Thin Equivalence Relations in Scaled Pointclasses

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Abstract

An inner model-theoretic proof that every thin $\Sigma_1^{J_\alpha(\mathbb{R})}$ equivalence relation is $\Delta_1^{J_\alpha(\mathbb{R})}$ in a certain parameter is presented for ordinals α beginning a Σ_1 gap in $L(\mathbb{R})$ where $\Sigma_1^{J_\alpha(\mathbb{R})}$ is closed under number quantification. We use the (optimal) hypothesis $\mathsf{AD}^{J_\alpha(\mathbb{R})}$.

Several results in descriptive set theory proved from determinacy have been studied from an inner model-theoretic perspective [3, 4]. In this line of research we present a new proof of a result of Harrington and Sami [1] on thin equivalence relations. The proof allows us to isolate the optimal hypothesis for the following type of equivalence relation.

Theorem 0.1. *Let $\alpha \geq 2$ begin a Σ_1 gap in $L(\mathbb{R})$ and assume $\mathsf{AD}^{J_\alpha(\mathbb{R})}$. Also, setting $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$, assume Γ to be closed under number quantification, i.e., $\forall^{\mathbb{N}}\Gamma \subset \Gamma$. Let E be a thin Γ equivalence relation and $\mathcal{N}$ an α-suitable mouse with a capturing term for a universal Γ set. Then E is $\breve{\Gamma}$ in any real coding $\mathcal{N}$ as a parameter.*

An equivalence relation E is called *thin* if there is no perfect set of pairwise E–inequivalent reals. The notion of α–suitable mice with capturing terms (which is due to Woodin) is described in our section 1 and in detail in [7]. Such α–suitable mice are in a sense analogues of $M_n^{\#}$ (capturing Σ^1_{n+2}) which capture more complicated sets of reals. The pointclass $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$ as in the statement of Theorem 0.1 is scaled under $\mathsf{AD}^{J_\alpha(\mathbb{R})}$ (cf. [8]).

The remaining cases for α which we address in this paper are subsumed in

Theorem 0.2. *Let $\Gamma = \boldsymbol{\Sigma}_n^{J_\alpha(\mathbb{R})}$ where $\alpha \geq 2$ begins a Σ_1 gap, $n = 1$, and α is a successor ordinal or $cf(\alpha) = \omega$, or else α ends a proper weak Σ_1 gap and n is least with $\rho_n(J_\alpha(\mathbb{R})) = \mathbb{R}$. Assume $\mathsf{AD}^{L(\mathbb{R})}$. Then every thin Γ equivalence relation is $\breve{\Gamma}$.*

In section 1 we collapse a substructure of a suitable premouse and prove upwards absoluteness for the preimages of capturing terms. In section 2 the method of term capturing is applied to prove Theorem 0.1 building on an argument of Hjorth [4, Lemma 2.5] for Σ^1_2 equivalence relations. Theorem 0.2 is proved in section 3.

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1 Weak term condensation

Let us fix an ordinal $\alpha \geq 2$ beginning a Σ_1 gap in $L(\mathbb{R})$ (cf. [8]) with the property that $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$ is closed under number quantification. Let us also assume $\mathsf{AD}^{J_\alpha(\mathbb{R})}$ throughout.

Definition 1.1. *For any bounded subset A of ω_1, the* (α–)lower-part closure $Lp^\alpha(A)$ *of A is the the union of all A-premice $\mathcal{N}$ which are sound above A, project to* $\sup(A)$, *and are ω_1-iterable in $J_\alpha(\mathbb{R})$ (i.e., there is an iteration strategy $\Sigma \in J_\alpha(\mathbb{R})$ with respect to countable iteration trees on $\mathcal{N}$).*

Under $\mathsf{AD}^{J_\alpha(\mathbb{R})}$, any two A-premice as in definition 1.1 are lined up,[1] so that $Lp^\alpha(A)$ is well–defined.

Definition 1.2. *An A-premouse $\mathcal{N}$ for bounded $A \subseteq \omega_1$ with a unique Woodin cardinal $\delta = \delta^{\mathcal{N}}$ is called* α-suitable *if*

- *δ is minimal such that δ is Woodin in $Lp^\alpha(\mathcal{N}|\delta)$, and*
- *$\mathcal{N}$ is the Lp^α closure of $\mathcal{N}|\delta$ up to its ω^{th} cardinal above δ, i.e. $\mathcal{N} = \bigcup_{k\in\omega} \mathcal{N}_k$ where $\mathcal{N}_0 := \mathcal{N}|\delta$ and $\mathcal{N}_{k+1} := Lp^\alpha(\mathcal{N}_k)$ for all k.*

In what follows, we let $\delta = \delta^{\mathcal{N}}$ always denote the Woodin cardinal of an α-suitable premouse $\mathcal{N}$.

Definition 1.3. *An ω_1-iteration strategy Σ for an α–suitable $\mathcal{N}$ is* fullness-preserving *if for every iteration tree $\mathcal{T}$ on $\mathcal{N}$ according to Σ which lives below δ[2] we have that*

- *if the branch to the last model $\mathcal{P}$ does not drop, then $\mathcal{P}$ is α-suitable, and*
- *if the branch to $\mathcal{P}$ drops, then $\mathcal{P}$ is ω_1-iterable in $J_\alpha(\mathbb{R})$.*

[1] Notice that if Σ and $\Sigma' \in J_\alpha(\mathbb{R})$ witness the countable A-premice $\mathcal{N}$ and $\mathcal{N}'$ to be ω_1-iterable, respectively, then $\omega_1^{L[\Sigma,\Sigma',\mathcal{N},\mathcal{N}']} < \omega_1^V$ by $\mathsf{AD}^{J_\alpha(\mathbb{R})}$, so that $\mathcal{N}$ and $\mathcal{N}'$ can be successfully compared in $L[\Sigma,\Sigma',\mathcal{N},\mathcal{N}']$.

[2] That $\mathcal{T}$ *lives below* ξ (or, *on* $\mathcal{N}|\xi$) means that $\mathcal{T}$ may be construed as an iteration tree on $\mathcal{N}|\xi$.

Definition 1.4. *Suppose Σ is an ω_1-iteration strategy for a countable α-suitable A-premouse $\mathcal{N}$ and $\mathbb{Q} \in \mathcal{N}$ is a forcing notion. A $\mathbb{Q}$–name $\dot{E} \in \mathcal{N}^{\mathbb{Q}}$ is said to* capture *a set $E \subseteq \mathbb{R}$ relative to Σ if*

$$\pi(\dot{E})^g = E \cap \mathcal{P}[g]$$

whenever $\pi : \mathcal{N} \to \mathcal{P}$ is a non-dropping iteration map produced by a countable iteration tree which is according to Σ and g is $\pi(\mathbb{Q})$-generic over $\mathcal{P}$. $\dot{E}$ is then also called a ($\mathbb{Q}$–)capturing term *for E (relative to Σ).*

Let $Col_\eta = Col(\omega, \eta)$ denote the forcing for collapsing η.

Theorem 1.5. *(Woodin, see [7]) Assume $\mathsf{AD}^{J_\alpha(\mathbb{R})}$ holds, where $\alpha \geq 2$ begins a Σ_1 gap in $L(\mathbb{R})$ and $\Sigma_1^{J_\alpha(\mathbb{R})}$ is closed under number quantification. Suppose $E \subseteq \mathbb{R}$ is a $\Sigma_1^{J_\alpha(\mathbb{R})}$ set. There is then a countable α-suitable A-premouse $\mathcal{N}$ and a fullness-preserving ω_1-iteration strategy Σ for $\mathcal{N}$ such that for every $\eta \geq \delta$ in $\mathcal{N}$ there is a Col_η–name capturing E relative to Σ.*

Let us fix such an A-premouse $\mathcal{N}$ together with a fullness-preserving ω_1-iteration strategy Σ. It is easy to see that Theorem 1.5 provides $\mathbb{Q}$-capturing terms for any forcing $\mathbb{Q}$ in $\mathcal{N}$, since the forcing is absorbed in Col_η for some η. We now show that A-premice which embed into an initial segment of $\mathcal{N}$ retain a weak capturing property.

Lemma 1.6. *Suppose $E \subseteq \mathbb{R}$ is a $\Sigma_1^{J_\alpha(\mathbb{R})}$ set and $\dot{E}, \dot{s}$ are Col_δ-capturing terms for E and its $\Sigma_1^{J_\alpha(\mathbb{R})}$ scale (relative to Σ). Let $\pi : \mathcal{M} \to \mathcal{N}|(\delta^{+n})^{\mathcal{N}}$ be sufficiently elementary with $\dot{E}, \dot{s} \in ran(\pi)$ and $\dot{F} = \pi^{-1}(\dot{E})$, where $n \geq 2$. Then $\dot{F}^g \subseteq E$ for every $Col_{\pi^{-1}(\delta)}$-generic filter g over $\mathcal{M}$.*

Proof. We argue that it is possible to replace $\dot{E}$ with the name for the projection of a tree and we then use upwards absoluteness for this name. Suppose h is Col_δ-generic over $\mathcal{N}$ and $\dot{T} \in \mathcal{N}$ is a Col_δ-name for the tree

$$T = \{(x \restriction k, (r_0(x)^{\mathcal{N}[h]}, ..., r_{k-1}(x)^{\mathcal{N}[h]})) : x \in \dot{E}^h = E \cap \mathcal{N}[h] \ \&\ k \in \omega\} \tag{1}$$

where the r_i are the ranks in the scale as computed in $\mathcal{N}[h]$ via the capturing term $\dot{s}$ for the scale. The tree T is the image of a countable subtree S of the tree from the scale on E in V via the map which collapses the set of ordinals occuring in S to an ordinal, so that $p[\dot{T}^h]^{\mathcal{N}[h]} = E \cap \mathcal{N}[h]$. This implies $\dot{E}^h = E \cap \mathcal{N}[h] = p[\dot{T}^h]^{\mathcal{N}[h]}$.

Notice that $T = \dot{T}^h$ is independent of the choice of the particular generic h, and hence $T \in \mathcal{N}$. Suppose $p, q \in Col_\delta$ were conditions with $p \Vdash^{\mathcal{N}} \check{(a, f)} \in \dot{T}$ and $q \Vdash^{\mathcal{N}} \check{(a, f)} \notin \dot{T}$. We may pick generics h_p and h_q over $\mathcal{N}$ with $p \in h_p$ and $q \in h_q$ such that $\mathcal{N}[h_p] = \mathcal{N}[h_q]$. As $\dot{E}$ and $\dot{s}$ capture E and its scale over $\mathcal{N}$, respectively, we have $\dot{E}^{h_p} = \dot{E}^{h_q}$ and $\dot{s}^{h_p} = \dot{s}^{h_q}$, so $(a, f) \in \dot{T}^{h_p} = \dot{T}^{h_q}$ by (1). This contradicts the choice of p, q.

Now as $p[T]^{\mathcal{N}[h]} = \dot{E}^h$,

$$\Vdash^{\mathcal{N}}_{Col_\delta} \; p[\check{T}] = \dot{E},$$

and therefore

$$\Vdash^{\mathcal{M}}_{Col_{\pi^{-1}(\delta)}} \; p[\pi^{-1}(\check{T})] = \dot{F}.$$

This yields that $\dot{F}^g = p[\pi^{-1}(T)]^{\mathcal{M}[g]} \subseteq p[T] \subseteq E$ for every $Col_{\pi^{-1}(\delta)}$-generic g over $\mathcal{M}$. $\square$

When $\mathcal{M}$ is iterated, the capturing term is still upwards absolute.

Lemma 1.7. *Suppose $E \subseteq \mathbb{R}$ is a $\Sigma_1^{J_\alpha(\mathbb{R})}$ set and $\dot{E}, \dot{s}$ are Col_δ-capturing terms for E and its $\Sigma_1^{J_\alpha(\mathbb{R})}$ scale (relative to Σ). Let $\pi : \mathcal{M} \to \mathcal{N}|(\delta^{+n})^{\mathcal{N}}$ be sufficiently elementary with $\dot{E}, \dot{s} \in ran(\pi)$ and $\dot{F} := \pi^{-1}(\dot{E})$, where $n \geq 2$. Let $\sigma : \mathcal{M} \to \mathcal{P}$ be a non-dropping iteration map via the pullback strategy. Then $\sigma(\dot{F})^g \subseteq E$ for every $Col_{\sigma(\pi^{-1}(\delta))}$-generic filter g over $\mathcal{P}$.*

Proof. Let $\sigma^\pi : \mathcal{N} \to \mathcal{R}$ denote the iteration map of the tree copied onto $\mathcal{N}$. There is an embedding $\pi^* : \mathcal{P} \to \mathcal{R}|\sigma^\pi((\delta^{+n})^{\mathcal{N}})$ such that the diagram

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\pi^*} & \mathcal{R}|\sigma^\pi((\delta^{+n})^{\mathcal{N}}) \\ \uparrow{\scriptstyle\sigma} & & \uparrow{\scriptstyle\sigma^\pi} \\ \mathcal{M} & \xrightarrow{\pi} & \mathcal{N}|(\delta^{+n})^{\mathcal{N}} \end{array}$$

commutes. Then $\sigma(\dot{F})^g \subseteq E$ by the previous Lemma applied to $\mathcal{R}$. $\square$

2 At the beginning of a gap

As in the previous section we shall assume that α begins a Σ_1 gap in $L(\mathbb{R})$, $\Sigma_1^{J_\alpha(\mathbb{R})}$ is closed under number quantification, and $\mathsf{AD}^{J_\alpha(\mathbb{R})}$ holds. Let $\mathcal{N}$ be an α-suitable A-premouse (for some A) as in definition 1.2, and let Σ be a fullness-preserving ω_1–iteration strategy for $\mathcal{N}$.

Definition 2.1. *Let $\mathcal{T}$ be a normal iteration tree of countable length on $\mathcal{N}$, and suppose that $\mathcal{T}$ lives below $\delta^{\mathcal{N}}$. We then say that $\mathcal{T}$ is* short *iff for all limit ordinals $\lambda < \mathrm{lh}(\mathcal{T})$, $Lp^\alpha(\mathcal{M}(\mathcal{T} \upharpoonright \lambda)) \models \delta(\mathcal{T} \upharpoonright \lambda)$ is not Woodin. Otherwise, we say that $\mathcal{T}$ is maximal.*

Lemma 2.2. *The restriction of the ω_1-iteration strategy Σ to short trees on $\mathcal{N}$ is $\Sigma_1^{J_\alpha(\mathbb{R})}$.*

Proof. Let $\mathcal{T}$ be a countable short iteration tree of limit length which is on $\mathcal{N}$ and according to Σ. We then have that $\Sigma(\mathcal{T}) = b$ if and only if there is a $\mathcal{Q}$–structure $\mathcal{Q} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$ such that $\mathcal{Q}$ is ω_1–iterable in $J_\alpha(\mathbb{R})$. This immediately shows that Σ, restricted to short trees, is in $\Sigma_1^{J_\alpha(\mathbb{R})}$. □

Lemma 2.3. *For all $n \geq 1$ there is $\mathcal{M} \triangleleft \mathcal{N}$ and a fully elementary map $\pi : \mathcal{M} \rightarrow \mathcal{N}|(\delta^{+n})^{\mathcal{N}}$ with $\gamma = \pi^{-1}(\delta) < \delta$ and $V_\gamma^{\mathcal{M}} = V_\gamma^{\mathcal{N}}$.*

Proof. Let us construct $(H_i\colon i < \omega) \in \mathcal{N}$ as follows. Let $\mathcal{P} = \mathcal{N}|(\delta^{+n})^{\mathcal{N}} + 1$. Set $H_0 = \emptyset$, and given H_i set

$$H_{i+1} = \mathrm{Hull}_{\Sigma_1}^{\mathcal{P}}(V^{\mathcal{N}}_{\sup(H_i \cap \delta)+1})$$

for $i < \omega$. Then $\gamma = \sup(\bigcup_{i<\omega} H_i \cap \delta) < \delta$ since δ is inaccessible in $\mathcal{N}$. Let

$$\pi^* : \mathcal{M}^* \rightarrow \bigcup_{i<\omega} H_i = \mathrm{Hull}_{\Sigma_1}^{\mathcal{P}}(V_\gamma^{\mathcal{N}})$$

be the inverse of the collapsing map. The construction ensures that $V_\gamma^{\mathcal{M}^*} = V_\gamma^{\mathcal{N}}$. We have $crit(\pi) = \gamma$ and $\rho_1(\mathcal{M}^*) = \gamma$. We easily get $\mathcal{M}^* \triangleleft \mathcal{N}$ by the Condensation Lemma (see [10, Theorem 5.5.1] or the remark after the proof of [6, Theorem 8.2]). Then

$$\pi^* \upharpoonright (\pi^*)^{-1}(\mathcal{N}|(\delta^{+n})^{\mathcal{N}})$$

is as desired. □

Let us fix a notation: Given a forcing $\mathbb{P}$ and a $\mathbb{P}$-name τ, let τ_i for $i = 0, 1$ denote $\mathbb{P} \times \mathbb{P}$-names such that $\tau^{g_i} = (\tau_i)^g$ for any $\mathbb{P} \times \mathbb{P}$-generic filter $g = g_0 \times g_1$.

Lemma 2.4. *Let E be a thin $\Sigma_1^{J_\alpha(\mathbb{R})}$ equivalence relation. Suppose $\dot{E}$ captures E over $\mathcal{N}$ for the forcing $\mathbb{Q} = \mathbb{P} \times \mathbb{P}$. Then for every $\mathbb{P}$-name $\tau \in \mathcal{N}$ for a real,*

$$(p, p) \Vdash_{\mathbb{Q}}^{\mathcal{N}} \tau_0 \dot{E} \tau_1$$

holds for a dense set of conditions $p \in \mathbb{P}$.

Proof. The proof is essentially that of [4, Lemma 2.2]. Suppose the set is not dense. In this case let $p_\emptyset$ be a condition such that for every $r \leq p_\emptyset$ there are conditions $p, q \leq r$ with

$$(p, q) \Vdash_{\mathbb{Q}}^{\mathcal{N}} \neg \tau_0 \dot{E} \tau_1.$$

Let $(D_i : i < \omega)$ enumerate the dense open subsets of $\mathbb{Q}$ in $\mathcal{N}$. We can inductively construct a family $(p_s : s \in 2^{<\omega})$ of conditions in $\mathbb{P}$ so that for all $s, t \in 2^n$ and $u \in 2^{<n}$

- $p_s \leq p_u$ if $u \subseteq s$,
- $(p_{u^\frown 0}, p_{u^\frown 1}) \Vdash^{\mathcal{N}}_{\mathbb{Q}} \neg\tau_0 \dot{E} \tau_1$,
- p_s decides $\tau \restriction n$, and
- $(p_s, p_t) \in D_0 \cap \cdots \cap D_n$ if $s \neq t$.

If we now define $g_x := \{p \in \mathbb{P} : \exists n < \omega(p_{x \restriction n} \leq p)\}$ for each $x \in \mathbb{R}$, then $g_x \times g_y$ is $\mathbb{Q}$-generic over $\mathcal{N}$ for all $x \neq y$ and

$$\dot{E}^{g_x \times g_y} = E \cap \mathcal{N}[g_x \times g_y]$$

since $\dot{E}$ $\mathbb{Q}$-captures E. We will then have

$$\mathcal{N}[g_x \times g_y] \models \neg\tau^{g_x} \dot{E}^{g_x \times g_y} \tau^{g_y}$$

and thus $\neg\tau^{g_x} E \tau^{g_y}$ for all $x \neq y$ because $\tau^{g_x} = (\tau_0)^{g_x \times g_y}$ and $\tau^{g_y} = (\tau_1)^{g_x \times g_y}$. Since τ^{g_x} depends continuously on x, the set $\{\tau^{g_x} : x \in \mathbb{R}\}$ is a perfect set of E-inequivalent reals, contradicting the assumption on E. □

By Theorem 1.5, we may assume that our $\mathcal{N}$ and Σ satisfy the hypothesis of the following theorem.

Theorem 2.5. *Let E be a thin $\Sigma_1^{J_\alpha(\mathbb{R})}$ equivalence relation and suppose $\mathcal{N}$ has capturing terms for E and a $\Sigma_1^{J_\alpha(\mathbb{R})}$ scale on E relative to Σ. Then E is $\Pi_1^{J_\alpha(\mathbb{R})}$ in (any real coding) $\mathcal{N}$.*

Proof. This is similar to the proof of [4, Lemma 2.5]. Let

$$\pi : \mathcal{M} = \mathcal{N}|\beta \to \mathcal{N}|\delta^{++\mathcal{N}}$$

be as in (the proof of) Lemma 2.3 with $Col_\delta \times Col_\delta$-capturing terms $\dot{E}, \dot{s}$ for E and a $\Sigma_1^{J_\alpha(\mathbb{R})}$ scale on E in $ran(\pi)$. This is possible since these capturing terms have size $\delta^{+\mathcal{N}}$. Let $\gamma := \pi^{-1}(\delta)$ and and $\dot{r}$ the preimage under π of the Col_δ-name for the generic real for the extender algebra at δ. Let $\rho, \tau \in \mathcal{M}$ be Col_γ-names for reals such that $\Vdash^{\mathcal{M}}_{Col_\gamma} \dot{r} = \rho \oplus \tau$, where $\rho \oplus \tau$ enumerates the bits of ρ and τ. As Col_γ is absorbed in $Col_\delta \times Col_\delta$, it is easy to see that $\mathcal{N}$ has a Col_γ-capturing term $\dot{E}_\gamma$ for E.

We claim that for $a, b \in \mathbb{R}$ the fact that $\neg aEb$ holds true is equivalent to the following condition.

Condition 2.6. *There are $a^*, b^* \in \mathbb{R}$, a non-dropping iteration map $\sigma : \mathcal{N} \to \mathcal{P}$ produced by an iteration tree $\mathcal{T}$ according to Σ which lives on $\mathcal{N}|\gamma$, and a $Col_{\sigma(\gamma)}$-generic filter h over $\mathcal{P}$ with*

- $a^* \oplus b^* = \sigma(\dot{r})^h$,
- $\mathcal{P}[h] \models \neg a^* \sigma(\dot{E}_\gamma)^h b^*$, *and*
- aEa^* & bEb^*.

Condition 2.6 clearly implies $\neg aEb$ by our hypotheses.

If on the other hand $\neg aEb$ holds, using Woodin's extender algebra argument (see [9, Theorem 7.14]) we may pick $\sigma : \mathcal{N} \to \mathcal{P}$ and g such that $\sigma : \mathcal{N} \to \mathcal{P}$ is a non-dropping iteration map produced by a tree $\mathcal{T}$ according to Σ which lives on $\mathcal{N}|\gamma$ and g is $Col_{\sigma(\gamma)}$-generic over $\sigma(\mathcal{M}) = \mathcal{P}|\sigma(\beta)$ with $a \oplus b = \sigma(\dot{r})^g$. This works since $V_\gamma^{\mathcal{M}} = V_\gamma^{\mathcal{N}}$; all that is needed is that $\sigma(\mathcal{M})$ be well-founded and there are no extenders left which violate the axioms of the extender algebra. Let $\dot{F} := \pi^{-1}(\dot{E})$. By Lemma 2.4 and the elementarity of π, if $\epsilon \in \mathcal{M}^{Col_\gamma}$ is a name for a real, then for a dense set of conditions $p \in Col_\gamma$,

$$(p, p) \Vdash^{\mathcal{M}}_{Col_\gamma \times Col_\gamma} \epsilon_0 \dot{F} \epsilon_1.$$

Using the elementarity of σ, there is hence some $p \in g$ such that

$$(p, p) \Vdash^{\sigma(\mathcal{M})}_{Col_{\sigma(\gamma)} \times Col_{\sigma(\gamma)}} \sigma(\rho)_0 \sigma(\dot{F}) \sigma(\rho)_1 \,\&\, \sigma(\tau)_0 \sigma(\dot{F}) \sigma(\tau)_1. \qquad (2)$$

Let h be $Col_{\sigma(\gamma)}$-generic below p over both $(\mathcal{P}|\sigma(\beta))[g]$ and $\mathcal{P}$ and let $a^* \oplus b^* = \sigma(\dot{r})^h$. We then have aEa^* and bEb^* by (2) and by Lemmas 1.6 and 1.7. As $\neg aEb$, this means that $\neg a^* E b^*$, so that $\mathcal{P}[h] \models \neg a^* \sigma(\dot{E}_\gamma)^h b^*$. We have shown that condition 2.6 holds.

Finally, it is true that Σ, restricted to short trees, is $\Sigma_1^{J_\alpha(\mathbb{R})}$ by Lemma 2.2, so that the reformulation of $\neg aEb$ given by condition 2.6 shows that $\neg E$ is $\exists^{\mathbb{R}}\, \forall^{\mathbb{N}}\, \Sigma_1^{J_\alpha(\mathbb{R})}$ in $\mathcal{N}$. As we assume $\Sigma_1^{J_\alpha(\mathbb{R})}$ to be closed under number quantification, this shows that E is $\Pi_1^{J_\alpha(\mathbb{R})}$ in $\mathcal{N}$, as desired. □

3 ω-cofinal pointclasses

The argument in the last section used that $\Sigma_1^{J_\alpha(\mathbb{R})}$ be closed under number quantification. We do not know how to drop this hypothesis, unless we replace the hypothesis $\mathsf{AD}^{J_\alpha(\mathbb{R})}$ by $\mathsf{AD}^{L(\mathbb{R})}$.

We thus now turn to the case that $\alpha \geq 2$ begins a gap and $\Sigma_1^{J_\alpha(\mathbb{R})}$ is not closed under number quantification. In this case α is either a successor or $cf(\alpha) = \omega$, since $cf(\alpha) > \omega$

and $J_\alpha(\mathbb{R}) \models \forall n\, \varphi(x,n)$ imply that there is some $\beta < \alpha$ with $J_\beta(\mathbb{R}) \models \forall n\, \varphi(x,n)$. Hence $A \in \Sigma_1^{J_\alpha(\mathbb{R})}$ if and only if A is a countable union of sets in $J_\alpha(\mathbb{R})$.

Lemma 3.1. *Assume* AD. *Let* Γ *be a scaled pointclass closed under* $\exists^{\mathbb{R}}$ *and continuous preimages. Suppose* $\Gamma_k \subseteq \Delta = \Gamma \cap \check{\Gamma}$ *for* $k \in \omega$ *are pointclasses such that for every* $A \in \Gamma$ *there are* $A_k \in \Gamma_k$ *with* $A = \bigcup_{k\in\omega} A_k$. *Then every thin* Γ *equivalence relation is* Δ.

Proof. Let E be a thin Γ equivalence relation. Then E is co-κ-Suslin via some tree T, since the class of κ-Suslin sets is closed under countable intersections: Trees T_k on $\omega \times \kappa$ with $A_k = p[T_k]$ can be amalgamated into a tree T with $\bigcap_{k\in\omega} A_k = p[T]$. There is no injective ω_1-sequence of reals under AD, so $L[T] \cap \mathbb{R}$ is countable and hence there is a Cohen real in V over $L[T]$. Harrington and Shelah [2] proved that if $E = \mathbb{R}^2 - p[T]$ is a thin equivalence relation and there is a Cohen real over $L[T]$, then the equivalence classes of E can be enumerated with order type $\delta \leq \kappa$. Let $(A_\gamma : \gamma < \delta)$ be such an enumeration.

Notice that Γ is closed under wellordered unions by [5, Lemma 2.18], since it is closed under $\exists^{\mathbb{R}}$ and has the prewellordering property. This shows that

$$\mathbb{R}^2 - E = \bigcup_{\beta \neq \gamma < \delta} (A_\beta \times A_\gamma)$$

is Γ. $\square$

Theorem 3.2. *Let* $\Gamma = \boldsymbol{\Sigma}_n^{J_\alpha(\mathbb{R})}$ *where* $\alpha \geq 2$ *begins a* Σ_1 *gap,* $n = 1$*, and* α *is a successor ordinal or* $cf(\alpha) = \omega$*, or else* α *ends a proper weak* Σ_1 *gap and* n *is least with* $\rho_n(J_\alpha(\mathbb{R})) = \mathbb{R}$. *Assume* $\mathsf{AD}^{L(\mathbb{R})}$. *Then every thin* Γ *equivalence relation is* $\check{\Gamma}$.

Proof. If α begins a gap and $\alpha = \beta + 1$, let $\Gamma_k = \boldsymbol{\Sigma}_k^{J_\beta(\mathbb{R})}$. If $cf(\alpha) = \omega$, let $\alpha = \sup \alpha_k$ and $\Gamma_k = J_{\alpha_k}(\mathbb{R})$. Finally, let $\Gamma_k = J_\alpha(\mathbb{R})$ for all k if α ends a gap. The previous lemma applies in each case. $\square$

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