A HIERARCHY OF RAMSEY-LIKE CARDINALS

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Abstract. We introduce a hierarchy of large cardinals between weakly compact and measurable cardinals, that is closely related to the Ramsey-like cardinals introduced by Victoria Gitman in [Git11], and is based on certain infinite *filter games*, however also has a range of equivalent characterizations in terms of elementary embeddings. The aim of this paper is to locate the Ramsey-like cardinals studied by Gitman, and other well-known large cardinal notions, in this hierarchy.

Contents

1. Introduction

Ramsey cardinals are a very popular and well-studied large cardinal concept in modern set theory. Like many, or perhaps most other large cardinal notions, they admit a characterization in terms of elementary embeddings, which is implicit in the work of William Mitchell ([Mit79]), and explicitly isolated by Victoria Gitman in [Git11, Theorem 1.3] – we provide the statement of this characterization in Theorem 4.3 below. However this embedding characterization does not lend itself very well to certain set theoretic arguments (for example, indestructibility arguments), as it is based on elementary embeddings between very weak structures. Therefore, Gitman considered various strengthenings of Ramsey cardinals in her [Git11], that she calls Ramsey-like cardinals, the definitions of which are based on the existence of certain elementary embeddings between stronger models of set theory – we will review her definitions in Section 4.

In this paper, we want to introduce a whole hierarchy of Ramsey-like cardinals, that have a uniform definition, and, as we will show, are closely related to the Ramsey-like cardinals defined by Gitman, but which may be seen, as we will try to argue, to give rise to more natural large cardinal concepts than Gitman's Ramsey-like cardinals.

We will also show that the Ramsey-like cardinals in our hierarchy are very robust in the sense that they have a range of equivalent characterizations, in particular one that is based on certain infinite games on regular and uncountable cardinals κ, where one of the players provides κ-models,

Date: April 13, 2017.

2010 *Mathematics Subject Classification.* 03E55,03E35.

Key words and phrases. Large Cardinals, Ramsey Cardinals, Filters, Infinite Games.

We would like to thank Joel Hamkins for a very helpful discussion on the subject matter, and we would like to thank Victoria Gitman for reading an early draft and pointing out several inaccuracies, which greatly helped us to improve this paper. The first author would also like to thank the Bristol logic group, and in particular Philip Welch and Charles Morgan, for giving him an opportunity to talk about an early version of this paper in their seminar, and for making helpful comments and remarks. Both authors were supported by DFG-grant LU2020/1-1.

and the other player has to measure the subsets of κ appearing in those models in a coherent way. These games will be introduced in Section 3. They are what actually led us to the discovery of our hierarchy of Ramsey-like cardinals, and they may also be of independent interest.

2. The filter property and some of its variants

In this section, we will introduce some basic notation, and make some related basic observations. Most of these will not be used much in the central sections of our paper – their use is mostly to provide a solid background on some obvious related questions, and they will provide ideas for many possible variations of our central defintions in Sections 3 and 5. Most of these variations have not been studied properly so far, and we will pick up some of them again in the final open questions section of our paper. In particular though, impatient readers may skip most of this section for now, and only turn back to it when necessary.

Since we will consider filters over subsets of $\mathcal{P}(\kappa)$, where κ is a cardinal, we use the following modified definitions of filters (one could also call these *partial filters*, but we would like to stick to the notion of filter also for the generalized versions below).

A *weak κ-model* is a set M of size κ with $\kappa+1\subseteq M$ and such that $\langle M,\in\rangle\models\mathsf{ZFC}^-$. A weak κ-model is a *κ-model* if additionally $M^{<\kappa}\subseteq M$.[1]

Definition 2.1. (a) A *filter* on κ is a subset F of $\mathcal{P}(\kappa)$ such that $|\bigcap_{i<n}A_i|=\kappa$ whenever $n\in\omega$ and $\langle A_i\mid i<n\rangle$ is a sequence of elements of F.[2]

(b) A filter F on κ *measures* a subset A of κ if $A\in F$ or $\kappa\setminus A\in F$. F *measures* a subset X of $\mathcal{P}(\kappa)$ if F measures every element of X. F is an *ultrafilter* on κ if it measures $\mathcal{P}(\kappa)$.

(c) A filter F on κ is *$<\kappa$-complete* if $|\bigcap_{i<\gamma}X_i|=\kappa$ for every sequence $\langle X_i\mid i<\gamma\rangle$ with $\gamma<\kappa$ and $X_i\in F$ for all $i<\gamma$.

(d) If M is a weak κ-model, a filter F on κ is M-complete if it measures $\mathcal{P}(\kappa)\cap M$ and $\bigcap_{i<\gamma}X_i\in F$ for every sequence $\langle X_i\mid i<\gamma\rangle\in M$ with $\gamma<\kappa$ and $X_i\in F$ for all $i<\gamma$.

(e) A filter F on κ is *normal* if for every sequence $\vec{X}=\langle X_\alpha\mid\alpha<\kappa\rangle$ of elements of F, the diagonal intersection $\triangle\vec{X}$ is a stationary subset of κ.

(f) If M is a weak κ-model, then a filter F on κ is *M-normal* if it measures $\mathcal{P}(\kappa)\cap M$ and $\triangle\vec{X}\in F$ whenever $\vec{X}=\langle X_\alpha\mid\alpha<\kappa\rangle\in M$ is a sequence of elements of F.

Definition 2.2. Suppose that κ is a cardinal. κ has the *filter property* if for every subset X of $P(\kappa)$ of size $\leq\kappa$, there is a $<\kappa$-complete filter F on κ which measures X.

It is well-known that an uncountable cardinal κ satisfying $\kappa=\kappa^{<\kappa}$ has the filter property if and only if κ is weakly compact. [3] If $\vec{X}=\langle X_\alpha\mid\alpha<\kappa\rangle$ is a sequence, we write $\triangle\vec{X}=\triangle_{\alpha<\kappa}X_\alpha$ for its diagonal intersection. Note that every normal filter on κ is easily seen to be $<\kappa$-complete and to only contain stationary subsets of κ. If F is a normal filter on κ and $\vec{X}=\langle X_\alpha\mid\alpha<\kappa\rangle$ is a sequence of elements of F, then $\triangle\vec{X}\in F$ whenever F measures $\triangle\vec{X}$. In particular, if a filter F is normal and measures $\mathcal{P}(\kappa)\cap M$, then F is M-normal. Moreover, every M-normal filter on κ is M-complete and contains the M-club filter. The reason for demanding that $\triangle\vec{X}$ be stationary in Definition 2.1, (e) is provided by the next lemma.

Lemma 2.3. *Suppose that F is a filter and $\vec{X}=\langle X_\alpha\mid\alpha<\kappa\rangle$ is a sequence of elements of F such that $\triangle\vec{X}$ is non-stationary. Then there is a subset $\mathcal{D}$ of $\mathcal{P}(\kappa)$ of size κ, such that every filter that extends F and measures $\mathcal{D}$, contains a sequence $\vec{Y}=\langle Y_\alpha\mid\alpha<\kappa\rangle$, such that $\triangle\vec{Y}=\emptyset$.*

Proof. Suppose that $\vec{X}=\langle X_\alpha\mid\alpha<\kappa\rangle$ is a sequence of elements of F and $\triangle\vec{X}$ is nonstationary. Suppose that C is a club subset of κ that is disjoint from $\triangle\vec{X}$. We consider the regressive function $f\colon\triangle\vec{X}\to\kappa$ defined by $f(\alpha)=\max(C\cap\alpha)$ for $\alpha\in\triangle\vec{X}$. Moreover, we consider the sequence $\vec{A}=\langle A_\alpha\mid\alpha<\kappa\rangle$ of bounded subsets $A_\alpha=f^{-1}[\{\alpha\}]$ of κ for $\alpha<\kappa$.

[1]Note that, unlike is often done, we do not require (weak) κ-models to be transitive.

[2]In particular, this implies that every element of a filter on κ has size κ.

[3]By results of Joel Hamkins and others, the assumption $\kappa=\kappa^{<\kappa}$ is indeed necessary here, and also for many other equivalences of weak compactness.

Let $\mathcal{D}$ denote the closure under finite intersections and relative complements in κ of the set consisting of the elements of F, $\triangle\vec{X}$, the sets A_α for $\alpha<\kappa$ and of $\triangle\vec{A}$. Suppose that $\bar{F}\subseteq\mathcal{D}$ extends F and measures $\mathcal{D}$. Note that this implies that $\bar{F}$ is closed under finite intersections.

Suppose first that $\kappa\setminus\triangle\vec{X}\in\bar{F}$. For every $\alpha<\kappa$, let $Y_\alpha=X_\alpha\setminus\triangle\vec{X}\in\bar{F}$ and let $\vec{Y}=\langle Y_\alpha\mid\alpha<\kappa\rangle$. Then $\triangle\vec{Y}=\varnothing$.

Now suppose that $\triangle\vec{X}\in\bar{F}$. Since each A_α is a bounded subset of κ, $\kappa\setminus A_\alpha\in\bar{F}$ for every $\alpha<\kappa$. But then $\triangle_{\alpha<\kappa}(\kappa\setminus A_\alpha)=\{\beta<\kappa\mid\beta\in\bigcap_{\gamma<\beta}(\kappa\setminus f^{-1}(\gamma))\}=\{\beta<\kappa\mid f(\beta)\geq\beta\vee\beta\notin\mathrm{dom}(f)\}=\kappa\setminus\triangle\vec{X}\notin\bar{F}$. But now making use of the sequence $\langle\kappa\setminus A_\alpha\mid\alpha<\kappa\rangle$ rather than $\vec{X}$, we are in the situation of the first case above, thus obtaining an empty diagonal intersection of elements of $\bar{F}$. □

Definition 2.4. A cardinal κ has the *normal filter property* if for every subset X of $P(\kappa)$ of size $\leq\kappa$, there is a normal filter F on κ measuring X. It has the *M-normal filter property* if there exists an M-normal filter on κ for every weak κ-model M.

Lemma 2.5. *Suppose that F is a filter on κ of size κ and that $\vec{X}=\langle X_\alpha\mid\alpha<\kappa\rangle$ is an enumeration of F. Then F is normal if and only if $\triangle\vec{X}$ is stationary.*

Proof. Suppose that $\triangle\vec{X}$ is stationary. Moreover, suppose that $\vec{Y}=\langle Y_\alpha\mid\alpha<\kappa\rangle$ and $g{:}\,\kappa\to\kappa$ is a function with $Y_\alpha=X_{g(\alpha)}$ for all $\alpha<\kappa$. Let $C_g=\{\alpha<\kappa\mid g[\alpha]\subseteq\alpha\}$ denote the club of closure points of g. Then

$$\triangle\vec{X}\cap C_g\subseteq\triangle\vec{Y}\cap C_g$$

and hence $\triangle\vec{Y}$ is stationary. □

It is immediate from the embedding characterization of weakly compact cardinals, that weak compactness implies the M-normal filter property. On the other hand, if $\kappa^{<\kappa}=\kappa$, every κ-sized subset of $\mathcal{P}(\kappa)$ is contained, as a subset, in some κ-model M. Thus if the M-normal filter property holds for $\kappa=\kappa^{<\kappa}$, then κ is weakly compact, as follows immediately fom the filter property characterization of weakly compact cardinals. For the normal filter property, the following is an immediate consequence of [DPZ80, Theorem 1] together with Lemma 2.5. Remember that a cardinal κ is *ineffable* if whenever $\langle A_\alpha\mid\alpha<\kappa\rangle$ is a κ-list, that is $A_\alpha\subseteq\alpha$ for every $\alpha<\kappa$, then there is $A\subseteq\kappa$ such that $\{\alpha<\kappa\mid A\cap\alpha=A_\alpha\}$ is stationary.

Lemma 2.6 (Di Prisco, Zwicker)**.** *A cardinal κ has the normal filter property if and only if it is ineffable.* □

Definition 2.7. A cardinal κ has the *filter extension property* if for every $<\kappa$-complete filter F on κ of size at most κ and for every subset X of $\mathcal{P}(\kappa)$ of size at most κ, there is a $<\kappa$-complete filter $\bar{F}$ with $F\subseteq\bar{F}$ that measures X.

κ has the *M-normal filter extension property* if for every weak κ-model M, every M-normal filter F on κ and every weak κ-model $N\supseteq M$, there is an N-normal filter $\bar{F}$ with $F\subseteq\bar{F}$.

κ has the *normal filter extension property* if for every normal filter F on κ of size at most κ and every $X\subseteq\mathcal{P}(\kappa)$ of size at most κ, there is a normal filter $\bar{F}\supseteq F$ that measures X.

Lemma 2.8. *Every weakly compact cardinal κ satisfies the filter extension property.*

Proof. Let F be a $<\kappa$-complete filter on κ of size at most κ and let X be a subset of $\mathcal{P}(\kappa)$ of size at most κ. We construct a subtree T of ${}^{<\kappa}2$ as follows. Suppose that $\langle A_i\mid i<\kappa\rangle$ is an enumeration of F and $\langle B_i\mid i<\kappa\rangle$ is an enumeration of X.

We define $\mathrm{Lev}_\alpha(T)$ for $\alpha<\kappa$ as follows. Let $B_{i,j}=B_i$ for $j=0$ and $B_{i,j}=\kappa\setminus B_i$ for $j=1$, where $i<\kappa$. If $t\in 2^\alpha$, let $A_\alpha=\bigcap_{i<\alpha}A_i$, let $B_{\alpha,t}=\bigcap_{i<\alpha}B_{i,t(i)}$ and let $t\in\mathrm{Lev}_\alpha(T)$ if $|A_\alpha\cap B_{\alpha,t}|=\kappa$. Then T is a subtree of $2^{<\kappa}$.

Since $|A_\alpha|=\kappa$ and $\langle B_{\alpha,t}\mid t\in 2^\alpha\rangle$ is a partition of κ, $\mathrm{Lev}_\alpha(T)\neq\varnothing$. Since κ has the tree property, there is a cofinal branch b through T. Let $\bar{F}=\{A\subseteq\kappa\mid\exists\alpha<\kappa\ A_\alpha\cap B_{\alpha,b\restriction\alpha}\subseteq A\}$. Then $\bar{F}$ is a $<\kappa$-complete filter that measures X and extends F. □

Lemma 2.9. *The normal filter extension property fails for every infinite cardinal.*

Proof. The property clearly fails for ω. Suppose for a contradiction that the normal filter extension property holds for some uncountable cardinal κ. Since this implies that the filter property holds for κ, we know that κ is weakly compact. Suppose that $S = S^\kappa_\omega$ and that $F_0 = \{S\}$. F_0 is a normal filter. Let M be a κ-model with $S \in M$. Assume that F_1 is a normal filter on κ that measures $\mathcal{P}(\kappa) \cap M$. Normality of F_1 easily implies that F_1 is M-normal and that the ultrapower N of M by F_1 is well-founded. By Łos' theorem, since κ is represented by the identity function in N, κ has cofinality ω in N, contradicting that κ is inaccessible. □

The counterexample of a normal filter that cannot be extended to a larger set in the above is somewhat pathological, and perhaps the more interesting question is whether the M-normal filter extension property is consistent for some (weakly compact) cardinal κ. We do not know the answer to this question, but would like to close this section with two remarks regarding this topic. The first remark is certainly a folklore observation, and shows that not every M-normal filter for some κ-model M is necessarily normal.

Remark 2.10. It is consistent, relative to a weakly compact cardinal, that κ is inaccessible, M is a κ-model, U is an M-normal ultrafilter on $P(\kappa)^M$, and there is a non-stationary subset X of κ that is an element of U.

Proof. Suppose that κ is weakly compact in L. Recall that a subset T of κ is fat stationary if for every club C in κ, the set $T \cap C$ contains closed subsets of arbitrary lengths $\gamma < \kappa$. It is well-known that there is a subset S of κ in L such that both S and its complement in κ are fat stationary in L (see [HL, Section 7]), and that the forcing $\mathbb{P}_T$ for adding a club subset of a fat stationary subset T of κ is $<\kappa$-distributive (see [AS83, Theorem 1]). Since κ is weakly compact in L, there is a κ-model M and an M-normal ultrafilter U on $P(\kappa)^M$ in L. Then either S or its complement is an element of U. We will assume that S is an element of U. Suppose that G is $\mathbb{P}_{\kappa \smallsetminus S}$-generic over L. Then S is non-stationary in $L[G]$. Hence the required statement holds in $L[G]$. □

The second remark shows that the M-normal filter extension property may consistently fail at a supercompact cardinal.

Remark 2.11. It is consistent, relative to a supercompact cardinal, that there is a supercompact cardinal κ such that the M-normal filter extension property fails for κ. [4]

Proof. Using Richard Laver's classical result on obtaining an indestructibly supercompact cardinal, suppose that κ is supercompact and its supercompactness is indestructible under $<\kappa$-directed closed forcing. Suppose that S is an $\mathrm{Add}(\kappa, 1)$-generic subset of κ. Since κ is supercompact, and hence weakly compact, in $V[S]$, there is a κ-model M and an M-normal M-ultrafilter U on $P(\kappa)^M$ in $V[S]$, such that either S or its complement is an element of U. We will assume that S is an element of U. Over $V[S]$, let C be generic for the standard forcing to adds a club subset of $\kappa \smallsetminus S$. By easy standard arguments, the two-step iteration of the above notions of forcing is $<\kappa$-directed closed, and therefore κ is weakly compact in $V[S, C]$.

Since S is non-stationary in $V[S, C]$, then by the same argument as in the proof of Lemma 2.3, there is a subset $\mathcal{D}$ of $P(\kappa)$ of size κ, such that every filter that extends U and measures $\mathcal{D}$ contains a κ-sequence $\vec{Y}$ with empty diagonal intersection. Moreover that proof shows that there are only two candidates for such witnessing sequences $\vec{Y}$, and both can be defined, in a very absolute way, using only S and U as parameters. Thus suppose that N is a κ-model with $P(\kappa)^M \cup \mathcal{D} \subseteq N$, such that both candidates for $\vec{Y}$ are elements of N as well. Then there is obviously no N-normal filter on $P(\kappa)^N$ that extends U. □

3. Filter games

Definition 3.1. Given an ordinal $\gamma \le \kappa^+$ and regular uncountable cardinals $\kappa = \kappa^{<\kappa} < \theta$, consider the following two-player game of perfect information $G^\theta_\gamma(\kappa)$. Two Players, the *challenger* and the *judge*, take turns to play $\subseteq$-increasing sequences $\langle M_\alpha \mid \alpha < \gamma\rangle$ of κ-models, and $\langle F_\alpha \mid \alpha \le \gamma\rangle$ of filters on κ, such that the following hold for every $\alpha < \gamma$.

- At any stage $\alpha < \gamma$, the challenger plays M_α, and then the judge plays F_α.
- $M_\alpha \prec H(\theta)$,
- $\langle M_{\bar\alpha} \mid \bar\alpha < \alpha\rangle, \langle F_{\bar\alpha} \mid \bar\alpha < \alpha\rangle \in M_\alpha$,

[4] Using more recent indestructibility results for smaller large cardinals, similar arguments work for considerably smaller notions of large cardinal.

- F_α is a filter on κ that measures $\mathcal{P}(\kappa) \cap M_\alpha$ and
- $F_\alpha \supseteq \bigcup_{\beta<\alpha} F_\beta$.

Let $M_\gamma := \bigcup_{\alpha<\gamma} M_\alpha$, and let $F_\gamma := \bigcup_{\alpha<\gamma} F_\alpha$. If F_γ is an M_γ-normal filter, then the judge wins. Otherwise, the challenger wins. [5]

We also define the following variation of the above games. For γ, κ and θ as above, let $\overline{G^\theta_\gamma}(\kappa)$ denote the variant of $G^\theta_\gamma(\kappa)$ where we additionally require the judge to play such that each $F_\alpha \subseteq M_\alpha$, that is she is not allowed to measure more sets than those in M_α in her α^{th} move, for every $\alpha < \gamma$.

Lemma 3.2. *Let $\gamma \leq \kappa^+$, let $\kappa = \kappa^{<\kappa}$ be an uncountable cardinal, and let $\theta > \kappa$ be a regular cardinal.*

1. *The challenger has a winning strategy in $G^\theta_\gamma(\kappa)$ iff he has a winning strategy in $\overline{G^\theta_\gamma}(\kappa)$.*
2. *The judge has a winning strategy in $G^\theta_\gamma(\kappa)$ iff she has a winning strategy in $\overline{G^\theta_\gamma}(\kappa)$.*

Proof. If the challenger has a winning strategy in $G^\theta_\gamma(\kappa)$, then he has one in $\overline{G^\theta_\gamma}(\kappa)$, as the latter game only gives less choice for the judge. Assume the challenger has a winning strategy $\bar{S}$ in $\overline{G^\theta_\gamma}(\kappa)$. Let S be the strategy for $G^\theta_\gamma(\kappa)$ where he pretends that the judge had played $F_i \cap M_i$ rather than F_i, at every stage i of a play of $G^\theta_\gamma(\kappa)$, and responds according to that, following the strategy $\bar{S}$. This yields a run of the game $\overline{G^\theta_\gamma}(\kappa)$ where the challenger follows his winning strategy, hence the judge loses this play, i.e. $F_\gamma \cap M_\gamma$ is not M_γ-normal. But then the same is the case for F_γ, i.e. S is a winning strategy for the challenger in the game $G^\theta_\gamma(\kappa)$.

If the judge has a winning strategy in $\overline{G^\theta_\gamma}(\kappa)$, then this is also a winning strategy in $G^\theta_\gamma(\kappa)$. If she has a winning strategy S in $G^\theta_\gamma(\kappa)$, let $\bar{S}$ be the modification where rather than playing F_i, she plays $F_i \cap M_i$, at each stage $i < \gamma$. Since S is a winning strategy, F_γ is M_γ-normal, whenever it is the outcome of a play of $G^\theta_\gamma(\kappa)$. But then also $F_\gamma \cap M_\gamma$ is M_γ-normal. Hence $\bar{S}$ is also a winning strategy for $G^\theta_\gamma(\kappa)$. But every play of $G^\theta_\gamma(\kappa)$ following $\bar{S}$ is also a run of the game $\overline{G^\theta_\gamma}(\kappa)$, i.e. $\bar{S}$ is a winning strategy for $\overline{G^\theta_\gamma}(\kappa)$. $\square$

Lemma 3.3. *Let $\gamma < \kappa^+$, let $\kappa = \kappa^{<\kappa}$ be an uncountable cardinal, and let θ_0 and θ_1 both be regular cardinals greater than κ.*

1. *The challenger has a winning strategy in $G^{\theta_0}_\gamma(\kappa)$ iff he has a winning strategy in $G^{\theta_1}_\gamma(\kappa)$.*
2. *The judge has a winning strategy in $G^{\theta_0}_\gamma(\kappa)$ iff she has a winning strategy in $G^{\theta_1}_\gamma(\kappa)$.*

Proof. For (1), assume that the challenger has a winning strategy σ_0 in $G^{\theta_0}_\gamma(\kappa)$. We show that he then has a winning strategy σ_1 in $G^{\theta_1}_\gamma(\kappa)$. σ_1 is obtained as follows. Whenever the challenger would play M_α in a run of the game $G^{\theta_0}_\gamma(\kappa)$, then he plays some M^*_α which is a valid move in the game $G^{\theta_1}_\gamma(\kappa)$ and such that $M^*_\alpha \supseteq \mathcal{P}(\kappa) \cap M_\alpha$. Every possible response of the judge in $G^{\theta_1}_\gamma(\kappa)$ is also a possible response in $G^{\theta_0}_\gamma(\kappa)$, where the challenger played M_α. So the challenger can continue to pretend playing both these games simultaneously. As he is following a winning strategy in the game $G^{\theta_0}_\gamma(\kappa)$, F_γ is not M_γ-normal. But this shows that σ_1 is a winning strategy for the challenger in the game $G^{\theta_1}_\gamma(\kappa)$.

Let γ be an ordinal, and assume that θ_0 and θ_1 are both regular cardinals greater than κ. For (2), assume that the judge has a winning strategy σ_0 in $G^{\theta_0}_\gamma(\kappa)$. We show that she then has a winning strategy σ_1 in $G^{\theta_1}_\gamma(\kappa)$. σ_1 is obtained by simply pretending that, if the challenger plays M_α at any stage α of the game $G^{\theta_1}_\gamma(\kappa)$, he in fact played some M^*_α in the game $G^{\theta_0}_\gamma(\kappa)$ with the property that $M^*_\alpha \supseteq M_\alpha \cap \mathcal{P}(\kappa)$, and respond according to that. Since σ_0 is a winning strategy for the judge in the game $G^{\theta_0}_\gamma(\kappa)$, F_γ is $\bigcup_{\alpha<\gamma} M^*_\alpha$-normal. But then F_γ will also be M_γ-normal. This shows that σ_1 is a winning strategy for the judge in $G^{\theta_1}_\gamma(\kappa)$. $\square$

In the light of the above lemma, we make the following definition.

[5] The following possible alternative definition of the games $G^\theta_\gamma(\kappa)$ was remarked by Joel Hamkins, and provides a very useful perspective. In each step $\alpha < \gamma$, in order to have a chance of winning, the judge has to play not only an M_α-normal filter F_α, but in fact has to play some F_α which is normal, as follows by Lemma 2.3. Thus by Lemma 2.5, one might assume that rather than playing filters, the judge is just playing stationary sets which correspond to diagonal intersections of enumerations of the relevant filters.

Definition 3.4. Suppose $\kappa = \kappa^{<\kappa}$ is an uncountable cardinal, $\theta > \kappa$ is a regular cardinal, and $\gamma \le \kappa^+$.

(1) κ has the *γ-filter property* if the challenger does not have a winning strategy in $G^\theta_\gamma(\kappa)$.
(2) κ has the *strategic γ-filter property* if the judge has a winning strategy in $G^\theta_\gamma(\kappa)$.

The 1-filter property follows from weak compactness by its embedding characterization, and implies the filter property, hence it is equivalent to weak compactness. Note that if $\gamma_0 < \gamma_1$, then the γ_1-filter property implies the γ_0-filter property. The following observation shows that assuming $2^\kappa = \kappa^+$, the κ^+-filter property is equivalent to κ being a measurable cardinal.

Observation 3.5. *The following are equivalent for any uncountable cardinal $\kappa = \kappa^{<\kappa}$ satisfying $2^\kappa = \kappa^+$.*

(1) κ satisfies the κ^+-filter property.
(2) κ satisfies the strategic κ^+-filter property.
(3) κ is measurable. [6]

Proof. For the implication from (1) to (3), suppose that κ has the κ^+-filter property, and that $\langle a_\alpha \mid \alpha < \kappa^+\rangle$ is an enumeration of $\mathcal{P}(\kappa)$. Let $\theta > \kappa$ be an arbitrary regular cardinal. We consider a run of the game $G^\theta_{\kappa^+}(\kappa)$ such that in each step α, the challenger plays a valid $M_\alpha \supseteq \{a_\beta \mid \beta \le \alpha\}$, however the judge wins. Then, F_γ is a normal ultrafilter on $\mathcal{P}(\kappa)$.

To see that (3) implies (2), suppose that κ is measurable and let F be a $<\kappa$-complete ultrafilter on $\mathcal{P}(\kappa)$. Then, for any regular $\theta > \kappa$, the judge wins any run of $G^\theta_{\kappa^+}(\kappa)$ by playing F in each of her moves.

Finally, the implication from (2) to (1) is immediate. □

We will show that the α-filter properties for infinite cardinals α with $\omega \le \alpha \le \kappa$ give rise to a proper hierarchy of large cardinal notions, that are closely related to the following Ramsey-like cardinals, that were introduced by Victoria Gitman in [Git11].

4. Victoria Gitman's Ramsey-like cardinals

Definition 4.1.

- An embedding $j\colon M \to N$ is *κ-powerset preserving* if it has critical point κ and M and N have the same subsets of κ.
- [Git11, Definition 1.2] A cardinal κ is *weakly Ramsey* if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a κ-powerset preserving elementary embedding $j\colon M \to N$.
- [Git11, Definition 1.4] A cardinal κ is *strongly Ramsey* if every $A \subseteq \kappa$ is contained, as an element, in a κ-model M for which there exists a κ-powerset preserving elementary embedding $j\colon M \to N$.
- [Git11, Definition 1.5] A cardinal κ is *super Ramsey* if every $A \subseteq \kappa$ is contained, as an element, in a κ-model $M \prec H(\kappa^+)$ for which there exists a κ-powerset preserving elementary embedding $j\colon M \to N$.

The following lemma is an immediate consequence of [Git11, Theorem 3.7], where Gitman shows that weakly Ramsey cardinals are limits of *completely ineffable* cardinals (see [Git11, Definition 3.4]). It yields in particular that weak Ramseyness is strictly stronger than weakly compactness.

Lemma 4.2. [Git11] *Weakly Ramsey cardinals are weakly compact limits of ineffable cardinals.*

The following Theorem from [Git11], which is already implicit in [Mit79], shows that strongly Ramsey cardinals are Ramsey cardinals, which in turn are weakly Ramsey. In fact, as is shown in [Git11, Theorems 3.9 and 3.11], strongly Ramsey cardinals are Ramsey limits of Ramsey cardinals, and Ramsey cardinals are weakly Ramsey limits of weakly Ramsey cardinals.

[6] One could extend our definitions in a natural way so to give rise to the concept of κ having the γ-filter property also for ordinals $\gamma > \kappa^+$, essentially dropping the requirement that the models played by the challenger have size κ. This would however make our definitions less elegant, and was omitted for we will mostly be interested in the case when $\gamma \le \kappa$ in what follows. However right now, these extended definitions would yield the more elegant observation that κ being measurable is equivalent to it having the (strategic) 2^κ-filter property.

Theorem 4.3. [Git11, Theorem 1.3] *A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is contained, as an element, in a weak κ-model M for which there exists a κ-powerset preserving elementary embedding $j\colon M \to N$ with the additional property that whenever $\langle A_n \mid n \in \omega\rangle$ is a sequence of subsets of κ (that is not necessarily an element of M) such that for each $n \in \omega$, $A_n \in M$ and $\kappa \in j(A_n)$, then $\bigcap_{n\in\omega} A_n \neq \emptyset$.*

Lemma 4.4. [Git11, Theorem 3.14] *Super Ramsey cardinals are strongly Ramsey limits of strongly Ramsey cardinals.*

A notion that is closely related to the above, that however was not introduced in [Git11], is the strengthening of weak Ramseyness where we additionally require the witnessing structures M to be elementary substructures of $H(\kappa^+)$, like Gitman does when strengthening strongly Ramsey to super Ramsey cardinals. We make the following definition.

Definition 4.5. A cardinal κ is *super weakly Ramsey* if every $A \subseteq \kappa$ is contained in a weak κ-model $M \prec H(\kappa^+)$ for which there exists a κ-powerset preserving elementary embedding $j\colon M \to N$.

Lemma 4.6. *Super weakly Ramsey cardinals are weakly Ramsey limits of weakly Ramsey cardinals.*

Proof. Suppose that κ is super weakly Ramsey, and pick a weak κ-model $M \prec H(\kappa^+)$ and a κ-powerset preserving elementary embedding $j\colon M \to N$. It suffices to show that κ is weakly Ramsey in N. But as we can assume that the models witnessing instances of weak Ramseyness of κ are all elements of $H(\kappa^+)$, M thinks that κ is weakly Ramsey by elementarity, and hence N thinks that κ is weakly Ramsey for j is κ-powerset preserving. □

As is observed in [Git11], since ineffable cardinals are $\mathbf{\Pi}^1_2$-indescribable and being Ramsey is a $\mathbf{\Pi}^1_2$-statement, ineffable Ramsey cardinals are limits of Ramsey cardinals. Thus in particular not every Ramsey cardinal is ineffable. However the following holds true.

Lemma 4.7. *Super weakly Ramsey cardinals are ineffable.*

Proof. Assume that κ is super weakly Ramsey. Let $\vec{A} = \langle A_\alpha \mid \alpha < \kappa\rangle$ be a κ-list, and let $j\colon M \to N$ be κ-powerset preserving with $M \prec H(\kappa^+)$ and $\vec{A} \in M$. Let $A = j(\vec{A})(\kappa)$. Then $A \in M$, since j is κ-powerset preserving. Let $S = \{\alpha < \kappa \mid A \cap \alpha = A_\alpha\} \in M$. Let C be a club subset of κ in M. Then $\kappa \in j(S) \cap j(C)$, and thus $C \cap S \neq \emptyset$ by elementarity of j, showing that S is a stationary subset of κ in M. But since $M \prec H(\kappa^+)$, S is indeed stationary, thus showing that κ is ineffable, as desired. □

5. A new hierarchy of Ramsey-like cardinals

We want to introduce the following hierarchy of Ramsey-like cardinals.

Definition 5.1. Let $\alpha \leq \kappa$ be regular cardinals. κ is *α-Ramsey* if for arbitrarily large regular cardinals θ, every $A \subseteq \kappa$ is contained, as an element, in some weak κ-model $M \prec H(\theta)$ which is closed under $<\alpha$-sequences, and for which there exists a κ-powerset preserving elementary embedding $j\colon M \to N$.

Note that, in the case when $\alpha = \kappa$, a weak κ-model closed under $<\kappa$-sequences is exactly a κ-model. It would have been more in the spirit of [Git11], and in stronger analogy to Gitman's super Ramsey cardinals, to only require the above for $\theta = \kappa^+$. However we will argue that asking for the existence of arbitrary large $\theta > \kappa$ as above results in a more natural (and strictly stronger) notion.

Lemma 5.2. *If κ is κ-Ramsey, then κ is a super Ramsey limit of super Ramsey cardinals.*

Proof. Assume that κ is κ-Ramsey, as witnessed by some large regular cardinal θ and $j\colon M \to N$ with $M \prec H(\theta)$. Since $\kappa^+ \in M$, it follows that the restriction of j to $H(\kappa^+)^M$ witnesses that κ is super Ramsey in V. It thus suffices to show that κ is super Ramsey in N.

By elementarity, M thinks that κ is super Ramsey. However, as the target structures of embeddings witnessing super Ramseyness can be assumed to be elements of $H(\kappa^+)$, this is a statement which is absolute between weak κ-models with the same subsets of κ (and thus the same $H(\kappa^+)$) that contain κ^+ as an element, hence κ is super Ramsey in N, using that j is κ-powerset preserving. □

Unsurprisingly, the same proof yields the analogous result for ω-Ramsey and super weakly Ramsey cardinals. Note that together with Lemma 4.7 and the remarks preceding it, the following lemma shows in particular that Ramsey cardinals are not provably ω-Ramsey.

Lemma 5.3. *If κ is ω-Ramsey, then κ is a super weakly Ramsey limit of super weakly Ramsey cardinals.* □

Lemma 5.4. *If κ is ω_1-Ramsey, then κ is a Ramsey limit of Ramsey cardinals.*

Proof. Suppose that κ is ω_1-Ramsey. Then κ is Ramsey, as the witnessing models for ω_1-Ramseyness are closed under countable sequences, and thus also witness the respective instances of Ramseyness. Pick a weak κ-model M and $j\colon M \to N$ witnessing ω_1-Ramseyness for $A = \emptyset$. Note that Ramseyness of κ is, considering only transitive weak κ-models, which suffices, a statement about $H(\kappa^+)$ and thus κ is Ramsey in M. Since j is κ-powerset preserving, κ is also Ramsey in N, for the same reason. But this implies, by elementarity, that κ is a limit of Ramsey cardinals, both in M and in V. □

In [Fen90], Feng introduces a hierarchy of Ramsey cardinals that he denotes as $\mathbf{\Pi}_\alpha$-Ramsey, for $\alpha \in \mathrm{Ord}$. This hierarchy is topped by the notion of what he calls a *completely Ramsey* cardinal. This hierarchy is not so much of interest to us here, as already ω_1-Ramsey cardinals are completely Ramsey limits of completely Ramsey cardinals. This follows from elementarity together with the proof of [Git11, Theorem 3.13], observing that rather than using a κ-model M, using a weak κ-model M that is closed under ω-sequences suffices to run the argument. Note that by [Fen90, Theorem 4.2], completely Ramsey cardinals are $\mathbf{\Pi}^2_0$-indescribable, thus in particular this implies that ω_1-Ramsey cardinals are $\mathbf{\Pi}^2_0$-indescribable as well.

Lemma 5.7 below will show that α-Ramseyness is a very robust notion, for any regular cardinal $\alpha \leq \kappa$. This will be given additional support by a filter game characterization of α-Ramseyness for uncountable cardinals α in Theorem 5.8 and Corollary 5.11 below.

Definition 5.5. Suppose that M is a weak κ-model. An M-normal filter U on κ is *weakly amenable* if for every $A \in M$ of size at most κ in M, the intersection $U \cap A$ is an element of M. An M-normal filter U on κ is *good* if it is weakly amenable and the ultrapower of M by U is well-founded.

We will make use of the following lemma, that is provided in [Kan09, Section 19] for transitive weak κ-models, however the same proofs go through for possibly non-transitive weak κ-models.

Lemma 5.6. *Suppose that M is a weak κ-model.*

(1) If $j\colon M \to N$ is the well-founded ultrapower map that is induced by a weakly amenable M-normal filter on κ, then j is κ-powerset preserving.

(2) If $j\colon M \to N$ is a κ-powerset preserving embedding, then the M-normal filter $U = \{A \in \mathcal{P}(\kappa)^M \mid \kappa \in j(A)\}$ is weakly amenable and induces a well-founded ultrapower of M.

Lemma 5.7. *Let $\alpha \leq \kappa$ be regular cardinals. The following properties are equivalent.*

(a) *κ is α-Ramsey.*

(b) *For arbitrarily large regular cardinals θ, every $A \subseteq \kappa$ is contained, as an element, in a weak κ-model $M \prec H(\theta)$ that is closed under $<\alpha$-sequences, and for which there exists a good M-normal filter on κ.*

(c) *Like (a) or (b), but A can be any element of $H(\theta)$.*

(d) *Like (a) or (b), but only for $A = \emptyset$.*

If $\alpha > \omega$, the following property is also equivalent to the above.

(e) *Like (a), (b) or (c), but only for a single regular $\theta \geq (2^\kappa)^+$.*

Proof. The equivalence of (a) and (b), as well as the equivalences of the versions of (c), (d) and (e) that refer to (a) to their respective counterparts that refer to (b) are immediate consequences of Lemma 5.6 together with [Git11, Proposition 2.3]. Clearly, (c) implies (a), and (a) implies each of (d) and (e). The proof of the implication from (e) to (a) for $\alpha > \omega$ will be postponed to Lemma 5.9 below. We will now show that (d) implies (c).

Therefore, suppose that (d) holds, and let us suppose for a contradiction that there is some regular $\theta > \kappa$ and some $A \in H(\theta)$, such that no M, N and j witnessing (c) for θ and A exist. Choose a regular cardinal θ', large enough so that this can be seen in $H(\theta')$, i.e.

$$H(\theta') \models \exists\theta > \kappa \text{ regular } \exists A \in H(\theta)\,\forall M\,\forall j\,\forall N$$

$$[(M \prec H(\theta) \text{ is a weak } \kappa\text{-model} \wedge j\colon M \to N \text{ is } \kappa\text{-powerset preserving}) \to A \notin M],$$

such that the above statement is absolute between $H(\theta')$ and V for the least witness θ and any A in $H(\theta)$, and such that (d) holds for θ'. The absoluteness statement can easily be achieved, noting that it suffices to consider transitive models N of size κ. Making use of Property (d), there is a weak κ-model $M_1 \prec H(\theta')$ and a κ-powerset preserving embedding $j\colon M_1 \to N_1$. By elementarity, M_1 models the above statement about $H(\theta')$, thus in particular we can find the least θ and some $A \in H(\theta)$ witnessing the above statement in M_1. Since $\theta \in M_1$, $M_1 \cap H(\theta) \prec H(\theta)$, $A \in M_1 \cap H(\theta)$ and $j \upharpoonright (H(\theta)^{M_1})\colon H(\theta)^{M_1} \to H(j(\theta))^{N_1}$ is κ-powerset preserving, contradicting our assumption about θ and A. □

Theorem 5.8. *Let $\alpha \leq \kappa$ be regular and uncountable cardinals. Then κ is α-Ramsey if and only if κ has the α-filter property.*

Proof. Assume first that κ has the α-filter property. Pick some large regular cardinal θ. Let $A \subseteq \kappa$ and pick any strategy for the challenger in the game $G^\theta_\alpha(\kappa)$, such that A is an element of the first model played. Since the challenger has no winning strategy in the game $G^\theta_\alpha(\kappa)$ by our assumption, there is a run of this game where the challenger follows the above strategy, however the judge wins. Let $\langle M_\gamma \mid \gamma < \alpha\rangle$ and $\langle F_\gamma \mid \gamma < \alpha\rangle$ be the moves made during such a run, let F_α and M_α be their unions. By the regularity of α, M_α is a weak κ-model that is closed under $<\alpha$-sequences. Since the judge wins, F_α is an M_α-normal filter. Since $\alpha > \omega$, F_α induces a well-founded ultrapower of M_α. It remains to show that F_α is weakly amenable for M_α. Therefore, assume that $X \subseteq \mathcal{P}(\kappa)$ is of size at most κ in M_α. By the definition of M_α, this is the case already in M_γ, for some $\gamma < \alpha$. But since $F_{\gamma+1} \in M_{\gamma+1}$, $F_\alpha \cap X = F_{\gamma+1} \cap X \in M_{\gamma+1} \subseteq M_\alpha$, showing that F_α is weakly amenable and hence good, i.e. κ is α-Ramsey.

Now assume that κ is α-Ramsey and let $\theta = (2^\kappa)^+$. Towards a contradiction, suppose that the challenger has a winning strategy σ in $\overline{G^\theta_\alpha}(\kappa)$. Then $\sigma \in H(\theta)$. Since κ is α-Ramsey, there is a weak κ-model $M \prec H(\theta)$ that is closed under $<\alpha$-sequences, with $\sigma \in M$, and a good M-normal filter U on κ. We define a partial strategy τ for the judge in $\overline{G^\theta_\alpha}(\kappa)$ as follows. If the challenger played $M_\gamma \prec H(\theta)$, with $M_\gamma \in M$, in his last move, then the judge answers by playing $F_\gamma = U \cap M_\gamma$. Note that $F_\gamma \in M$, since U is weakly M-amenable. Since $\sigma \in M$, the above together with closure of M under $<\alpha$-sequences implies that the run of σ against τ has length α, since all its initial segments of length less than α are elements of M. Note that F_α is an M_α-normal filter. Thus using her (partial) strategy τ, the judge wins against σ, contradicting the assumption that σ is a winning strategy for the challenger in $\overline{G^\theta_\alpha}(\kappa)$. □

We can now use the above to fill in the missing part of the proof of Lemma 5.7.

Lemma 5.9. *For regular and uncountable cardinals $\alpha \leq \kappa$, Property (e) implies Property (a) in the statement of Lemma 5.7.*

Proof. Note that when showing that κ being α-Ramsey implies the α-filter property in the proof of Theorem 5.8, we only used the case when $\theta = (2^\kappa)^+$, and in fact it would have worked for any regular $\theta \geq (2^\kappa)^+$ in the very same way. Thus our assumption implies the α-filter property. But then again by Lemma 5.8, κ is α-Ramsey, as desired. □

We think that the above results in particular show κ-Ramseyness to be a more natural large cardinal notion than the closely related concept of super Ramseyness defined by Gitman - in particular, super Ramseyness corresponds to Property (e) for $\theta = \kappa^+$ in Lemma 5.7 below, while what may seem to be a hierarchy for different $\theta \geq (2^\kappa)^+$ in Property (e) of Lemma 5.7 below, actually collapses to the single notion of κ-Ramseyness.

To obtain a version of Theorem 5.8 for ω-Ramsey cardinals, we make the following, somewhat ad hoc definitions.

Definition 5.10. Suppose $\kappa = \kappa^{<\kappa}$ is an uncountable cardinal, $\theta > \kappa$ is a regular cardinal, and $\gamma \le \kappa^+$. We define the *well-founded filter games* $wfG^\theta_\gamma(\kappa)$ just like the filter games $G^\theta_\gamma(\kappa)$ in Definition 3.1, however for the judge to win, we additionally require that the ultrapower of M_γ by F_γ be well-founded. [7] We say that κ has the *well-founded* (γ,θ)*-filter property* if the challenger does not have a winning strategy in $wfG^\theta_\gamma(\kappa)$. We say that κ has the *well-founded* γ*-filter property* iff it has the well-founded (γ,θ)-filter property for every regular $\theta > \kappa$.

The proof of Theorem 5.8 also shows the following, where in the backward direction, well-foundedness of the ultrapower of M_ω by F_ω now follows from the well-founded ω-filter property rather than the (now missing) closure properties of M_ω.

Corollary 5.11. *κ is ω-Ramsey iff it has the well-founded ω-filter property.* □

While α-Ramseyness for singular cardinals α is not a very useful property, as it implies α^+-Ramseyness (since weak κ-models closed under $<\alpha$-sequences are also closed under $<\alpha^+$-sequences), the α-filter property makes perfect sense also when α is singular. We may thus define, for singular cardinals α, that κ is α-Ramsey if it has the α-filter property. For the cases when α has cofinality ω, we may rather want to consider the well-founded α-filter property instead.

We now want to show that the α-Ramsey cardinals (including those we just defined for singular cardinals α) form a strict hierarchy for cardinals $\omega \le \alpha \le \kappa$, and moreover that κ-Ramsey cardinals are strictly weaker than measurable cardinals.

Lemma 5.12. *If $\omega \le \alpha_0 < \alpha_1 \le \kappa$, both α_0 and α_1 are cardinals and κ is α_1-Ramsey, then it is a limit of α_0-Ramsey cardinals.*

Proof. Pick a regular cardinal $\theta > \kappa$. We may assume that α_1 is regular, for we may replace it with a regular $\bar{\alpha_1}$ that lies strictly between α_0 and α_1 otherwise. Using that κ is α_1-Ramsey, pick an ultrapower embedding $j\colon M \to N$ where $M \prec H(\theta)$ is a weak κ-model that is closed under $<\alpha_1$-sequences, and j is κ-powerset preserving. We may also assume that N is transitive, since we can replace it by its transitive collapse in case it is not. Using that j is an ultrapower embedding, it follows by standard arguments that N is closed under $<\alpha_1$-sequences as well. Moreover, j induces a weakly amenable M-normal filter F, by Lemma 5.6, (2). By κ-powerset preservation of j, F is also weakly amenable for N and N-normal. We show that κ has the well-founded α_0-filter property in N.

Let $\nu > \kappa$ be a regular cardinal of N. Suppose for a contradiction that the challenger has a winning strategy for $wfG^\nu_{\alpha_0}(\kappa)$ in N, and let him play according to this strategy. Whenever he plays a κ-model $X \prec H(\nu)$, let the judge answer by playing $F \cap X \in N$. By closure of N under $<\alpha_1$-sequences, this yields a run of the game $wfG^\nu_{\alpha_0}(\kappa)$ that is an element of N. Moreover, the judge wins this run: If Y denotes the union of the models played by the challenger, potential ill-foundedness of the ultrapower of Y by $F \cap Y$ would be witnessed by a sequence $\langle f_i \mid i < \omega\rangle$ of functions $f_i\colon\kappa \to Y$ in Y, for which $F_i = \{\alpha < \kappa \mid f_{i+1}(\alpha) \in f_i(\alpha)\} \in F$ for every $i < \omega$. Now by transitivity of N and since N is closed under ω-sequences, $\langle f_i \mid i < \omega\rangle \in N$. But then since F is N-normal, $\bigcap_{i<\omega} F_i \in F$, yielding a decreasing ω-sequence of ordinals in N, a contradiction. This means that the ultrapower of Y by $F \cap Y$ is well-founded, i.e. the judge wins the above run of the game $wfG^\nu_{\alpha_0}(\kappa)$. However this contradicts that the challenger followed his winning strategy. □

Lemma 5.13. *If κ is measurable, then it is a limit of regular cardinals $\alpha < \kappa$ which are α-Ramsey.*

Proof. Assume that κ is measurable, as witnessed by $j\colon V \to M$. Using that M is closed under κ-sequences, the proof now proceeds as the proof of Lemma 5.12. □

6. Filter sequences

In this section, we show that the filter properties, which are based on (the non-existence of) winning strategies for certain games, are closely related to certain principles that are solely based on the existence of certain sequences of models and filters.

[7] Note that in case γ has uncountable cofinality, M_γ will always be closed under countable sequences and thus this extra condition becomes vacuous.

Definition 6.1. Let α be an ordinal and let κ be a cardinal. Suppose that $\vec{M} = \langle M_i \mid i < \alpha\rangle$ is a $\subseteq$-increasing $\in$-chain of κ-models, and let $M = \bigcup_{i<\alpha} M_i$. An M-ultrafilter F on κ is *amenable for* $\vec{M}$ if $F \cap M_i \in M_{i+1}$ for all $i < \alpha$. If such an α-sequence $\vec{M}$ and such an M-ultrafilter F exist, we say that κ *has an α-filter sequence.* If additionally the ultrapower of M by F is well-founded, we say that κ *has a well-founded α-filter sequence.* [8]

A first observation is that if α is a limit ordinal and F is a filter on κ that is amenable for an $\in$-chain $\vec{M} = \langle M_i \mid i < \alpha\rangle$ of weak κ-models, then letting $M = \bigcup_{i<\alpha} M_i$, F is weakly amenable for M and measures $\mathcal{P}(\kappa) \cap M$.

The following is immediate by Theorem 5.8 and Corollary 5.11.

Observation 6.2. *Assume that $\alpha \leq \kappa$ are both cardinals, and κ is α-Ramsey. Then κ has a well-founded α-filter sequence.*

The next lemma shows that consistency-wise, the existence of (well-founded) α-filter sequences forms a proper hierarchy for infinite cardinals $\alpha \leq \kappa$, that interleaves with the hierarchy of α-Ramsey cardinals. Its proof is similar to the proof of Lemma 5.12.

Lemma 6.3. *Suppose that $\omega \leq \alpha < \beta \leq \kappa$ are cardinals, and that κ has a β-filter sequence. Then there are stationarily many α-Ramsey cardinals below κ.*

Proof. We may assume that β is regular, for we may replace it with a regular $\bar{\beta}$ that lies strictly between α and β otherwise. Suppose that κ has a β-filter sequence, as witnessed by $\vec{M} = \langle M_i \mid i < \beta\rangle$, $M = \bigcup_{i<\beta} M_i$, and by the M-ultrafilter F. Let N be the well-founded ultrapower of M by F, using that M is closed under $<\beta$-sequences, and note that since $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^N$, F is weakly amenable for N and N-normal. Note that N is also closed under $<\beta$-sequences. Then κ has the α-filter property in N, since the judge can win any relevant (well-founded) filter game in N by playing appropriate κ-sized pieces of F, just like in the proof of Lemma 5.12. By elementarity, and since $V_\kappa \subseteq M$, the statement of the lemma follows. □

Observation 6.4. *The existence of a κ-filter sequence does not imply that κ is weakly compact.*

Proof. Start in a model with a κ-filter sequence in which κ is also weakly compact. Perform some forcing of size less than κ. This preserves both these properties of κ. Now by [Ham98, Main Theorem], there is a $<\kappa$-closed forcing that destroys the weak compactness of κ over this model. Clearly this forcing preserves the existence of the κ-filter sequence that we started with. □

However for regular cardinals α, we can actually characterize α-Ramsey cardinals by the existence of certain filter sequences. Note that this lemma is highly analogous to Lemma 5.7, and that some more equivalent characterizations of α-Ramseyness could be extracted from that lemma similar to the ones below.

Lemma 6.5. *The following are equivalent, for regular cardinals $\alpha \leq \kappa$.*

(a) *κ is α-Ramsey.*
(b) *For every regular $\theta > \kappa$, κ has an α-filter sequence, as witnessed by $\vec{M} = \langle M_i \mid i < \alpha\rangle$ and F, where each $M_i \prec H(\theta)$.*

If $\alpha > \omega$, the following property is also equivalent to the above.

(c) *For some regular $\theta > 2^\kappa$ and every $A \subseteq \kappa$, κ has an α-filter sequence, as witnessed by $\vec{M} = \langle M_i \mid i < \alpha\rangle$ and F, where $A \in M_0$ and each $M_i \prec H(\theta)$.*

Proof. If κ is α-Ramsey, then both (b) and (c) are immediate by the proof of Theorem 5.8.

Now assume that (b) holds. Thus fix some regular $\theta > \kappa$, and let (b) be witnessed by $\vec{M}$ and by F. Then $M = \bigcup_{i<\alpha} M_i \prec H(\theta)$ is a weak κ-model closed under $<\alpha$-sequences, F is weakly amenable for M and the ultrapower of M by F is well-founded. This shows that κ is α-Ramsey by Lemma 5.7, (d).

Assuming that (c) holds and that $\alpha > \omega$, the same argument shows that κ is α-Ramsey, this time making use of Lemma 5.7, (e). □

[8] As before this additional assumption becomes vacuous if α has uncountable cofinality.

7. Absoluteness to L

Weakly Ramsey cardinals are downward absolute to L by [GW11, Theorem 3.12]. Since ω_1-Ramsey cardinals are Ramsey by Lemma 5.4, they cannot exist in L. We want to show that ω-Ramsey cardinals are downwards absolute to L. This proof is a variation of the proof of [GW11, Theorem 3.4]. We will make use of a slight adaption of what is known as the *ancient Kunen lemma.*

Lemma 7.1. *Let $M \models \mathsf{ZFC}^-$, let $j\colon M \to N$ be an elementary embedding with critical point κ, such that $\kappa + 1 \subseteq M \subseteq N$. Assume that $|X|^M = \kappa$. Then $j \upharpoonright X \in N$.*

Proof. Note that $j \upharpoonright X$ is definable from an enumeration f of X in M in order-type κ, together with $j(f)$, both of which are elements of N by our assumptions. Namely, for $x \in X$,

$$j(x) = y \iff \exists \alpha < \kappa\ x = f(\alpha) \wedge y = j(f)(\alpha).$$

The lemma follows as $\kappa + 1 \subseteq N$ implies that this definition is absolute between N and V. □

We will make use of the standard lemma that if $0^\sharp$ exists and κ is a Silver indiscernible, then $\operatorname{cof}((\kappa^+)^L) = \omega$ (see [BJW82] for a proof).

Theorem 7.2. *If $0^\sharp$ exists, then the Silver indiscernibles are ω-Ramsey in L.*

Proof. Let $I = \{i_\xi \mid \xi \in \mathrm{Ord}\}$ be the Silver indiscernibles, enumerated in increasing order. Fix $\kappa \in I$, let $\lambda = (\kappa^+)^L$, let $\theta = ((2^\kappa)^+)^L$, and let A be a subset of κ in L. Define $j\colon I \to I$ by $j(i_\xi) = i_\xi$ for all $i_\xi < \kappa$ and $j(i_\xi) = i_{\xi+1}$ for all $i_\xi \geq \kappa$ in I. The map j extends, via the Skolem functions, to an elementary embedding $j\colon L \to L$ with critical point κ. Let U be the weakly amenable L_λ-ultrafilter on κ generated by j. Since every $\alpha < \lambda$ has size κ in L_λ, each $U \cap L_\alpha \in L_\lambda$ by weak amenability of U. Using the standard lemma above, construct sequences $\langle M_i \mid i \in \omega\rangle$ and $\langle \lambda_i \mid i \in \omega\rangle$ such that each M_i satisfies $M_i \prec L_\theta$ is a weak κ-model with $M_i \cap \lambda = \lambda_i$, such that the λ_i are cofinal in λ, such that $A \in M_0$, and such that $M_i, U \cap M_i \in M_{i+1}$.[9] For each $i < \omega$, let j_i be the restriction of j to M_i. Each $j_i\colon M_i \to j(M_i)$ has a domain of size κ in L_θ, and is hence an element of $L_{j(\theta)} \subseteq L$ by Lemma 7.1.

To show that κ is ω-Ramsey in L, we need to construct in L, a weak κ-model $M^* \prec L_\theta$ containing A as an element, and a κ-powerset preserving elementary embedding $j\colon M^* \to N^*$. Define in L, the tree T of finite sequences of the form

$$s = \langle h_0\colon M_0^* \to N_0^*, \ldots, h_n\colon M_n^* \to N_n^*\rangle$$

ordered by extension and satisfying the following properties:

(1) $A \in M_0^*$, each $M_i^* \prec L_\theta$ is a weak κ-model,
(2) $h_i\colon M_i^* \to N_i^*$ is an elementary embedding with critical point κ,
(3) $N_i^* \subseteq L_{j(\theta)}$.
Let W_i be the M_i^*-ultrafilter on κ generated by h_i.
(4) For $i < j \leq n$, we have $M_i^*, W_i \in M_j^*$, $N_i^* \prec N_j^*$ and $h_j \supseteq h_i$.

Consider the sequences $s_n = \langle j_0\colon M_0 \to j(M_0), \ldots, j_n\colon M_n \to j(M_n)\rangle$. Each s_n is clearly an element of T and $\langle s_n \mid n \in \omega\rangle$ is a branch through T in V. Hence the tree T is ill-founded, and by absoluteness of this property, T is ill-founded in L. Let $\langle h_i\colon M_i^* \to N_i^* \mid i \in \omega\rangle$ be a branch of T in L, and let W_i denote the M_i^*-ultrafilter on κ induced by h_i. Let

$$h = \bigcup_{i\in\omega} h_i,\ \ M^* = \bigcup_{i\in\omega} M_i^* \text{ and } N^* = \bigcup_{i\in\omega} N_i^*.$$

It is clear that $M^* \prec L_\theta$, $h\colon M^* \to N^*$ is an elementary embedding with critical point κ and that M^* is a weak κ-model containing A as an element. If $x \subseteq \kappa$ in N^*, then $x = [f]_{W_i} \in N_i^*$ for some $i < \omega$ and some $f\colon \kappa \to M_i^*$ in M_i^*. But then $x = \{\alpha < \kappa \mid \{\beta < \kappa \mid \alpha \in f(\beta)\} \in W_i\} \in M_{i+1}^* \subseteq M^*$ by Property (4). This shows that h is κ-powerset preserving and thus that κ has the ω-filter property in L, as desired. □

[9]Note that we can achieve $M_i \in L_\theta$ by picking first – externally – a sufficiently large $\xi_i < \theta$ such that $L_{\xi_i} \prec L_\theta$ and then picking $M_i \prec L_{\xi_i}$ within M in each step i of our construction. Thus we can satisfy the condition $M_i \in M_{i+1}$ in each step of our construction.

To show that ω-Ramsey cardinals are downwards absolute to L, we need yet another characterization of ω-Ramsey cardinals.

Lemma 7.3. *κ is ω-Ramsey if and only if for arbitrarily large regular cardinals θ and every subset C of θ, every $A \subseteq \kappa$ is contained, as an element, in some weak κ-model M such that $\langle M, C\rangle \prec \langle H(\theta), C\rangle$, for which $\operatorname{cof}(M \cap \kappa^+) = \omega$, and for which there exists a κ-powerset preserving elementary embedding $j: M \to N$.*

Proof. The backward direction of the lemma is immediate. For the forward direction, assume that κ is ω-Ramsey, and let θ and C be as in the statement of the lemma. By Corollary 5.11, κ has the well-founded ω-filter property. Now adapt the proof that the well-founded ω-filter property implies ω-Ramseyness, that is provided for Theorem 5.8. Namely, let the challenger simply play structures M_γ which satisfy $\langle M_\gamma, C\rangle \prec \langle H(\theta), C\rangle$. Note that the resulting structure M_ω witnessing ω-Ramseyness will satisfy $\langle M_\omega, C\rangle \prec \langle H(\theta), C\rangle$ and moreover, as is immediate to observe from the proof in Theorem 5.8, that $\operatorname{cof}(M_\omega \cap \kappa^+) = \omega$. □

We are finally ready to show that ω-Ramsey cardinals are downwards absolute to L.

Theorem 7.4. *ω-Ramsey cardinals are downwards absolute to L.*

Proof. Let κ be an ω-Ramsey cardinal. We may assume that $0^\sharp$ does not exist, by Theorem 7.2, and thus that $(\kappa^+)^L = \kappa^+$ by a classic observation of Kunen for weakly compact cardinals (see e.g. [Jec03, Exercise 18.6]). Fix $A \subseteq \kappa$ in L, and a regular cardinal $\theta \geq (2^\kappa)^+$. Let $C \subseteq \theta$ be the club of $\gamma < \theta$ for which $L_\gamma \prec L_\theta$. Using Lemma 7.3, we may pick a weak κ-model M such that $\langle M, C\rangle \prec \langle H(\theta), C\rangle$, containing A as an element, with a κ-powerset preserving elementary embedding $j: M \to N$, such that $\operatorname{cof}(M \cap \kappa^+) = \omega$.

Let $\lambda = \kappa^+$ and let $\bar\lambda = M \cap \kappa^+ = L^M \cap \kappa^+$ and note that $\operatorname{cof}(\bar\lambda) = \omega$ by the above. Restrict j to $j: L^M \to L^N$. It is easy to see that κ-powerset preservation of the original embedding j implies that $L^M_{\kappa^+} = L^N_{\kappa^+}$, and hence that the restricted embedding j is again κ-powerset preserving. Moreover $L^M \prec L^{H(\theta)} = L_\theta = H(\theta)^L$.

Let U be the weakly amenable L^M_λ-ultrafilter on κ generated by j. Since every $\alpha < \lambda$ in M has size κ in L^M, each $U \cap L_\alpha \in L^M_\lambda$ by weak amenability of U. Using that $\operatorname{cof}(\bar\lambda) = \omega$ and that $L^M_\lambda = L_{\bar\lambda}$, construct sequences $\langle M_i \mid i \in \omega\rangle$ and $\langle \lambda_i \mid i \in \omega\rangle$ such that each $M_i \prec L^M$ is a weak κ-model in L^M with $M_i \cap \lambda = \lambda_i$, such that the λ_i are cofinal in $\bar\lambda$, such that $A \in M_0$, and such that $M_i, U \cap M_i \in M_{i+1}$. Note that we can achieve $M_i \in L^M$ since C is unbounded in $M \cap \theta$ by elementarity, by picking first – externally – a sufficiently large $\xi_i \in M \cap C$, and then picking $M_i \prec L^M_{\xi_i}$ within L^M in each step i of our construction.

For each $i < \omega$, let j_i be the restriction of j to M_i. Each $j_i: M_i \to j(M_i)$ has a domain of size κ in L^M, and is hence an element of L^N by Lemma 7.1. Moreover since L is $\Delta_1^{\mathsf{ZF}^-}$-definable, $L^N \subseteq L$, hence $j_i \in L$ for every $i < \omega$.

To show that κ is ω-Ramsey in L, we need to construct in L, a weak κ-model $M^* \prec L_\theta$ containing A as an element, and a κ-powerset preserving elementary embedding $j: M^* \to N^*$. In order to do so, we now continue verbatim as in the proof of Theorem 7.2. □

8. The strategic filter property versus measurability

Note that we have not only introduced the γ-filter properties, but also the strategic γ-filter properties in Definition 3.4. While we have already provided a variety of results about the γ-filter properties, we do not know a lot about their strategic counterparts. However we want to close our paper with the following result, that was suggested to us by Joel Hamkins. We originally had a similar result, however with a much more complicated proof, starting from a much stronger large cardinal hypothesis.

Definition 8.1. A cardinal κ is *λ-tall* if there is an embedding $j: V \to M$ with critical point κ such that $j(\kappa) > \theta$ and $M^\kappa \subseteq M$.

Lemma 8.2 (Hamkins)**.** *Starting from a κ^{++}-tall cardinal κ, it is consistent that there is a cardinal κ with the strategic κ-filter property, however κ is not measurable.*

Proof. By an unpublished result of Woodin (see [Hamkins - Tall Cardinals, Theorem 1.2]), if κ is κ^{++}-tall, then there is a forcing extension in which κ is measurable and the GCH fails at κ (this improves a classic result of Silver, where the same is shown under the assumption of a κ^{++}-supercompact cardinal). Now we may perform the standard reverse Easton iteration of length κ, to force the GCH below κ, in each step adding a Cohen subset to the least regular cardinal of the intermediate model which has not been considered in the iteration so far. By the Π^2_1-indescribability of measurable cardinals, κ can not be measurable in the resulting model, since if it were, the failure of the GCH at κ would reflect below κ. But clearly, the measurability of κ is resurrected after adding a Cohen subset to κ, by standard lifting arguments.

Assume that κ is not measurable, but it is so in a further $\mathrm{Add}(\kappa^+, 1)$-generic extension (we may assume this situation starting from a κ^{++}-tall cardinal by the above). Let $\dot{U}$ be an $\mathrm{Add}(\kappa^+, 1)$-name for a measurable filter on κ. Let $\theta > \kappa$ be a regular cardinal. We define a strategy for the judge in $G^\theta_\kappa(\kappa)$ as follows. Provided the challenger plays some κ-model $M_\alpha \prec H(\theta)$, the judge picks a condition p_α deciding whether $\check{x} \in \dot{U}$ for every $x \subseteq \kappa$ in M_α, and then plays F_α such that p_α decides that $\dot{U} \cap X_\alpha = F_\alpha$. She does this so that $\langle p_\alpha \mid \alpha < \kappa \rangle$ forms a decreasing sequence of conditions. Let M_κ be the union of the models played by the challenger. By the κ^+-closure of $\mathrm{Add}(\kappa^+, 1)$, this sequence has a lower bound p, which decides $\dot{U} \cap \check{M}_\kappa$ to equal some good M_κ-normal filter U in the ground model. For example, to check that U is weakly amenable for M_κ, note that if $A \subseteq M_\kappa$ is of size κ in M_κ, then it is an element, and hence also a subset, of some M_α played by the challenger, but then $U \cap A = F_\alpha \cap A \in M_{\alpha+1}$, by the definition of the game $G^\theta_\kappa(\kappa)$. This shows that κ has the strategic κ-filter property. □

9. Questions

This paper opens up many possible directions for further research, that are still left to be investigated. The following collection of questions is merely a sample of what could be asked.

A very basic question that was left open in Section 2 is the following.

Question 9.1. *Can the M-normal filter extension property hold at a weakly compact cardinal κ?*

While for uncountable cardinals α, we obtained a direct correspondence between α-Ramseyness and the α-filter property, the issue of potential ill-foundedness forced us to introduce the concept of the well-founded ω-filter property, in order to characterize ω-Ramseyness in terms of filter games. The following should have a negative answer.

Question 9.2. *Does the ω-filter property imply the well-founded ω-filter property?*

We would expect the filter games $G^\theta_\gamma(\kappa)$ from Section 3 not to be determined in case γ is an uncountable cardinal (note that these are open games, so they will be determined in case $\gamma = \omega$), and ask the following question, for which we expect a negative answer.

Question 9.3. *If γ is an uncountable cardinal and the challenger does not have a winning strategy in the game $G^\theta_\gamma(\kappa)$, does it follow that the judge has one?*

Our definitions allow for many variations, some of which we have partially studied, and some of which we haven't yet looked at at all.

Question 9.4. *What properties does one obtain by considering variants of the games $G^\theta_\gamma(\kappa)$, where rather than M-normal filters for κ-models $M \prec H(\theta)$, we consider either*

- *$<\kappa$-complete filters on subsets of $\mathcal{P}(\kappa)$ of size κ,*
- *M-normal filters for arbitrary κ-models M, weak κ-models M, or*
- *normal filters on subsets of $\mathcal{P}(\kappa)$ of size κ?*

We showed in Theorem 7.4 that ω-Ramsey cardinals are downwards absolute to L, and a positive answer seems highly likely for the following.

Question 9.5. *If $\omega \leq \alpha \leq \kappa$, are α-Ramsey cardinals downwards absolute to the Dodd-Jensen core model?*

What is the relationship between ω-Ramsey cardinals and other cardinals that are compatible with L. For example:

Question 9.6. *Does 2-iterability imply ω-Ramseyness, or conversely?*

A direction of possible research that we have not looked into so far at all is the following.

Question 9.7. *The notions of Ramsey-like cardinals are connected to measurable cardinals in talking about filters on κ. Can we obtain interesting variants of other filter-based large cardinals, for example supercompact cardinals, in a similar way? Do they have similar connections to generalized filter games?*

Lemma 3 shows that the strategic κ-filter property does not imply that κ is measurable, and we expect the following question to have a negative answer.

Question 9.8. *Does κ having the strategic κ-filter property have the consistency strength of a measurable cardinal?*

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