ZFC without parameters (A note on a question of Kai Wehmeier)

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Abstract

It is shown that the parameter free formulation of ZFC is as strong as ZFC itself.

The current note only reports on results which have been known since the 70ies, see the remarks below.

It is well–known that if the induction schema of Peano arithmetic, PA, is formulated without parameters, then the resulting theory is as strong as PA itself in that both theories prove the same theorems. The reason is basically that every natural number is definable in the language of PA, so that parameters may be replaced by their definition.

As it is certainly not true that every set is definable (cf. [1], though), we may ask if the "parameter free version" of ZFC is weaker than ZFC itself. Zermelo–Fraenkel set theory has the follwing axioms: Extensionality (Ext), Foundation (Fund), Pairing (Par), Union (Union), Power set (Pow), Infinity (Inf), choice (AC), as well as the schemas Aus and Repl of Aussonderung (separation) and replacement. Aus says that for every formula $\varphi(x, v_1, \ldots, v_n)$ in which the variable b does not occur,

$$(1) \qquad \forall v_1 \ldots \forall v_n \forall a \exists b \forall x (x \in b \longleftrightarrow x \in a \wedge \varphi(x, v_1, \ldots, v_n)),$$

and Repl says that for every formula $\varphi(x, y, v_1, \ldots, v_n)$,

$$(2) \qquad \forall v_1 \ldots \forall v_n (\forall x \exists y \forall y' (\varphi(x, y', v_1, \ldots, v_n) \leftrightarrow y' = y) \longrightarrow$$
$$\forall a \exists b \forall y (y \in b \leftrightarrow \exists x \in a \varphi(x, y, v_1, \ldots, v_n))).$$

Repl plus $\exists x(x = \emptyset)$ proves Aus. We may also phrase Repl by saying that if $F \colon x \mapsto y$ is a class function definable by the formula $\varphi(x, y, v_1, \ldots, v_n)$, then for all sets a, $F''a = \{y \colon \exists x \in a F(x) = y\}$ exists.

In (1) and (2), $v_1, \ldots, v_n$ play the role of parameters. We may let $\mathsf{Aus}^{\mathrm{o}}$ be the statement that for every formula $\varphi(x)$ (with $x \neq b$ as variables),

$$(1^{\mathrm{o}}) \qquad \forall a \exists b \forall x (x \in b \longleftrightarrow x \in a \wedge \varphi(x)),$$

and we may let $\mathsf{Repl}^{\mathrm{o}}$ be the statement that for every formula $\varphi(x, y)$,

$$(2^{\mathrm{o}}) \qquad \forall x \exists y \forall y' (\varphi(x, y') \leftrightarrow y' = y) \longrightarrow$$
$$\forall a \exists b \forall y (y \in b \leftrightarrow \exists x \in a \varphi(x, y)).$$

We let ZFC^o be the theory with the axioms Ext, Fund, Par, Union, Pow, Inf, AC, Aus^o, and Repl^o. We now prove the following.[1] The argument for Theorem 0.1 which we present here is due to the first author, while the argument for Theorem 0.7 which we present here is due to the second author.

Theorem 0.1 ZFC^o *proves both* Aus *and* Repl.

Naive attempts of proving Aus and Repl in ZFC^o all go through trying to first show that the cross product $a \times b$ exists, which seems to require Aus. We basically follow this route and cook up a version of such a cross product which can be shown to exist in ZFC^o with not much effort.

Lemma 0.2 (ZFC^o) *For all sets a, both $a \times \{0\}$ and $a \times \{1\}$ exist.*

PROOF. $a \times \{0\} = F''a$, where $F(x) = (x, 0)$, and $a \times \{1\} = G''a$, where $G(x) = (x, 1)$. □

Lemma 0.2 immediately yields

Lemma 0.3 (ZFC^o) *For all sets a and b, $(a \times \{0\}) \cup \{(b, 1)\}$ exists.* □

Lemma 0.4 (ZFC^o) *For all sets a and b, $\{((u, 0), (b, 1)) \colon u \in a\}$ exists.*

PROOF. By Lemma 0.3 and Pow, $d = \mathcal{P}(\mathcal{P}((a \times \{0\}) \cup \{(b, 1)\}))$ exists. We then have that

$$\{((u, 0), (b, 1)) \colon u \in a\} = \{x \in d \colon \exists u \exists v x = ((u, 0), (v, 1))\},$$

so that the desired set exists by Aus^o. □

Lemma 0.5 *Let $\varphi(x, v)$ be a formula. For all sets a and b, $\{x \in a \colon \varphi(x, b)\}$ exists.*

PROOF. Let $F(x) = 0$ unless there are u, c with $x = ((u, 0), (c, 1))$ and $\varphi(x, c)$ in which case $F(((u, 0), (c, 1))) = u$. Then

$$\{x \in a \colon \varphi(x, b)\} \cup \{0\} = F''\{((u, 0), (b, 1)) \colon u \in a\}$$

exists by Lemma 0.4 and Repl^o. We may then easily use Aus^o to get the desired set $\{x \in a \colon \varphi(x, b_1, \ldots, b_n)\}$. □

As finitely many parameters may be easily amalgamated into one, using Par, this shows that Aus follows from ZFC^o. To prove Repl from ZFC^o, it also suffices to consider just one parameter and prove the following by a slight generalization of the argument for Lemma 0.5.

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[1]We thank Kai Wehmeier for asking us the question as to whether ZFC^o is weaker than ZFC. The first author also thanks Rene Schipperus for discussions about this question. A previous version of this paper contained the phrase "Theorem 0.1 may be part of the set theoretic folklore. If so, then the [first] autor of this note just displays his ignorance."
In fact, after having written that version in 2011, Ali Enayat in 2012 established the ignorance of the first author by pointing him to the paper [2].

Lemma 0.6 *Let $\varphi(x, y, v)$ be a formula, Let b be a set such that for every x there is exactly one y with $\varphi(x, y, b)$. Then for all sets a, $\{y \colon \exists x \in a \varphi(x, y, b)\}$ exists.*

PROOF. We may let $F(z) = 0$ unless there are x, c with $z = ((x, 0), (c, 1))$ and there is a unique y with $\varphi(x, y, c)$ in which case $F(z) = y$. Then

$$\{y \colon \exists x \in a \varphi(x, y, b)\} \cup \{0\} = F''\{((x, 0), (b, 1)) \colon x \in a\}$$

exists by Lemma 0.4 and $\mathsf{Repl}^{\mathrm{o}}$. We may then easily use $\mathsf{Aus}^{\mathrm{o}}$ to get the desired set $\{y \colon \exists x \in a \varphi(x, y, b)\}$. □

It was of course redundant to prove Lemma 0.5 before Lemma 0.6. In any event, we have verified Theorem 0.1.

The proof of Lemma 0.5 makes use of $\mathsf{ZFC}^{\mathrm{o}}$, in particular, $\mathsf{Repl}^{\mathrm{o}}$ gets used. Let ZC^{o} be the theory with the axioms Ext, Fund, Par, Union, Pow, Inf, AC, and $\mathsf{Aus}^{\mathrm{o}}$. We don't need $\mathsf{Repl}^{\mathrm{o}}$ to prove Aus:

Theorem 0.7 ZC^{o} *proves* Aus.

PROOF.[2] Let $\varphi(x, v)$ be a formula, and let a and b be arbitrary sets. We need to verify in ZF^{o} that $\{x \in a \colon \varphi(x, b)\}$ exists.

We may assume without loss of generality that $b \neq \emptyset$, as $\emptyset$ is definable from no parameters. In other words, $\{0, b\}$ has exactly two elements.

Using $\mathsf{Aus}^{\mathrm{o}}$, both sets

$$a^1 = \{x \in a \colon \mathrm{Card}(x) = 1\}$$

and

$$a^{\neq 1} = \{x \in a \colon \mathrm{Card}(x) \neq 1\}$$

exist.

$\mathcal{P}(a^1 \cup \{\{0, b\}\})$ exists, and so does, with the help of $\mathsf{Aus}^{\mathrm{o}}$, the set A^1 of all elements of $\mathcal{P}(a^1 \cup \{\{0, b\}\})$ which have exactly two elements one of which has exactly two elements. A^1 is the set $\{\{x, \{0, b\}\} \colon x \in a^1\}$. Using $\mathsf{Aus}^{\mathrm{o}}$ again, we may separate from A^1 the set $A^{1,*}$ of all $\{x, \{0, y\}\}$ from A^1 such that $\varphi(x, y)$. Then

$$\bigcup A^{1,*} = \{x \in a^1 \colon \varphi(x, b)\} \cup \{\{0, b\}\}$$

also exists. Using $\mathsf{Aus}^{\mathrm{o}}$ one more time, we get that

$$\tilde{a}^1 = \{x \in a^1 \colon \varphi(x, b)\}$$

exists as the set of all elements of $\bigcup A^{1,*}$ which have exactly one element.

Exploiting the same argument, with $\{b\}$ playing the role of $\{0, b\}$, we get that

$$\tilde{a}^{\neq 1} = \{x \in a^{\neq 1} \colon \varphi(x, b)\}$$

exists. But then $\tilde{a}^1 \cup \tilde{a}^{\neq 1}$ exists and is the set $\{x \in a \colon \varphi(x, b)\}$. □

[2] We follow the argument the second author sent to the first author of the current note by email on May 20, 2011.

References

[1] Hamkins, J., *Pointwise definable models of set theory*, talk at Oberwolfach, 2011, cf. `http://www.logic.univie.ac.at/~holy/ow2011/joelhamkins.pdf`

[2] Levy, Azriel, *Parameters in comprehension axiom schemas of set theory.* Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971), pp. 309324. Amer. Math. Soc., Providence, R.I., 1974