CANONICAL TRUTH

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Abstract. We introduce and study a notion of canonical set theoretical truth, which means truth in a transitive class model that is uniquely characterized by some $\in$-formula.

1. Introduction

It is an old logical dream to devise an effectively describable axiomatic system for mathematics that uniquely describes 'mathematical reality'; in modern logical language, this should mean at least that it uniquely fixes a model. It is well-known that this dream is unattainable in first-order logic: By the Löwenheim-Skolem-theorem, we get models of all infinite cardinalities once there is one infinite model; and by Gödel's incompleteness theorem, if the theory is strong enough to express elementary arithmetic, it will have different models that are not even elementary equivalent.

Focusing on ZFC set theory, one of the main foundational frameworks for mathematics, these two effects can in a certain sense be cancelled out by asking not for arbitrary models, but for transitive models that are proper class-sized, i.e. contain all ordinals. When we restrict the allowed models in this way, there are extensions of ZFC that uniquely fix a model. The most prominent example is $V = L$: It is well-known (and provable in ZFC) that ZFC+V=L has exactly one transitive class-model, provided that ZFC is consistent.

This form of canonicity gives the axiom of constructibility a certain attractivity: It seems to describe, up to the unavoidable weakness of first-order logic, a unique 'mathematical reality'. However, it is usually seen as too restrictive since many objects of set-theoretical interest are ruled out under this assumption.

However, $V = L$ is by far not the only theory that uniquely fixes a transitive class model: Other example include $V = L[0^\sharp]$, or $V = L[x]$, where x is an absolute Π^1_2-singleton (see below). The 'true mathematical reality' that the adherents of the logical dream mentioned in the beginning believe in would have to be one of those 'canonical' models. Hence, whatever holds in all of these 'canonical' models will have to be believed by someone who believes in a uniquely describable mathematical reality. We call such statements 'canonically necessary'. If there are no such statements that go beyond what is derivable from ZFC, then this kind of mathematical realism would be mathematically neutral:

the belief in a uniquely describable mathematical reality would merely be a way of interpreting set theory, without influencing it. On the other hand, if there are statements that hold in all canonical models without following from ZFC, this realistic mindset would be mathematically informative.

In this paper, we investigate statements that hold in all 'canonical' models of ZFC, i.e. in all transitive class models that are uniquely fixed by some extension of ZFC by finitely many extra statements. It turns out that the realistic mindset is indeed mathematically informative: Examples of canonically necessary statements that do not follow from ZFC are the ground model axiom of ([R]) (Theorem 10) and the non-existence of measurable cardinals (Theorem 11).

We conclude with various open questions; in particular, we do not know whether there are canonical models of ZF+AD (i.e. whether there are canonical models of the axiom of determinacy) or even whether there are canonical models of ZF+¬AC (i.e. whether the axiom of choice is canonically necessary over ZF).

2. Basics Definitions

We start by giving a formal counterpart to the intuitive idea that a theory T 'uniquely fixes a transitive class model'. This is not straightforward, as quantifying over proper classes is not possible in ZFC. This might be solvable by instead working in NBG, but we prefer to stick to ZFC for the moment, partly because the methods we intend to use (forcing, class forcing and inner models) are commonly developed for ZFC models. Thus, a proper class model of ZFC will always be an inner model of V. Of course, this will immediately trivialize our analysis when one assumes $V = L$, so that L is the only transitive class model. To get a sufficient supply of inner models, we will hence assume sufficient large cardinals in our metatheory.

Still, we need to deal with our inability, due to the lack of a truth predicate, to quantify over all inner models. This will be solved by formulating the uniqueness not as a single statement, but as a scheme. Still, we need to express that the class defined by a formula ϕ is a model of ZFC. Again, this is not trivial, since ZFC is not finitely axiomatizable. Fortunately, for the case we are interested in, there is a workaround:

Lemma 1. [See [Je], Theorem 13.9.] A transitive class C is a model of ZF if and only if C is closed under Gödel operations and almost universal (i.e. for every subset $x \subseteq C$, there is $y \in C$ with $x \subseteq y$).

We fix a natural enumeration $(\psi_i : i \in \omega)$ of the $\in$-formulas in order type ω.

Definition 2. Let ϕ be an $\in$-formula, $i, j \in \omega$. Let $\mathrm{TM}_i(\phi, y)$ ('transitive model') abbreviate the statement '$M_{\psi_i,y} := \{x : \psi_i(x, y)\}$ is transitive, is almost universal, closed under Gödel operations, contains all ordinals and satisfies $\mathrm{AC}^{M_{\psi_i,y}}$ and $\phi^{M_{\psi_i,y}}$'.

The uniqueness statement U_{ij}^{ϕ} is the following $\in$-formula: $\forall y, y'[(\mathrm{TM}_i(\phi, y) \wedge \mathrm{TM}_j(\phi, y')) \rightarrow \forall x(\psi_i(x, y) \leftrightarrow \psi_j(x, y'))]$.

Now, ϕ is a uniqueness statement if and only if all elements of $U_\phi := \{U_{ij}^{\phi} : i, j \in \omega\}$ are provable in ZFC[1].

Moreover, for T an extension of ZFC, ϕ is a T-canonical statement if and only if there is some $i \in \omega$ such that T proves $\exists y \mathrm{TM}_i(\phi, y)$.

Remark: Typically, T will consist of ZFC together with appropriate large cardinal assumptions.

Definition 3. Let T be an extension of ZFC.

If M is a transitive class model of ZFC, then M is T-canonical if and only if there is a T-canonical statement ϕ such that $M \models \phi$.

If A is any $\in$-theory and ϕ is an $\in$-statement, then ϕ canonically follows from A if and only if ϕ holds in all canonical models in which A holds.

Remark: In the last definition, we can drop the dependence of ZFC and talk e.g. about canonical consequences of KP or ZF in the obvious manner.

3. Examples of Canonical Truth

Obvious examples for uniqueness statements are $V = L$ or $V = L[0^\sharp]$ with corresponding canonical models L and $L[0^\sharp]$. These actually give rise to a larger class of examples:

Definition 4. A real number x is a relative Π^1_2-singleton if and only if there is a Π^1_2-statement ϕ such that x is the unique element y of $L[x]$ with $L[x] \models \phi(y)$.

A real number x is an absolute Π^1_2-singleton if and only if there is a Π^1_2-statement ϕ such that x is the unique element y of V with $\phi(y)$.

Corollary 5. An absolute Π^1_2-singleton x is the unique element satisfying its defining Π^1_2-formula ϕ in each transitive inner model that contains x, while all other models will not contain such a witness.

Proof. By Shoenfield absoluteness, if M is a transitive class model, $x \in M$ and $M \models \phi(x)$, then $V \models \phi(x)$. Hence, if some transitive inner model had two distinct elements satisfying ϕ, the same would hold for V, contradicting uniqueness. Similarly, if M was some transitive inner model with $x \notin M$ but $M \models \phi(y)$ for some $y \in M$, then $V \models \phi(x) \wedge \phi(y) \wedge x \neq y$, again contradicting the uniqueness. ☐

[1]Alternatively, we could also demand that all elements of U_ϕ hold in V. We will not follow this idea here.

The existence of $0^\sharp$ has consistency strength. However, no such assumption is needed to obtain canonical models beyond the constructible universe:

Proposition 6. It is consistent relative to ZFC that there are canonical models besides L.

Proof. (Sketch) Force a Π^1_2-singleton over L as described in chapter 6 of [Fr], the generic extension satisfies that there is a real number r satisfying the Π^1_2-statement ψ (which is unique) and $V = L[r]$ and is unique with this property. □

Definition 7. A statement ϕ is canonically necessary (c.n.) if and only if ϕ holds in all canonical models.

A statement ϕ is canonically possible (c.p.) if and only if there is a canonical model $M \models \phi$, i.e. if and only if its negation is not canonically necessary.

Our first observation is that there are canonically necessary statements that are not provable in ZFC:

Lemma 8. There is some $\in$-formula ϕ such that ϕ does not hold in all transitive class models of ZFC, but ϕ is canonically necessary.

Proof. Let ϕ be the statement: 'It is not the case that there is a Cohen-generic filter G over L such that $V = L[G]$'. (Thus, intuitively, ϕ says: 'I am not a Cohen-extension of L'). This is an $\in$-statement. Clearly, ϕ is false in a Cohen-extension $L[G]$ of L.

On the other hand, let M be canonical and assume that $M \models \phi$. Let ψ be a uniqueness statement for M. Then there is some G Cohen-generic over L with $M = L[G]$. Moreover, as $M \models \psi$, there is some condition p such that $p \Vdash \psi$. Let G' be Cohen-generic over L relative to G such that $p \in G'$. Then $L[G'] \models \psi$ but $L[G'] \neq L[G]$, a contradiction to the assumption that ψ is a uniqueness statement. □

This example can be considerably strengthened: In fact, no set forcing extension can be canonical. It is not clear that the statement 'I am not a set forcing extension' is expressable in the first-order language of set theory at all, but by [R], where it is introduced under the name 'ground model axiom' or 'ground axiom', it turns out to be so.

Definition 9 (See [R])**.** The Ground Model Axiom (GMA) is the statement that there is no transitive class model M of ZFC such that, for some forcing $\mathbb{P} \in M$ and some $\mathbb{P}$-generic filter G over M, we have $V = M[G]$. It is proved in [R] that GMA is expressible as an $\in$-formula.

Theorem 10. The ground model axiom GMA is canonical necessary.

Proof. Assume that M is canonical, witnessed by ϕ, and M does not satisfy the ground axiom, e.g. $M = N[G]$, where N is an inner model of M and G is a generic filter for a forcing $\mathbb{P} \in N$. As ϕ holds in M, there is some $p \in \mathbb{P}$ such that $p \Vdash \phi$ over N.

We pass from M to a generic extension $M[H]$ in which $\mathfrak{P}^M(\mathbb{P})$ is countable (via some Levy collapse). (The generic filter H need not exist, but the argument here still suffices for showing that ZFC can't prove that there is a unique transitive class model of ZFC+ϕ.) In $M[H]$, everything we care about is sufficiently countable so that we find two mutually $\mathbb{P}$-generic filters containing p, namely G_1, G_2 over M, a fortiori over N. Hence $N[G_1] \models \phi$ and $N[G_2] \models \phi$, but $N[G_1] \neq N[G_2]$, as e.g. $G_1 \in N[G_1] \setminus N[G_2]$, so $M = N[G]$ cannote be unique with this property, a contradiction.

□

Given this, one might wonder whether GMA captures the full strength of canonical necessity, i.e. whether there are canonically necessary statements that do not follow from GMA. This also turns out to be true:

Theorem 11. The statement that 'There is no measurable cardinal' is canonically necessary.

Proof. Assume otherwise, and let M be a canonical model with a measurable cardinal κ and a normal ultrafilter U on κ. Furthermore, let ϕ be a statement that witnesses the canonicity of M. Then $\mathrm{Ult}(M, U)$ is a transitive class that is elementary equivalent to M, hence in particular satisfies ZFC+ϕ and is different from M, a contradiction. □

Remark: By the same reasoning, no ultrapower can be a canonical model. This, however, does not lead to another c.n. statement, as 'I am no ultrapower' is not first-order expressable. (If it was, then so was 'I am no ultrapower', but the truth value of this statement would have to change when passing e.g. from L to $\mathrm{Ult}(L, U)$.)

Corollary 12. There are c.n. statements that do not follow from GMA.

Proof. By results of J. Reitz (see [R]), the finestructural models for measurable cardinals satisfy GMA. Hence, the nonexistence of measurable cardinals does not follow from GMA. □

3.1. **Strong canonicity.**

Definition 13. If Φ is a recursive set of $\in$-sentences, then Φ is a uniqueness set iff there is only one transitive class model M of ZFC with $M \models \Phi$. In this case, the model M is called weakly canonical. ϕ is called weakly canonically iff ϕ holds in a least one weakly canonical model. If it holds in all such models, ϕ is called strongly canonically necessary.

Strong canonicity may approximate the idea of a 'true' axiomatic system that fixes a unique model of set theory (up to the inevitable shortcomings of first-order logic that require the restriction to transitive class models) better, as recursive axiom systems are quite common.

The Cohen example above of a nonprovable, but canonically necessary statement may not work for strong canonicity, as there may not be a single condition forcing all elements of Φ at once. However, we still get an example when we use a different forcing:

Theorem 14. There is a strongly canonically necessary statement ϕ that does not hold in all transitive class models of ZFC.

Proof. This time, we take 'I am not a $\mathbb{P}$-extension of L', where $\mathbb{P}$ is a countably closed and non-trivial notion of forcing (such as the Levy collapse). Now, if this would hold in a weakly canonical model M with uniqueness set Φ, then each element of Φ would be forced by some condition; the set of these conditions would belong to the ground model as Φ is recursive; and hence, by countable closure, there would be a common strengthening p to all of them, which forces all elements of Φ. But now again, we can take a mututally generic filter and obtain a contradiction as above. □

3.2. **Canonical Choices for the Continuum Function.** A tempting question is to determine the status of the continuum hypothesis under our notions of canonicity. Clearly, CH is canonically possible, as it holds in L. We conjecture that it is not canonically necessary, and we plan to prove this in subsequent work. The idea is to iterate Friedman's forcing for adding a Π^1_2-singleton in chapter 6 of [Fr] ω_2 many times to generate a canonical model in which CH fails.

Here, we restrict ourselves to a simple-minded observation which suffices to exclude many values for the size of the continuum.

Definition 15. We define the continuum function $c : \mathrm{On} \to \mathrm{On}$ as that function that takes an ordinal α to $2^{|\alpha|}$.

Lemma 16. (i) Every canonically possible value of $2^{\aleph_0}$ is definable.

(ii) Every canonically possible continuum function $\kappa \mapsto 2^\kappa$ is definable without parameters. (I.e. if M is a canonical model, then c^M is definable in V.)

Proof. (ii) implies (i).

(ii) Let M be a canonical model, and let ϕ be a formula such that, for some parameter x, $M = \{y : \phi(x, y)\}$. Moreover, let ψ be a formula such that M is the unique transitive class model of ZFC+ψ. Then c^M is definable in V as: $c^M(\alpha) = \beta$ if and only if $\exists x[\{y : \phi(x, y)\} \models \mathrm{ZFC} + \psi \wedge \{y : \phi(x, y)\}$ is transitive $\wedge \{y : \phi(x, y)\} \models 2^{|\alpha|} = \beta$. □

3.3. **Weak Canonicity.** We briefly touch the question mentioned above whether CH is canonically necessary.

Here, we pursue this question by considering a weakened version of canonical necessity: In our definition above, we required that ZFC must be capable of proving the existence and uniqueness of a model of ZFC+ϕ. A reasonable weaker requirement would be that ZFC+ϕ proves this.

Definition 17. ϕ is weakly canonical if and only if ZFC+ϕ proves the uniqueness and existence statements in the definition of canonicity. If M is a model of ZFC+ϕ for some weakly canonical ϕ, then M is weakly canonical. If ψ holds in all weakly canonical models, then ψ is weakly canonically necessary (weakly c.n.).

Theorem 18. $\neg CH$ is weakly c.p.

Proof. In Theorem 19 of [G], a set forcing extension M of L is constructed such that $M \models \neg$CH, but CH holds in every transitive class N such that $L \subseteq N \subseteq M$ (and this is provable in ZFC). The forcing $\mathbb{P}$ used is definable over M without parameters.

Consider the $\in$-statement ϕ 'I am a $\mathbb{P}$-extension of L'. Then every proper inner model of a transitive class model M of ZFC+ϕ will satisfy CH, so that M is the only inner model of M in which CH fails (and all of this is provable in ZFC+ϕ). Hence M is a weakly canonical model of ZFC+$\neg$CH.

□

4. Further Ideas and Questions

Question 19. Is $V = HOD$ canonically necessary?

Question 20. (Dominik Klein): Is AC canonical for models of ZF? That is, is there a formula ϕ such that there is a unique transitive class model M of ZF+ϕ and such that AC fails in M?

Question 21. Is there a canonical model of ZF+AD? (This would answer the last question in the negative.)

Question 22. In general, are some ZFC axioms canonical over the others? Or over KP? Are there e.g. canonical models for ZFC$^-$ in which power set fails? Or of ZFC without replacement in which replacement is false?

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