^{n-1}(e_i, e_{i+1}) + (d_1, d)$ is a finite sum of intervals that (by induction hypothesis) have FC-rank at most $i - 1$. Thus, $\mathsf{FC}((c, d)) \leq i$ as desired. ☐

Let us collect one more fact about $\mathcal{C}_{n+1}$. Assume that α is an order with $\mathsf{FC}(\alpha) = \mathsf{FC}_*(\alpha) = n+1$. This implies that $\alpha = \sum_{\gamma \in \Gamma} \alpha_\gamma$ where $\Gamma \in \{\omega, \omega^*, \mathbb{Z}\}$ and $\mathsf{FC}(\alpha_\gamma) \leq n$ where for infinitely many $\gamma \in \Gamma$ we have $\mathsf{FC}(\alpha_\gamma) = n$. Thus, $\mathcal{C}_n$ has an accepting run on each α_γ that agrees with the run of $\mathcal{C}_{n+1}$ on α on the interval α_γ. Hence, the run of $\mathcal{C}_{n+1}$ assumes only finitely many often a state from $M_n := \{n\} \times \{1, 2, \dots, n\} \cup \{1, 2, \dots, n\} \times n\}$ on each α_γ. The next lemma follows immediately.

Lemma 35. *Let α be an order with $\mathsf{FC}(\alpha) = n + 1$. Let r be the accepting run of $\mathcal{C}_{n+1}$ on some α-word. The suborder induced by the cuts $\{c \mid r(c) \in M_n\}$ form a suborder of $\mathbb{Z}$.*

Then there is an accepting run of $\mathcal{C}_{n+1}$ on every α-word but no run of $\mathcal{C}_n$ on some α-word.

Lemma 36. *Suppose that α is a scattered linear order with $\mathsf{FC}(\alpha) < \omega$. Then there is an α-oracle-automatic scattered linear order on the set of finite α-words.*

Proof. We define automata $\mathcal{A}_n$ (and $\mathcal{B}_n$, respectively) for each $n < \omega$ which uniformly define α-automatic scattered linear orders on the finite α-words over a fixed alphabet Σ for all scattered linear orders α with $\mathsf{FC}(\alpha) \leq n$ (and $\mathsf{FC}_*(\alpha) \leq n$, respectively). Note that for α with $\mathsf{FC}(\alpha) = 0$ there is a finite number of α-words over Σ whence the construction of $\mathcal{A}_0$ is trivial.

Suppose that we have constructed $\mathcal{A}_n$. We define $\mathcal{B}_n$ as follows. If $\mathsf{FC}_*(\alpha) \leq n$, the run of the automaton $\mathcal{C}_n$ partitions α uniquely into a finite sum of intervals $\alpha = \alpha_1 + \alpha_2 + \dots, \alpha_m$ of FC-rank $\leq n$ by taking the states from $\{n\} \times \{0, 1, \dots, n\} \cup \{0, 1, \dots, n\} \times \{n\}$ as splitting points. Then $\mathcal{B}_n$ orders α-words lexicographically by comparing the restrictions to the intervals α_i via $\mathcal{A}_n$. If $\mathcal{A}_n$ orders α_i-words as some order L_i, then $\mathcal{B}_n$ orders α-words as the scattered sum $\sum_{a_1 \in L_1} \sum_{a_2 \in L_2} \cdots \sum_{a_m \in L_m} 1$ of one element orders which clearly is scattered again.

Suppose that $\mathsf{FC}(\alpha) \leq n + 1$. Let r be the accepting run of $\mathcal{C}_{n+1}$ on every α-word. Recall that from Lemma 35, we conclude that the cuts of rank n embed into $\mathbb{Z}$, Thus, the cuts

$$C := \{c \mid c \text{ minimal or maximal or } r(c) \in M_n\}$$

are a suborder of $1 + \mathbb{Z} + 1$. Given an α-word w we define $c(w)$ to be maximal element $c \in C$ such that $c < \mathsf{supp}(w)$ and define $d(w)$ to be the minimal element $c \in C$ such that $\mathsf{supp}(w) < c$. We define $\mathcal{A}_{n+1}$ as follows. Given finite α-words v, w, let $v \leq w$ if

1. $c(v) < c(w)$, or
2. $c(v) = c(w)$ and $d(v) < d(w)$, or
3. $c(v) = c(w)$, $d(v) = d(w)$ and $\mathcal{B}_n$ applied to the interval between $c(v)$ and $d(v)$ reports $v < w$. Note that $\mathsf{FC}_*((c(v), d(v)) \leq n$ because the accepting run of $\mathcal{C}_{n+1}$ on α assumes only finitely many states of rank n on this subinterval. Thus, also $\mathcal{C}_n$ accepts $(c(v), d(v))$-words.

This defines an α-automatic linear order $(W_\alpha, \preceq)$ on the set of finite α-words W_α. Since $\preceq$ embeds into a $(1+\mathbb{Z}+1)^2$-sum of scattered linear orders (induced by $\mathcal{B}_n$), where $(1+\mathbb{Z}+1)^2$ is ordered lexicographically, $(W_\alpha, \preceq)$ is scattered. □

Lemma 37. *Let α be some ordinal. Then there is an α-automatic well-order of all finite α-words over an alphabet Σ.*

Proof. Fix a linear order $\leq_\Sigma$ on Σ. Let w, v be α-words. We set $w < v$ if either $\max(\mathsf{supp}(w)) < \max(\mathsf{supp}(v))$ or $\max(\mathsf{supp}(w)) = \max(\mathsf{supp}(v))$ and there is a $\beta < \max(\mathsf{supp}(w))$ such that $w(\beta) <_\Sigma v(\beta)$ and for all $\alpha > \beta' > \beta$, $w(\beta') = v(\beta')$. Apparently this order is α-automatic. Note that for $\alpha = \omega$ this is a the length-backward-lexicographic order (we first compare words with respect to size and words of the same size are compared lexicographically from the last letter to the first one). In order to show that this defines a well-order, first note that it is reflexive, transitive and antisymmetric, i.e., a linear order. Heading for a contradiction, assume that there is some ordinal α such that the order on α-words contains an infinite descending chain $w_1 > w_2 > w_3 > \dots$. The chain $\alpha_i := \max(\mathsf{supp}(w_i))$ is a monotone decreasing sequence in α. Since α is an ordinal, it stabilises at some $k \in \mathbb{N}$. We conclude that the sequence $v_j := w_{k+j}$ satisfies $\max(\mathsf{supp}(v_j)) = \alpha_k$ for all $j \in \mathbb{N}$.

We now iterate the following argument: let $\alpha' < \alpha_k$ be maximal such that there are v_j, v_k such that $v_j(\alpha') \neq v_k(\alpha')$. Since Σ is finite, there is an infinite subsequence $v_{i_1} > v_{i_2} > \dots$ such that v_{i_k} and v_{i_j} agree at α', i.e., $v_{i_k}(\alpha') = v_{i_j}(\alpha')$. Replace the sequence v_k by the sequence v_{i_k}. Since this is an decreasing chain and above α' all v_{i_k} agree, we can repeat this argument with some smaller $\alpha'' < \alpha'$ which is maximal such that some v_{i_k} do not agree on α''. Since α is an ordinal and since $\alpha' > \alpha'' > \alpha''' > \dots$, this sequence must be finite. But this process terminates if and only if $v_{i_k} = v_{i_j}$ for all $j, k \in \mathbb{N}$. This contradicts the assumption that the v_{i_j} form a strictly decreasing infinite chain. □

D.2 Kleene-Brouwer Orders of Trees

Let $\mathfrak{T} = (T, \sqsubseteq)$ be a tree and let $\mathfrak{L} = (T, \preceq)$ be a linear order. Then we can define the *Kleene-Brouwer order* (also called Lusin-Sierpiński order) $\mathsf{KB}(\mathfrak{T}, \mathfrak{L}) := (T, \lessdot)$ given by $t \lessdot t'$ if either $t \sqsubseteq t'$ or there are $t \sqsubseteq s, t' \sqsubseteq s'$ such that $\{r \in T \mid s \sqsubset r\} = \{r \in T \mid s' \sqsubset r\}$ and $s \prec s'$. This generalises the order induced by postorder traversal to infinitely branching trees where the children of each node are ordered corresponding to the linear order $\preceq$. Since α-oracle-automatic structures are closed under first-order definitions, the following observation is immediate.

Proposition 38. *If $\mathfrak{T}$ is an tree and $\mathfrak{L}$ a linear order such that both are α-oracle-automatic, then* $\mathsf{KB}(\mathfrak{T}, \mathfrak{L})$ *is α-oracle-automatic.*

For the following section, it is important that $(T, \lessdot)$ is scattered if $(T, \preceq)$ is a scattered linear order.

Lemma 39. *Let $\mathfrak{T} = (T, \sqsubseteq)$ be a tree and $\mathfrak{L} = (T, \preceq)$ a scattered linear order, then* $\mathsf{KB}(\mathfrak{T}, \mathfrak{L}) = (T, \lessdot)$ *is scattered.*

Proof. The proof is by induction on the rank of $\mathfrak{T}$. If $\mathfrak{T}$ has rank 1, it consists only of the root whence $\mathsf{KB}(\mathfrak{T}, \mathfrak{L})$ is the linear order of 1 element which is scattered. Otherwise, let T_0 be the set of children of the root and let t_0 be the root of $\mathfrak{T}$. T_0 induces a scattered suborder $(T_0, \preceq)$ of $\mathfrak{L}$. Now (abusing notation slightly) $\mathsf{KB}(\mathfrak{T}, \mathfrak{L}) = \left(\sum_{t \in (T_0, \preceq)} \mathsf{KB}(\mathfrak{T}(t), \mathfrak{L})\right) + t_0$ which is a scattered sum of scattered orders. Proposition 2.17 in [16] shows that $\mathsf{KB}(\mathfrak{T}, \mathfrak{L})$ is scattered. □

D.3 Bounds for Forests on Scattered Orders of Finite Rank

In this section, we prove the main theorem in the case that $\mathsf{FC}(\alpha)$ is finite, α is an ordinal, or in general, the set of finite α-words allows a scattered linear order. In the next Section we then prove the other cases.

Lemma 40. *Let $\mathfrak{T} =$ be a nonempty α-oracle-automatic order tree with domain T and $\mathfrak{L}$ a scattered α-oracle-automatic order with domain T. If $\mathsf{FC}_*(\mathsf{KB}(\mathfrak{T}, \mathfrak{L})) < \beta$, then ∞-$\mathsf{rank}(\mathfrak{T}) < \beta$.*

Proof. The proof is by contraposition and induction on β.

- If $\beta = 0$, there is nothing to show.
- Assume that ∞-$\mathsf{rank}(\mathfrak{T}) = \beta = \beta' + 1$ and for each tree $\mathfrak{T}'$ with ∞-$\mathsf{rank}(\mathfrak{T}') = \beta'$ we have $\mathsf{FC}_*(\mathsf{KB}(\mathfrak{T}', \mathfrak{L})) \geq \beta'$. By definition of ∞-$\mathsf{rank}(\mathfrak{T})$ there is an infinite antichain $d_1, d_2, d_3, \dots$ in $\mathfrak{T}$ such that the subtree $\mathfrak{T}(d_i)$ rooted at d_i satisfies ∞-$\mathsf{rank}(\mathfrak{T}(d_i)) = \beta'$. By induction hypothesis, $\mathsf{FC}_*(\mathsf{KB}(\mathfrak{T}(d_i), \mathfrak{L})) \geq \beta'$. Moreover, $\mathfrak{L}$ orders $\{d_i \mid i \in \mathbb{N}\}$ as order type $\gamma \in \{\omega, \omega^*, \zeta\}$ Thus, $\mathsf{KB}(\mathfrak{T}, \mathfrak{L})$ contains a suborder of the form $\sum_{x \in \gamma} \mathsf{KB}(\mathfrak{T}(d_x), \mathfrak{L}))$ with $\mathsf{FC}_*(\mathsf{KB}(\mathfrak{T}(d_x), \mathfrak{L})) = \beta'$. Due to Lemma 21, we conclude that
$$\mathsf{FC}_*(\mathsf{KB}(\mathfrak{T}, \mathfrak{L})) \geq \mathsf{FC}_*(\sum_{x \in \gamma} \mathsf{KB}(\mathfrak{T}(d_x), \mathfrak{L})) = \beta' + 1 = \beta.$$
- Assume that ∞-$\mathsf{rank}(\mathfrak{T}) = \beta$ is a limit ordinal. By definition for each $\beta' < \beta$ there is $d \in \mathfrak{T}$ such that ∞-$\mathsf{rank}(\mathfrak{T}(d)) \geq \beta'$ whence $\mathsf{FC}_*(\mathsf{KB}(\mathfrak{T}(d), \mathfrak{L})) \geq \beta'$ by induction. Thus, $\mathsf{FC}_*(\mathsf{KB}(\mathfrak{T}, \mathfrak{L})) \geq \sup\{\beta' \mid \beta' < \beta\} = \beta$.

□

Corollary 41. *Let $\mathfrak{T}$ be a nonempty α-automatic order tree. If $\mathsf{FC}(\mathsf{KB}(\mathfrak{T}, \mathfrak{L})) < \beta$, then ∞-$\mathsf{rank}(\mathfrak{T}) < \beta + 1$.*

Combining this result with our bound on the FC ranks of α-oracle-automatic we can now prove the first part of Theorem 29.

Proof (Proof of Theorem 29 part (1)). Assume that $\mathfrak{T} = (T, \leq)$ is an α-oracle-automatic order tree such that $\mathfrak{L}$ is an α-oracle-automatic scattered order with domain T. Since $\mathsf{KB}(\mathfrak{T}, \mathfrak{L})$ is an α-oracle-automatic scattered linear order, $\mathsf{FC}(\mathsf{KB}(\mathfrak{T}, \mathfrak{L})) < \omega^{\gamma+1}$ due to Theorem 15. Due to Corollary 41, ∞-$\mathsf{rank}(\mathfrak{T}) < \omega^{\gamma+1}$. By application of Lemma 28 we finally obtain $\mathsf{rank}(\mathfrak{T}) < \omega^{1+\gamma+1}$.

Note that this result easily extends to forests because for each α-oracle-automatic forest, we can turn it into a α-oracle-automatic tree by adding a new root. This tree has the same ∞-rank as the forest we started with. □

D.4 Bounds for Forests on Scattered Orders of Infinite Rank

Since we do not know whether there is an α-automatic scattered linear ordering of all finite α-words for all linear orders α with $\mathsf{FC}(\alpha) \geq \omega$, we have to do a direct analysis of the sum-of-box decompositions of α-automatic forests. Fortunately, we can rely on the analogous analysis in the case of tree-automatic structures from [9]. The essence of this analysis can be rewritten as the following result.

Theorem 42. *[9] If $\mathfrak{F}$ is a forest that is tamely-colourable sum-augmentation of box-augmentations of classes $\mathcal{C}_1, \dots, \mathcal{C}_k$ such that for all structures $\mathfrak{F}' \in \bigcup_{i=1}^{k} \mathcal{C}_1$ we have $\infty\text{-}\mathsf{rank}(\mathfrak{F}') \neq \omega^\alpha$, then $\infty\text{-}\mathsf{rank}(\mathfrak{F}) \neq \omega^\alpha$.*

Using this decomposition result, the second part of Theorem 29 is obtained by induction.

Proof (Proof of Theorem 29 part (2)). Because of the first part of this theorem and Lemma 36, the claim for orders α with $\mathsf{FC}(\alpha) < \omega$ has already been proved.

We now establish the following claim. Assume that α is a scattered linear order of rank $\mathsf{FC}(\alpha) = \gamma \geq \omega$. Let $\delta \geq \omega$ be an ordinal such that for all α' with $\mathsf{FC}(\alpha') < \gamma$ and all α'-oracle-automatic forests $\mathfrak{F}'$, $\infty\text{-}\mathsf{rank}(\mathfrak{F}') < \omega^\delta$. Then every α-oracle-automatic forest $\mathfrak{F}$ satisfies $\infty\text{-}\mathsf{rank}(\mathfrak{F}) < \omega^{\delta+\omega}$.

Heading for a contradiction assume that $\mathfrak{F}$ is an α-oracle-automatic forest with $\infty\text{-}\mathsf{rank}(\mathfrak{F}) \geq \omega^{\delta+\omega}$. Then there is a parameter automaton $\mathcal{A}$ (corresponding to the formula $x < y$ and parameters p_n for $n \in \mathbb{N}$ such that $\infty\text{-}\mathsf{rank}(\mathfrak{F}_{p_n}) = \omega^{\delta+n}$. Assume that $\mathcal{A}$ has q many states and the order automaton of $\mathfrak{F}$ has $q_<$ many states. Now fix $n_0 > 2\exp(q^2 + 2q_<^2)$. Due to Theorem 14, there are sets $\mathcal{C}_0, \mathcal{C}_1$ of size $\exp(q^2 + 2q_<^2)$ such that for each $n \leq n_0$, $\mathfrak{F}_{p_n}$ is a tamely-colourable sum-of-box augmentation of $\mathcal{C}_0, \mathcal{C}_1$ and some sets of α_i-oracle-automatic structures where $\mathsf{FC}(\alpha_i) < \gamma$ for each i. By choice of n_0, there is some $1 \leq m \leq n_0$ such that

$$\infty\text{-}\mathsf{rank}(\mathfrak{A}) \neq \omega^{\delta+m}$$

for all structures $\mathfrak{A} \in \mathcal{C}_0 \cup \mathcal{C}_1$. Moreover, by definition of δ every α_i-oracle-automatic forest has ∞-rank strictly below ω^δ. Thus, $\mathfrak{F}_{p_m}$ is a tamely-colourable sum-of-box augmentation of classes of structures such that none of these structures has ∞-rank $\omega^{\delta+m}$. But this contradicts directly Theorem 42 because $\infty\text{-}\mathsf{rank}(\mathfrak{F}_{p_m}) = \omega^{\delta+m}$.

Using Lemma 28 this claim carries over from ∞-rank to rank because for $\gamma \geq \omega$ some forest has rank strictly below ω^γ if and only if it has ∞-rank strictly below ω^γ (note that $\omega \cdot \omega^\gamma = \omega^\gamma$).

The proof of the theorem now follows by a straightforward induction on $\mathsf{FC}(\alpha)$ using the claim proved above. ◻

D.5 Optimality of the Bounds on Forests

The upper bounds on the ranks of trees stated in the first part of Theorem 29 are optimal in the sense that we can reach all lower ranks as stated in the following theorem.

Theorem 43. *1. For all $i, c \in \mathbb{N}$ there is an ω^i-automatic tree $\mathfrak{T}_{i,c}$ with $\mathsf{rank}(\mathfrak{T}_{i,c}) = \omega^i \cdot c$.*

2. *For all ordinals $\gamma \geq \omega$ and all $c \in \mathbb{N}$, there is an $\omega^{1+\gamma}$-automatic tree $\mathfrak{T}_{\gamma,c}$ with $\mathsf{rank}(\mathfrak{T}_{\gamma,c}) = \omega^\gamma \cdot c$.*

In order to prove the first part of Theorem 43, we want to construct for all $i \in \mathbb{N}$ and $c \in \mathbb{N}$ an ω^i-automatic tree of ∞-rank $\omega^{i-1} \cdot c$ and rank $\omega^i \cdot c$.

We define a finite word ω-automatic tree as follows. Let $T = (\{\varepsilon\} \cup \{(n,m) \mid n \leq m\}$ and $\mathfrak{T}_0 = (T, \leq)$ where

$$\varepsilon \leq t \text{ for all } t \in T,$$
$$(n,m) \leq (n',m') \text{ if } m = m' \text{ and } n \leq n'.$$

$\mathfrak{T}_0$ is clearly well-founded, finite word ω-automatic, and satisfies ∞-$\mathsf{rank}(\mathfrak{T}_0) = 1$ and $\mathsf{rank}(\mathfrak{T}_0) = \omega$.

Next, we show that for any $i, c \in \mathbb{N}$ and any given ω^i-automatic tree $\mathfrak{T}$ there is also an ω^i-automatic tree $\mathfrak{T}'$ such that ∞-$\mathsf{rank}(\mathfrak{T}') = \infty$-$\mathsf{rank}(\mathfrak{T}) \cdot c$ and $\mathsf{rank}(\mathfrak{T}) = \mathsf{rank}(\mathfrak{T}) \cdot c$.

Lemma 44. *Let $c \in \mathbb{N}$ and $\mathfrak{T}$ an α-automatic tree. Then there is an α-automatic tree $\mathfrak{T}_c$ such that ∞-$\mathsf{rank}(\mathfrak{T}_c) = \infty$-$\mathsf{rank}(\mathfrak{T}) \cdot c$ and $\mathsf{rank}(\mathfrak{T}) = \mathsf{rank}(\mathfrak{T}) \cdot c$.*

Proof. Let $\mathfrak{T} = (T, \leq)$ and $L \subseteq T$ be the set of leaves of $\mathfrak{T}$ (L is α-automatic because it is first-order definable if $\mathfrak{T}$). Set $T_c = \bigcup_{i=0}^{c-1} L^{\otimes i} \otimes T$ where $L^{\otimes 1} = L$ and $L^{\otimes i+1} = L^{\otimes i} \otimes L$. The order of $\mathfrak{T}_c$ is given by

$$l_1 \otimes l_2 \otimes \cdots \otimes l_i \otimes t \leq_c l_1' \otimes l_2' \otimes \cdots \otimes l_j' \otimes t'$$

iff either $i = j, l_1 = l_1', \ldots, l_i = l_i'$, and $t \leq t'$ or $i < j$ and $t \leq l_{i+1}'$.

Note that $\mathfrak{T}_1 = \mathfrak{T}$ and $\mathfrak{T}_{c+1}$ is obtained from $\mathfrak{T}_c$ by attaching a copy of $\mathfrak{T}$ to each leaf of $\mathfrak{T}_c$. Thus, an easy induction on c proves the claim. □

In the case $\alpha = \omega^i$ By replacing the convolution by composition of ω^i-words, we construct a finite word ω^{i+1}-automatic representation of the forest $\bigsqcup_{c \in \mathbb{N}} \mathfrak{T}_c$ for any ω^i-automatic tree $\mathfrak{T}$.

Lemma 45. *For $\mathfrak{T}$ a finite word ω^i-automatic tree, the forest $\mathfrak{F} := \bigsqcup_{c \in \mathbb{N}} \mathfrak{T}_c$ is finite word ω^{i+1}-automatic.*

Proof. Let $\perp$ be a fresh symbol not occurring in the alphabet Σ of the representation of $\mathfrak{T}$. Let W_c be the set of finite ω^{i+1}-words whose letters all occur before position $\omega^i \cdot c$ and that have $\perp$ exactly at position $\omega^i \cdot c$. We write $\perp_c$ for the word of W_c whose only letter is $\perp$. We identify an element of $\mathfrak{T}_c$ with a word in W_c as follows. Assume that $t_c \in \mathfrak{T}_c$ has the form $t_c = l_1 \otimes \cdots \otimes l_k \otimes t$ where each l_i is a ω^i-word denoting a leaf of $\mathfrak{T}$ and t is an ω^i-word denoting an arbitrary element of $\mathfrak{T}$. Now let t_c' be the word $l_1 + l_2 + \cdots + l_k + t + \perp_{c-k-1}$ where $+$ denotes the concatenation of α-words. Note that $t_c' \in W_c$. Since the order of two elements of $\mathfrak{T}_c$ is defined by componentwise comparisons on the convolutions, this results in an ω^{i+1}-automatic presentation of $\mathfrak{T}_c$ whose domain is a subset of W_c. It is easy to see that the union of all these representations is an ω^{i+1}-automatic forest. □

Of course, we can add a new root to $\mathfrak{F}$ and obtain an ω^{i+1}-automatic tree $\mathfrak{T}'$ with $\infty\text{-}\mathsf{rank}(\mathfrak{T}') = \sup\{\infty\text{-}\mathsf{rank}(\mathfrak{T}) \cdot c \mid c \in \mathbb{N}\}$ and $\mathsf{rank}(\mathfrak{T}') = \sup\{\mathsf{rank}(\mathfrak{T}) \cdot c \mid c \in \mathbb{N}\}$.

Iterated application of this lemma to the tree $\mathfrak{T}_1$ shows that for each $i \in \mathbb{N}$ there is an ω^i-automatic tree of rank ω^{i+1} (and ∞-rank ω^i). Application of Lemma 44 then proves the first part of Theorem 43.

We now use a variant of the previous construction in order to prove the second part of Theorem 43, i.e., we construct α-automatic trees of high ranks for ordinals $\alpha \geq \omega^\omega$.

Definition 46. *Let α be an ordinal. Let D_α be the set of finite α-words w over $\{\diamond, 1\}$ such that for all limit ordinals $\beta < \alpha$ and all $c \in \omega$ the implication*

$$w(\beta + c) = 1 \Rightarrow w(\beta) = w(\beta + 1) = \cdots = w(\beta + c) = 1$$

holds. We define a partial order on D_α via the suffix relation: for $w_1, w_2 \in D_\alpha$ let $w_1 \overrightarrow{\sqsupseteq}_\alpha w_2$ if and only if for $\beta \leq \alpha$ maximal such that for all $0 \leq \gamma < \beta \quad w_2(\gamma) = \diamond$ we have that $\forall \beta \leq \delta < \alpha \quad w_1(\delta) = w_2(\delta)$, i.e., the domain of w_2 is an upwards closed subset of the domain of w_1 and both agree on the domain of w_2.

Note that $\mathcal{T}_\alpha := (D_\alpha, \overrightarrow{\sqsupseteq}_\alpha)$ is α-automatic.

Lemma 47. *$\mathcal{T}_\alpha := (D_\alpha, \overrightarrow{\sqsupseteq}_\alpha)$ is a tree.*

Proof. Since D_α contains finite α-words w there are only finitely many positions $\beta < \gamma$ with $w(\beta) = 1$. Thus, there are also only finitely many suffixes of w that are undefined up to some position in $\mathsf{supp}(w)$. This implies that all ascending chains are finite. Moreover, the suffix relation is a linear order when restricted to the suffixes of a fixed word w. □

The following lemma combined with Lemma 44 proves the second part of Theorem 43.

Lemma 48. *For all ordinals α, α' such that $\alpha = \omega \cdot \alpha' \geq \omega$, $\mathsf{rank}(\mathcal{T}_\alpha) = \alpha'$.*

Proof. The proof is by induction on α'. For $\alpha = \omega \cdot 1 = \omega$ note that D_α consists of all words $1^m\diamond^\omega$, $m \in \mathbb{N}$ where the word $\diamond^\omega$ is suffix of all other elements. Moreover, these others are pairwise incomparable. Thus, $\mathcal{T}_\omega$ is the infinite tree of depth 1 which has rank 1 as desired. We now proceed by induction.

1. Assume that α' is a successor ordinal, i.e., there is some β' such that $\alpha = \omega \cdot \alpha' = \omega \cdot \beta' + \omega$. Note that the words directly below $\diamond^\alpha$ are those of the form $w = \diamond^\gamma 1^m \diamond^\delta$ such that $\gamma + \delta = \alpha$ and γ is some limit ordinal and $m < \omega$. Fix such a word and note that $D_\alpha \cap \{w' \mid w' \overrightarrow{\sqsupseteq}_\alpha w\}$ induces a suborder isomorphic to $(D_\gamma, \overrightarrow{\sqsupseteq}_\gamma)$ which by induction hypothesis has rank γ' for γ' such that $\gamma = \omega \cdot \gamma'$. Thus, the suborders of maximal rank β' are induced by the elements $w_m = \diamond^{\omega \cdot \beta'} 1^m \diamond^\omega$ for each $m < \omega$. Since these are infinitely many nodes of ∞-rank β', the rank of $\mathcal{T}_\alpha$ is $\beta' + 1 = \alpha'$.
2. Assume that α' is a limit ordinal and $(\beta_i)_{i \in \omega}$ converges to α' and $\beta_i < \alpha$ for each $i \in \omega$. Then each $w_i^m := \diamond^{\beta_i} 1^m \diamond^\alpha$ for $m, i \in \omega$ is directly below $\diamond^\alpha$ and induces a suborder isomorphic to $(D_{\beta_i}, \overrightarrow{\sqsupseteq}_{\beta_i})$ of ∞-rank β_i. Thus, $\infty\text{-}\mathsf{rank}(\mathcal{T}_\alpha) \geq \alpha'$. But as in the previous case we see that all proper suborders have ∞-rank $< \alpha$ whence $\infty\text{-}\mathsf{rank}(\mathcal{T}_\alpha) \leq \alpha'$. Thus, its ∞-rank is exactly α'.

□

E Finite-Rank-Scattered-Automatic Ordinals

In this section we prove that for every countable scattered linear $\mathfrak{L}$ order such that $\mathsf{FC}(\mathfrak{L}) = \mathsf{FC}_*(\mathfrak{L}) = n < \omega$, the $\mathfrak{L}$-oracle-automatic ordinals are exactly those below $\omega^{\omega^{n+1}}$. For this purpose, it suffices to show that ω^{ω^n} is $\mathfrak{L}$-oracle-automatic. Since $\mathfrak{L}$-oracle-automatic structures are closed under finite (lexicographically ordered) products, it follows that for each k the ordinal $(\omega^{\omega^n})^k = \omega^{\omega^n \cdot k}$ is $\mathfrak{L}$-oracle-automatic. Since the $\mathfrak{L}$-oracle-automatic ordinals are closed under definable substructures we conclude that all ordinals below $\omega^{\omega^{n+1}}$ are $\mathfrak{L}$-oracle-automatic.

Theorem 49. *Let $\mathfrak{L}$ be a scattered linear order with $\mathsf{FC}(\mathfrak{L}) = \mathsf{FC}_*(\mathfrak{L}) = 1 + n < \omega$. The ordinal ω^{ω^n} is finite word $\mathfrak{L}$-oracle-automatic.*

Proof. We inductively prove the following claim: For each n there is are finite automata $\mathcal{A}_n$ and $\mathcal{B}_n$ such that for every scattered linear order $\mathfrak{L}$ with $\mathsf{FC}(\mathfrak{L}) = \mathsf{FC}_*(\mathfrak{L}) = 1 + n$ there is an $\mathfrak{L}$-oracle $o_{\mathfrak{L}}$ such that ω^{ω^n} is $\mathfrak{L}$-$o_{\mathfrak{L}}$-automatic where $\mathcal{A}_n$ recognises the domain and $\mathcal{B}_n$ the order $<$ in this representation. Moreover the empty $\mathfrak{L}$-word represents 0).

In the base $n = 0$, we distinguish two cases:

1. $\mathfrak{L} = \omega$ or $\mathfrak{L} = \mathbb{Z}$: Let $o_{\mathfrak{L}} : \mathfrak{L} \to \{\diamond, 1\}$ be an oracle such that $\mathsf{supp}(o_{\mathfrak{L}})$ is isomorphic to ω. Let $\mathcal{A}_0$ accept all $\mathfrak{L}$-words w such that $\mathsf{supp}(w) \subseteq \mathsf{supp}(o_{\mathfrak{L}})$ and $|\mathsf{supp}(w)| = 1$. The order is given by $w < v$ iff $\mathsf{supp}(w)$ is to the left of $\mathsf{supp}(v)$.
2. $\mathfrak{L} = \omega^*$: Let $o_{\mathfrak{L}} : \omega^* \to \{\diamond, 1\}$ be the constant 1 oracle. Again, the domain recognised by $\mathcal{A}_0$ consists of all $\mathfrak{L}$-words w such that $\mathsf{supp}(w) \subseteq \mathsf{supp}(o_{\mathfrak{L}})$ and $|\mathsf{supp}(w)| = 1$. The order is given by $w < v$ iff $\mathsf{supp}(w)$ is to the right of $\mathsf{supp}(v)$.

There is an automaton $\mathcal{B}_1$ recognising the order independent of the shape of $\mathfrak{L}$. If $\mathcal{B}_1$ applies a right limit transition it guesses whether $\mathsf{supp}(o_{\mathfrak{L}})$ is defined arbitrarily close to the minimal cut. This guess can be checked at the successor transitions. Depending on its guess, it recognises the correct order according to the case distinction. Since both orders are automatic, this combined order is also automatic.

For the induction step assume that the claim was proved for all $n' < n$. Let $\mathfrak{L}$ be some scattered linear order with $\mathsf{FC}(\mathfrak{L}) = \mathsf{FC}_*(\mathfrak{L}) = 1 + n$. Recall the automaton $\mathcal{C}_{n+1}$ from definition 31 which determines the left and right order of each cut of $\mathfrak{L}$. Using those cuts where $\mathcal{C}_n$ is in a state from $M_n = \{n\} \times \{0, 1, \dots, n\} \cup \{0, 1, \dots, n\} \times \{n\}$, we obtain a decomposition $\mathfrak{L} = \sum_{i \in \mathbb{Z}} \mathfrak{L}_i$ such that $\mathsf{FC}(\mathfrak{L}_i) = n$ and there is an infinite ascending (or descending) sequence

$$i_0 < i_1 < i_2 < \dots \text{ such that } \mathsf{FC}(\mathfrak{L}_i) = n \text{ and } \forall j \geq 1 \quad i_j - i_{j-1} \geq 2. \tag{3}$$

By this we mean that $\mathcal{C}_n$ upon reading any $\mathfrak{L}$-word is not in a state from M_n on any cut strictly in $\mathfrak{L}_i$ but it is in one of the states from M_n at the last cut before and the first cut after $\mathfrak{L}_i$.

We now describe the case of an ascending chain, but the descending case is analogous. Let $o_{\mathfrak{L}}$ be the oracle defined by $o_{\mathfrak{L}}(x) = (1, o_{\mathfrak{L}_{i_j}}(x))$ if $x \in \mathfrak{L}_{i_j}$ and $o(x) = \diamond$ if for all $j \in \mathbb{N}$ we have $x \notin \mathfrak{L}_{i_j}$. The domain of our presentation of ω^{ω^n} consists of those finite $\mathfrak{L}$-words w such that $\mathsf{supp}(w) \subseteq \bigcup_{j\in\mathbb{N}} \mathfrak{L}_{i_j}$ and for each j $\mathcal{A}_{n-1}$ accepts w restricted to $\mathfrak{L}_{i_j}$. This set is recognised by an $\mathfrak{L}$-$o_{\mathfrak{L}}$-automaton $\mathcal{A}_n$ as follows. $\mathcal{A}_n$ simulates $\mathcal{C}_n$. At the initial state and whenever $\mathcal{C}_n$ is in a state from M_n, it guesses whether the next part of $\mathfrak{L}$ is one of the $\mathfrak{L}_{i_j}$ where $o_{\mathfrak{L}}$ is defined. In this case, it starts a simulation of $\mathcal{A}_{n-1}$. This simulation is stopped when $\mathcal{C}_n$ is again in a state from M_n. If it starts a simulation of $\mathcal{A}_{n-1}$ and $o_{\mathfrak{L}}$ turns out to be undefined on this part, then the run is aborted. Analogously, the run is aborted if we did not start a simulation of $\mathcal{A}_{n-1}$ and reach a position in $\mathsf{supp}(o_{\mathfrak{L}})$.

We identify each word w accepted by $\mathcal{A}$ with a sequence $(\alpha_j)_{j\in\omega}$ of ordinals in $\omega^{\omega^{n-1}}$ such that all but finitely many α_j are 0 and α_j is the ordinal represented by the restriction of w to the $\mathfrak{L}_{i_j}$ (with respect to the order induced by the order automaton $\mathcal{B}_{n-1}$). Of course there is an automaton $\mathcal{B}'_n$ that orders the sequences $(\alpha_j)_{j\in\omega}$ backwards lexicographically, i.e., $(\alpha_j)_{j\in\omega} < (\beta_j)_{j\in\omega}$ if and only if there is $j_0 \in \mathbb{N}$ such that $\alpha_j = \beta_j$ for all $j > j_0$ and $\alpha_{j_0} < \beta_{j_0}$. This order is $\mathfrak{L}$-$o_{\mathfrak{L}}$-automatic (just apply order automaton $\mathcal{B}_{n-1}$ on the part corresponding to $\mathfrak{L}_{i_j}$ indicated by the oracle and remember the last outcome different from '$=$'). This gives a presentation of $\sum_{i\in\omega}(\omega^{\omega^{n-1}})^i = \sum_{i\in\omega}(\omega^{\omega^{n-1}\cdot i}) = \omega^{\omega^n}$. Note that the definition of the oracle $o_{\mathfrak{L}}$ depends on $\mathfrak{L}$ but the automata do not depend on $\mathfrak{L}$. We only need a a slightly different order in the case of a descending sequence instead of the ascending sequence in (3). In the case of a descending sequence the order automaton uses lexicographic ordering instead of backwards lexicographic order because the domain of the presentation can be identified with $(\alpha_j)_{j\in\omega^*}$. Of course, we can define an automaton $\mathcal{B}_n$ that guesses (and verifies) whether $\mathsf{supp}(o_{\mathfrak{L}})$ is cofinal and depending on this guess, simulates $\mathcal{B}'_n$ or the variant $\mathcal{B}''_n$ performing lexicographic ordering (since $o_{\mathfrak{L}}$ is either coinitial or cofinal, the correctness of this guess can be checked immediately in the transitions leaving the initial states). □

Problem 50. Can one lift the previous theorem to orders of transfinite rank?

F The countable atomless Boolean algebra is not ordinal automatic

If η is an ordinal, there is an apparent bijection between $\mathsf{Cuts}(\eta)$ and the ordinal $\eta+1 = \{\alpha \mid \alpha \leq \eta\}$ which we will use to identify cuts. Let $\mathsf{Cuts}^-(\eta) = \mathsf{Cuts}(\eta) \setminus \{(\eta, \emptyset)\}$. We call $w : \eta \to \{\diamond\}$ the *empty input*. If $r \colon \mathsf{Cuts}^-(\eta) \to \mathcal{S}$ is a run of $\mathcal{A}$ with $\gamma \leq \eta$ and $\gamma < \eta$ is a limit ordinal, let as above $\lim_{\gamma^-} r$ denote the set of states appearing unboundedly often before γ.

Lemma 51. *(Pumping) Let $\mathcal{A}$ be a non-deterministic ordinal automaton with state set S, $m \in \mathbb{N}$, and γ some ordinal with $\gamma \neq 0$. Suppose $S_{\lim} \subseteq S^+ \subseteq S$ and $s \in S^+$ with $|S_{lim}| \leq m$.*

If there is a run

$$r \colon \mathsf{Cuts}(\omega^m) \to S^+$$

on empty input with $r(0) = s$, $\lim_{\omega^m-} r = S_{\mathsf{lim}}$, *and* $r(\mathsf{Cuts}(\omega^m)) = S^+$, *then there is a run*

$$\bar{r}\colon \mathsf{Cuts}(\omega^m\gamma) \to S^+$$

on empty input with $\bar{r}(0) = s$, $\lim_{\omega^m\gamma-} \bar{r} = S_{\mathsf{lim}}$, *and* $\bar{r}(\mathsf{Cuts}(\omega^m\gamma)) = S^+$.

Proof. The proof is by induction on m and $|S_{lim}|$.

- First suppose that $S_{lim} = S^+$. Then $r(\omega^m) = r(\alpha_0)$ for some $\alpha_0 < \omega^m$. Let $\bar{r}(\alpha) = r(\alpha)$ and $\bar{r}(\omega^m\beta + \alpha) = r(\alpha_0 + \alpha)$ for $\alpha < \omega^m$ and $1 \leq \beta < \gamma$. Let $\bar{r}(\omega^m\gamma) = r(\omega^m)$.
- Now suppose that $S_{lim} \subsetneq S^+$. Choose n_0 with $r([\omega^{m-1}n_0, \omega^m)) \subseteq S_{lim}$.
  - If there is $n \geq n_0$ with $r([\omega^{m-1}n, \omega^{m-1}(n+1))) = S_{lim}$, choose $\beta_0 \in [\omega^{m-1}n, \omega^{m-1}(n+1)$ with $r(\beta_0) = r(\omega^{m-1}(n+1))$. Let $\bar{r}(\alpha) = r(\alpha)$ for $\alpha \leq \omega^{m-1}(n+1)$, let $\bar{r}(\omega^{m-1}\beta+\alpha) = r(\beta_0+\alpha)$ for $\alpha < \omega^{m-1}$ and $\omega\beta < \gamma$, and let $\bar{r}(\omega^m\gamma) = r(\omega^m)$.
  - If there is no such n, find $n_0 = \beta_0 < \beta_1 < \ldots$ with $\sup_{i\in\omega} \omega^{m-1}\beta_i = \omega^m\gamma$. Let $\bar{r}(\alpha) = r(\alpha)$ for $\alpha \leq \omega^{m-1}n_0$. We can pump $r \restriction [\omega^{m-1}n, \omega^{m-1}(n+1)]$ to a run $\bar{r}\colon [\omega^{m-1}\beta_n, \omega^{m-1}\beta_{n+1}] \to S^+$ for $n \geq n_0$ by the induction hypothesis for smaller S_{lim}.

□

Lemma 52. *(Shrinking) Let $\mathcal{A}$ be a non-deterministic ordinal automaton with state set S, $m \in \mathbb{N}$, and γ some ordinal with $\gamma \neq 0$. Suppose $S_{\lim} \subseteq S^- \subseteq S$ and $s \in S^-$ with $|S^-| \leq m$. If there is a run*

$$r\colon \mathsf{Cuts}^-(\omega^m\gamma) \to S^-$$

on empty input with $r(0) = s$, $\lim_{\omega^m\gamma-} r = S_{\mathsf{lim}}$, *and* $r(\mathsf{Cuts}^-(\omega^m\gamma)) = S^-$, *then there is a run*

$$\bar{r}\colon \mathsf{Cuts}^-(\omega^m) \to S^-$$

on empty input with $\bar{r}(0) = s$, $\lim_{(\omega^m)-} \bar{r} = S_{\mathsf{lim}}$, *and* $\bar{r}(\mathsf{Cuts}^-(\omega^m)) = S^-$.

Proof. The proof is by induction on m, γ, and the size of S^-. The claim is obvious for $m = 1$ or $\gamma = 1$. Thus, we assume that $\gamma \geq 2$ and $m \geq 2$.

- First suppose that $S_{\mathsf{lim}} = S^-$ and that there is some $\beta < \gamma$ with $\lim_{(\omega^m\beta)-} r = S^-$. Then we can shrink the run $r \restriction \omega^m\beta$ to a run $\bar{r}\colon \omega^m \to S^-$ by the induction hypothesis for β.
- Next suppose that $S_{\mathsf{lim}} = S^-$ and that for each $\beta < \gamma$, there is an $s \in S^-$ such that $s \notin \lim_{(\omega^m\beta)-} r$. There are the following subcases:
  - First suppose that $\gamma = \bar{\gamma} + 1$. Choose $\beta_0 \in [\omega^m\bar{\gamma}, \omega^m\gamma))$ with $r(\beta_0) = r(0)$. Let $\bar{r}(\alpha) = r(\beta_0 + \alpha)$ for $\alpha < \omega^m$.

- Suppose that $\gamma = \omega$. By assumption, for each i, there is some α_i with $\omega^m i \leq \alpha_i < \omega^m(i+1)$ and a state $s_i \in S^-$ such that $s_i \neq r(\beta)$ for all $\alpha_i \leq \beta < \omega^m(i+1)$. Thus, we can apply the induction hypothesis for smaller S^- to each $r{\restriction}[\alpha_i, \omega^m(i+1))$ and shrink it to a run of size ω^{m-1}. Note that the length of $r{\restriction}[\omega^m i, \alpha_i]$ is also bounded by some $\omega^{m-1} \cdot k_i$. Thus, composition of these runs yields the desired run of length ω^m.
- Finally, suppose that $\gamma > \omega$ is a limit ordinal and that $\gamma_1 < \gamma_2 < \cdots < \gamma$ are ordinals such that $\lim \gamma_i = \gamma$. By induction hypothesis for smaller γ, we can shrink each run $r{\restriction}[\omega^m\gamma_i, \omega^m\gamma_{i+1})$ to a run of length ω^m such that each state of S^- appears in one of these runs. Composition of the resulting runs reduces this case to the previous case.

– Finally, suppose that $S_{\mathsf{lim}} \subsetneq S^-$. Let α_0 denote the least $\alpha < \omega^m\gamma$ such that only states $s \in S_{\mathsf{lim}}$ appear in $[\alpha, \omega^m\gamma)$. There are two subcases:
  - First suppose that $\alpha_0 < \omega^m\beta$ for some $\beta < \gamma$. Note that $[\alpha_0, \omega^m\beta)$ and $[\alpha_0, \omega^m\gamma)$ are of the form $\omega^m \cdot \delta$ with $\delta \leq \gamma$. Since the image of $r \restriction [\alpha_0, \omega^m\gamma)$ is contained in $S_{\mathsf{lim}} \subsetneq S^-$, we can shrink $r \restriction [\alpha_0, \omega^m\gamma)$ to a run $\bar{r} \restriction [\alpha_0, \omega^m\beta)$ by the induction hypothesis for smaller S^-. Since $\beta < \gamma$ we conclude by application of the induction hypothesis to this shorter run.
  - Second suppose that $\alpha_0 \geq \omega^m\beta$ for all $\beta < \gamma$. We conclude immediately that γ is a successor, i.e., $\gamma = \bar{\gamma} + 1$ and $\alpha_0 \geq \omega^m\bar{\gamma}$. Now we distinguish the following cases.
    1. Assume that $\bar{\gamma} = 1$ and that $r(\omega^m) \in \lim_{(\omega^m)^-} r$. Then there is a $\beta < \omega^m$ such that $r(\beta) = r(\omega^m)$ and for each state $s \in S^-$ such that s occurs in r strictly before ω^m also occurs before β. Then the composition of $r{\restriction}[0, \beta)$ with $r{\restriction}[\omega^m, \omega^m\gamma]$ yields the desired run.
    2. Assume that $\bar{\gamma} = 1$ and that $r(\omega^m) \notin \lim_{(\omega^m)^-} r$. Thus, there is some $\beta < \omega^m$ such that $r([\beta, \omega^m)) \subseteq S^- \setminus \{r(\omega^m)\}$. Thus, we can apply the induction hypothesis for smaller m and S^- shrinking $r{\restriction}[\beta, \omega^m)$ to a run $\hat{r}$ on domain $[\beta, \beta + \omega^{m-1})$ with $\hat{r}([\beta, \beta + \omega^{m-1})) = r([\beta, \omega^m))$ and $\lim_{(\beta+\omega^{m-1})^-} \hat{r} = \lim_{(\omega^m)^-} r$. Since $\beta < \omega^{m-1} \cdot k$ for some $k \in \mathbb{N}$, Composition of $r{\restriction}[0, \beta)$ with $\hat{r}$ and $r{\restriction}[\omega^m, \omega^m\gamma)$ yields the desired run $\bar{r}$ of length ω^m.
    3. If $\bar{\gamma} > 1$, we apply the induction hypothesis (for smaller γ) to $r{\restriction}[0, \omega^m\bar{\gamma})$ and shrink this run to a run $\bar{r}$ of length ω^m. The composition of $\bar{r}$ and $r{\restriction}[\omega^m\bar{\gamma}, \omega^m\gamma)$ is a run of length $\omega^m \cdot 2$ and we can apply the induction hypothesis for smaller γ.

□

We directly obtain the following corollary.

Corollary 53. *Let $\gamma \geq 1$ be an ordinal and let $\mathcal{A}_\gamma = \mathcal{A}_1 = (S, \Sigma, I, F, \Delta)$ be an automaton (where we interpret $\mathcal{A}_i$ as $\omega^m i$-automaton). For all $s_0, s_1 \in S$,*

$$s_0 \xrightarrow[\mathcal{A}_1]{\diamond^{\omega^m}} s_1 \iff s_0 \xrightarrow[\mathcal{A}_\gamma]{\diamond^{\omega^m\gamma}} s_1$$

where $\diamond^\alpha$ denotes the empty input of length α.

A formula is Σ_0 if it is quantifier-free. A formula is Π_i if it is logically equivalent to the negation of a Σ_i-formula. Formulas of the form $\exists x_0 ... \exists x_n \varphi(x_0, ..., x_n, y_0, ..., y_k)$ for some Π_i-formula φ are Σ_{i+1}.

Lemma 54. *Let $L_0, L_1, \ldots, L_k$ be linear orders and let $\delta_1, \delta_2, \ldots, \delta_k, \eta_1, \eta_2, \ldots, \eta_k$ be ordinals all strictly greater than 0. Let $\mathcal{A}, \mathcal{A}_{R_1}, \ldots, \mathcal{A}_{R_n}$ be automata such that $m \in \mathbb{N}$ is a bound on the number of states of any of these. Let $n_0 \in \mathbb{N}$ be some number. Setting $K_i^j := \omega^{m+n_0} \cdot j_i$ for $j \in \{\delta, \eta\}$ define the maps*

$$\begin{aligned} f_\delta : \prod_{i=0}^{k} L_i \to \delta := L_0 + \sum_{i=1}^{k} (K_i^\delta + L_i), \\ (w_1, \ldots, w_k) \mapsto w_1 + \diamond^{\omega^{m+i} \cdot \delta_1} + w_2 + \cdots + \diamond^{\omega^{m+i} \cdot \delta_k} + w_k, \text{ and} \\ f_\eta : \prod_{i=0}^{k} L_i \to \eta := L_0 + \sum_{i=1}^{k} (K_i^\eta + L_i), \\ (w_1, \ldots, w_k) \mapsto w_1 + \diamond^{\omega^{m+i} \cdot \eta_1} + w_2 + \cdots + \diamond^{\omega^{m+i} \cdot \eta_k} + w_k. \end{aligned}$$

Let M_i be the finite word i-automatic structure induced by $\mathcal{A}, \mathcal{A}_1 \ldots, \mathcal{A}_n$ for $i \in \{\delta, \eta\}$. For every $\Sigma_i \cup \Pi_{n_0}$-formula $\varphi(\boldsymbol{x})$ and all $\boldsymbol{w} = (\boldsymbol{w_0}, \ldots, \boldsymbol{w_k}) \in (\prod_{i=0}^{k} W_i)^{<\omega}$ (where W_i denotes the set of finite L_i-words), $M_\delta \models \varphi(f_\delta(\boldsymbol{w}))$ if and only if $M_\eta \models \varphi(f_\eta(\boldsymbol{w}))$.

Proof. The claim for $n_0 = 0$ follows from the Pumping and Shrinking Lemmas because we can translate any run on $\diamond^{\omega^m \cdot \gamma}$ into a run on $\diamond^{\omega^m \cdot \gamma'}$ with same initial and final state for all ordinals $\gamma, \gamma' \geq 1$.

For the inductive step, assume that the claim holds for all $n' < n_0 \in \mathbb{N}$.

Due to symmetry of the claim and since every Π_{n_0}-formula is the negation of a Σ_{n_0}-formula it suffices to prove that $M_\eta \models \varphi(f_\eta(\boldsymbol{w}))$ if $M_\delta \models \varphi(f_\delta(\boldsymbol{w}))$ for a Σ_{n_0} formula φ.

Let φ be some Π_{n_0-1}-formula and $\boldsymbol{v} = f^\delta(\boldsymbol{w})$ for some $\boldsymbol{w}_i \in (\prod_{i=0}^{k} L_i)^{<\omega}$ such that $N \vDash \exists \boldsymbol{x}\, \varphi(\boldsymbol{x}, \boldsymbol{v})$. Choose $\boldsymbol{t} \in {M_\delta}^{<\omega}$ with $M_\delta \vDash \varphi(t, \boldsymbol{v})$.

Since $\boldsymbol{t}$ has finite support, for each $1 \leq i \leq k$, $\mathsf{supp}(\boldsymbol{t}) \cap K_i^\delta$ induces a decomposition $K_i^\delta = L_0^i + \sum_{j=1}^{m} (\bar{K}_j^\delta + L_j^i)$ such that

- $\bar{K}_j^\delta = \omega^{n_0-1} \cdot \kappa$ for some ordinal $\kappa \geq 1$,
- $L_j^i = \omega^{n_0-1}$, and
- $\mathsf{supp}(\boldsymbol{t}) \cap K_i^\delta \subseteq \bigcup_{j=1}^{m} L_j^i$,

Fix ordinals $\bar{K}_j^\eta$ such that $K_i^\eta = L_0^i + \sum_{j=1}^{m} (\bar{K}_j^\eta + L_j^i)$ (these exist because $K_i^\eta = \omega^{m+n_0-1} \cdot (\omega \cdot \kappa)$ for some ordinal $\kappa \geq 1$).

Application of the inductive hypothesis to φ and the functions

$$g^\delta : L_0 \times \prod_{i=1}^{k} (\prod_{j=1}^{m} L_j^i \times L_i) \to \delta, \text{ and}$$

$$g^\eta : L_0 \times \prod_{i=1}^{k} (\prod_{j=1}^{m} L_j^i \times L_i) \to \eta$$

defined in the apparent way shows that there are words $\boldsymbol{w_1}, \boldsymbol{w_2}$ such that $g^\delta(\boldsymbol{w_1}) = \boldsymbol{v}, g^\delta(\boldsymbol{w_2}) = \boldsymbol{t}$, and

$$M_\delta \models \varphi(g^\delta(\boldsymbol{w_1}), g^\delta(\boldsymbol{w_2})) \text{ if and only if } M_\eta \models \varphi(g^\eta(\boldsymbol{w_1}), g^\eta(\boldsymbol{w_2})).$$

By definition, one easily sees that $g^\eta(\boldsymbol{w_1}) = f^\eta(\boldsymbol{w}) = \boldsymbol{v}$. Thus,

$$M_\eta \models \exists \boldsymbol{x}\, \varphi(\boldsymbol{x}, f^\eta(\boldsymbol{w})).$$

□

Definition 55. *For any ordinal α, let $\bar{\alpha}$ be the ordinal of the form $\bar{\alpha} = \omega^{m+1}\beta$ for some ordinal β such that $\alpha = \bar{\alpha} + \omega^m n_m + \omega^{m-1} n_{m-1} + ... + n_0$ and*

1. a. *Let $U_m(\alpha)$ denote the set of ordinals $\gamma = \bar{\alpha} + \omega^m l_m + \omega^{m-1} l_{m-1} + ... + l_0$ such that either*
   - *$\gamma = \alpha$ or*
   - *$l_k \leq n_k + m$ and $l_i \leq m$ for all $i < k$,*

   where k is maximal with $l_k \neq n_k$.

   b. *Let $U_m(X) = \bigcup_{\gamma \in X \cup \{0\}} U_m(\gamma)$.*

   c. *Let $U_m(X, \delta) = U_m(X \cup \{\delta\}) \cap \delta$.*
2. a. *Let $c_m(\alpha) = \max_{i \leq m} n_i$.*

   b. *Let $c_m(X) = \max_{\gamma \in X} c_m(\gamma)$.*
3. *Let $d_m(X) = |\{\bar{\gamma} \mid \gamma \in X \cup \{0\}\}|$.*

Let $U_m^1(X) = U_m(X)$ and $U_m^{i+1}(X) = U_m(U_m^i(X))$ for $i \in \mathbb{N}$, and similarly let $U_m^1(X, \delta) = U_m(X, \delta)$ and $U_m^{i+1}(X, \delta) = U_m(U_m^i(X, \delta), \delta)$ for $i \in \mathbb{N}$. A rough upper bound for the sizes of these sets is given in the following lemma.

Lemma 56. *Suppose that X is a finite set of ordinals and $i \geq 1$. Then*

$$|U_m^i(X)| \leq (c_m(X) + im)^{m+1} d_m(X), \text{ and also}$$
$$|U_m^i(X, \delta)| \leq (c_m(X \cup \{\delta\}) + im)^{m+1} d_m(X \cup \{\delta\}).$$

Proof. The coefficient of ω^j of an element of $U_m^i(\gamma)$ can take at most $(c_m(w)+im)^{m+1}$ many different values for any fixed $j \leq m$. Hence $|U_m^i(\alpha)| \leq (c_m(w) + im)^{m+1}$ for all ordinals α and all $i \geq 1$. Moreover $d_m(U_m^i(X)) = d_m(X)$ for all $i \geq 1$. □

It follows that there are at most $|\Sigma_\diamond|^{(c_m(X)+im)^{m+1} d_m(X)}$ many finite words w over alphabet $\Sigma_\diamond$ with $\mathsf{supp}(w) \subseteq U_m^i(X)$ for $i \geq 1$, where $\Sigma_\diamond$ is an alphabet with $\diamond \in \Sigma$.

A relation $R \subseteq X \times Y$ is called *locally finite* if for every $x \in X$, there are at most finitely many $y \in Y$ with $(x, y) \in R$.

Lemma 57. *(Growth lemma) Suppose η is an ordinal and $R \subseteq (\Sigma^*)^k \times (\Sigma^*)^l$ is a locally finite relation of finite η-words. Suppose R is recognised by an η-automaton $\mathcal{A}$ with at most m states. Then $\mathsf{supp}(w) \subseteq U_{m+1}(\mathsf{supp}(v), \eta)$ for all $(v, w) \in R$.*

Proof. Suppose $\alpha \in \mathsf{supp}(w) \setminus U_{m+1}(\mathsf{supp}(v), \eta)$ is minimal. Let $k \in \mathbb{N}$ be least such that there are $\beta \in \mathsf{supp}(w) \cup \{0, \eta\}$ and δ with $\omega^{k+1}\delta \leq \alpha, \beta < \omega^{k+1}(\delta+1)$. It follows from the Pumping Lemma that $k \leq m$. Choose the maximal such β for this k. If $\beta \neq \delta$ then $\mathsf{supp}(w) \cap (\beta, \omega^{k+1}(\delta+1)) = \emptyset$. At most m different states can appear at the ordinals $\beta + \omega^k l$ for $l \in \mathbb{N}$. Since R is locally finite, none of the states appears twice between β and α if $\beta \neq \delta$, since otherwise it is possible to shrink the run, and hence $\omega^{k+1}\delta \leq \alpha < \beta + \omega^k m$. If $\beta = \delta$ then $\omega^{k+1}\delta \leq \alpha < \beta$. Let α_j denote the coefficient of ω^j in the Cantor normal form of α for $j \in \mathbb{N}$. Since there are at most m states and R is locally finite, $\alpha_j < m$ for all $j < k$ and hence $\alpha \in U_{m+1}(\beta)$. □

Let $\lceil x \rceil$ denote the least $n \in \omega$ with $x \leq n$, and $\log$ the logarithm with base 2.

Lemma 58. *(Growth lemma for monoids) Suppose the multiplication of the monoid $(M, \cdot)$ is recognised by an automaton with $\leq m$ states. Suppose $s_1, ..., s_n \in M$ and $\mathsf{supp}(s_i) \subseteq X$ for $1 \leq i \leq n$ where $n \geq 2$. Then $\mathsf{supp}(s_1 \cdot ... \cdot s_n) \subseteq U_{m+1}^{\lceil \log n \rceil}(X)$.*

Proof. We follow the proof of [12, Lemma 3.2]. The statement follows from the Growth Lemma for $n \geq 2$. For $n > 2$ let $k = \lceil \frac{n}{2} \rceil$ and $l = n - k$. Then $\lceil \log k \rceil, \lceil \log l \rceil < \lceil \log n \rceil$. Let $t = s_1 \cdot ... \cdot s_k$ and $u = s_{k+1} \cdot ... \cdot s_n$. Then $\mathsf{supp}(t) \cup \mathsf{supp}(u) \subseteq U_{m+1}^{\lceil \log n \rceil - 1}(X)$ by the induction hypothesis for $\lceil \frac{n}{2} \rceil$. Thus, $\mathsf{supp}(t \cdot u) \subseteq U_{m+1}^{\lceil \log n \rceil}(X)$ by the Growth Lemma applied to t and u. □

We prove that the countable atomless Boolean algebra is not δ automatic for any ordinal δ. We first conclude by Lemma 54 that it suffices to consider ordinals of the form $\delta = \omega^k$ with $k \in \mathbb{N}$.

Corollary 59. *Let η, κ, δ be ordinals such that $\eta \geq 1$, $\kappa < \omega^\omega$, and $\delta = \omega^\omega \cdot \eta + \kappa$. If the countable Boolean algebra is finite word δ-automatic then it is finite word ω^k-automatic for some $k \in \mathbb{N}$.*

Proof. Let $\bar{\mathcal{A}} = (\mathcal{A}, \mathcal{A}_\cup, \mathcal{A}_\cap, \mathcal{A}_0, \mathcal{A}_1)$ be δ-automata representing the countable atomless Boolean algebra $\mathfrak{M} = (M, \cup, \cap, \mathbf{0}, \mathbf{1})$. Let $n \in \mathbb{N}$ be a bound on the number of states of any of these automata. δ can be written as a sum $\omega^{n+2} + \omega^\omega \cdot \eta + \kappa$. Let m' be a finite δ-word such that $m' \in M$. Due to the Shrinking Lemma 52, there is an $m \in M$ with $\mathsf{supp}(m) \subseteq \omega^{n+2} \cup \kappa$, i.e., a word whose support has a $\omega^\omega \eta$ gap at ω^{n+2}. Now let $\mathcal{A}_\varphi$ be the automaton that corresponds in $\mathfrak{M}$ to the Π_2 formula $\varphi(x)$ saying $x \in M \wedge$ *M forms a Boolean algebra without atoms*.[6] Due to Lemma 54 and since m satisfies φ in $\mathfrak{M}$, in the structure $\mathfrak{M}' = (M', \cup', \cap, \mathbf{o}', \mathbf{1}')$ induced by $\bar{\mathcal{A}}$ seen as $(\omega^{m+2} + \omega^{m+2} + \kappa)$-automata, there is a word $m' \in M'$ satisfying φ. Thus, $\mathfrak{M}'$ forms a countable Boolean algebra without atoms. By definition of κ, there is a $k' \in \mathbb{N}$ such that $\kappa < \omega^{k'}$ Set $k := \max(k'+1, m+3)$. Since there is an ω^k-automaton marking position $\omega^{m+2}2 + \kappa$ by a unique state, the countable atomless Boolean algebra $\mathfrak{M}'$ also has an ω^k-automatic presentation. □

We finally show that the countable atomless Boolean algebra has no ω^k-automatic presentation.

[6] Note that associativity, commutativity, identity, distributivity are Π_1-statements, existence of complements is Π_2 and absence of atoms is a Π_2-statement.

Theorem 60. *The countable atomless Boolean algebra is not δ-automatic for any ordinal δ.*

Proof. Assume that the countable atomless Boolean algebra has an ω^k-automatic presentation $\mathfrak{M} = (M, \cup, \cap, \mathbf{0}, \mathbf{1})$. Suppose the automata have at most m states.

We follow the proof of [12, Lemma 3.4]. We construct trees T_n with nodes a_σ for all $\sigma \in 2^{\leq n}$ such that T_n has exactly 2^n leaves and $u \cap v = \mathbf{0}$ for any any two leaves $u \neq v$ of T_n.[7] The partial functions which determine the successor nodes $a_{\sigma 0}, a_{\sigma 1}$ from a_σ are definable in $\mathfrak{M}$ by first-order formulas with the quantifier $\exists^\infty$ and hence recognisable by automata by the closure of ω^k-automatic relations under first-order definable relations[8] (see [2]). Suppose each of these automata has at most $l \geq m$ states. Then $\mathsf{supp}(a_\sigma) \subseteq U^n_{l+1}(\mathsf{supp}(a_\emptyset))$ for all $\sigma \in \{0,1\}^{\leq n}$. If $s = s_1 \cup ... \cup s_j$ with pairwise different leaves $s_1, ..., s_j \in T_n$, then $j \leq 2^n$ and $\mathsf{supp}(s) \subseteq U^{n\lceil \log(2^n) \rceil}_{l+1}(\mathsf{supp}(a_\emptyset), \omega^l)$ by the growth lemma for monoids. There are at most $|\Sigma_\diamond|^{(c_l(\mathsf{supp}(a_\emptyset) \cup \{\omega^l\}) + n\lceil \log(2^n) \rceil l)^{l+1}}$ many such s. However, since the leaves of T_n are pairwise incompatible, there are $2^{(2^n)}$ many s. This is a contradiction for large n. □

Note that the growth argument in the previous proof can also be applied directly to the δ-automatic presentation.

Lemma 61. *Suppose $\mathfrak{L}$ is of the form $\mathbb{Z} \cdot \mathfrak{M}$ for some linear order $\mathfrak{M}$. Then the countable atomless Boolean algebra is not $\mathfrak{L}$-automatic.*

Proof. Suppose that R is a locally finite $\mathfrak{L}$-automatic relation on an $\mathfrak{L}$-automatic structure M such that R and the domain and relations of M are recognised by automata with $\leq m$ states. Then for any $(x, y) \in R$ and any $t \in \mathsf{supp}(y)$, there are $u \leq v \in \mathsf{supp}(x)$ with finite distance such that $\bar{u} \leq t \leq \bar{v}$ for the n^{th} predecessor $\bar{u}$ of u and the m^{th} successor $\bar{v}$ of v. The same growth rate argument as for finite automata (see [12, Theorem 3.4]) shows that every infinite $\mathfrak{L}$-automatic Boolean algebra is a finite product of the Boolean algebra of finite and co-finite subsets of $\mathbb{N}$ with inclusion. □

Question 62. Is the countable atomless Boolean algebra $\mathfrak{L}$-o-automatic for any linear order $\mathfrak{L}$ and any oracle o?

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[7] In this construction we replace Khoussainov et al.'s use of the length-lexicographic order by the use of an ω^k-automatic well-order of the finite ω^k-words.

[8] as usual in automatic structures $\exists^\infty$ can be replaced by a first-order statement