PERFECT SUBSETS OF GENERALIZED BAIRE SPACES AND LONG GAMES

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Abstract. We extend Solovay's theorem about definable subsets of the Baire space to the generalized Baire space ${}^\lambda\lambda$, where λ is an uncountable cardinal with $\lambda^{<\lambda} = \lambda$. In the first main theorem, we show that that the perfect set property for all subsets of ${}^\lambda\lambda$ that are definable from elements of ${}^\lambda\mathrm{Ord}$ is consistent relative to the existence of an inaccessible cardinal above λ. In the second main theorem, we introduce a Banach-Mazur type game of length λ and show that the determinacy of this game, for all subsets of ${}^\lambda\lambda$ that are definable from elements of ${}^\lambda\mathrm{Ord}$ as winning conditions, is consistent relative to the existence of an inaccessible cardinal above λ. We further obtain some related results about definable functions on ${}^\lambda\lambda$ and consequences of resurrection axioms for definable subsets of ${}^\lambda\lambda$.

Contents

1. Introduction

The *perfect set property* for a subset of the Baire space states that it either contains a perfect subset, i.e. a nonempty, closed subset without isolated points, or is countable. By a classical result, this property is provable for the analytic subsets of the Baire space [Kan09, Corollary 14.8], but not for their complements [Kan09, Theorem 13.12]. Moreover, by an important result of Solovay, it is consistent relative to the existence of an inaccessible cardinal, that all subsets of ${}^\omega\omega$ that are definable from countable sequences of ordinals have the perfect set property [Sol70, Theorem 2].

It is natural to ask whether the last result extends to uncountable cardinals λ. In the uncountable setting, a *perfect subset of* ${}^\lambda\lambda$ is defined as the set of all cofinal branches of some $<\lambda$-closed subtree of the set ${}^{<\lambda}\lambda$ of all sequences in λ of length strictly less than λ. Accordingly, a subset of ${}^\lambda\lambda$ has the *perfect set property* if it either contains a perfect subset or has size at most λ. The next question (and variants thereof) was asked by Mekler and Väänänen [MV93], Kovachev [Kov09], Friedman and others.

Question 1.1. *Is it consistent, relative to the existence of large cardinals, that for some uncountable cardinal λ, the perfect set property holds for all subsets of ${}^\lambda\lambda$ that are definable from λ?*

The author was partially supported by DFG-grant LU2020/1-1 during the preparation of this paper.

The first main result, which we prove in Theorem 2.19 below, gives a positive answer to this question.

Theorem 1.2. *For any uncountable regular cardinal λ with an inaccessible cardinal above it, there is a generic extension by a $<\lambda$-closed forcing in which every subset of ${}^{\lambda}\lambda$ that is definable from a λ-sequence of ordinals has the perfect set property.*

Assuming that there is a proper class of inaccessible cardinals, this can be extended to the next result, which is proved in Theorem 2.20 below.

Theorem 1.3. *Assume that there is a proper class of inaccessible cardinals. Then there is a class generic extension in which for every infinite regular cardinal λ, every subset of ${}^{\lambda}\lambda$ that is definable from a λ-sequence of ordinals has the perfect set property.*

We will further obtain the next result about definable functions in Theorem 2.22. In the statement, let $[X]^{\gamma}_{\neq}$ denote the set of sequences $\langle x_i \mid i < \gamma\rangle$ of distinct elements of X for any set X and any ordinal γ.

Theorem 1.4. *For any uncountable regular cardinal λ with $\lambda^{<\lambda} = \lambda$, there is a generic extension by a $<\lambda$-closed forcing in which for every $\gamma < \lambda$ and every function $f\colon [{}^{\lambda}\lambda]^{\gamma}_{\neq} \mapsto {}^{\lambda}\lambda$ that is definable from a λ-sequence of ordinals, there is a perfect subset C of ${}^{\lambda}\lambda$ such that $f\upharpoonright[C]^{\gamma}$ is continuous.*

We now turn to the Baire property and generalizations thereof, which we study in the second part of this paper. It is provable that analytic and co-analytic subsets of ${}^{\omega}\omega$ have the Baire property [Kec95, Theorem 21.6]. Moreover, Solovay proved that it is consistent, relative to the existence of an inaccessible cardinal, that all subsets of ${}^{\omega}\omega$ that are definable from elements of ${}^{\omega}\mathrm{Ord}$ have the Baire property [Sol70, Theorem 2].

The direct generalization of the Baire property, which we here call λ-*Baire*, is given in Definition 1.12 below. However, the situation for this property in the uncountable setting is very different compared to both the Baire property in the countable setting and the perfect set property in the uncountable setting, since there are always Σ^1_1 subsets of ${}^{\lambda}2$ that are not λ-Baire by the next example. To state the example, we consider the set

$$\mathrm{Club}_{\lambda} = \{x \in {}^{\lambda}2 \mid \exists C \subseteq \lambda \text{ club } \forall i \in C\ x(i) \neq 0\}$$

of functions coding elements of the club filter as characteristic functions.

Example 1.5. [HS01, Theorem 4.2] Suppose that λ is a cardinal with $\mathrm{cof}(\lambda) > \omega$. Then the set Club_{λ} is not a λ-Baire subset of ${}^{\lambda}2$.

Moreover, this counterexample is generalized to subsets of ${}^{\lambda}\lambda$ in [FHK14] as follows. If S is a subset of λ, we consider the set

$$\mathrm{Club}^{S}_{\lambda} = \{x \in {}^{\lambda}\lambda \mid \exists C \subseteq \lambda \text{ club } \forall i \in C\ x(i) \in S\}.$$

Example 1.6. [FHK14, Theorem 3.10] Suppose that λ is an uncountable cardinal with $\lambda^{<\lambda} = \lambda$ and S is a bi-stationary subset of λ. Then the set $\mathrm{Club}^{S}_{\lambda}$ is not a λ-Baire subset of ${}^{\lambda}\lambda$.

It is worthwhile to mention that there are further strengthenings of this failure that can be found in [LS15, Proposition 3.7].

Since the Baire property for subsets of ${}^{\omega}\omega$ is characterized by the Banach-Mazur game [Kec95, Theorem 8.33], it is useful to consider a generalization of this game of uncountable length (see Definition 3.5 below). However, because of the asymmetry of the game at limit times, the condition that a given subset A of ${}^{\lambda}\lambda$ is λ-Baire is stronger than the determinacy of the Banach-Mazur game of length λ for the set A as a winning condition. This motivates the following question, which was asked in [Kov09].

Question 1.7. *Is it consistent, relative to the existence of large cardinals, that for some uncountable cardinal λ, the Banach-Mazur game of length λ is determined for all subsets of ${}^{\lambda}\lambda$ that are definable from λ as winning conditions?*

The second main result, which we prove in Theorem 3.28 below, gives a positive answer to this question.

Theorem 1.8. *For any uncountable regular cardinal λ with an inaccessible cardinal above it, there is a generic extension by a $<\lambda$-closed forcing in which the Banach-Mazur game of length λ is determined for any subset of ${}^{\lambda}\lambda$ that is definable from a λ-sequence of ordinals.*

We will moreover use the Banach-Mazur game to define a generalization of the Baire property, which we call *almost Baire*, in Section 3.1, and show that it is consistent that this property holds for the same class of definable sets that is considered above.

We now turn to the question whether the conclusions of the above results follow from strong axioms. In the countable setting, it is well known that $M_n^{\#}$ is absolute to all set generic extensions for all natural numbers n and that therefore, the theory of $(H_{\omega_1}, \in)$ is absolute to all generic extensions if there is a proper class of Woodin cardinals (see [Ste10, Sch14]). Hence the conclusion of Solovay's theorem [Sol70, Theorem 1] for projective sets is provable from a proper class of Woodin cardinals.[1]

In the uncountable setting, the theory of $(H_{\omega_2}, \in)$ is not absolute to all generic extensions that preserve ω_1, since both the existence and non-existence of ω_1-Kurepa trees can be forced by $<\omega_1$-closed forcings, assuming the existence of an inaccessible cardinal. Therefore, we will consider a variant ot the *resurrection axiom* that was introduced by Hamkins and Johnstone [HJ14]. The idea for such axioms is to postulate that certain properties of the ground model which might be lost in a generic extension can be resurrected by passing to a further extension.

We will see that variants of the conclusions of the above results follow from such an axioms for a class of $<\lambda$-closed forcings. If λ is a regular cardinal, we say that ν is λ-*inaccessible* if $\nu > \lambda$ is regular and $\mu^{<\lambda} < \nu$ holds for all cardinals $\mu < \nu$. The following result is proved in Theorem 4.4 below.

Theorem 1.9. *Suppose that λ is an uncountable regular cardinal, and the resurrection axiom* RA^{λ} *(see Definition 4.2 below) holds for the class of forcings* $\mathrm{Col}(\lambda, <\nu)$*, where ν is λ-inaccessible. Then the following statements hold for every subset A of ${}^{\lambda}\lambda$ that is definable over $(H_{\lambda^+}, \in)$ with parameters in H_{λ^+}.*

(1) *A has the perfect set property.*
(2) *The Banach-Mazur game of length λ with A as a winning condition is determined.*

This paper is organized as follows. In the remainder of this section, we will collect several definitions and facts about trees and forcings. In Section 2, we will prove among other results the consistency of the perfect set property for definable subsets of ${}^{\lambda}\lambda$. In Section 3, we will prove among other results the consistency of the almost Baire property for definable subsets of ${}^{\lambda}\lambda$. Finally, in Section 4, we will derive variants of the conclusions of the main results from resurrection axioms.

For notation, we will assume throughout this paper that κ is an uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$ and λ is an uncountable regular cardinal.

We would like to thank Peter Holy for discussions about the presentation and the referee for various helpful comments. The results in this paper are motivated by work of Solovay [Sol70], Mekler and Väänänen [MV93], Donder and Kovachev [Kov09] and some ideas from this work have already been applied in subsequent work [Lag15, LMRS16].

1.1. **Trees and perfect sets.** We always assume that λ is a regular uncountable cardinal. The *standard topology* (or *bounded topology*) on ${}^{\lambda}\lambda$ is generated by the basic open sets

$$N_t = \{x \in {}^{\lambda}\lambda \mid t \subseteq x\}$$

for $t \in {}^{<\lambda}\lambda$. The *generalized Baire space for* λ is the set ${}^{\lambda}\lambda$ of functions $f\colon \lambda \to \lambda$ with the standard topology.

Since we will work with definable subsets of ${}^{\lambda}\lambda$, we will use the following notation.

Definition 1.10. If $\varphi(x, y)$ is a formula with the two free variables x, y and z is a set, let

$$A^{\lambda}_{\varphi,z} = \{x \in {}^{\lambda}\lambda \mid \varphi(x, z)\}.$$

If $\varphi(x)$ is a formula with the free variable x, let

$$A^{\lambda}_{\varphi} = \{x \in {}^{\lambda}\lambda \mid \varphi(x)\}.$$

[1] Infinitely many Woodin cardinals are sufficient by [MS89].

The following definition generalizes perfect trees and perfect sets to the uncountable setting.

Definition 1.11. Suppose that T is a subtree of ${}^{<\lambda}\lambda$, that is a downwards closed subset of ${}^{<\lambda}\lambda$.

(a) $\mathrm{pred}_T(t) = \{s \in T \mid s \subsetneq t\}$ and $\mathrm{l}(t) = \mathrm{dom}(t)$.
(b) A node s in T is *terminal* if it has no direct successors in T and *splitting* if it has at least two direct successors in T.
(c) A *branch* in T is a sequence $b \in {}^{\lambda}\lambda$ with $b{\restriction}\alpha \in T$ for all $\alpha < \lambda$.
(d) The *body* of T is the set $[T]$ of branches in T.
(e) T is *closed* (*$<\lambda$-closed*) if every strictly increasing sequence in T of length $<\lambda$ has an upper bound in T.
(f) T is *perfect* (*λ-perfect*) if T is closed and the set of splitting nodes in T is cofinal in the tree order of T, that is, above every node there is some splitting node.
(g) A subset A of ${}^{\lambda}\lambda$ is *perfect* (*λ-perfect*, *superclosed*) if $A = [T] = \{x \in {}^{\lambda}\lambda \mid \forall\alpha < \lambda (x{\restriction}\alpha \in T)\}$ for some perfect tree T.
(h) A subset A of ${}^{\lambda}\lambda$ has the *perfect set property* if $|A| \leq \lambda$ or A has a perfect subset.

Väänänen [Vää91, Section 2] introduced a different notion of λ-perfect sets based on a game of length λ. We will see in Section 2 that the perfect set property associated to this notion is equivalent to our definition. Moreover, Kanamori [Kan80] introduced a variant of Sacks forcing for λ, leading to a corresponding stronger notion of perfect sets (see also [FKK16]), but our results do not hold for this notion.

In the following definition, a *λ-algebra* of subsets of ${}^{\lambda}\lambda$ is a set of subsets of ${}^{\lambda}\lambda$ that is closed under complements, unions of length λ and intersections of length λ.

Definition 1.12. Suppose that A, B are subsets of ${}^{\lambda}\lambda$.

(a) A is *λ-Borel* (*Borel*) if it is an element of the smallest λ-algebra containing the open subsets of ${}^{\lambda}\lambda$.
(b) A is *λ-meager* (*meager*) *in* B if $A \cap B$ is the union of λ many nowhere dense subsets of B, and *λ-comeager* (*comeager*) *in* B if its complement is λ-meager in B. Moreover, we will omit B if it is equal to ${}^{\lambda}\lambda$.
(c) A is *λ-Baire* (*Baire*) if there is an open subset U of ${}^{\lambda}\lambda$ such that $A \triangle U$ is λ-meager.

1.2. **Forcings.** A forcing $\mathbb{P} = (P, \leq)$ consists of a set P and a transitive reflexive relation (also called a pre-order) $\leq$ with domain P. We will also write $p \in \mathbb{P}$ for conditions $p \in P$ by identifying $\mathbb{P}$ with its domain. If $\mathbb{P}$ is a separative partial order, we will assume that $\mathbb{B}(\mathbb{P})$ denotes a fixed Boolean completion such that $\mathbb{P}$ is a dense subset of $\mathbb{B}(\mathbb{P})$.

Definition 1.13. (a) An *atom* in a forcing $\mathbb{P}$ is a condition $p \in \mathbb{P}$ with no incompatible extensions. Moreover, a forcing $\mathbb{P}$ is *non-atomic* if it has no atoms.
(b) A forcing $\mathbb{P}$ is *homogeneous* if for all $p, q \in \mathbb{P}$, there is an automorphism $\pi: \mathbb{P} \to \mathbb{P}$ such that $\pi(p)$ and q are compatible.

The sub-equivalence in the next definition is stronger than the standard notion of equivalence for separative partial orders, which states that the Boolean completions are isomorphic. This specific definition is used in several constructions in the proofs below.

Definition 1.14. Suppose that $\mathbb{P}$, $\mathbb{Q}$ are forcings.

(a) A *sub-isomorphism* $\iota: \mathbb{P} \to \mathbb{Q}$ is an isomorphism between $\mathbb{P}$ and a dense subset of $\mathbb{Q}$.
(b) $\mathbb{P}$, $\mathbb{Q}$ are *sub-equivalent* ($\mathbb{P} \approx \mathbb{Q}$) if there are sub-isomorphisms $\iota: \mathbb{R} \to \mathbb{P}$, $\nu: \mathbb{R} \to \mathbb{Q}$ for some forcing $\mathbb{R}$.
(c) $\mathbb{P}$, $\mathbb{Q}$ are *equivalent* ($\mathbb{P} \simeq \mathbb{Q}$) if there are sub-isomorphisms $\iota: \mathbb{P} \to \mathbb{R}$, $\nu: \mathbb{Q} \to \mathbb{R}$ for some forcing $\mathbb{R}$.
(d) $\mathbb{P}$, $\mathbb{Q}$ are *isomorphic* ($\mathbb{P} \cong \mathbb{Q}$) if there is an isomorphism $\iota: \mathbb{P} \to \mathbb{Q}$.
(e) if $\iota: \mathbb{P} \to \mathbb{Q}$ is a sub-isomorphism, we define a $\mathbb{P}$-name τ^ι for each $\mathbb{Q}$-name τ by induction on the rank as
$$\tau^\iota = \{(\sigma^\iota, p) \mid p \in \mathbb{P},\ \exists q \in \mathbb{Q}\ (\sigma, q) \in \tau,\ \iota(p) \leq q\}.$$

It is easy to check that in Definition 1.14 (e), for any $\mathbb{P}$-generic filter G and for the upwards closure H of $\iota[G]$ in $\mathbb{Q}$, $(\tau^\iota)^G = \tau^H$.

Lemma 1.15. *Suppose that $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$ are forcings.*

1. *If $\iota: \mathbb{P} \to \mathbb{Q}$, $\nu: \mathbb{P} \to \mathbb{R}$ are sub-isomorphisms, then there is a partial order $\mathbb{S}$ and isomorphisms onto dense subforcings $\iota^*: \mathbb{Q} \to \mathbb{S}$, $\nu^*: \mathbb{R} \to \mathbb{S}$ with $\iota^* \iota = \nu^* \nu$.*
2. *If $\mathbb{P} \approx \mathbb{Q}$, then $\mathbb{P} \simeq \mathbb{Q}$.*
3. *The relation $\simeq$ is transitive.*

Proof. For the first claim, let $\leq_{\mathbb{P}}$, $\leq_{\mathbb{Q}}$, $\leq_{\mathbb{R}}$ be the given forcing preorders. We can assume that $\mathbb{Q}$, $\mathbb{R}$ are disjoint and let $S = \mathbb{Q} \cup \mathbb{R}$. Moreover, we define the relation $\leq_S$ on S by $u \leq_S v$ if $u \leq_{\mathbb{Q}} v$, $u \leq_{\mathbb{R}} v$ or for some $p \in \mathbb{P}$,

$$(u \leq_{\mathbb{Q}} \iota(p) \text{ and } \nu(p) \leq_{\mathbb{R}} v) \text{ or } (u \leq_{\mathbb{R}} \nu(p) \text{ and } \iota(p) \leq_{\mathbb{Q}} v).$$

It is then easy to check that $\leq_S$ is transitive and reflexive, $\leq_S \upharpoonright \mathbb{Q} = \leq_{\mathbb{Q}}$ and ι, ν are sub-isomorphisms into S.

We now let $u \equiv_S v$ if $u \leq_S v$ and $v \leq_S u$ and let $\mathbb{S}$ denote the poset that is obtained as a quotient of S by $\equiv_S$ with the partial order induced by $\leq_S$. Let further $\iota^*: \mathbb{Q} \to \mathbb{S}$, $\iota(q) = [q]$ and $\nu^*: \mathbb{R} \to \mathbb{S}$, $\nu^*(r) = [r]$, where $[p]$ denotes the equivalence class of $p \in S$ with respect to $\equiv_S$. By the definitions, ι^*, ν^* are sub-isomorphisms into $\mathbb{S}$ that commute in the required fashion.

Moreover, this immediately implies the second claim.

For the last claim, suppose that $\mathbb{P} \simeq \mathbb{Q}$ and $\mathbb{Q} \simeq \mathbb{R}$ are witnessed by sub-isomorphisms $\iota: \mathbb{P} \to \mathbb{S}$, $\lambda: \mathbb{Q} \to \mathbb{S}$, $\mu: \mathbb{Q} \to \mathbb{T}$ and $\nu: \mathbb{R} \to \mathbb{T}$. By the first claim, there is a partial order $\mathbb{U}$ and sub-isomorphisms $\lambda^*: \mathbb{S} \to \mathbb{U}$, $\mu^*: \mathbb{T} \to \mathbb{U}$ with $\lambda^* \lambda = \mu^* \mu$. Then $\lambda^* \iota$, $\mu^* \nu$ witness that $\mathbb{P} \simeq \mathbb{R}$. □

Definition 1.16. Suppose that $\mathbb{P}$, $\mathbb{Q}$ are forcings.

(a) A *complete subforcing* $\mathbb{P}$ *of* $\mathbb{Q}$ ($\mathbb{P} \lessdot \mathbb{Q}$) is a subforcing of $\mathbb{Q}$ such that every maximal antichain in $\mathbb{P}$ is maximal in $\mathbb{Q}$.

(b) A *complete embedding* $i: \mathbb{P} \to \mathbb{Q}$ is a homomorphism with respect to $\leq$ and $\perp$ with the property that for every $q \in \mathbb{Q}$, there is a condition $p \in \mathbb{P}$ (called a *reduction of* q) such that for every $r \leq p$ in $\mathbb{P}$, $i(r)$ is compatible with q.

(c) Suppose that $i: \mathbb{P} \to \mathbb{Q}$ is a complete embedding and G is $\mathbb{P}$-generic over V. The *quotient forcing* $\mathbb{Q}/G$ *for* G *in* $\mathbb{Q}$ is defined as the subforcing

$$\mathbb{Q}/G = \{q \in \mathbb{Q} \mid \forall p \in G \ i(p) \parallel q\}$$

of $\mathbb{Q}$. Moreover, we fix a $\mathbb{P}$-name $\mathbb{Q}/\mathbb{P}$ for for the quotient forcing for $\mathbb{P}$ in $\dot{G}$, where $\dot{G}$ is a $\mathbb{P}$-name for the $\mathbb{P}$-generic filter, and also refer to this as (a name for) the quotient forcing for $\mathbb{P}$ in $\mathbb{Q}$.

It is a standard fact that a subforcing $\mathbb{P}$ of $\mathbb{Q}$ is a complete subforcing if and only if the identity on $\mathbb{P}$ is a complete embedding.

Definition 1.17. (see [Abr83, Definition 0.1], [Cum10, Definition 5.2]) Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are forcings.

(a) A *projection* $\pi: \mathbb{Q} \to \mathbb{P}$ is a homomorphism with respect to $\leq$ such that $\pi[\mathbb{Q}]$ is dense in $\mathbb{P}$ and for all $q \in \mathbb{Q}$ and all $p \leq \pi(q)$, there is a condition $\bar{q} \leq q$ with $\pi(\bar{q}) \leq p$.

(b) Suppose that $\pi: \mathbb{Q} \to \mathbb{P}$ is a projection and G is a $\mathbb{P}$-generic filter over V. The *quotient forcing* $\mathbb{Q}/G$ *for* G *in* $\mathbb{Q}$ *relative to* π is defined as the subforcing

$$\mathbb{Q}/G = \{q \in \mathbb{Q} \mid \pi(q) \in G\}$$

of $\mathbb{Q}$. Moreover, we fix a $\mathbb{P}$-name $(\mathbb{Q}/\mathbb{P})^\pi$ for the quotient forcing for $\dot{G}$ in $\mathbb{Q}$ relative to π, where $\dot{G}$ is a $\mathbb{P}$-name for the $\mathbb{P}$-generic filter, and will refer to this as (a name for) the quotient forcing for $\mathbb{P}$ in $\mathbb{Q}$ relative to π.

In Definition 1.17, by standard facts about quotient forcing, for any $\mathbb{P}$-generic filter G over V, any $\mathbb{Q}/G$-generic filter H over $V[G]$ is $\mathbb{Q}$-generic over V. Moreover, any $\mathbb{Q}$-generic filter H over V induces the $\mathbb{P}$-generic filter $G = \pi[H]$ over V and H is $[(\mathbb{Q}/\mathbb{P})^\pi]^H$-generic over $V[G]$ with $V[H] = V[G * H]$. Assuming that $\mathbb{P}$ and $\mathbb{Q}$ have weakest elements $\mathbb{1}_{\mathbb{P}}$ and $\mathbb{1}_{\mathbb{Q}}$, respectively, it is easy to see that the condition that $\pi[\mathbb{Q}]$ is dense in $\mathbb{P}$ in Definition 1.17 is equivalent to the condition that $\pi(\mathbb{1}_{\mathbb{Q}}) = \mathbb{1}_{\mathbb{P}}$.

It is easy to check that the following map is actually a projection.

Definition 1.18. Suppose that $\mathbb{P}$, $\mathbb{Q}$ are complete Boolean algebras and $\mathbb{P}$ is a complete subalgebra of $\mathbb{Q}$. We define the *natural projection* $\pi{:}\,\mathbb{Q} \to \mathbb{P}$ by

$$\pi(q) = \inf_{p \in \mathbb{P},\ p \geq q} p.$$

We will further use the following notation when working with quotient forcings induced by names.

Definition 1.19. If $\mathbb{P}$ is a complete Boolean algebra and σ is a $\mathbb{P}$-name for a subset of a set x, let $\mathbb{B}(\sigma) = \mathbb{B}^{\mathbb{P}}(\sigma)$ denote the complete Boolean subalgebra of $\mathbb{P}$ that is generated by the Boolean values $[\![y \in \sigma]\!]_{\mathbb{P}}$ for all $y \in x$. Moreover, we will also use this notation if σ is a name for a set that can be coded as a subset of a ground model set in an absolute way.

Moreover, we will often add Cohen subsets to a regular cardinal κ with $\kappa^{<\kappa} = \kappa$. The following definition of the forcing for adding Cohen subsets is non-standard, but essential in several proofs below. In the following definitions, let Succ denote the class of successor ordinals.

Definition 1.20. Suppose that λ is a regular uncountable cardinal.

(a) $\mathrm{Add}(\lambda, 1)$ is defined as the forcing

$$\mathrm{Add}(\lambda, 1) = \{p{:}\,\alpha \to \lambda \mid \alpha < \lambda\},$$

ordered by reverse inclusion.

(b) $\mathrm{Add}^*(\lambda, 1)$ is defined as the dense subforcing

$$\mathrm{Add}^*(\lambda, 1) = \{p \in \mathrm{Add}(\lambda, 1) \mid \mathrm{dom}(p) \in \mathrm{Succ}\}$$

of $\mathrm{Add}(\lambda, 1)$.

(c) $\mathrm{Add}(\lambda, \gamma)$ is defined as the $<\lambda$-support product $\prod_{i<\gamma} \mathrm{Add}(\lambda, 1)$ for any ordinal γ.

We will often use the following standard facts about adding Cohen subsets and collapse forcings.

Lemma 1.21. *Suppose that λ is a regular uncountable cardinal.*

(1) *If $\lambda^{<\lambda} = \lambda$ and $\mathbb{P}$ is a non-atomic $<\lambda$-closed forcing of size λ, then $\mathbb{P}$ has a dense subset that is isomorphic to* $\mathrm{Add}^*(\lambda, 1)$. *In particular, $\mathbb{P}$ is sub-equivalent to* $\mathrm{Add}(\lambda, 1)$.

(2) [Fuc08, Lemma 2.2] *Suppose that $\nu > \lambda$ is a cardinal with $\nu^{<\lambda} = \nu$, $\mathbb{P}$ is a separative $<\lambda$-closed forcing of size ν and $\mathbb{1}_{\mathbb{P}}$ forces that ν has size λ. Then $\mathbb{P}$ has a dense subset that is isomorphic to the dense subforcing*

$$\mathrm{Col}^*(\lambda, \nu) = \{p \in \mathrm{Col}(\lambda, \nu) \mid \mathrm{dom}(p) \in \mathrm{Succ}\}$$

of $\mathrm{Col}(\lambda, \nu)$. *In particular, $\mathbb{P}$ is sub-equivalent to* $\mathrm{Col}(\lambda, \nu)$.

Proof. Since the proof of the first claim is straightforward and well-known, we do not include it here. The proof of the second claim is an adaptation of the proof of [Jec03, Lemma 26.7] that can be found in [Fuc08, Lemma 2.2]. □

1.3. **A counterexample for quotient forcings.** The following well-known example of a quotient of $\mathrm{Add}(\kappa, 1)$ that does not preserve stationary subsets of κ shows that the proofs of regularity properties for definable subsets of ${}^{\omega}\omega$ in Solovay's model do not generalize to any uncountable regular cardinal.

For any uncountable regular cardinal κ with $\kappa^{<\kappa} = \kappa$, we define a complete subforcing of the Boolean completion $\mathbb{B}(\mathrm{Add}(\kappa, 1))$ of $\mathrm{Add}(\kappa, 1)$ such that its quotient forcing in $\mathbb{B}(\mathrm{Add}(\kappa, 1))$ does not preserve stationary subsets of κ.

For any condition $p \in \mathrm{Add}(\kappa, 1)$, we consider the set $s_p = \{\alpha \in \mathrm{dom}(p) \mid p(\alpha) \neq 0\}$. Suppose that $\dot{G}$ is an $\mathrm{Add}(\kappa, 1)$-name for the generic filter G and $\dot{S}$ is an $\mathrm{Add}(\kappa, 1)$-name for $\bigcup_{p \in G} s_p$. Then $\mathbb{1}_{\mathrm{Add}(\kappa,1)}$ forces that $\dot{S}$ is a bi-stationary subset of κ. Moreover, if S is a subset of κ, we define

$$\mathbb{Q}_S = \{p \in \mathrm{Add}(\kappa, 1) \mid s_p \text{ is a closed subset of } S\}.$$

Lemma 1.22. *Suppose that $\kappa^{<\kappa} = \kappa$ and $\dot{\mathbb{Q}}$ is an* $\mathrm{Add}(\kappa, 1)$*-name for $\mathbb{Q}_{\dot{S}}$, where $\dot{S}$ is defined as above. Then* $\mathrm{Add}(\kappa, 1) * \dot{\mathbb{Q}}$ *is sub-equivalent to* $\mathrm{Add}(\kappa, 1)$.

Proof. The set

$$\{(p,\check{q}) \mid p,q \in \mathrm{Add}(\kappa,1),\ \mathrm{dom}(p)=\mathrm{dom}(q),\ p \Vdash_{\mathrm{Add}(\kappa,1)} \check{q} \in \dot{\mathbb{Q}}\}$$

is a non-atomic $<\kappa$-closed dense subset of $\mathrm{Add}(\kappa,1) * \mathbb{Q}$, hence $\mathrm{Add}(\kappa,1) * \mathbb{Q}$ is sub-equivalent to $\mathrm{Add}(\kappa,1)$ by Lemma 1.21. □

It is forced by $\mathbb{1}_{\mathbb{P}}$ that $\dot{\mathbb{Q}}$ shoots a club through $\dot{S}$ and hence $\dot{\mathbb{Q}}$ is not stationary set preserving. Thus, $\dot{\mathbb{Q}}$ is a name for the required quotient forcing. In particular, such a forcing fails to be $<\kappa$-closed.

Now suppose that κ is an uncountable regular cardinal and $\lambda > \kappa$ is inaccessible. An argument analogous to the proof of [Sol70, Theorem 1] shows that after forcing with $\mathrm{Col}(\kappa,<\lambda)$, all $\mathbf{\Sigma}^1_1$ subsets of ${}^\kappa\kappa$ have the perfect set property. However, this proof fails to work for $\mathbf{\Pi}^1_1$ subsets of ${}^\kappa\kappa$ precisely because some quotient forcings, such as the ones appearing in Lemma 1.22, are not necessarily $<\kappa$-closed.

Remark 1.23. In the situation of Lemma 1.22, $\mathbb{1}_{\mathbb{P}}$ forces that $\mathbb{Q}_{\dot{S}}$ is $<\kappa$-distributive, since it appears in a two-step iteration which is $<\kappa$-distributive. However, in general one needs to require more conditions on S to ensure that $\mathbb{Q}_S$ is $<\kappa$-distributive. For instance, assuming that the GCH holds, it is sufficient that S is a *fat stationary subset of* κ in the sense that for every club C in κ, $S \cap C$ contains closed subsets of arbitrarily large order types below κ (see [AS83, Theorem 1 & Theorem 2]).

2. The perfect set property

We always assume that κ is an uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$ and that λ is an uncountable regular cardinal. We define the *length* of various types of objects in the next definition.

Definition 2.1. (a) Let $\mathrm{l}(s) = \mathrm{dom}(s)$ for any function s.
(b) Let $\mathrm{l}(t) = \sup_{s\in t} \mathrm{l}(s)$ for $t \subseteq {}^{<\lambda}\lambda$.
(c) Let $\mathrm{l}(p) = \mathrm{l}(t)$ for $p = (t,s)$ and $t,s \subseteq {}^{<\lambda}\lambda$.

2.1. **Perfect set games.** The perfect set property is characterized by the *perfect set game.*

Definition 2.2. The *perfect set game* $F_\lambda(A)$ of length λ for a subset A of ${}^\lambda 2$ is defined as follows. The first (even) player, player I, plays some $s_\alpha \in {}^{<\lambda}2$ in all even rounds α. The second (odd) player, player II, plays some $s_\alpha \in {}^{<\lambda}2$ in all odd rounds α. Together, they play a strictly increasing sequence $\vec{s} = \langle s_\alpha \mid \alpha < \lambda\rangle$ with $s_\alpha \in {}^{<\lambda}2$ for all $\alpha < \lambda$. Player II has to satisfy the additional requirement that $\mathrm{l}(s_{\alpha+1}) = \mathrm{l}(s_\alpha)+1$ for all even ordinals $\alpha < \lambda$. The combined sequence $\vec{s}$ of moves of both players defines a sequence

$$\bigcup_{\alpha<\lambda} s_\alpha = x = \langle x(i) \mid i < \lambda\rangle \in {}^\lambda 2.$$

Player I wins if $x \in A$. Moreover, if $t \in {}^{<\lambda}2$, the game $F^t_\lambda(A)$ is defined as $F_\lambda(A)$ with the additional requirement that $t \subseteq s_0$ for the first move s_0 of player I.

The perfect set game characterizes the perfect set property for subsets of ${}^\lambda 2$ in the following sense.

Lemma 2.3. [Kov09, Lemma 7.2.2] *Suppose that A is a subset of ${}^\lambda 2$ and $t \in {}^{<\lambda}2$.*
(1) *Player I has a winning strategy in $F^t_\lambda(A)$ if and only if $A \cap N_t$ has a perfect subset.*
(2) *Player II has a winning strategy in $F^t_\lambda(A)$ if and only if $|A \cap N_t| \leq \lambda$.*

The perfect set property is equivalent to the following variant defined in [Vää91, Section 2].

Definition 2.4. The game $V_\lambda(A)$ of length λ for a subset A of ${}^\lambda 2$ is defined as follows. The first (even) player, player I, plays an ordinal α_i in all even rounds i. The second (odd) player, player II, plays an element x_i of A in all odd rounds i. Moreover, the sequence $\langle \alpha_i \mid i < \lambda\rangle$ of moves of player I has to be continuous. Player II wins if for all $i < j < \lambda$, $x_i \restriction \alpha_i = x_j \restriction \alpha_i$ and $x_i \neq x_j$.

Lemma 2.5. *Suppose that A is a subset of ${}^\lambda\lambda$. Then A has a perfect subset if and only there is a closed subset C of A such that player II has a winning strategy in $V_\lambda(C)$.*

Proof. If A has a perfect subset C, then it is straightforward to define a winning strategy for player II in $V_\lambda(C)$.

Now suppose that C is a closed subset of A and that player II has a winning strategy σ in $V_\lambda(C)$. Using σ, we can inductively construct $\langle x_s, t_s, \gamma_s \mid s \in {}^{<\lambda}2\rangle$ such that the following conditions hold for all $r, s \in {}^{<\lambda}2$.

(1) (a) $t_r \subsetneq t_s$ if $r \subsetneq s$.
   (b) $t_{r^\frown\langle 0\rangle} \perp t_{r^\frown\langle 1\rangle}$.
   (c) $t_s = \bigcup_{u\subsetneq s} t_u$ if $\mathrm{l}(s)$ is a limit.

(2) $\mathrm{l}(t_s) = \gamma_s$.

(3) $t_s \subseteq x_s$.

(4) Let c_s denote the closure of the set $\{\alpha < \mathrm{l}(s) \mid \exists\bar\alpha\ \alpha = \bar\alpha + 1,\ s(\bar\alpha) = 1\}$ and $\pi{:}\, c_s \to \delta_s$ its transitive collapse. Then

$$\langle \gamma_{s\restriction\pi^{-1}(\alpha)}, x_{s\restriction\pi^{-1}(\alpha)} \mid \alpha < \delta_s\rangle$$

is a partial run of $V_\lambda(C)$ according to σ.

The last condition ensures the existence of partial runs that split exactly at the times α where s has successor length α and the last value 1 and at the limits of such times. In particular, whenever $\alpha < \delta_s$, $\pi^{-1}(\alpha) = \bar\alpha + 1$ and $s(\bar\alpha) = 1$, the partial run for s is extended by player I playing an ordinal $\gamma_{s\restriction(\bar\alpha+1)}$ and player II responding with an element $x_{s\restriction(\bar\alpha+1)}$ of A that splits from $x_{s\restriction\bar\alpha}$, and whenever $s(\bar\alpha) = 0$, the partial run for s is not extended.

We thus obtain a perfect tree $T = \{t \in {}^{<\lambda}2 \mid \exists s \in {}^{<\lambda}2\ t \subseteq t_s\}$. Since C is closed, it follows from the construction that $[T] \subseteq C$, proving the claim. □

2.2. **The perfect set property for definable sets.** We will show that forcing with $\mathrm{Add}(\kappa, 1)$ adds a perfect set of $\mathrm{Add}(\kappa, 1)$-generic elements of ${}^\kappa\kappa$ whose quotient forcings are sub-equivalent to $\mathrm{Add}(\kappa, 1)$. More precisely, each of these elements will have an $\mathrm{Add}(\kappa, 1)$-name that generates a complete subalgebra of $\mathbb{B}(\mathrm{Add}(\kappa, 1))$ whose quotient forcing in $\mathbb{B}(\mathrm{Add}(\kappa, 1))$ is sub-equivalent to $\mathrm{Add}(\kappa, 1)$.

This will be proved by considering the following forcing $\mathbb{P}$. The forcing adds a perfect subtree of ${}^{<\kappa}\kappa$ by approximations of size $<\kappa$.

Definition 2.6. Let $\mathbb{P}$ denote the set of pairs (t, s) such that

(a) $t \subseteq {}^{<\kappa}\kappa$ is a tree of size $<\kappa$,

(b) every node $u \in t$ has at most two direct successors in t,

(c) $s \subseteq t$ and if $u \in t$ is non-terminal in t, then $u \in s$ if and only if u has exactly one successor in t.

Let $(t, s) \leq (u, v)$ if $u \subseteq t$ and $s \cap u = v$.

The set s marks the non-branching nodes in the tree. It follows from the definition of $\mathbb{P}$ that the forcing adds a perfect binary splitting subtree of ${}^{<\kappa}\kappa$. Since every decreasing sequence of length $< \kappa$ in $\mathbb{P}$ has an infimum, $|\mathbb{P}| = \kappa$ and $\mathbb{P}$ is non-atomic, the forcing is sub-equivalent to $\mathrm{Add}(\kappa, 1)$ by Lemma 1.21.

In the remainder of this section, we write $T_G = \bigcup_{(t,s)\in G} t$ if G is a $\mathbb{P}$-generic filter over V.

Lemma 2.7. *Suppose that G is $\mathbb{P}$-generic over V and $T = T_G$. Then $V[G] = V[T]$.*

Proof. Since $T \in V[G]$, it is sufficient to show that $G \in V[T]$. Since G is generic, for all $(t, s) \in \mathbb{P}$, $(t, s) \in G$ if and only if (t, s) is compatible with all conditions in G. Hence the elements (t, s) of G are exactly the pairs (t, s) such that $s \subseteq t \subseteq T$ and s is the set of $u \in t$ such that u has exactly one direct successor in T. Hence $G \in V[T]$. □

If $b = \cup g$ for some $\mathrm{Add}(\kappa, 1)$-generic filter g over V as in the next lemma, we will also say that b is $\mathrm{Add}(\kappa, 1)$-generic over V.

Lemma 2.8. *Suppose that G is $\mathbb{P}$-generic and b, c are distinct branches in $T = T_G$. Then there is an $\mathrm{Add}(\kappa, 1) \times \mathrm{Add}(\kappa, 1)$-generic filter $g \times h$ over V in $V[T]$ such that $b = \bigcup g$ and $c = \bigcup h$.*

Proof. Suppose that b, c are distinct branches in T and σ, τ are $\mathbb{P}$-names for b, c in the sense that $\sigma^G = b$ and $\tau^G = c$. Moreover, let $\dot T$ be a $\mathbb{P}$-name for T. Then there is a condition $p_0 \in G$ with

$p_0 \Vdash_{\mathbb{P}} \sigma, \tau \in [\dot{T}]$ and $p_0 \Vdash_{\mathbb{P}} \sigma \neq \tau$. We can assume that $p_0 = \mathbb{1}_{\mathbb{P}}$ by replacing σ, τ with names that satisfy these conditions for $p_0 = \mathbb{1}_{\mathbb{P}}$.

Now suppose that D is a dense open subset of $\mathrm{Add}(\kappa, 1) \times \mathrm{Add}(\kappa, 1)$ and let

$$E = \{q \in \mathbb{P} \mid \exists (u, v) \in D,\ q \Vdash_{\mathbb{P}} u \subseteq \sigma,\ v \subseteq \tau\}.$$

Claim. *E is dense.*

Proof. Suppose that $p \in \mathbb{P}$. Since $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \sigma \neq \tau$, we can assume by extending p that for some $\alpha < \mathrm{l}(p)$, p decides $\sigma(\alpha)$, $\tau(\alpha)$ and these values are different. We let $q_0 = p$ and choose successively for each $n \in \omega$ an extension $q_{n+1} \leq q_n$ such that $\mathrm{l}(q_n) < \mathrm{l}(q_{n+1})$ and q_{n+1} decides both $\sigma{\upharpoonright}\mathrm{l}(q_n)$ and $\tau{\upharpoonright}\mathrm{l}(q_n)$. Finally, let $q = \inf_{n \in \omega} q_n$ and suppose that $q = (t_q, s_q)$. Since the lengths $\mathrm{l}(q_n)$ form a strictly increasing sequence, $\gamma = \mathrm{l}(q) = \sup_{n \in \omega} \mathrm{l}(q_n)$ is a limit ordinal. Moreover, by the choice of the sequence of conditions, there are $u, v \in {}^{\gamma}\kappa$ with $u \neq v$ and $q \Vdash_{\mathbb{P}} \sigma{\upharpoonright}\gamma = u,\ \tau{\upharpoonright}\gamma = v$.

We first claim that $(u{\upharpoonright}\alpha),\ (v{\upharpoonright}\alpha) \in t_q$ for all $\alpha < \gamma$. It is sufficient to prove that $(u{\upharpoonright}\alpha) \in t_q$ for all $\alpha < \gamma$ by symmetry. To see this, suppose towards a contradiction that $u{\upharpoonright}\alpha \notin t_q$ for some $\alpha < \gamma$. Suppose that α is minimal. We extend $q = (t_q, s_q)$ to $r = (t_r, s_r)$ as follows. We choose $\beta < \kappa$ with $u(\alpha) \neq \beta$ and let $t_r = t_q \cup \{u{\upharpoonright}\alpha, (u{\upharpoonright}\alpha)^\frown\langle\beta\rangle\}$ and $s_r = s_q \cup \{u{\upharpoonright}\alpha\}$. Then $r \Vdash_{\mathbb{P}} u(\alpha) = \beta$ and hence $r \Vdash_{\mathbb{P}} u \notin \sigma$, contradicting the fact that $q \Vdash_{\mathbb{P}} \sigma{\upharpoonright}\gamma = u$ by the choice of q and u. This shows that $u{\upharpoonright}\alpha \in t_q$ for all $\alpha < \gamma$.

Since D is dense in $\mathrm{Add}(\kappa, 1) \times \mathrm{Add}(\kappa, 1)$, there are conditions $\bar{u} \leq u$, $\bar{v} \leq v$ with $(\bar{u}, \bar{v}) \in D$. Since D is open, we can assume that $\mathrm{l}(\bar{u}) = \mathrm{l}(\bar{v}) = \delta$ for some limit ordinal δ with $\gamma < \delta < \kappa$. Now let

$$x = \{\bar{u}{\upharpoonright}\eta \mid \gamma \leq \eta < \delta\} \cup \{\bar{v}{\upharpoonright}\eta \mid \gamma \leq \eta < \delta\}.$$

Moreover, let $\bar{t} = t_q \cup x$, $\bar{s} = s_q \cup x$ and $r = (\bar{t}, \bar{s})$. Then $r \in \mathbb{P}$ and $r \leq p$.

Subclaim. *$r \Vdash_{\mathbb{P}} \bar{u} \subseteq \sigma,\ \bar{v} \subseteq \tau$.*

Proof. It is sufficient to prove $r \Vdash_{\mathbb{P}} \bar{u} \subseteq \sigma$ by symmetry. Since $r \leq q$ and $q \Vdash_{\mathbb{P}} \sigma{\upharpoonright}\gamma = u$ by the choice of u, we have $r \Vdash_{\mathbb{P}} \sigma{\upharpoonright}\gamma = u$. Since $u = \bar{u}{\upharpoonright}\gamma \in x \subseteq s_q \cup x = \bar{s}$ by the definition of x and $\bar{s}$ and since $r = (\bar{t}, \bar{s}) \in \mathbb{P}$, the node $u = \bar{u}{\upharpoonright}\gamma$ has the unique direct successor $\bar{u}{\upharpoonright}(\gamma + 1)$ in t. Hence $r \Vdash_{\mathbb{P}} \sigma{\upharpoonright}(\gamma+1) = \bar{u}{\upharpoonright}(\gamma+1)$. An analogous argument shows inductively that $r \Vdash_{\mathbb{P}} \sigma{\upharpoonright}(\eta+1) = \bar{u}{\upharpoonright}(\eta+1)$ for all η with $\gamma \leq \eta < \delta$. Hence $r \Vdash_{\mathbb{P}} \sigma{\upharpoonright}\delta = \bar{u}$. □

This implies that $r \leq p$ and $r \in E$, proving the claim. □

Let $g = \{s \in {}^{<\kappa}\kappa \mid s \subseteq b\}$, $h = \{s \in {}^{<\kappa}\kappa \mid s \subseteq h\}$. The previous claim implies that $g \times h$ is $\mathrm{Add}(\kappa, 1) \times \mathrm{Add}(\kappa, 1)$-generic over V. □

We obtain the same result for $<\kappa$ many branches in T_G.

Lemma 2.9. *Suppose that G is $\mathbb{P}$-generic and $\langle b_i \mid i < \gamma\rangle$ is a sequence of distinct branches in $T = T_G$ for some $\gamma < \kappa$. Then there is an $\mathrm{Add}(\kappa, \gamma)$-generic filter $\prod_{i<\gamma} g_i$ over V in $V[G]$ with $b_i = \bigcup g_i$ for all $i < \gamma$.*

Proof. The proof is as the proof of Lemma 2.8, but instead of working with names σ, τ for branches in T_G with $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \sigma \neq \tau$, we work with a sequence $\langle \sigma_i \mid i < \gamma\rangle$ of names for branches in T_G with $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \sigma_i \neq \sigma_j$ for all $i < j < \gamma$. □

We will show that for every branch b of $T = T_G$, the quotient forcing relative to a name for b is equivalent to $\mathrm{Add}(\kappa, 1)$. Suppose that $\dot{T}$ is a $\mathbb{P}$-name for T and $\dot{b}$ is a $\mathbb{P}$-name for a branch in $\dot{T}$, in the sense that these properties are forced by $\mathbb{1}_{\mathbb{P}}$. Moreover, if $p \in \mathbb{P}$, let $\dot{b}_p = \{(\alpha, \beta) \mid p \Vdash \dot{b}(\alpha) = \beta\}$.

Lemma 2.10. *If $p = (t, s) \in \mathbb{P}$ and $\gamma \subseteq \mathrm{dom}(\dot{b}_p)$, then*

(1) *$\dot{b}_p{\upharpoonright}\beta \in t$ for all $\beta < \gamma$, if γ is a limit, and*
(2) *$\dot{b}_p{\upharpoonright}\gamma \in t$ if γ is a successor.*

Proof. Suppose that γ is least such that the claim fails. First suppose that γ is a limit. In this case, we define $q = (u, v) \leq p$ by $u = t \cup \{\dot{b}_p{\upharpoonright}\gamma, (\dot{b}_p{\upharpoonright}\gamma)^\frown\langle\eta\rangle\}$ for some $\eta \neq \dot{b}_p(\gamma)$ and $v = s \cup \{\dot{b}_p{\upharpoonright}\gamma\}$. Then $q \Vdash_{\mathbb{P}} \dot{b}(\gamma) = \eta$, contradicting the definition of $\dot{b}_p$. Now suppose that γ is a successor. Then $\gamma = \beta + 1$ and $\dot{b}_p{\upharpoonright}\gamma = r^\frown\langle\alpha\rangle$ for some $r \in t$ with $\mathrm{l}(r) = \beta$. In particular, $p \Vdash_{\mathbb{P}} \dot{b}(\beta) = \alpha$. We distinguish two cases.

First suppose that $r \in s$. If r has a successor $r^\frown\langle\eta\rangle$ in t, then this successor is unique and $\alpha \neq \eta$, since we have $r^\frown\langle\alpha\rangle = \dot{b}_p{\upharpoonright}\gamma \notin t$ by the assumption on γ. Then $p \Vdash_{\mathbb{P}} \dot{b}(\beta) = \eta$, contradicting the fact that $p \Vdash_{\mathbb{P}} \dot{b}(\beta) = \alpha$. If r has no successor in t, let η be an ordinal below κ with $\eta \neq \alpha$. Let $u = t \cup \{r^\frown\langle\eta\rangle\}$, $v = s$ and $q = (u, s)$. Then $q \Vdash_{\mathbb{P}} \dot{b}(\beta) = \eta$, contradicting the fact that $p \Vdash_{\mathbb{P}} \dot{b}(\beta) = \alpha$.

Second, suppose that $r \notin s$. If r is non-terminal in t, then r has exactly two successors $r^\frown\langle\zeta\rangle$, $r^\frown\langle\eta\rangle$ in t with $\zeta, \eta \neq \alpha$. Then $p \Vdash_{\mathbb{P}} \dot{b}{\upharpoonright}\gamma \in t$, contradicting the fact that $p \Vdash_{\mathbb{P}} \dot{b}(\beta) = \alpha$. If r is terminal in t, let ζ, η be distinct ordinals below κ with $\zeta, \eta \neq \alpha$. Let $u = t \cup \{r^\frown\langle\zeta\rangle, r^\frown\langle\eta\rangle\}$, $v = s$ and $q = (u, s)$. Then $q \Vdash_{\mathbb{P}} \dot{b}(\beta) \neq \alpha$, contradicting the fact that $p \Vdash_{\mathbb{P}} \dot{b}(\beta) = \alpha$. □

Let $\mathbb{P}^*$ denote the set of conditions $p = (t, s) \in \mathbb{P}$ such that $\mathrm{l}(t)$ is a limit ordinal and $\mathrm{l}(\dot{b}_p) = \mathrm{l}(t)$.

Lemma 2.11. *$\mathbb{P}^*$ is dense in $\mathbb{P}$.*

Proof. Suppose that $p \in \mathbb{P}$ and let $p_0 = p = (t_0, s_0)$. We choose successively for each $n \in \omega$ a condition $p_{n+1} = (t_{n+1}, s_{n+1})$ that decides $\dot{b}{\upharpoonright}\mathrm{l}(t_n)$ with $p_{n+1} \leq p_n$ and $\mathrm{l}(t_n) < \mathrm{l}(t_{n+1})$. Let $t = \bigcup_{n\in\omega} t_n$, $s = \bigcup_{n\in\omega} s_n$ and $q = (t, s)$. By the construction, $\mathrm{l}(q)$ is a limit and $\mathrm{l}(q) \leq \mathrm{l}(\dot{b}_q)$. Moreover, we have $\mathrm{l}(b_q) \leq \mathrm{l}(q)$ by Lemma 2.10 and hence $q \in \mathbb{P}^*$. □

We will expand $\mathbb{P}$ to determine the quotient forcing in $V[G]$ for a branch $\dot{b}^G \in [T_G]$. The precise statement is given in Lemma 2.16 below.

Suppose that $\dot{b}$ is a $\mathbb{P}$-name for a branch in $T_{\dot{G}}$, where $\dot{G}$ is a name for the $\mathbb{P}$-generic filter, in the sense that this is forced by $\mathbb{1}_{\mathbb{P}}$. Let

$$\mathbb{Q} = \{(\dot{b}_p, q) \mid p \in \mathbb{P}^* \text{ and } (q = p \text{ or } q = 1_{\mathbb{P}})\}$$

and for all $(u, p), (v, q) \in \mathbb{P}$, let $(u, p) \leq (v, q)$ if $v \subseteq u$ and $p \leq_{\mathbb{P}} q$. Moreover, let

$$\mathbb{Q}_0 = \{(\dot{b}_p, \mathbb{1}_{\mathbb{P}}) \mid p \in \mathbb{P}^*\}$$

$$\mathbb{Q}_1 = \{(\dot{b}_p, p) \mid p \in \mathbb{P}^*\}.$$

Then $\mathbb{Q} = \mathbb{Q}_0 \cup \mathbb{Q}_1$, $\mathbb{Q}_1$ is a dense subforcing of $\mathbb{Q}$ and $\mathbb{Q}_0 \cap \mathbb{Q}_1$ contains at most $\mathbb{1}_{\mathbb{Q}}$. We further consider the map $e\colon \mathbb{P}^* \to \mathbb{Q}_1$, $e(p) = (\dot{b}_p, p)$. Since e an isomorphism, $\mathbb{P}^*$ is dense in $\mathbb{P}$ and $\mathbb{Q}_1$ is dense in $\mathbb{Q}$, it follows that the forcings $\mathbb{P}$, $\mathbb{Q}$ are sub-equivalent.

Lemma 2.12. *The map $\pi = \pi_{\mathbb{Q},\mathbb{Q}_0}\colon \mathbb{Q} \to \mathbb{Q}_0$, $\pi(\dot{b}_p, r) = (\dot{b}_p, \mathbb{1}_{\mathbb{P}})$ is a projection.*

Proof. By the definition, π is a homomorphism with respect to $\leq$ and it is surjective onto $\mathbb{Q}_0$.

To prove the remaining requirement for projections, first suppose that $u = (\dot{b}_p, p) \in \mathbb{Q}_1$ and $v = (\dot{b}_q, 1_{\mathbb{P}}) \in \mathbb{Q}_0$ are conditions with $v \leq \pi(u)$. In particular, $\dot{b}_p \subseteq \dot{b}_q$ and hence $\mathrm{l}(p) \leq \mathrm{l}(q)$. It is sufficient to show that u, v are compatible in $\mathbb{Q}$, since for any extension $w \leq u, v$, we have $\pi(w) \leq v$ by the definition of π and since $v \in \mathbb{Q}_0$.

To see that u, v are compatible, suppose that $p = (t, s)$. Since $p \in \mathbb{P}^*$, $\mathrm{l}(p)$ is a limit and $\dot{b}_p$ is cofinal in t by the definition of $\mathbb{P}^*$. Let

$$\bar{t} = t \cup \{\dot{b}_q \upharpoonright \alpha \mid \mathrm{l}(\dot{b}_p) \leq \alpha < \mathrm{l}(\dot{b}_q)\}$$

$$\bar{s} = s \cup \{\dot{b}_q \upharpoonright \alpha \mid \mathrm{l}(\dot{b}_p) \leq \alpha < \mathrm{l}(\dot{b}_q)\}$$

and $\bar{p} = (\bar{t}, \bar{s})$. Then $\bar{p} \in \mathbb{P}$ and $\bar{p} \leq p$. Moreover, it follows from Lemma 2.10 that $\dot{b}_q \subseteq \dot{b}_{\bar{p}}$.

We can choose a condition $r \leq \bar{p}$ with $r \in \mathbb{P}^*$, since $\mathbb{P}^*$ is dense in $\mathbb{P}$, and let $w = (\dot{b}_r, r) \in \mathbb{Q}_1$. Since $u = (\dot{b}_p, p)$ and $r \leq p$, we have $w \leq u$. Since $v = (\dot{b}_q, 1_{\mathbb{P}})$ and $\dot{b}_q \subseteq \dot{b}_{\bar{p}} \subseteq \dot{b}_r$, we have $w \leq v$, and in particular, u, v are compatible.

Second, suppose that $u = (\dot{b}_p, \mathbb{1}_{\mathbb{P}}) \in \mathbb{Q}_0$ and v is as above. Since $(\dot{b}_p, \mathbb{1}_{\mathbb{P}}) \leq u$, the required statement follows from the property of $(\dot{b}_p, p)$ that we just proved. □

Lemma 2.13. *$\mathbb{Q}_0$ is a complete subforcing of $\mathbb{Q}$.*

Proof. It is sufficient to show that every maximal antichain A in $\mathbb{Q}_0$ is maximal in $\mathbb{Q}$. Let

$$D_0 = \{p \in \mathbb{Q}_0 \mid \exists q \in A\ p \leq q\}$$

$$D = \{p \in \mathbb{Q} \mid \exists q \in A\ p \leq q\}.$$

It is sufficient to show that D is dense in $\mathbb{Q}$, since this implies that A is a maximal antichain in $\mathbb{Q}$. To see that D is dense, suppose that $u \in \mathbb{Q}$. If $u \in \mathbb{Q}_0$, then there is a condition $v \leq u$

in $D_0 \subseteq D$, since D_0 is dense in $\mathbb{Q}_0$ by the assumption that A is maximal in $\mathbb{Q}_0$. Now suppose that $u = (\dot{b}_p, p) \in \mathbb{Q}_1$. Since D_0 is dense in $\mathbb{Q}_0$, there is some $v = (\dot{b}_q, 1_{\mathbb{P}}) \in D_0$ with $\dot{b}_p \subseteq \dot{b}_q$. Since $v \leq \pi(u)$ and π is a projection by Lemma 2.12, there is some $w \leq u$ with $\pi(w) \leq v$. Then $w \leq \pi(w) \leq v \in D_0$ and hence $w \in D$ by the definition of D, proving that D is dense in $\mathbb{Q}$. □

Let $e : \mathbb{P}^* \to \mathbb{Q}_1$, $e(p) = (\dot{b}_p, p)$ be the isomorphism between $\mathbb{P}^*$ and $\mathbb{Q}_1$ that was given after the definition of $\mathbb{Q}$ above. If G is a $\mathbb{P}$-generic filter over V, then the upwards closure

$$H = \{q \in \mathbb{Q} \mid \exists p \in G \; e(p) \leq q\}$$

of $e[G]$ in $\mathbb{Q}$ is a $\mathbb{Q}$-generic filter over V. In the following, we will write $T_H = T_G$, where T_G is the perfect tree adjoined by G that is given after the definition of $\mathbb{P}$ above.

Since it is convenient to work with complete Boolean algebras, we will now check that $\mathbb{P}$ is separative.

Lemma 2.14. *$\mathbb{P}$ is a separative partial order.*

Proof. It is easy to see that $\mathbb{P}$ is a partial order. To show that $\mathbb{P}$ is separative, suppose that (t, s), (v, u) are conditions in $\mathbb{P}$ with $(t, s) \nleq (v, u)$.

We first assume that $v \subseteq t$. Then $s \cap v \neq u$. We claim that (t, s), (v, u) are already incompatible. Otherwise there is a common extension (y, x), so that $x \cap t = s$ and $y \cap v = u$. However, this implies that $s \cap v = (x \cap t) \cap v = x \cap v = u$, contradicting the fact that $s \cap v \neq u$.

We now assume that $v \nsubseteq t$ and choose some $w \in v \setminus t$. We can assume that (t, s), (v, u) are compatible, so that $(t \cup v, s \cup u)$ is a condition. We define $y \subseteq t \cup v$ by removing all nodes strictly above w. To define x, we first let $\bar{x} = (y \setminus \{w\}) \cap (s \cup u)$. Let $x = \bar{x}$ if $w \in u$ and $x = \bar{x} \cup \{w\}$ otherwise. The choice of x implies that (y, x), (v, u) are incompatible, since $w \in x \Leftrightarrow w \notin u$. This is sufficient, since $(y, x) \leq (t, s)$. □

Moreover, it follows from the previous lemma and Lemma 2.10 that $\mathbb{Q}$ is also a separative partial order.

If $\mathbb{R}$ is a complete Boolean algebra and σ is an $\mathbb{R}$-name for an element of ${}^{\kappa}\kappa$, as a special case of the notation given in Definition 1.19, we will write $\mathbb{B}(\sigma) = \mathbb{B}^{\mathbb{R}}(\sigma)$ for the complete Boolean subalgebra of $\mathbb{R}$ that is generated by the Boolean values $[\![\sigma(\alpha) = \beta]\!]_{\mathbb{R}}$ for ordinals $\alpha, \beta < \kappa$.

We will use the following terminology for quotient forcings relative to elements of ${}^{\kappa}\kappa$ in a generic extension.

Definition 2.15. Suppose that $\mathbb{R}$ is a separative forcing, $\mathbb{S}$ is any other forcing, G is $\mathbb{R}$-generic over V and $c \in V[G]$ is a set that can be coded as a subset of a ground model set in an absolute way. We say that *c has $\mathbb{S}$ as a quotient in $V[G]$* if there is a $\mathbb{R}$-name $\dot{c}$ with $\dot{c}^G = c$ such that for the $\mathbb{B}(\dot{c})$-generic filter $G_0 = G \cap \mathbb{B}(\dot{c})$, the quotient forcing $[\mathbb{B}(\mathbb{R})/\mathbb{B}(\dot{c})]^{G_0}$ is equivalent to $\mathbb{S}$ in $V[G_0]$.

Lemma 2.16. *$\mathbb{1}_{\mathbb{B}(\dot{b})}$ forces that the quotient forcing $\mathbb{B}(\mathbb{P})/\mathbb{B}(\dot{b})$ is sub-equivalent to* $\mathrm{Add}(\kappa, 1)$.

Proof. Let $\dot{b}_{\mathbb{Q}}$ denote the $\mathbb{Q}$-name induced by the $\mathbb{P}$-name $\dot{b}$ via the sub-isomorphism $e : \mathbb{P}^* \to \mathbb{Q}$ defined above. Since e induces an isomorphism $\mathbb{B}(\mathbb{P}) \cong \mathbb{B}(\mathbb{Q})$ on the Boolean completions, it is sufficient to prove the claim for $\mathbb{Q}$, $\dot{b}_{\mathbb{Q}}$ instead of $\mathbb{P}$, $\dot{b}$. Moreover, it follows from the definition of $\mathbb{Q}_0$ that $\mathbb{B}(\dot{b}_{\mathbb{Q}})$ is equal to the complete subalgebra of $\mathbb{B}(\mathbb{Q})$ generated by $\mathbb{Q}_0$. Since $\mathbb{Q}_0$ is a complete subforcing of $\mathbb{Q}$ by Lemma 2.13, it is therefore sufficient to prove that $\mathbb{Q}_0$ forces that the quotient forcing $\mathbb{Q}/\mathbb{Q}_0$ is equivalent to $\mathrm{Add}(\kappa, 1)$.

It follows from Lemma 2.8 that $\mathbb{Q}$ forces that there is an $\mathrm{Add}(\kappa, 1)$-generic filter over $V[\dot{b}_{\mathbb{Q}}]$ in $V[\dot{G}]$, where $\dot{G}$ is a name for the $\mathbb{Q}$-generic filter, and therefore $\mathbb{Q}$ forces that the quotient forcing $\mathbb{Q}/\mathbb{Q}_0$ is non-atomic.

We have that $\pi : \mathbb{Q} \to \mathbb{Q}_0$ is a projection (with $\pi \upharpoonright \mathbb{Q}_0 = \mathrm{id}_{\mathbb{Q}_0}$) by Lemma 2.12 and $\mathbb{Q}_0$ is a complete subforcing of $\mathbb{Q}$ by Lemma 2.13. Since moreover $\pi(q) \geq q$ for all $q \in \mathbb{Q}$, it is easy to check that $\mathbb{Q}_0$ forces that the quotient forcing $\mathbb{Q}/\mathbb{Q}_0$ given in Definition 1.16 and the quotient forcing $(\mathbb{Q}/\mathbb{Q}_0)^{\pi}$ with respect to π given in Definition 1.17 are equal. Hence we can consider $(\mathbb{Q}/\mathbb{Q}_0)^{\pi}$ instead of $\mathbb{Q}/\mathbb{Q}_0$.

Now suppose that G_0 is $\mathbb{Q}_0$-generic over V and $b = \dot{b}^{G_0}$. By the definition of the quotient forcing with respect to π in Definition 1.17, we have

$$[(\mathbb{Q}/\mathbb{Q}_0)^\pi]^{G_0} = \{(\dot{b}_p, q) \in \mathbb{Q} \mid \pi(\dot{b}_p, q) \in G_0\} = \{(\dot{b}_p, q) \in \mathbb{Q} \mid \dot{b}_p \subseteq b\}.$$

It follows from the definitions of $\mathbb{P}^*$ and $\mathbb{Q}$ that the last set in the equation is a $<\kappa$-closed subset of $\mathbb{Q}$. Since we already argued that the quotient forcing is non-atomic, it is sub-equivalent to $\mathrm{Add}(\kappa, 1)$ by Lemma 1.21. □

The next result shows that the statement of the previous lemma also holds for names for sequences of length $<\kappa$ of branches in T_G. For the statement of the result, we assume that $\gamma < \kappa$, $\dot{G}$ is a $\mathbb{P}$-name for the $\mathbb{P}$-generic filter and σ is a $\mathbb{P}$-name for a sequence of length γ of distinct branches in $T_{\dot{G}}$, in the sense that this is forced by $\mathbb{1}_{\mathbb{P}}$.

Lemma 2.17. $\mathbb{1}_{\mathbb{B}(\sigma)}$ *forces that the quotient forcing* $\mathbb{B}(\mathbb{P})/\mathbb{B}(\sigma)$ *is sub-equivalent to* $\mathrm{Add}(\kappa, 1)$.

Proof. The proof is analogous to the proof of Lemma 2.16, but instead of working with a name $\dot{b}$ for a branch in $T_{\dot{G}}$, we work with the name σ for a sequence of branches in $T_{\dot{G}}$. As in the definitions of $\mathbb{Q}$, $\mathbb{Q}_0$ before Lemma 2.12, we can define variants of these forcings with respect to σ instead of $\dot{b}$ and thus obtain the required properties as in the proofs of Lemma 2.12 and Lemma 2.13. □

The previous two lemmas imply that $\dot{b}^G$ and σ^G have $\mathrm{Add}(\kappa, 1)$ as a quotient in $V[G]$ for every $\mathbb{P}$-generic filter G over V.

Lemma 2.18. *Suppose that λ is an uncountable regular cardinal, $\mu > \lambda$ is inaccessible and G is $\mathrm{Add}(\lambda, 1)$-generic over V. Then in $V[G]$, there is a perfect subtree T of ${}^{<\lambda}\lambda$ such that for every $\gamma < \lambda$, every sequence $\langle x_i \mid i < \gamma\rangle$ of distinct branches of T is $\mathrm{Add}(\lambda, \gamma)$-generic over V and has $\mathrm{Add}(\lambda, 1)$ as a quotient in $V[G]$.*

Proof. Since $\mathbb{P}$ is sub-equivalent to $\mathrm{Add}(\lambda, 1)$, there is a $\mathbb{P}$-generic filter H over V with $V[G] = V[H]$. Let $C = [T_H]^{V[H]}$, where T_H is the tree given after Definition 2.6.

We first assume that $\gamma = 1$. By Lemma 2.8, every $x \in C$ is $\mathrm{Add}(\lambda, 1)$-generic over V and by Lemma 2.16, every $x \in C$ has $\mathrm{Add}(\lambda, 1)$ as a quotient in $V[G]$.

The proof is analogous for arbitrary $\gamma < \lambda$. By Lemma 2.9, any sequence $\vec{x} = \langle x_i \mid i < \gamma\rangle$ of distinct elements of C is $\mathrm{Add}(\lambda, \gamma)$-generic over V and by Lemma 2.17, $\vec{x}$ has $\mathrm{Add}(\lambda, 1)$ as a quotient in $V[G]$. □

In the next proof, we will use the following notation $\mathrm{Col}(\lambda, X)$ for subforcings of the Levy collapse $\mathrm{Col}(\lambda, <\mu)$. Suppose that $\lambda < \mu$ are cardinals and $X \subseteq \mu$ is not an ordinal (to avoid a conflict with the notation for the standard collapse). We then write

$$\mathrm{Col}(\lambda, X) = \{p \in \mathrm{Col}(\lambda, <\mu) \mid \mathrm{dom}(p) \subseteq X \times \lambda\}.$$

Let further $G_X = G \cap \mathrm{Col}(\lambda, X)$ and $G_\gamma = G \cap \mathrm{Col}(\lambda, <\mu)$ for any $\mathrm{Col}(\lambda, <\mu)$-generic filter G over V and any $\gamma < \mu$.

The notation $\mathrm{Col}(\lambda, X)$ will be used for intervals X, for which we use the standard notation

$$(\alpha, \gamma) = \{\beta \in \mathrm{Ord} \mid \alpha < \beta < \gamma\}$$

$$[\alpha, \gamma) = \{\beta \in \mathrm{Ord} \mid \alpha \leq \beta < \gamma\}.$$

Moreover, we will use the following consequence of Lemma 1.21 in the next proof. Suppose that λ is regular and $\mu > \lambda$ is inaccessible. If $\mathbb{R}$ is a separative $<\lambda$-closed forcing of size $<\mu$ and $\gamma < \mu$ is an ordinal, then $\mathbb{R} \times \mathrm{Col}(\lambda, <\mu)$ and $\mathrm{Col}(\lambda, [\gamma, \mu))$ are sub-equivalent.

Theorem 2.19. *Suppose that λ is an uncountable regular cardinal, $\mu > \lambda$ is inaccessible and G is $\mathrm{Col}(\lambda, <\mu)$-generic over V. Then in $V[G]$, every subset of ${}^\lambda\lambda$ that is definable from an element of ${}^\lambda V$ has the perfect set property.*

Proof. Suppose that $\varphi(x, y)$ is a formula with two free variables and $z \in \mathrm{Ord}^\lambda$. Using the set $A^\lambda_{\varphi,z}$ given in Definition 1.10, let

$$(A^\lambda_{\varphi,z})^{V[G]} = \{x \in ({}^\lambda\lambda)^{V[G]} \mid V[G] \vDash \varphi(x, z)\}.$$

Moreover, for any subclass M of $V[G]$, let

$$A^M = (A^\lambda_{\varphi,z})^{V[G]} \cap M.$$

To prove the perfect set property for $A^{V[G]}$ in $V[G]$, suppose that in $V[G]$, $A^{V[G]}$ has size λ^+. We will show that A has a perfect subset in $V[G]$.

Since $\mathrm{Col}(\lambda, {<}\mu)$ has the μ-cc, there is some $\gamma < \mu$ with $z \in V[G_\gamma]$. Since $A^{V[G]}$ has size λ^+ in $V[G]$, there is some ordinal ν with $\gamma < \nu < \mu$ and $A^{V[G_\gamma]} \neq A^{V[G_\nu]}$. Moreover, it follows from the definition of A^M that this inequality remains true when ν increases. Let ν be a cardinal with $\gamma < \nu < \mu$, $\nu^{<\lambda} = \nu$ and $A^{V[G_\gamma]} \neq A^{V[G_\nu]}$.

The forcing $\mathrm{Col}(\lambda, [\nu+1, \mu))$ is sub-equivalent to $\mathrm{Add}(\lambda, 1) \times \mathrm{Col}(\lambda, {<}\mu)$ by the remarks before the statement of this theorem. Hence there is an $\mathrm{Add}(\lambda, 1) \times \mathrm{Col}(\lambda, {<}\mu)$-generic filter $g \times h$ over $V[G_{\nu+1}]$ with $V[G] = V[G_{\nu+1} \times g \times h]$.

Claim. $A^{V[G_{\nu+1}]} \neq A^{V[G_{\nu+1} \times g]}$.

Proof. We will prove the claim by writing the extension $V[G]$ with the generic filters added in a different order. For the original generic filter G, we have

$$V[G] = V[G_\gamma \times G_{[\gamma,\nu+1)} \times G_{(\nu,\mu)}],$$

but we can also write $V[G]$ as

$$V[G] = V[G_\gamma \times G_{[\gamma,\nu+1)} \times g \times h]$$

by the choice of g, h above.

Since $\nu^{<\lambda} = \nu$ and ν has size λ in $V[G_{\nu+1}]$, $\mathrm{Col}(\lambda, [\gamma, \nu+1))$ is a non-atomic ${<}\lambda$-closed forcing of size λ. Hence it is sub-equivalent to $\mathrm{Add}(\lambda, 1)$ in $V[G_{\nu+1}]$ by Lemma 1.21. It follows that there is a $\mathrm{Col}(\lambda, [\gamma, \nu+1))$-generic filter k over $V[G_{\nu+1}]$ with

$$V[G_{\nu+1} \times g] = V[G_{\nu+1} \times k].$$

Hence we can write $V[G]$ as

$$V[G] = V[G_\gamma \times G_{[\gamma,\nu+1)} \times k \times h]$$

by replacing g with k in the factorization above. By changing the order, we trivially obtain

$$V[G] = V[G_\gamma \times k \times G_{[\gamma,\nu+1)} \times h]$$

We have $A^{V[G_\gamma]} \neq A^{V[G_{\nu+1}]}$ by the choice of ν. By the last factorization of $V[G]$, this implies that

$$A^{V[G_\gamma]} \neq A^{V[G_\gamma \times k]}$$

by homogeneity of the forcings. Hence we can find some $x \in A^{V[G_\gamma \times k]} \setminus A^{V[G_\gamma]} = A^{V[G_\gamma \times k] \setminus V[G_\gamma]}$. In particular, $x \notin V[G_\gamma]$. Since the filters $G_{[\gamma,\nu+1)}$ and k are mutually generic over $V[G_\gamma]$ by the choice of k, we have $V[G_{\nu+1}] \cap V[G_\gamma \times k] = V[G_\gamma]$. However, this implies that x cannot be in $V[G_{\nu+1}]$, since it is not in $V[G_\gamma]$. Since we also have

$$x \in V[G_\gamma \times k] \subseteq V[G_{\nu+1} \times k] = V[G_{\nu+1} \times g],$$

it now follows that $x \in V[G_{\nu+1} \times g] \setminus V[G_{\nu+1}]$ and thus $x \in A^{V[G_{\nu+1} \times g]} \setminus A^{V[G_{\nu+1}]}$, proving the claim. □

We have

$$V[G] = V[G_{\nu+1} \times g \times h]$$

by the choice of g, h above. We now choose an $\mathrm{Add}(\lambda, 1)$-name σ witnessing the previous claim. More precisely, σ is an $\mathrm{Add}(\lambda, 1)$-name in $V[G_{\nu+1}]$ for a new element of ${}^\lambda\lambda$ such that $\mathbb{1}_{\mathrm{Add}(\lambda,1)}$ forces that $\sigma \in A_{\varphi,y}$ in every further $\mathrm{Col}(\lambda, {<}\mu)$-generic extension. Such a name exists by the maximality principle applied to $\mathrm{Add}(\lambda, 1)$.

Since the forcing $\mathbb{P}$ given in Definition 2.6 is sub-equivalent to $\mathrm{Add}(\lambda, 1)$, we can replace the $\mathrm{Add}(\lambda, 1)$-generic filter g with a $\mathbb{P}$-generic filter. Since the definition of $\mathbb{P}$ is absolute between models with the same V_λ, the definition of $\mathbb{P}$ yields the same forcing in V and $V[G_{\nu+1} \times h]$. Let $g_{\mathbb{P}}$ be a $\mathbb{P}$-generic filter over $V[G_{\nu+1} \times h]$ with

$$V[G_{\nu+1} \times g_{\mathbb{P}} \times h] = V[G_{\nu+1} \times g \times h].$$

Claim. *In $V[G]$, the set $[T_{g_{\mathbb{P}}}]$ is a perfect subset of $A^{V[G]}$.*

Proof. Since $T_{g_{\mathbb{P}}}$ is a perfect tree and therefore $[T_{g_{\mathbb{P}}}]$ is a perfect set, it is sufficient to show that it is a subset of $A^{V[G]}$.

Every branch b in $T_{g_{\mathbb{P}}}$ is $\mathrm{Add}(\lambda,1)$-generic over $V[G_{\nu+1} \times h]$ by Lemma 2.8 applied to forcing with $\mathbb{P}$ over the model $V[G_{\nu+1} \times h]$. Moreover, every branch b in $T_{g_{\mathbb{P}}}$ has $\mathrm{Add}(\lambda,1)$ as a quotient in $V[G_{\nu+1} \times g_{\mathbb{P}} \times h]$ over $V[G_{\nu+1} \times h]$ by Lemma 2.16 applied to the same situation. It follows that every branch b in $T_{g_{\mathbb{P}}}$ has $\mathrm{Add}(\lambda,1) \times \mathrm{Col}(\lambda,{<}\mu)$ and hence also $\mathrm{Col}(\lambda,{<}\mu)$ as a quotient in $V[G]$ over $V[G_{\nu+1} \times h]$.

Since we identify the branch b with an $\mathrm{Add}(\lambda,1)$-generic filter over $V[G_{\nu+1} \times h]$ that is given by Lemma 2.8, we will also write σ^b. By the choice of σ and by the previous statements, we have

$$\sigma^b \in (A^{\kappa}_{\varphi,y})^{V[G_{\nu+1} \times g_{\mathbb{P}} \times h]} = A^{V[G]},$$

proving the claim. □

The last claim completes the proof of Theorem 2.19, since the set $[T_{g_{\mathbb{P}}}]$ witnesses the perfect set property of $A^{V[G]}$. □

From the last result, we immediately obtain the consistency of the perfect set property for all subsets of ${}^{\lambda}\lambda$ with DC_{λ}. For instance, it is consistent relative to the existence of an inaccessible cardinal that this is the case in the λ-Chang model $\mathsf{C}^{\lambda} = L(\mathrm{Ord}^{\lambda})$. We further obtain the following global version of the perfect set property.

Theorem 2.20. *Suppose that there is a proper class of inaccessible cardinals. Then there is a class generic extension of V in which for every infinite regular cardinal λ, the perfect set property holds for every subset of ${}^{\lambda}\lambda$ that is definable from an element of ${}^{\lambda}V$.*

Proof. Let C be the closure of the class of inaccessible cardinals and ω and let $\langle \kappa_\alpha \mid \alpha \geq 1\rangle$ be the order-preserving enumeration of C.

We define the following Easton support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ with bounded support at regular limits and full support at singular limits. Let $\mathbb{P}_0 = \{\mathbb{1}\}$. If $\alpha > 0$, let $\dot{\nu}_\alpha$ be a $\mathbb{P}_\alpha$-name for the least regular cardinal $\nu \geq \kappa_\alpha$ that is not collapsed by $\mathbb{P}_\alpha$ and let $\dot{\mathbb{P}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathrm{Col}(\dot{\nu}_\alpha, {<}\kappa_{\alpha+1})$. Moreover, we can assume that the names $\dot{\mathbb{P}}_\alpha$ are chosen in a canonical fashion, so that the iteration is definable.

Let $\mathbb{P}$ be the iterated forcing defined by this iteration and let further $\dot{\mathbb{P}}^{(\alpha)}$ be a $\mathbb{P}_\alpha$-name for the tail forcing of the iteration at stage α. It follows from the definition of the iteration that $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{P}}^{(\alpha)}$ is ${<}\kappa_\alpha$-closed and that $\mathbb{P}_\alpha$ is strictly smaller than $\kappa_{\alpha+1}$ for all $\alpha \in \mathrm{Ord}$.

Now suppose that G is $\mathbb{P}$-generic over V. We will write $G_\alpha = G \cap \mathbb{P}_\alpha$ and $\mathbb{P}^{(\alpha)} = (\dot{\mathbb{P}}^{(\alpha)})^{G_\alpha}$ for $\alpha \in \mathrm{Ord}$. Moreover, let $\nu_\alpha = \dot{\nu}^{G_\alpha}$ for $\alpha \geq 1$ and $\vec{\nu} = \langle \nu_\alpha \mid \alpha \geq 1\rangle$.

Claim. (1) *If κ_α is inaccessible in V, then κ_α remains regular in $V[G]$, $\nu_\alpha = \kappa_\alpha$ and $\kappa_\alpha^{+V[G]} = \kappa_{\alpha+1}$.*
(2) *If κ_α is a singular limit in V, then $\nu_\alpha > \kappa_\alpha$ and $\kappa_\alpha^{+V[G]} = \nu_\alpha$.*

Proof. If κ_α is inaccessible in V, it follows from the Δ-system lemma that $\mathbb{P}_\alpha$ has the κ_α-cc. The remaining claims easily follow from this and the fact that $\mathbb{P}^{(\alpha+1)}$ is ${<}\kappa_{\alpha+1}$-closed.

If κ_α is a singular limit in V, then ν_α is the least regular cardinal strictly above κ_α in $V[G_\alpha]$ by the definition of $\dot{\nu}_\alpha$. Moreover, ν_α is not collapsed in $V[G]$, since $\mathbb{P}^{(\alpha+1)}$ is ${<}\kappa_{\alpha+1}$-closed. □

By the previous claim, $\vec{\nu}$ enumerates the class of infinite regular cardinals in $V[G]$. Therefore, we suppose that $\alpha \geq 1$, $\kappa = \nu_\alpha$ and A is a subset of ${}^{\kappa}\kappa$ in $V[G]$ that is definable from an element of ${}^{\kappa}V$.

Claim. *A has the perfect set property in $V[G]$.*

Proof. Since $\mathbb{1}_{\mathbb{P}_\alpha}$ forces that $\dot{\mathbb{P}}_\beta$ is homogeneous for all $\beta \in \mathrm{Ord}$, the tail forcing $\mathbb{P}^{(\beta)}$ is homogeneous for all $\beta \in \mathrm{Ord}$. Since $\mathbb{P}^{(\alpha+1)}$ is homogeneous, A is an element of $V[G_{\alpha+1}]$. Since $\kappa = \nu_\alpha$, $\dot{\mathbb{P}}_\alpha$ is a name for $\mathrm{Col}(\kappa, {<}\kappa_{\alpha+1})$ and hence A has the perfect set property in $V[G_{\alpha+1}]$ by [Sol70, Theorem 2] for $\kappa = \omega$ and by Theorem 2.19 for $\kappa > \omega$. Since $\mathbb{P}^{(\alpha+1)}$ is $\kappa_{\alpha+1}$-closed, this implies that A has the perfect set property in $V[G]$. □

The last claim completes the proof of Theorem 2.20. □

We further remark that the conclusion of Theorem 2.19 has the following consequence. We define the *Bernstein property* for a subset A of ${}^{\lambda}\lambda$ to mean that A or its complement in ${}^{\lambda}\lambda$ have a perfect subset.

Lemma 2.21. *Suppose that λ is an uncountable regular cardinal and all subsets of ${}^{\lambda}\lambda$ that are definable from elements of ${}^{\lambda}\mathrm{Ord}$ have the perfect set property. Then the following statements hold.*

1. *All subsets of ${}^{\lambda}\lambda$ that are definable from elements of ${}^{\lambda}\mathrm{Ord}$ have the Bernstein property.*
2. *There is no well-order on ${}^{\lambda}\lambda$ that is definable from an element of ${}^{\lambda}\mathrm{Ord}$.*

Proof. The first claim is immediate. To prove the second claim, suppose towards a contradiction that there is a well-order on ${}^{\lambda}\lambda$ that is definable from an element of ${}^{\lambda}\mathrm{Ord}$. Using a standard construction, one can then construct a definable Bernstein set by induction. □

We finally use the previous results to prove a result about definable functions on ${}^{\kappa}\kappa$. In the statement of the next result, let $[X]^{\gamma}_{\neq}$ denote the set of sequences $\langle x_i \mid i < \gamma\rangle$ of distinct elements of X for any set X and any ordinal γ.

Theorem 2.22. *Suppose that λ is an uncountable regular cardinal, $\mathbb{R}$ is a $<\lambda$-distributive forcing and $\gamma < \lambda$.*

1. *Suppose that G is $\mathrm{Add}(\lambda,1)\times\mathbb{R}$-generic over V. Then in $V[G]$, for every function $f\colon [{}^{\lambda}\lambda]^{\gamma}_{\neq} \mapsto {}^{\lambda}\lambda$ that is definable from an element of V, there is a perfect subset C of ${}^{\lambda}\lambda$ such that $f{\upharpoonright}[C]^{\gamma}_{\neq}$ is continuous.*
2. *Suppose that G is $\mathrm{Add}(\lambda,\lambda^+)\times\mathbb{R}$-generic over V. Then in $V[G]$, for every function $f\colon [{}^{\lambda}\lambda]^{\gamma}_{\neq} \mapsto {}^{\lambda}\lambda$ that is definable from an element of ${}^{\lambda}V$, there is a perfect subset C of ${}^{\lambda}\lambda$ such that $f{\upharpoonright}[C]^{\gamma}_{\neq}$ is continuous.*

Proof. We can assume that $\lambda^{<\lambda} = \lambda$ by replacing V with an intermediate model.

To prove the first claim, it suffices to consider the trivial forcing $\mathbb{R} = \{\mathbb{1}\}$, since it is easy to see that this implies the claim for arbitrary $<\lambda$-distributive forcings $\mathbb{R}$.

By Lemma 2.18, there is a perfect subset C of ${}^{\lambda}\lambda$ in $V[G]$ such that for every sequence $\vec{x} = \langle x_i \mid i < \gamma\rangle$ of distinct elements of C, $\vec{x}$ is $\mathrm{Add}(\lambda,\gamma)$-generic over V and has $\mathrm{Add}(\lambda,1)$ as a quotient in $V[G]$ over V.

Suppose that in $V[G]$, we have a function $f\colon [{}^{\lambda}\lambda]^{\gamma}_{\neq} \to {}^{\lambda}\lambda$ that is definable from an element of V. Then there is a formula $\varphi(\vec{x},y,\alpha,t)$ and some $y \in V$ such that for all $\vec{x} \in [{}^{\lambda}\lambda]^{\gamma}_{\neq}$ in $V[G]$, $\alpha < \lambda$ and $t \in {}^{<\lambda}\lambda$, we have
$$f(\vec{x}){\upharpoonright}\alpha = t \Leftrightarrow V[G] \vDash \varphi(\vec{x},y,\alpha,t).$$
Moreover, let $\psi(\vec{x},y,\alpha,t)$ denote the formula
$$\mathbb{1}_{\mathrm{Add}(\lambda,1)} \Vdash_{\mathrm{Add}(\kappa,1)} \varphi(\vec{x},y,\alpha,t).$$

For each sequence of distinct elements of C of length γ, we consider the $\mathrm{Add}(\lambda,\gamma)$-generic extension $V[\vec{x}]$ of V. Since $\vec{x}$ has $\mathrm{Add}(\lambda,1)$ as a quotient in $V[G]$, we have for all $\alpha < \lambda$ and $t \in {}^{<\lambda}\lambda$ that
$$f(\vec{x}){\upharpoonright}\alpha = t \Leftrightarrow \mathbb{1}_{\mathrm{Add}(\lambda,1)} \Vdash^{V[\vec{x}]}_{\mathrm{Add}(\lambda,1)} \varphi(\vec{x},y,\alpha,t) \Leftrightarrow V[\vec{x}] \vDash \psi(\vec{x},y,\alpha,t).$$
In particular, it follows that $f(\vec{x}) \in V[\vec{x}]$.

Claim. *$f{\upharpoonright}[C]^{\gamma}_{\neq}$ is continuous.*

Proof. Let σ be an $\mathrm{Add}(\lambda,\gamma)$-name for the sequence of $\mathrm{Add}(\lambda,1)$-generic subsets of λ added by the $\mathrm{Add}(\lambda,\gamma)$-generic filter.

For every $\vec{x} \in [C]^{\gamma}_{\neq}$ and every $\alpha < \lambda$, there is a condition $\vec{p} = \langle p_i \mid i < \gamma\rangle$ in the $\mathrm{Add}(\lambda,\gamma)$-generic filter added by $\vec{x}$ with
$$\vec{p} \Vdash^{V}_{\mathrm{Add}(\lambda,\gamma)} \psi(\sigma,y,\alpha,f(\vec{x}){\upharpoonright}\alpha).$$
Since $\vec{p}$ is in the generic filter added by $\vec{x}$, we have $p_i \subseteq x_i$ for all $i < \gamma$. Now suppose that $\vec{y} = \langle y_i \mid i < \gamma\rangle$ is a sequence of distinct elements of C with $p_i \subseteq y_i$ for all $i < \gamma$. By the choice of $\vec{p}$ and the fact that $\vec{y}$ is $\mathrm{Add}(\lambda,\gamma)$-generic over V and has $\mathrm{Add}(\lambda,1)$ as a quotient in $V[G]$, we have $f(\vec{x}){\upharpoonright}\alpha = f(\vec{y}){\upharpoonright}\alpha$. It follows that $f{\upharpoonright}[C]^{\gamma}_{\neq}$ is continuous. □

To prove the second claim, it suffices to consider the trivial forcing $\mathbb{R}$, as in the first claim. Suppose that f is defined from the parameter $y \in {}^{\lambda}V$. We write $G_\alpha = G \cap \mathrm{Add}(\lambda, \alpha)$ for $\alpha < \lambda^+$. Since $\mathrm{Add}(\lambda, \lambda^+)$ is λ^+-cc, there is some $\alpha < \lambda^+$ with $y \in V[G_\alpha]$. Since $V[G]$ is an $\mathrm{Add}(\lambda, \lambda^+)$-generic extension of $V[G_\alpha]$, the claim now follows from the first claim. $\square$

3. The almost Baire property

In the first part of this section, we define an analogue to the Baire property. This property is characterized by a Banach-Mazur type game (see [Kec95, Section 8.H]) of uncountable length.

3.1. **Banach-Mazur games.** In this section, we assume that ν is an infinite regular cardinal with $\nu^{<\nu} = \nu$ (we write ν instead of κ, since $\nu = \omega$ is allowed). The *standard topology* (or *bounded topology*) on ${}^{\nu}\nu$ is generated by the basic open sets

$$N_t = \{x \in {}^{\nu}\nu \mid t \subseteq x\}$$

for $t \in {}^{<\nu}\nu$. The analogue to the Baire property will be defined using the following types of functions.

Definition 3.1. Suppose that $f\colon {}^{<\nu}\nu \to {}^{<\nu}\nu$ is given.

(1) f is a *homomorphism* if for all $s \subsetneq t$ in ${}^{<\nu}\nu$, we have $f(s) \subsetneq f(t)$.
(2) f is *continuous* if for every limit $\gamma < \nu$ and every strictly increasing sequence $\langle s_\alpha \mid \alpha < \gamma\rangle$ in ${}^{<\nu}\nu$, we have
$$f(\bigcup_{\alpha<\gamma} s_\alpha) = \bigcup_{\alpha<\gamma} f(s_\alpha).$$
(3) f is *dense* if for all $s \in {}^{<\nu}\nu$, the set
$$\{f(s^\frown\langle\alpha\rangle) \mid \alpha < \nu\}$$
is dense above $f(s)$ in the sense that for any $t \supseteq f(s)$, there is some $\alpha < \nu$ with $f(s^\frown\langle\alpha\rangle) \supseteq t$.
(4) If f is a homomorphism, let f^* denote the function $f^*\colon {}^{\nu}\nu \to {}^{\nu}\nu$ defined by
$$f^*(x) = \bigcup_{\alpha<\nu} f(x \restriction \alpha).$$

By using such functions on ${}^{<\nu}\nu$, we can characterize comeager subsets of ${}^{\nu}\nu$, which were defined in Definition 1.12, as follows.

Lemma 3.2. *Suppose that ν is an infinite cardinal with $\nu^{<\nu} = \nu$ and $t \in {}^{<\nu}\nu$. A subset A of ${}^{\nu}\nu$ is comeager in N_t if and only if there is a dense continuous homomorphism $f\colon {}^{<\nu}\nu \to {}^{<\nu}\nu$ with $f(\varnothing) = t$ and $\mathrm{ran}(f^*) \subseteq A$.*

Proof. To prove the first implication, suppose that A is comeager in N_t. Then there is a sequence $\langle U_\alpha \mid \alpha < \nu\rangle$ of dense open subsets of N_t with $\bigcap_{\alpha<\nu} U_\alpha \subseteq A$. Since any intersection of strictly less than ν many dense open subsets of N_t is again dense open in N_t, we can assume that $U_\beta \subseteq U_\alpha$ for all $\alpha < \beta < \nu$.

We now define $f(s)$ by induction on $\mathrm{l}(s)$. Let $f(\varnothing) = t$. In the successor case, suppose that $\mathrm{l}(s) = \gamma$ and that $f(s)$ is defined. Since U_γ is a dense open subset of N_t, the set

$$K = \{u \supsetneq f(s) \mid N_u \subseteq U_\gamma\}$$

is dense above $f(s)$ in the sense that for every $t \supseteq f(s)$, there is some $v \in K$ with $t \subseteq v$. Since $\nu^{<\nu} = \nu$, we can choose an enumeration $\langle t_\alpha \mid \alpha < \nu\rangle$ of K. We then define $f(s^\frown\langle\alpha\rangle) = t_\alpha$ for all $\alpha < \nu$. In the limit case, suppose that $\mathrm{l}(s) = \gamma$ is a limit and that $f(s\restriction\bar{\gamma})$ is defined for all $\bar{\gamma} < \gamma$. We then define $f(s) = \bigcup_{\bar{\gamma}<\gamma} f(s\restriction\bar{\gamma})$.

It follows from the construction that f satisfies the required properties and that $\mathrm{ran}(f^*) \subseteq \bigcap_{\alpha<\nu} U_\alpha \subseteq A$.

To prove the reverse implication, suppose that f satisfies the conditions stated above. For any $x \in {}^{\nu}\nu$, let

$$K_x = \{s \in {}^{<\nu}\nu \mid t \subseteq s,\ f(s) \subseteq x\}.$$

Since $f(\varnothing) = t$, K_x is nonempty for any $x \in N_t$.

Claim. *For any $x \in N_t$, if K_x has no maximal elements, then $x \in A$.*

Proof. Since K_x is nonempty and has no maximal elements, we can build a strictly increasing sequence $\langle s_\alpha \mid \alpha < \nu\rangle$ in ${}^{<\nu}\nu$ with $s_0 = t$ and $f(s_\alpha) \subseteq x$ for all $\alpha < \nu$. By the definition of f^*, this implies that $x \in \operatorname{ran}(f^*) \subseteq A$. □

For any $s \in {}^{<\nu}\nu$ with $t \subseteq s$, we now consider the set C_s of $x \in N_t$ such that s is a maximal element of K_x. It is easy to see that C_s is a closed nowhere dense subset of $N_{f(s)}$. Since $N_t \setminus A \subseteq \bigcup_{s \supseteq t} C_s$ by the previous claim, it follows that A is comeager in N_t. □

We now define an asymmetric version of the Baire property, using the functions above.

Definition 3.3. A subset A of ${}^{\nu}\nu$ is *almost ν-Baire* (*almost Baire*) if there is a dense homomorphism $f: {}^{<\nu}\nu \to {}^{<\nu}\nu$ with one of the following properties.

(a) $\operatorname{ran}(f^*) \subseteq A$.
(b) $\operatorname{ran}(f^*) \subseteq {}^{\nu}\nu \setminus A$, f is continuous and $f(\emptyset) = \emptyset$.

Since every homomorphism is continuous for $\nu = \omega$, it follows immediately from Lemma 3.2 that a subset A of ${}^{\omega}\omega$ is almost Baire if and only if there is some $t \in {}^{<\omega}\omega$ such that A is comeager in N_t or ${}^{\omega}\omega \setminus A$ is comeager. It can be easily seen that this implies that for every class Γ of subsets of ${}^{\omega}\omega$ that is closed under continuous preimages, the almost Baire property for all sets in Γ is equivalent to the Baire property for all sets in Γ.

The continuity in the definition of almost Baire is necessary by the next result. To state this result, let Club_ν denote the set

$$\mathrm{Club}_\nu = \{x \in {}^{\nu}\nu \mid \exists C \subseteq \nu \text{ club } \forall i \in C\ x(i) \neq 0\}$$

of functions coding elements of the club filter on ν as characteristic functions, and

$$\mathrm{NS}_\nu = \{x \in {}^{\nu}\nu \mid \exists C \subseteq \nu \text{ club } \forall i \in C\ x(i) = 0\}$$

the set of functions coding elements of the non-stationary ideal on ν.

Lemma 3.4. *Club_ν and NS_ν are almost Baire subsets of ${}^{\nu}\nu$, but for every dense continuous homomorphism $f: {}^{<\nu}\nu \to {}^{<\nu}\nu$, we have $\operatorname{ran}(f^*) \cap \mathrm{Club}_\nu \neq \emptyset$ and $\operatorname{ran}(f^*) \cap \mathrm{NS}_\nu \neq \emptyset$.*

Proof. It is easy to see that Club_ν and NS_ν are almost Baire subsets of ${}^{\nu}\nu$.

Since the remaining claims are symmetric, it is sufficient to prove that $\operatorname{ran}(f^*) \cap \mathrm{Club}_\nu \neq \emptyset$. We define a sequence $\langle x(\gamma) \mid \gamma < \nu\rangle$ with values in ν by the following induction. Suppose that $\gamma < \nu$, $s = \langle x(\alpha) \mid \alpha < \gamma\rangle$ is already defined and $\mathrm{l}(f(s)) = \delta$. If γ is a successor, since f is dense, there is some $\eta < \nu$ such that $f(s^\frown\langle\eta\rangle)(\delta) = 1$. If γ is a limit, the same conclusion follows from the additional assumption that f is continuous. In both cases, we let $x(\gamma) = \eta$.

By the construction, we have $x \in \operatorname{ran}(f^*) \cap \mathrm{Club}_\nu$ and hence $\operatorname{ran}(f^*) \cap \mathrm{Club}_\nu \neq \emptyset$, proving the claim. □

The motivation for the definition of the almost Baire property comes from its connection with the following game.

Definition 3.5. The *Banach-Mazur game* $G_\nu(A)$ of length ν for a subset A of ${}^{\nu}\nu$ is defined as follows. The first (even) player, player I, plays an element of ${}^{<\nu}\nu$ in each even round. The second (odd) player, player II, plays an element of ${}^{<\nu}\nu$ in each odd round. Together, they play a strictly increasing sequence $\vec{s} = \langle s_\alpha \mid \alpha < \nu\rangle$ with $s_\alpha \in {}^{<\nu}\nu$ for all $\alpha < \nu$. Thus the sequence of moves of both players defines a sequence

$$\bigcup_{\alpha<\nu} s_\alpha = x = \langle x(i) \mid i < \nu\rangle \in {}^{\nu}\nu$$

and the first player wins this run if $x \in A$.

The Banach-Mazur game of length ν with these rules, but without a specific winning set, is denoted by G_ν. Moreover, for any $t \in {}^{<\nu}\nu$, the game $G^t_\nu(A)$ is defined as $G_\nu(A)$ but with the additional requirement that $t \subseteq s_0$ for the first move s_0 of player I.

We will also consider the games $G^2_\nu(A)$ and $G^{2,(s,t)}_\nu(A)$ for $(s,t) \in ({}^{<\nu}\nu)^2$ with $\mathrm{l}(s) = \mathrm{l}(t)$ that are defined in analogy with $G_\nu(A)$. In these games, the players play elements (u,v) of $({}^{<\nu}\nu)^2$ with $\mathrm{l}(u) = \mathrm{l}(v)$ and A is a subset of $({}^{\nu}\nu)^2$. It is easy to check that all results for G_ν in this section also hold for G^2_ν, since the proofs can be easily modified to work for this game.

The next two results show the equivalence between the determinacy of $G_\nu(A)$ and the almost Baire property for A.

Lemma 3.6. *The following are pairs of equivalent statements for any subset A of ${}^\nu\nu$.*

(1) (a) *Player I has a winning strategy in $G_\nu(A)$.*
    (b) *There is a dense homomorphism $f\colon {}^{<\nu}\nu \to {}^{<\nu}\nu$ with $\mathrm{ran}(f^*) \subseteq A$.*

(2) (a) *Player II has a winning strategy in $G_\nu(A)$.*
    (b) *There is a dense continuous homomorphism $f\colon {}^{<\nu}\nu \to {}^{<\nu}\nu$ with $\mathrm{ran}(f^*) \subseteq {}^\nu\nu \setminus A$ and $f(\varnothing) = \varnothing$.*

Proof. We will only prove the first equivalence, since the proof of the second equivalence is analogous.

To prove the first implication, suppose that player I has a winning strategy σ in $G_\nu(A)$. For all $t \in {}^{<\nu}\nu$, by induction on $\mathrm{l}(t)$, we will define $f(t)$ and partial runs

$$\vec{s}_t = \langle s_t(\alpha) \mid \alpha < 2 \cdot \mathrm{l}(t) + 1 \rangle$$

according to σ such that $\vec{s}_t \subseteq \vec{s}_u$ for all $t \subseteq u$ and $f(t)(\alpha) = s_t(2 \cdot \alpha)$ for all $\alpha < \mathrm{l}(t)$.

We begin by considering the first move $v = \sigma(\varnothing)$ of player I according to σ and defining $f(\varnothing) = v$ and $\vec{s}_\varnothing = \langle v \rangle$. In the successor step, suppose that $t \in {}^{<\nu}\nu$ and $f(t)$, $\vec{s}_t$ are defined. Moreover, suppose that $\langle u_\alpha \mid \alpha < \nu \rangle$ is an enumeration of the possible responses of player II to $\vec{s}_t$ and that for each $\alpha < \nu$, v_α is the response of player I to $\vec{s}_t^\frown \langle u_\alpha \rangle$ according to σ. Let $\vec{s}_{t^\frown\langle\alpha\rangle} = \vec{s}_t^\frown \langle u_\alpha, v_\alpha \rangle$ and $f(t^\frown\langle\alpha\rangle) = v_\alpha$.

In the limit step, suppose that $\mathrm{l}(t)$ is a limit and that $\vec{s}_{t\restriction\alpha}$ and $f(t\restriction\alpha)$ are defined for all $\alpha < \mathrm{l}(t)$. If v is the response of player I to $\bigcup_{\alpha<\mathrm{l}(t)} \vec{s}_{t\restriction\alpha}$ according to σ, let $\vec{s}_t = (\bigcup_{\alpha<\mathrm{l}(t)} \vec{s}_{t\restriction\alpha})^\frown\langle v \rangle$ and $f(t) = v$. This completes the definition of f and by the construction, f is a dense homomorphism with $\mathrm{ran}(f^*) \subseteq A$.

To prove the second implication, suppose that $f\colon {}^{<\nu}\nu \to {}^{<\nu}\nu$ is a dense homomorphism with $\mathrm{ran}(f^*) \subseteq A$. We will define a winning strategy σ for player I in $G_\nu(A)$. To this end, by induction on $\mathrm{l}(\vec{s})$, we will define $t_{\vec{s}}, \sigma(\vec{s}) \in \nu^{<\nu}$ for all partial runs $\vec{s}$ of even length according to σ such that $\mathrm{l}(t_{\vec{s}\restriction 2\cdot\alpha}) = \alpha$, $t_{\vec{s}\restriction 2\cdot\alpha} \subseteq t_{\vec{s}\restriction 2\cdot\beta}$ and $\sigma(\vec{s}\restriction 2\cdot\alpha) = f(t_{\vec{s}\restriction 2\cdot\alpha})$ for all α, β with $2\cdot\alpha \leq 2\cdot\beta \leq \mathrm{l}(\vec{s})$.

We begin by defininig $\sigma(\varnothing) = f(\varnothing)$. In the successor step, suppose that $\mathrm{l}(\vec{s})$ is even and that $t_{\vec{s}\restriction\alpha}$, $\sigma(\vec{s}\restriction\alpha)$ are defined for all even $\alpha \leq \mathrm{l}(\vec{s})$. Moreover, suppose that u is a possible move of player II extending the partial run $\vec{s}^\frown\langle\sigma(\vec{s})\rangle$, so that $\sigma(\vec{s}) \subsetneq u$. Since f is dense, there is some $\alpha < \nu$ with $u \subsetneq f(t_{\vec{s}}^\frown\langle\alpha\rangle)$. Let $t_{\vec{s}^\frown\langle\sigma(\vec{s}),u\rangle} = t_{\vec{s}}^\frown\langle\alpha\rangle$ and $\sigma(\vec{s}^\frown\langle\sigma(\vec{s}),u\rangle) = f(\vec{s}^\frown\langle\sigma(\vec{s}),u\rangle)$.

In the limit step, suppose that $l(\vec{s}) = \gamma$ is a limit and $t_{\vec{s}\restriction\alpha}$, $\sigma(\vec{s}\restriction\alpha)$ are defined for all even $\alpha < \mathrm{l}(\vec{s})$. Let $t_{\vec{s}} = \bigcup_{\alpha<\gamma} t_{\vec{s}\restriction\alpha}$ and $\sigma(\vec{s}) = f(t_{\vec{s}})$. It is now easy to check that σ is a a winning strategy for player I in $G_\nu(A)$. □

In the next result, we will consider the following stronger type of strategy for G_ν that only relies on the union of the previous moves.

Definition 3.7. A *tactic* in G_ν is a strategy σ such that there is a map $\bar{\sigma}\colon {}^{<\nu}\nu \to {}^{<\nu}\nu$ with the property that

$$\sigma(\vec{s}) = \bar{\sigma}(\bigcup_{\alpha<\gamma} s_\alpha)$$

for all $\vec{s} = \langle s_\alpha \mid \alpha < \gamma \rangle \in \mathrm{dom}(\sigma)$.

The next result, which follows from [Kov09, Lemma 7.3.2], relates the Banach-Mazur game of length ν with the ν-Baire property.

Lemma 3.8. *Suppose that A is a subset of ${}^\nu\nu$ and $t \in {}^{<\nu}\nu$.*

(1) *(Kovachev) The following conditions are equivalent.*
  (a) *A is meager in N_t.*
  (b) *Player II has a winning strategy in $G^t_\nu(A)$.*
  (c) *Player II has a winning tactic in $G^t_\nu(A)$.*

(2) *If $A \cap N_t$ is ν-Baire, then the following conditions are equivalent.*
  (a) *A is meager in N_t.*
  (b) *Player I does not have a winning strategy in $G^t_\nu(A)$.*

(3) *If $\nu = \omega$, then the following conditions are equivalent.*

(a) *A is comeager in N_u for some $u \supseteq t$.*
(b) *Player I has a winning strategy in $G^t_\omega(A)$.*

Proof. The first claim is proved in [Kov09, Lemma 7.3.2]. Since the remaining claims are easy consequences of this, we only sketch the proofs.

For the second claim, suppose that A is not meager in N_t. Since A is ν-Baire, $A \cap N_u$ is comeager in N_u for some $u \supseteq t$. By the first claim, there is a winning strategy σ for player II in $G^u_\nu(N_u \setminus A)$. This means that player II succeeds with playing in A. Since it is harder for player II to win because she or he does not play at limits, we easily obtain a winning strategy τ for player I in $G^t_\nu(A)$ with the first move u from σ.

For the third claim, suppose that A is comeager in N_u for some $u \supseteq t$. Since player II has a winning strategy in $G^u_\omega({}^\nu\nu \setminus A)$ by the first claim, we obtain a winning strategy for player I in $G^t_\omega(A)$ with the first move u by switching the roles of the players. The reverse implication follows similarly from the first claim. □

This shows together with Lemma 3.2 that for any class Γ of subsets of the Baire space ${}^\omega\omega$ that is closed under continuous preimages, the statement that $G_\omega(A)$ is determined for all sets $A \in \Gamma$ is equivalent to the statement that all sets in Γ have the property of Baire.

Moreover, the previous result shows that $G_\nu(A)$ is determined for every ν-Baire subset A of ${}^\nu\nu$. The game is also determined for some Σ^1_1 subsets of ${}^\nu\nu$ that are not ν-Baire, since it is easy to see that player I has a winning strategy in $G_\nu(A)$ if A is one of the sets Club_ν, NS_ν that are defined after Definition 3.3. This leads to the question for which definable subsets A of ${}^\kappa\kappa$ the Banach-Mazur game is determined. We study this question in the next section.

3.2. **The almost Baire property for definable sets.** As before, we always assume that κ is an uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$.

In this section, we will prove that it is consistent for the Banach-Mazur game G_κ to be determined for all subsets of ${}^\kappa\kappa$ that are definable from elements of ${}^\kappa\mathrm{Ord}$. This will also imply that it is consistent that the almost Baire property holds for all such sets by the results in the previous section.

The following notions will be used to construct strategies for the first player in G_κ.

Definition 3.9. (i) An *almost strategy* for player I in G_κ is a partial strategy σ such that $\mathrm{dom}(\sigma)$ is dense in the following sense. Suppose that $\gamma < \kappa$ is odd, $\vec{s} = \langle s_\alpha \mid \alpha < \gamma\rangle$ is a strictly increasing sequence in ${}^{<\kappa}\kappa$ according to σ and $\bigcup_{\alpha<\gamma} s_\alpha \subsetneq v$. Then there is some $w \in {}^{<\kappa}\kappa$ with $v \subseteq w$ and $\vec{s}^\frown\langle w\rangle \in \mathrm{dom}(\sigma)$.
(ii) If σ, τ are partial strategies for player I in G_κ, then τ *expands* σ if for every run $\vec{s} = \langle s_\alpha \mid \alpha < \kappa\rangle$ according to τ, there is a run $\vec{t} = \langle t_\alpha \mid \alpha < \kappa\rangle$ according to σ with the same *outcome* $\bigcup_{\alpha<\kappa} s_\alpha = \bigcup_{\alpha<\kappa} t_\alpha$.
(iii) Suppose that A is a subset of ${}^\kappa\kappa$. A partial strategy σ for player I in $G_\kappa(A)$ is *winning* if for every run $\vec{s} = \langle s_\alpha \mid \alpha < \kappa\rangle$ according to σ, the outcome $\bigcup_{\alpha<\kappa} s_\alpha$ is in A.

The next result shows that to construct a winning strategy for player I in $G_\kappa(A)$, it is sufficient to construct a winning almost strategy. In the statement, we call a definition or a formula V_κ-*absolute* if it is absolute to outer models $W \supseteq V$ with $(V_\kappa)^W = V_\kappa$.

Lemma 3.10. *There is a V_κ-absolute definable function that maps every almost strategy σ for player I in G_κ to a strategy τ that expands σ and moreover, this property of σ, τ is V_κ-absolute.*

Proof. We fix a wellordering $\prec$ of ${}^{<\kappa}\kappa$. We will define τ by induction on the length of partial runs. To this end, for any partial run $\vec{t} = \langle t_\alpha \mid \alpha < \gamma\rangle$ that is according to τ, as defined up to this stage, we will define a *revised partial run* $\mathrm{rev}(\vec{t}) = \langle r_\alpha \mid \alpha < \gamma\rangle$ according to σ with $r_\alpha = t_\alpha$ for all even $\alpha < \gamma$ and let $\tau(\vec{t}) = \sigma(\mathrm{rev}(\vec{t}))$.

In the successor step, suppose that the construction has been carried out for some even ordinal $\gamma < \kappa$ and that $\vec{t} = \langle t_\alpha \mid \alpha < \gamma + 2\rangle$ is a partial run. If $\vec{t}$ is not according to τ, then we give $\tau(\vec{t})$ the $\prec$-least possible value. If $\vec{t}$ is according to τ, then $\vec{t}\restriction\gamma$ is according to τ and hence $\sigma(\mathrm{rev}(\vec{t}\restriction\gamma)) = \tau(\vec{t}\restriction\gamma) = t_\gamma$ by the induction hypothesis for γ. Since σ is an almost strategy, there is some $u \supsetneq t_{\gamma+1}$ with $\vec{t}^\frown\langle t_\gamma, u\rangle \in \mathrm{dom}(\sigma)$. For the $\prec$-least such u, we let

$$\mathrm{rev}(\vec{t}) = \mathrm{rev}(\vec{t}\restriction\gamma)^\frown\langle t_\gamma, u\rangle.$$

In the limit step, suppose that $\vec{t} = \langle t_\alpha \mid \alpha < \gamma \rangle$ is a partial run of limit length $\gamma < \kappa$ and that the construction has been carried out strictly below γ. If $\vec{t}$ is not according to τ, then we give $\tau(\vec{t})$ the $\lessdot$-least possible value. If $\vec{t}$ is according to τ, then we let

$$\mathrm{rev}(\vec{t}) = \bigcup_{2\cdot\alpha<\gamma} \mathrm{rev}(\vec{t}\upharpoonright 2\cdot\alpha).$$

Moreover, let $\tau(\vec{t}) = \sigma(\mathrm{rev}(\vec{t}))$.

It is easy to see that the construction of the function and its required properties are absolute to any model of set theory with the same V_κ that also contains $\lessdot$. □

We now collect some definitions that are relevant for the following proofs. The subsets S of $\mathrm{Add}(\kappa,1)^2$ introduced below will represent two-step iterated forcings that are sub-equivalent to $\mathrm{Add}(\kappa,1)$.

Definition 3.11. A set S is called a *level subset* of $\mathrm{Add}(\kappa,1)^2$ if it consists of pairs $(s,t) \in \mathrm{Add}(\kappa,1)^2$ with $\mathrm{l}(s) = \mathrm{l}(t)$. We further define the following properties, which such a set might have.

(a) S is *closed* if for every strictly increasing sequence $\langle (s_\alpha,t_\alpha) \mid \alpha < \gamma\rangle$ in S, there is some $(s,t) \in S$ with $s \supseteq \bigcup_{\alpha<\gamma} s_\alpha$ and $t \supseteq \bigcup_{\alpha<\gamma} t_\alpha$.
(b) S is *limit-closed* if for every strictly increasing sequence $\langle (s_\alpha,t_\alpha) \mid \alpha < \gamma\rangle$ in S, $s = \bigcup_{\alpha<\gamma} s_\alpha$ and $t = \bigcup_{\alpha<\gamma} t_\alpha$, we have $(s,t) \in S$.
(c) S is *perfect* if it is closed and every element of S has incompatible successors in S.

Moreover, we let $\mathrm{split}(S)$ denote the set of *splitting nodes*, i.e. the elements of S with incompatible direct successors in S, and $\mathrm{succsplit}(S)$ the set of direct successors of splitting nodes.

Note that for subtrees, the notions of closure and limit closure that we have just defined are equivalent.

The next definitions will be used below to define a forcing that adds a winning set for player I in G_κ.

Definition 3.12. Suppose that S is a level subset of $\mathrm{Add}(\kappa,1)^2$. An *S-tree* p consists of pairs (s,t) such that s, t are strictly increasing sequences with $\mathrm{l}(s) = \mathrm{l}(t)$ and the following conditions hold for all $(s,t),(u,v) \in p$ and all $\alpha < \mathrm{l}(s)$.

(a) $(s(\alpha),t(\alpha)) \in S$.
(b) $(s\upharpoonright\alpha, t\upharpoonright\alpha) \in p$.
(c) If $\gamma,\delta < \mathrm{l}(s)$ are even, $\bigcup\mathrm{ran}(s\upharpoonright\gamma) = \bigcup\mathrm{ran}(u\upharpoonright\delta)$ and $\bigcup\mathrm{ran}(t\upharpoonright\gamma) = \bigcup\mathrm{ran}(v\upharpoonright\delta)$, then $s(\gamma) = u(\delta)$ and $t(\gamma) = v(\delta)$.

Remark 3.13. The condition in Definition 3.12 (c) can be replaced with the following statement. If $\gamma < \mathrm{l}(s)$ is even, $s\upharpoonright\gamma = u\upharpoonright\gamma$ and $t\upharpoonright\gamma = v\upharpoonright\gamma$, then $s(\gamma) = u(\gamma)$ and $t(\gamma) = v(\gamma)$. Using this alternative definition, one can prove analogous results to all that follows.

The S-trees of size $<\kappa$ will be the conditions in a forcing that adds an S-tree with the following properties.

Definition 3.14. Suppose that S is a level subset of $\mathrm{Add}(\kappa,1)^2$ and p is an S-tree.

(a) Let $\mathrm{l}(p) = \sup_{(s,t)\in p} \mathrm{l}(s)$ and
$$\mathrm{ht}(p) = \sup_{(s,t)\in p,\alpha<\mathrm{l}(s)} \mathrm{l}(s(\alpha)).$$
(b) An S-tree p is called *superclosed* if
  (i) if $\langle (s_\alpha,t_\alpha) \mid \alpha < \gamma\rangle$ is a strictly increasing sequence in p, then there is some $(s,t) \in p$ which extends (s_α,t_α) for all $\alpha < \gamma$.
  (ii) p has no maximal elements.
(c) An S-tree p is called *strategic* if it is superclosed and the following condition holds. If $(s,t) \in p$, $\mathrm{l}(s) = \mathrm{l}(t) = \gamma+1$, γ is even and $u \supsetneq s(\gamma)$, then there are $v,w \in {}^{<\kappa}\kappa$ with $v \supseteq u$ and $(s^\frown\langle v\rangle, t^\frown\langle w\rangle) \in p$.

Note that we have $\mathrm{l}(s) \leq \mathrm{ht}(s)$ for all $(s,t) \in p$, since s is strictly increasing by the definition of S-trees. We will further work with the following weak projection of superclosed S-trees, which differs from the standard notion of projection.

Definition 3.15. If S is a level subset of $\mathrm{Add}(\kappa,1)^2$ and T is a superclosed S-tree, we define the following objects.

(a) The *body* $[T]$ of T is the set of $(x,y)\in\mathrm{Add}(\kappa,1)^2$ such that there are $\vec{s}=\langle s_\alpha\mid\alpha<\kappa\rangle$ and $\vec{t}=\langle t_\alpha\mid\alpha<\kappa\rangle$ with $\langle(s_\alpha,t_\alpha)\mid\alpha<\gamma\rangle\in T$ for all $\gamma<\kappa$ and
$$x=\bigcup_{\alpha<\kappa}s_\alpha,\ y=\bigcup_{\alpha<\kappa}t_\alpha.$$

(b) The *projection* $\mathrm{p}[T]$ of T is the set of $x\in{}^\kappa\kappa$ such that $(x,y)\in[T]$ for some $y\in{}^\kappa\kappa$.

The strategic S-trees are defined for the following purpose.

Lemma 3.16. *Suppose that S is a perfect level subset of* $\mathrm{Add}(\kappa,1)^2$ *and T is a strategic S-tree. Then there is a winning strategy for player I in $G_\kappa(\mathrm{p}[T])$ that remains so in all outer models $W\supseteq V$ with $(V_\kappa)^W=V_\kappa$.*

Proof. We fix a wellordering $\prec$ of ${}^{<\kappa}\kappa$. It is sufficient to construct a winning almost strategy for player I in $G_\kappa(\mathrm{p}[T])$ by Lemma 3.10, and this will be done as follows, by induction on $\delta<\kappa$. We will define σ for partial runs of length strictly below δ, and will simultaneously, for each partial run $\vec{s}$ according to σ with odd length $\mathrm{l}(\vec{s})\le\delta$, define a sequence $\vec{t}_{\vec{s}}$ with $(\vec{s},\vec{t}_{\vec{s}})\in T$ and $\vec{t}_{\vec{s}\restriction\alpha}\subseteq\vec{t}_{\vec{s}}$ for all odd $\alpha<\mathrm{l}(\vec{s})$.

In the successor step, we assume that the construction has been carried out up to $\delta=2\gamma+1$ for some $\gamma<\kappa$ and that $\vec{s}=\langle s_\alpha\mid\alpha<\delta\rangle$ is a partial run according to σ. Let
$$\Psi(u)\iff u\supseteq s_{2\gamma}\text{ and }\exists v\supseteq\vec{t}_{\vec{s}}(2\gamma)\ (\vec{s}^\frown\langle u\rangle,\vec{t}_{\vec{s}}^\frown\langle v\rangle)\in T\}$$
Since T is strategic, the set $D=\{u\mid\Psi(u)\}$ is dense above $s_{2\gamma}$, in the sense that for every $u\supseteq s_{2\gamma}$, there is some $v\supseteq u$ with $\Psi(v)$. Since we are constructing an almost strategy, it is sufficient to define $\sigma(\vec{s}^\frown\langle u\rangle)$ for all $u\in D$.

Given $u\in D$, let $v\supseteq\vec{t}_{\vec{s}}(2\gamma)$ be $\prec$-least with $(\vec{s}^\frown\langle u\rangle,\vec{t}_{\vec{s}}^\frown\langle v\rangle)\in T\}$. Since T is an S-tree and by Definition 3.12 (c), there is a unique pair (u^*,v^*) with
$$(\vec{s}^\frown\langle u,u^*\rangle,\vec{t}_{\vec{s}}{}^\frown\langle v,v^*\rangle)\in T.$$
Now let $\sigma(\vec{s}^\frown\langle u\rangle)=u^*$ and $\vec{t}_{\vec{s}^\frown\langle u,u^*\rangle}=\vec{t}_{\vec{s}}^\frown\langle v,v^*\rangle$.

In the limit step, we assume that the construction has been carried out strictly below γ for some $\gamma\in\mathrm{Lim}$ and that $\vec{s}=\langle s_\alpha\mid\alpha<\gamma\rangle$ is a partial run according to σ.

We first let $\vec{t}=\bigcup_{\alpha<\gamma}\vec{t}_{\vec{s}\restriction 2\alpha+1}$. Since T is superclosed, there is a pair (u,v) with $(\vec{s}^\frown\langle u\rangle,\vec{t}^\frown\langle v\rangle)\in T$, and moreover this pair is unique, since T is an S-tree and by Definition 3.12 (c). Let $\sigma(\vec{s})=u$ and $\vec{t}_{\vec{s}^\frown\langle u\rangle}=\vec{t}_{\vec{s}}^\frown\langle v\rangle$.

This completes the construction of σ. To prove that σ wins, suppose that $\vec{s}=\langle s_\alpha\mid\alpha<\kappa\rangle$ is a run according to σ and let $\vec{t}=\bigcup_{\alpha<\kappa}\vec{t}_{\vec{s}\restriction 2\alpha+1}$. Then $\langle(\vec{s}\restriction 2\alpha+1,\vec{t}_{\vec{s}\restriction 2\alpha+1})\mid\alpha<\kappa\rangle$ witnesses that the outcome $\bigcup_{\alpha<\kappa}s_\alpha$ is in $\mathrm{p}[T]$ and hence player I wins, proving the claim. □

Definition 3.17. Suppose that S is a perfect level subset of $\mathrm{Add}(\kappa,1)^2$. The forcing $\mathbb{P}_S$ consists of all S-trees of size strictly less than κ, ordered by reverse inclusion.

If G is a $\mathbb{P}_S$-generic filter over V, we will write $T_G=\bigcup G$. Moverover, for any perfect level subset S of $\mathrm{Add}(\kappa,1)^2$, we will write $\pi_S\colon S\to\mathrm{Add}(\kappa,1)$ for the projection to the first coordinate.

In the situation below, we will additionally assume that $\pi_S\colon S\to\mathrm{Add}(\kappa,1)$ is a projection. It is then easy to see that the forcing $\mathbb{P}_S$ is non-atomic, $<\kappa$-closed and has size κ, and is hence sub-equivalent to $\mathrm{Add}(\kappa,1)$ by Lemma 1.21.

Lemma 3.18. *If S is a perfect level subset of* $\mathrm{Add}(\kappa,1)^2$ *such that $\pi_S\colon S\to$* $\mathrm{Add}(\kappa,1)$ *is a projection and G is $\mathbb{P}_S$ -generic over V, then T_G is a strategic S-tree.*

Proof. Since every condition in $\mathbb{P}_S$ is an S-tree, it follows immediately that T_G is again an S-tree. Moreover, since S is perfect, it can be shown by a straightforward density argument that T_G is superclosed.

To see that T_G is strategic, suppose that $(s,t)\in T_G$, $\mathrm{l}(s)=\mathrm{l}(t)=\gamma+1$, γ is even and $u\supsetneq s(\gamma)$. Then there is some $p\in G$ with $(s,t)\in p$. Since $\pi_S\colon S\to\mathrm{Add}(\kappa,1)$ is a projection by our assumption, there is some $(v,w)\in S$ with $u\subseteq v$. We now claim that the set
$$D=\{q\le p\mid(s^\frown\langle v\rangle,t^\frown\langle w\rangle)\in q\}$$

is dense below p. To see this, suppose that $q \leq p$. Since γ is even, it is easy to check that $q \cup \{s^\frown\langle v\rangle, t^\frown\langle w\rangle\}$ is again a condition in $\mathbb{P}_S$, and thus D is dense below p. It follows immediately that T_G is strategic. □

In the next lemma, we will write $\mathbb{Q}_p$ for the subforcing

$$\mathbb{Q}_p = \{q \in \mathbb{Q} \mid q \leq p\}$$

of a forcing $\mathbb{Q}$ below a condition $p \in \mathbb{Q}$.

Lemma 3.19. *Suppose that $\mathbb{R}$ is a complete Boolean algebra and $\mathbb{Q}$ is a complete subalgebra such that $\mathbb{Q}$, $\mathbb{R}$, $\mathrm{Add}(\kappa,1)$ are sub-equivalent. Moreover, suppose that $p \in \mathbb{Q}$, $r \in \mathrm{Add}(\kappa,1)$ and $\iota\colon \mathrm{Add}(\kappa,1)_r \to \mathbb{Q}_p$ is a sub-isomorphism. Then there is a perfect limit-closed level subset S of $\mathrm{Add}(\kappa,1)_r^2$ such that π_S is a projection and*

$$\Vdash_{\mathrm{Add}(\kappa,1)_r} \mathbb{R}_p/\mathbb{Q}_p^{(\iota)} \simeq S/\mathrm{Add}(\kappa,1)_r^{\pi_S}.$$

Proof. Since $\mathrm{Add}(\kappa,1)_r$ is isomorphic to $\mathrm{Add}(\kappa,1)$, we can assume that $r = \mathbb{1}_{\mathrm{Add}(\kappa,1)}$ and $p = \mathbb{1}_{\mathbb{Q}}$. Let $\mathbb{Q}_0 = \iota[\mathbb{Q}]$ (note that ι necessarily preserves infima) and fix an arbitrary sub-isomorphism $\nu\colon \mathrm{Add}(\kappa,1) \to \mathbb{R}$. Moreover, we let $\pi\colon \mathbb{R} \to \mathbb{Q}$ denote the natural projection as given in Definition 1.18. Since $\pi(r) \geq r$ for all $r \in \mathbb{R}$, it is then easy to show that $\mathbb{R}/\mathbb{Q} = \mathbb{R}/\mathbb{Q}^\pi$.

Since π, ν are projections, it follows that $\pi\nu\colon \mathrm{Add}(\kappa,1) \to \mathbb{Q}$ is also a projection. Hence we can define

$$\dot{\mathbb{Q}} = \mathrm{Add}(\kappa,1)/\mathbb{Q}^{\pi\nu}.$$

Moreover, since ν is a sub-isomorphism, $\mathbb{Q}$ forces that $\nu\colon \dot{\mathbb{Q}} \to (\mathbb{R}/\mathbb{Q})^\pi$ is a sub-isomorphism. Thus by Lemma 1.15, it is sufficient to prove the existence of a set S as above with

$$\Vdash_{\mathrm{Add}(\kappa,1)} \dot{\mathbb{Q}}^{(\iota)} \simeq S/\mathrm{Add}(\kappa,1)^{\pi_S}$$

and we will prove this in the following claims.

We will write Lim for the class of limit ordinals. For any pair $(s,t) \in \mathrm{Add}(\kappa,1)^2$ with $\mathrm{l}(s) = \mathrm{l}(t) \in \mathrm{Lim}$, we further say that

$$\langle (s_\alpha, t_\alpha) \mid \alpha < \operatorname{cof} \mathrm{l}(s)\rangle$$

is an *intertwined sequence for* (s,t) if

$$s = \bigcup_{\alpha < \operatorname{cof} \mathrm{l}(s)} s_\alpha, \quad t = \bigcup_{\alpha < \operatorname{cof} \mathrm{l}(s)} t_\alpha$$

and $\pi\nu(t_{\alpha+1}) \leq \iota(s_\alpha) \leq \pi\nu(t_\alpha)$ for all for all $\alpha < \operatorname{cof} \mathrm{l}(s)$. We now consider the subset S of $\mathrm{Add}(\kappa,1)^2$ that consists of all pairs $(s,t) \in \mathrm{Add}(\kappa,1)^2$ with $\mathrm{l}(s) = \mathrm{l}(t) \in \mathrm{Lim}$ such that there is an intertwined sequence for (s,t).

Claim. *For every $(s,t) \in S$, there is some $(u,v) \leq (s,t)$ with $(\iota(u), \check{v}) \in \mathbb{Q}_0 * \dot{\mathbb{Q}}$.*

Proof. Suppose that $\langle (s_\alpha, t_\alpha) \mid \alpha < \operatorname{cof} \mathrm{l}(s)\rangle$ is an intertwined sequence for (s,t). Since $\pi\nu$ is order-preserving, we have

$$\pi\nu(t) \leq \pi\nu(t_{\alpha+1}) \leq \iota(s_\alpha)$$

for all $\alpha < \operatorname{cof} \mathrm{l}(s)$ and hence $\pi\nu(t) \leq \iota(s)$ by the assumption that ι preserves infima.

Since $\mathbb{Q}_0$ is dense in $\mathbb{Q}$, there is some $u \in \mathrm{Add}(\kappa,1)$ with $\iota(u) \leq \pi\nu(t)$. Then

$$\iota(u) \leq \pi\nu(t) \leq \iota(s)$$

and since ι is a sub-isomorphism, this implies that $u \leq s$ and hence $(u,t) \leq (s,t)$. Thus by the remark before the claim, (u,t) witnesses the conclusion of the claim. □

Claim. *For every $(u,v) \in \mathrm{Add}(\kappa,1)^2$ with $(\iota(u), \check{v}) \in \mathbb{Q}_0 * \dot{\mathbb{Q}}$, there is some $(s,t) \leq (u,v)$ in S.*

Proof. We can assume that $\mathrm{l}(u) > \mathrm{l}(v)$ by extending u. We will construct an intertwined sequence $\langle (s_n, t_n) \mid n < \omega\rangle$ by induction.

We choose $(s_0, t_0) = (u,v)$, so that $\iota(s_0) \leq \pi\nu(t_0)$ by the remark before the first claim. Now suppose that we have already constructed (s_n, t_n) with $\iota(s_n) \leq \pi\nu(t_n)$. Since $\pi\nu$ is a projection, there is some $t_{n+1} \leq t_n$ with $\pi\nu(t_{n+1}) \leq \iota(s_n)$, and we can further assume that $\mathrm{l}(t_{n+1}) > \mathrm{l}(s_n)$.

Moreover, since $\mathbb{Q}_0$ is dense in $\mathbb{Q}$, there is some $s_{n+1} \in \mathrm{Add}(\kappa,1)$ with $\iota(s_{n+1}) \leq \pi\nu(t_n)$, and we can further assume that $\mathrm{l}(s_{n+1}) > \mathrm{l}(t_{n+1})$. Then

$$\iota(s_{n+1}) \leq \pi\nu(t_n) \leq \iota(s_n)$$

and since ι is a sub-isomorphism, this implies that $s_{n+1} \leq s_n$ and hence $(s_{n+1}, t_{n+1}) \leq (s_n, t_n)$.

Letting $s = \bigcup_{n<\omega} s_n$, $t = \bigcup_{n\in\omega} t_n$, we have $\mathrm{l}(s) = \mathrm{l}(t)$ and there is an intertwined sequence for (s,t) by the construction. Thus $(s,t) \leq (u,v)$ and $(s,t) \in S$. □

Since $\mathbb{Q}_0$ is non-atomic, it follows immediately from the two previous claims that S is perfect. Moverover, since the projection onto the first coordinate of $\mathbb{Q} * \dot{\mathbb{Q}}$ is a projection in the sense of Definition 1.17, the claims show that $\pi_S \colon S \to \mathrm{Add}(\kappa,1)$ is also a projection.

Claim. *S is limit-closed.*

Proof. Suppose that $\langle (s_\alpha, t_\alpha) \mid \alpha < \operatorname{cof}\gamma \rangle$ is a strictly increasing sequence in S and

$$s = \bigcup_{\alpha < \operatorname{cof} \mathrm{l}(s)} s_\alpha, \quad t = \bigcup_{\alpha < \operatorname{cof} \mathrm{l}(s)} t_\alpha.$$

For each $\alpha < \operatorname{cof}\gamma$, we choose an element (u_α, v_α) of an intertwined sequence for $(s_{\alpha+1}, t_{\alpha+1})$ with $\mathrm{l}(u_\alpha) > \mathrm{l}(s_\alpha)$. It follows that $\langle (u_\alpha, v_\alpha) \mid \alpha < \operatorname{cof}\mathrm{l}(\gamma) \rangle$ is an intertwined sequence for (s,t). □

Claim. $\Vdash_{\mathrm{Add}(\kappa,1)} \dot{\mathbb{Q}}^{(\iota)} \simeq (S/\mathrm{Add}(\kappa,1))^{\pi_S}$.

Proof. We consider the forcing

$$T = \{(s,t) \in \mathrm{Add}(\kappa,1) \mid (s,t) \in S \text{ or } (\iota(s), \check{t}) \in \mathbb{Q}_0 * \dot{\mathbb{Q}}\}.$$

We first claim that $\mathrm{Add}(\kappa,1)$ forces that $S/\mathrm{Add}(\kappa,1)^{\pi_S}$ is a dense subforcing of $T/\mathrm{Add}(\kappa,1)^{\pi_T}$. To prove this, assume that G is $\mathrm{Add}(\kappa,1)$-generic over V and

$$(s,t) \in [S/\mathrm{Add}(\kappa,1)^{\pi_S}]^G,$$

so that $\pi_S(s,t) = s \in G$. Since π_S is a projection and by the claims above, the set

$$D = \{u \leq s \mid \exists v \ (u,v) \leq (s,t), \ (\iota(u), \check{v}) \in \mathbb{Q}_0 * \dot{\mathbb{Q}}\}$$

is dense below s in $\mathrm{Add}(\kappa,1)$. Letting $u \in G \cap D$, there is some v with $(u,v) \leq (s,t)$ and $(\iota(u), \check{v}) \in \mathbb{Q}_0 * \dot{\mathbb{Q}}$. Since $(\iota(u), \check{v}) \in \mathbb{Q}_0 * \dot{\mathbb{Q}}$ and $\mathbb{Q}_0$ is separative, we have $\iota(u) \leq \pi\nu(v)$ by the definition of $\mathbb{Q}_0$. We now write $G^{(\iota)}$ for the upwards closure of $\iota[G]$ in $\mathbb{Q}$. Since $u \in G$, we have $\iota(u) \in G^{(\iota)}$, $\pi\nu(v) \in G^{(\iota)}$ and hence

$$(u,v) \in [\dot{\mathbb{Q}}^{(\iota)}]^G = [\mathrm{Add}(\kappa,1)/\mathbb{Q}^{\pi\nu}]^G.$$

An analogous argument shows that $\mathrm{Add}(\kappa,1)$ also forces that $\dot{\mathbb{Q}}^{(\iota)}$ is a dense subforcing of $T/\mathrm{Add}(\kappa,1)^{\pi_T}$. □

The last claim completes the proof of Lemma 3.19. □

We now fix a perfect level subset S of $\mathrm{Add}(\kappa,1)^2$ such that $\pi_S \colon S \to \mathrm{Add}(\kappa,1)$ is a projection and let $\mathbb{P} = \mathbb{P}_S$. Since S is perfect, it is easy to see that $\mathbb{P}$ is a non-atomic ${<}\kappa$-closed forcing of size κ and hence $\mathbb{P}$ and $\mathrm{Add}(\kappa,1)$ are sub-equivalent by Lemma 1.21.

In the remainder of this section, we will consider $\mathbb{P}$-names $\dot{f}$, $\dot{g}$ such that

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \dot{f}, \dot{g} \colon \kappa \to {}^{<\kappa}\kappa, \ \forall \alpha < \kappa \ (\dot{f}{\restriction}\alpha, \dot{g}{\restriction}\alpha) \in T_{\dot{G}},$$

where $\dot{G}$ is a fixed name for the $\mathbb{P}$-generic filter. We will call such pairs $(\dot{f}, \dot{g})$ *adequate* and will always assume below that $(\dot{f}, \dot{g})$, $(\dot{h}, \dot{k})$ are such pairs.

The aim of the next lemmas is to show that for any adequate pair $(\dot{f}, \dot{g})$, there is a dense subforcing of $\mathbb{P}$ that projects onto a forcing for adding $\bigcup \mathrm{ran}(\dot{f})$, $\bigcup \mathrm{ran}(\dot{g})$ with a nice quotient forcing. This follows a similar line of reasoning as the arguments for the perfect set property in Section 2.2.

Definition 3.20. Let $\mathbb{P}^*_{\dot{f},\dot{g}}$ be the subforcing of $\mathbb{P}$ consisting of the conditions p such that the following statements hold for some $\gamma_p < \kappa$ and some $f_p, g_p \in {}^{<\kappa}\mathrm{Add}(\kappa,1)$.

(a) $\mathrm{l}(p) = \mathrm{ht}(p) = \gamma_p \in \mathrm{Lim}$.
(b) $p \Vdash_{\mathbb{P}} \dot{f}{\restriction}\gamma_p = f_p, \ \dot{g}{\restriction}\gamma_p = g_p$.

(c) $(f_p{\restriction}\alpha, g_p{\restriction}\alpha) \in p$ for all $\alpha < \gamma_p$.

Let further $\mathbb{P}^{\diamond}_{\dot f,\dot g}$ be the subforcing of $\mathbb{P}$ consisting of the conditions p that satisfy requirements (a) and (b). Moreover, let $s_p = \bigcup \mathrm{ran}(f_g{\restriction}\gamma_p)$ and $t_p = \bigcup \mathrm{ran}(g_p{\restriction}\gamma_p)$ for any $p \in \mathbb{P}^{\diamond}_{\dot f,\dot g}$.

We will also denote the corresponding values for an adequate pair $(\dot h, \dot k)$ and any $q \in \mathbb{P}^{\diamond}_{\dot h,\dot k}$ by $h_p, k_p \in {}^{<\kappa}\mathrm{Add}(\kappa,1)$ and $u_p, v_p \in {}^{<\kappa}\kappa$.

Lemma 3.21. *$\mathbb{P}^{\diamond}_{\dot f,\dot g} \cap \mathbb{P}^{\diamond}_{\dot h,\dot k}$ is a dense subforcing of $\mathbb{P}$.*

Proof. Note that in general, we have $\mathrm{l}(p) \leq \mathrm{ht}(p)$ for all $p \in \mathbb{P}$ by the definition of the length and the height. To prove the claim, we assume that p in $\mathbb{P}$ and construct a strictly decreasing sequence $\langle p_n \mid n \in \omega\rangle$ in $\mathbb{P}$ with $p_0 = p$ as follows.

If p_n is defined and $\mathrm{ht}(p_n) = \alpha$, we choose a condition p_{n+1} with $\mathrm{l}(p_{n+1}) > \alpha$ that decides $\dot f{\restriction}\alpha$, $\dot g{\restriction}\alpha$, $\dot h{\restriction}\alpha$ and $\dot k{\restriction}\alpha$. Then $p^{\diamond} = \bigcup_{n\in\omega} p_n$ is a condition in $\mathbb{P}$ with $p^{\diamond} \leq p$ that satisfies requirements (a) and (b) in Definition 3.20 for both $(\dot f, \dot g)$ and $(\dot h, \dot k)$, and thus $p^{\diamond} \in \mathbb{P}^{\diamond}_{\dot f,\dot g} \cap \mathbb{P}^{\diamond}_{\dot h,\dot k}$. □

Using the following lemma, we will see that $\mathbb{P}^{*}_{\dot f,\dot g}$ is also a dense subforcing of $\mathbb{P}$.

Lemma 3.22. *Suppose that p is a condition in $\mathbb{P}^{\diamond}_{\dot f,\dot g}$ and $\beta, \gamma \leq \gamma_p$ are even. Moreover, suppose that $q \leq p$ is a condition in $\mathbb{P}$ and $(s,t) \in q$ with $\mathrm{l}(s) = \mathrm{l}(t) > \beta$ and*

$$\bigcup \mathrm{ran}(s{\restriction}\beta) = \bigcup \mathrm{ran}(f_p{\restriction}\gamma),\ \bigcup \mathrm{ran}(t{\restriction}\beta) = \bigcup \mathrm{ran}(g_p{\restriction}\gamma).$$

Then $q \Vdash_{\mathbb{P}} \dot f(\gamma) = s(\beta),\ \dot g(\gamma) = t(\beta)$.

Proof. We assume that G is any $\mathbb{P}$-generic filter over V with $q \in G$ and let $(u,v) = (\dot f^G{\restriction}\gamma + 1, \dot g^G{\restriction}\gamma + 1)$. Since $(\dot f, \dot g)$ is an adequate pair, it follows that $(u,v) \in T_G$. Thus $(s,t), (u,v)$ are elements of the same S-tree T_G and therefore $s(\beta) = u(\gamma)$ and $t(\beta) = v(\gamma)$ by Definition 3.12 (c), as required. □

Lemma 3.23. *$\mathbb{P}^{*}_{\dot f,\dot g} \cap \mathbb{P}^{*}_{\dot h,\dot k}$ is a dense subforcing of $\mathbb{P}$.*

Proof. We will derive the conclusion from the next claim.

Claim. *For any condition $p \in \mathbb{P}^{\diamond}_{\dot f,\dot g}$, we have that $p \cup \{(f_p{\restriction}\alpha, g_p{\restriction}\alpha) \mid \alpha < \gamma_p\}$ is again a condition in $\mathbb{P}^{\diamond}_{\dot f,\dot g}$.*

Proof. We fix a condition $p \in \mathbb{P}^{\diamond}_{\dot f,\dot g}$. For any even ordinal $\gamma < \gamma_p$, let Ψ_γ denote the statement that there exist an even ordinal $\beta < \gamma_p$ and some $(s,t) \in p$ with $\mathrm{l}(s) = \mathrm{l}(t) > \beta$ that satisfy the following conditions.

(a) $\bigcup \mathrm{ran}(s{\restriction}\beta) = \bigcup \mathrm{ran}(f_p{\restriction}\gamma)$ and $\bigcup \mathrm{ran}(t{\restriction}\beta) = \bigcup \mathrm{ran}(g_p{\restriction}\gamma)$.
(b) $s(\beta) = f_p(\gamma)$ and $t(\beta) = g_p(\gamma)$.

Subclaim. *If $\delta \leq \gamma_p$ is an even ordinal and Ψ_γ holds for all even ordinals $\gamma < \delta$, then $q = p \cup \{(f_p{\restriction}\gamma, g_p{\restriction}\gamma) \mid \gamma < \delta\}$ is a condition in $\mathbb{P}^{\diamond}_{\dot f,\dot g}$.*

Proof. It is sufficient to check that q satisfies Definition 3.12 (c). To this end, suppose that $\gamma < \delta$ is even, $(u,v) \in p$, $\mathrm{l}(u) = \delta$ is even, $\bigcup \mathrm{ran}(u{\restriction}\alpha) = \bigcup \mathrm{ran}(f_p{\restriction}\gamma)$ and $\bigcup \mathrm{ran}(v{\restriction}\alpha) = \bigcup \mathrm{ran}(g_p{\restriction}\gamma)$. Now let $\beta < \gamma_p$ and $(s,t) \in p$ witness Ψ_γ. It follows from condition (a) and Definition 3.12 (c) for p that $u(\alpha) = s(\beta)$ and $v(\alpha) = t(\beta)$. Moreover, by condition (b), $u(\alpha) = s(\beta) = f_p(\gamma)$ and $v(\alpha) = t(\beta) = g_p(\gamma)$, as required. □

Subclaim. *Ψ_γ holds for all even ordinals $\gamma < \gamma_p$.*

Proof. Towards a contradiction, we assume that $\gamma < \gamma_p$ is the least even ordinal such that Ψ_γ fails. Since Ψ_α holds for all even ordinals $\alpha < \gamma$ by the minimality of γ, the previous subclaim implies that

$$q = p \cup \{(f_p{\restriction}\alpha, g_p{\restriction}\alpha) \mid \alpha < \gamma\}$$

is a condition in $\mathbb{P}$.

Since S is perfect, there is some $(u,v) \in S$ with $u \supseteq \bigcup \mathrm{ran}(f_p{\restriction}\alpha)$ and $v \supseteq \bigcup \mathrm{ran}(g_p{\restriction}\alpha)$. We can further assume that $(u,v) \neq (f_p(\gamma), g_p(\gamma))$ by extending u, v.

If (a) holds for an even ordinal $\beta < \gamma_p$ and some $(s,t) \in p$ with $\mathrm{l}(s) = \mathrm{l}(t) > \beta$, we also have (b) by Lemma 3.22. Hence we can assume that there are no such $\beta < \gamma_p$ and $(s,t) \in p$. It follows that $q \cup \{(u,v)\}$ is a condition in $\mathbb{P}$ by Definition 3.12 (c) and further $q \Vdash_{\mathbb{P}} (\dot{f}(\gamma), \dot{g}(\gamma)) = (u,v)$ by Lemma 3.22. However, since $q \leq p$, this contradicts the fact that $(u,v) \neq (f_p(\gamma), g_p(\gamma))$. □

The previous subclaims show that $r = p \cup \{(f_p\restriction\alpha, g_p\restriction\alpha) \mid \alpha < \gamma_p\}$ is a condition in $\mathbb{P}$. Since moreover $p \in \mathbb{P}^{\diamond}_{\dot{f},\dot{g}}$ and $\mathrm{l}(r) = \mathrm{ht}(r) = \gamma_p$, we have $r \in \mathbb{P}^{\diamond}_{\dot{f},\dot{g}}$. □

To see that $\mathbb{P}^{*}_{\dot{f},\dot{g}} \cap \mathbb{P}^{*}_{\dot{h},\dot{k}}$ is a dense subforcing of $\mathbb{P}$, assume that p is an arbitrary condition in $\mathbb{P}$. By Lemma 3.21, there is some $q \leq p$ in $\mathbb{P}^{\diamond}_{\dot{f},\dot{g}} \cap \mathbb{P}^{\diamond}_{\dot{h},\dot{k}}$. By the previous claim applied to $(\dot{f},\dot{g})$ and q, we obtain some $r \leq q$ in $\mathbb{P}^{*}_{\dot{f},\dot{g}} \cap \mathbb{P}^{\diamond}_{\dot{h},\dot{k}}$, and by then applying the claim to $(\dot{h},\dot{k})$ and r, we obtain the required condition $s \leq r$ in $\mathbb{P}^{*}_{\dot{f},\dot{g}} \cap \mathbb{P}^{*}_{\dot{h},\dot{k}}$. □

As for $\mathbb{P}$, it is easy to see that $\mathbb{P}^{*}_{\dot{f},\dot{g}}$ is a non-atomic $<\kappa$-closed forcing of size κ and hence $\mathbb{P}^{*}_{\dot{f},\dot{g}}$ and $\mathrm{Add}(\kappa,1)$ are sub-equivalent by Lemma 1.21.

As defined before Lemma 3.19, we will write $\mathbb{Q}_p$ for the subforcing

$$\mathbb{Q}_p = \{q \in \mathbb{Q} \mid q \leq p\}$$

of a forcing $\mathbb{Q}$ below a condition $p \in \mathbb{Q}$ in the following lemmas.

Lemma 3.24. *Letting $\mathbb{P}^* = \mathbb{P}^{*}_{\dot{f},\dot{g}} \cap \mathbb{P}^{*}_{\dot{h},\dot{k}}$, for any condition r in $\mathbb{P}^*$ with $(s_r,t_r) \neq (u_r,v_r)$, the map*

$$\tau_r : \mathbb{P}^*_r \to S_{(s_r,t_r)} \times S_{(u_r,v_r)},\ \ \tau_r(p) = ((s_p,t_p),(u_p,v_p))$$

is a projection.

Proof. It follows from the definition of s_p, t_p, u_p, v_p that ρ_r is order-preserving. For the remaining requirement on projections, suppose that $p \in \mathbb{P}^*$, $p \leq r$ and $((s,t),(u,v)) \in S_{(s_r,t_r)} \times S_{(u_r,v_r)}$ are given with $s_p \subseteq s$, $t_p \subseteq t$, $u_p \subseteq u$, $v_p \subseteq v$. We can moreover assume that these subsets are strict by extending s, t, u, v.

By the definition of $\mathbb{P}^{*}_{\dot{f},\dot{g}}$ and $\mathbb{P}^{*}_{\dot{h},\dot{k}}$, we have $(f_p\restriction\alpha, g_p\restriction\alpha), (h_p\restriction\alpha, k_p\restriction\alpha) \in p$ for all $\alpha < \gamma_p$. Since moreover $(s_r,t_r) \neq (u_r,v_r)$,

$$q = p \cup \{(f_p,g_p), (f_p^\frown(\gamma_p,s), g_p^\frown(\gamma_p,t)), (h_p,k_p), (h_p^\frown(\gamma_p,s), k_p^\frown(\gamma_p,t))\}$$

is downwards closed and satisfies Definition 3.12 (c), hence it is a condition in $\mathbb{P}$. Finally,

$$q \Vdash_{\mathbb{P}} \dot{f}(\gamma_p) = s,\ \dot{g}(\gamma_p) = t,\ \dot{h}(\gamma_p) = u,\ \dot{k}(\gamma_p) = v$$

by Lemma 3.22. Now any condition $r \leq q$ in $\mathbb{P}^*$ is as required. □

In the next two lemmas, we let $\mathbb{P}^* = \mathbb{P}^{*}_{\dot{f},\dot{g}}$.

Lemma 3.25. *Letting $\mathbb{P}^* = \mathbb{P}^{*}_{\dot{f},\dot{g}}$, for any condition r in $\mathbb{P}^*$, the map*

$$\rho_r : \mathbb{P}^*_r \to S_{(s_r,t_r)},\ \ \rho_r(p) = (s_p,t_p)$$

is a projection and (s_r,t_r) forces that the quotient forcing $[\mathbb{P}^*_r / S_{(s_r,t_r)}]^{\rho_r}$ and $\mathrm{Add}(\kappa,1)$ are sub-equivalent.

Proof. It can be proved as in the proof of Lemma 3.24 that ρ_r is a projection and moreover, it follows from Lemma 3.24 that the quotient forcing $[\mathbb{P}^*_r / S_{(s_r,t_r)}]^{\rho_r}$ is non-atomic. Since the quotient forcing had size κ and is $<\kappa$-closed by the definitions of s_p, t_p and $\mathbb{P}^{*}_{\dot{f},\dot{g}}$, it is sub-equivalent to $\mathrm{Add}(\kappa,1)$ by Lemma 1.21. □

Our next aim is to calculate a quotient forcing for a given branch in the superclosed S-tree that is added by $\mathbb{P}$. Since it is convenient to work with a separative forcing, but $\mathbb{P}$ and $\mathbb{P}^*$ are not separative, we will assume that $\mathbb{T}$ is a dense subforcing of $\mathbb{P}^*$ that is isomorphic to $\mathrm{Add}^*(\kappa,1)$ and that $\dot{T}_{\mathbb{T}}$ is a name for the superclosed S-tree added by $\mathbb{T}$. We will further assume that $\dot{b}$ is a $\mathbb{T}$-name with $\mathbb{1}_{\mathbb{P}} \Vdash \dot{b} = \mathrm{ran}(\bigcup \dot{f})$ for the adequate pair $(\dot{f},\dot{g})$ considered above.

If moreover r is any condition in $\mathbb{T}$, then

$$\pi_S\rho_r : \mathbb{P}^*_r \to \mathrm{Add}(\kappa,1)_{s_r},\ \ \pi_S\rho_r(p) = s_p$$

is a projection, since $\rho_r: \mathbb{P}^*_r \to S_{(s_r,t_r)}$ is a projection by Lemma 3.25 and π_S is a projection by the assumption on S.

For any $r \in \mathbb{T}$, we further choose a $\mathbb{T}_r$-name $\dot{b}_r$ with $r \Vdash_{\mathbb{T}} \dot{b} = \dot{b}_r$. It follows from the definition of s_p that $\mathbb{1}_{\mathbb{T}}$ forces that $\dot{b}_r = \bigcup_{p \in \dot{G}} s_p$, where $\dot{G}$ is a name for the $\mathbb{T}$-generic filter. Using the fact that $\pi_S \rho_r$ is a projection, it then follows easily that r forces that $\dot{b}_r$ is $\mathrm{Add}(\kappa,1)$-generic over V. Moreover, since this holds for every condition r in $\mathbb{T}$, it follows that $\mathbb{1}_{\mathbb{T}}$ forces that $\dot{b}$ is $\mathrm{Add}(\kappa,1)$-generic over V.

In the next lemma, we will fix a condition r in $\mathbb{T}$ and let $\mathbb{R} = \mathbb{B}(\mathbb{T}_r)$, $\mathbb{Q} = \mathbb{B}^{\mathbb{R}}(\dot{b})$. It is clear that the map

$$\iota: \mathrm{Add}(\kappa,1)_{s_r} \to \mathbb{Q},\ \iota(s) = [\![s \subseteq \dot{b}]\!]$$

preserves $\leq$ and $\perp$, and since $\pi_S \rho_r$ is a projection, we have that $\iota(s) \neq 0_{\mathbb{Q}}$ for all $s \in \mathrm{Add}(\kappa,1)_{s_r}$ and that $\mathrm{ran}(\iota)$ is dense in $\mathbb{Q}$, so that ι is a sub-isomorphism.

We will further consider the natural projection $\pi: \mathbb{R} \to \mathbb{Q}$, $\pi(p) = \inf_{p \leq q \in \mathbb{Q}} q$. Since $\mathbb{T}$ is dense in $\mathbb{P}^*$, $\pi \restriction \mathbb{T}_r$ and $\pi_S \rho_r \restriction \mathbb{T}_r$ are projections and it can be checked from the definitions of π_s, ρ_r that $\pi \restriction \mathbb{T}_r = \iota \pi_S \rho_r \restriction \mathbb{T}_r$.

Lemma 3.26. *Suppose that $\mathbb{T}$ and $\dot{b}$ are as above and $r \in \mathbb{T}$.*

(1) *If $\pi: \mathbb{R} \to \mathbb{Q}$ and $\iota: \mathrm{Add}(\kappa,1)_{s_r} \to \mathbb{Q}$ are as above, then*

$$\Vdash_{\mathrm{Add}(\kappa,1)_{s_r}} (\mathbb{R}_r/\mathbb{Q}^\pi)^{(\iota)} \simeq [S_{(s_r,t_r)}/\mathrm{Add}(\kappa,1)_{s_r}]^{\pi_S} \times \mathrm{Add}(\kappa,1).$$

(2) *If G is $\mathbb{T}$-generic over V with $r \in G$, then there is an $([S_{(s_r,t_r)}/\mathrm{Add}(\kappa,1)_{s_r}]^{\pi_S})^{\dot{b}^G} \times \mathrm{Add}(\kappa,1)$-generic filter h over $W = V[\dot{b}^G]$ with $W[h] = V[G]$.*

Proof. Since we argued before this lemma that $\pi \restriction \mathbb{T}_r = \iota \pi_S \rho_r \restriction \mathbb{T}_r$, we have

$$\Vdash_{\mathrm{Add}(\kappa,1)_{s_r}} (\mathbb{T}_r/\mathbb{Q}^{\pi \restriction \mathbb{T}_r})^{(\iota)} = [\mathbb{T}_r/\mathrm{Add}(\kappa,1)_{s_r}]^{\pi_S \rho_r \restriction \mathbb{T}_r}. \tag{3.1}$$

Moreover, since $\mathbb{T}_r$ is dense in both $\mathbb{R}_r$ and $\mathbb{P}^*_r$, $\mathrm{Add}(\kappa,1)_{s_r}$ forces that

$$(\mathbb{T}_r/\mathbb{Q}^{\pi \restriction \mathbb{T}_r})^{(\iota)} \subseteq (\mathbb{R}_r/\mathbb{Q}^\pi)^{(\iota)}$$

$$[\mathbb{T}_r/\mathrm{Add}(\kappa,1)_{s_r}]^{\pi_S \rho_r \restriction \mathbb{T}_r} \subseteq [\mathbb{P}^*_r/\mathrm{Add}(\kappa,1)_{s_r}]^{\pi_S \rho_r}$$

are dense subforcings. With equation 3.1, this shows that

$$\Vdash_{\mathrm{Add}(\kappa,1)_{s_r}} (\mathbb{R}_r/\mathbb{Q}^\pi)^{(\iota)} \simeq [\mathbb{P}^*_r/\mathrm{Add}(\kappa,1)_{s_r}]^{\pi_S \rho_r}. \tag{3.2}$$

Using Lemma 3.25 and the properties of projections, one can now show that

$$\Vdash_{\mathrm{Add}(\kappa,1)_{s_r}} [\mathbb{P}^*_r/\mathrm{Add}(\kappa,1)_{s_r}]^{\pi_S \rho_r} \simeq [S_{(s_r,t_r)}/\mathrm{Add}(\kappa,1)_{s_r}]^{\pi_S} \times \mathrm{Add}(\kappa,1). \tag{3.3}$$

By equations 3.2 and 3.3 and Lemma 1.15,

$$\Vdash_{\mathrm{Add}(\kappa,1)_{s_r}} (\mathbb{R}_r/\mathbb{Q}^\pi)^{(\iota)} \simeq [S_{(s_r,t_r)}/\mathrm{Add}(\kappa,1)_{s_r}]^{\pi_S} \times \mathrm{Add}(\kappa,1). \tag{3.4}$$

For the second claim, it follows from the definition of ι that

$$[(\mathbb{R}_r/\mathbb{Q}^\pi)^{(\iota)}]^G = (\mathbb{R}_r/\mathbb{Q}^\pi)^{G^{(\iota)}} = (\mathbb{R}_r/\mathbb{Q}^\pi)^{\dot{b}^G},$$

where $G^{(\iota)}$ denotes the upwards closure of $\iota[G]$ in $\mathbb{Q}$. Then by equation 3.4,

$$(\mathbb{R}_r/\mathbb{Q}^\pi)^{\dot{b}^G} \simeq ([S_{(s_r,t_r)}/\mathrm{Add}(\kappa,1)_{s_r}]^{\pi_S})^{\dot{b}^G} \times \mathrm{Add}(\kappa,1).$$

The claim now follows from the standard properties of quotient forcings. $\square$

Lemma 3.27. *Suppose that $\mathbb{S}$ is a $<\kappa$-distributive forcing and $F = G \times H \times I$ is $\mathrm{Add}(\kappa,1) \times \mathrm{Add}(\kappa,1) \times \mathbb{S}$-generic over V. Moreover, suppose that*

$$x \in (A^\kappa_{\varphi,z})^{V[F]} \cap {}^\kappa\kappa \cap V[G]$$

is $\mathrm{Add}(\kappa,1)$-generic over V, where $\varphi(u,v)$ is a formula and $z \in V[I]$. Then in $V[F]$, there is a winning strategy for player I in $G_\kappa((A^\kappa_{\varphi,z})^{V[F]})$.

Proof. We first note that x is $\mathrm{Add}(\kappa,1)$-generic over $V[I]$, since G, I are mutually generic. Therefore, by replacing $V[I]$ with V, the claim follows from the claim for the special case where $\mathbb{S}$ does not add any new sets, which we assume in the following.

Suppose that $\dot{x}$ is an $\mathrm{Add}(\kappa,1)$-name for x such that $\mathbb{1}_{\mathrm{Add}(\kappa,1)}$ forces that $\Vdash_{\mathrm{Add}(\kappa,1)} \dot{x} \in A_{\varphi,z}$ holds and that $\dot{x}$ is $\mathrm{Add}(\kappa,1)$-generic over V. Let further $\mathbb{R} = \mathbb{B}(\mathrm{Add}(\kappa,1))$, $\mathbb{Q} = \mathbb{B}(\dot{x})^{\mathbb{R}}$ and

$$\nu\colon \mathrm{Add}(\kappa,1) \to \mathbb{Q},\ \nu(s) = [\![s \subseteq \dot{x}]\!]^{\mathbb{R}}.$$

Claim. *There is a condition $r \in \mathrm{Add}(\kappa,1)$ such that $\nu \restriction \mathrm{Add}(\kappa,1)_r\colon \mathrm{Add}(\kappa,1)_r \to \mathbb{Q}_{\nu(r)}$ is a sub-isomorphism.*

Proof. We first claim that there is a condition $r \in \mathrm{Add}(\kappa,1)$ such that for all $s \leq r$ in $\mathrm{Add}(\kappa,1)$ and all $\alpha < \kappa$, $\nu(s) \neq \nu(s^\frown\langle\alpha\rangle)$. Otherwise

$$D = \{s^\frown\langle\alpha\rangle \mid s \in \mathrm{Add}(\kappa,1),\ \alpha < \kappa,\ \exists\beta \neq \alpha\ \nu(s) = \nu(s^\frown\langle\beta\rangle)\}$$

is dense in $\mathrm{Add}(\kappa,1)$. However, by the definition of ν, this contradicts the assumption that $\dot{x}$ is a name for an $\mathrm{Add}(\kappa,1)$-generic over V.

We now fix such a condition $r \in \mathrm{Add}(\kappa,1)$. To prove the claim, it is sufficient to show that the subforcing $\mathbb{U} = \{\nu(s) \mid s \in \mathrm{Add}(\kappa,1),\ s \leq r\}$ is dense in $\mathbb{Q}_{\nu(r)}$.

Subclaim. $\mathbb{U} \lessdot \mathbb{R}_{\nu(r)}$.

Proof. Otherwise, there is a subset A of $\mathbb{U}$ that is an antichain in $\mathbb{R}_{\nu(r)}$ and is maximal in $\mathbb{U}$, but not in $\mathbb{R}_{\nu(r)}$. We can then choose some $q \in \mathbb{R}_{\nu(r)}$ that is incompatible with all elements of A. However, if J is $\mathbb{R}_{\nu(r)}$-generic over V with $q \in J$, then $\dot{x}^J$ cannot be $\mathrm{Add}(\kappa,1)$-generic over V by the choice of A and q, contradicting the choice of $\dot{x}$. □

Let $\mathbb{V}$ denote the Boolean subalgebra of $\mathbb{Q}_{\nu(r)}$ generated by $\mathbb{U}$. Since $\mathbb{U}$ is closed under finite conjunctions, $\mathbb{U}$ is dense in $\mathbb{V}$ and it hence follows from the previous subclaim that $\mathbb{V} \lessdot \mathbb{R}_{\nu(r)}$. It then follows from [Jec03, Exercise 7.31] applied to $\mathbb{V}$ and $\mathbb{R}_{\nu(r)}$ that $\mathbb{Q}_{\nu(r)}$ is a Boolean completion of $\mathbb{V}$, in particular $\mathbb{V}$ is dense in $\mathbb{Q}_{\nu(r)}$. □

Suppose that $r \in \mathrm{Add}(\kappa,1)$ is chosen as in the previous claim and let $\iota = \nu \restriction \mathrm{Add}(\kappa,1)_r$. We can further assume that $r = \mathbb{1}_{\mathrm{Add}(\kappa,1)}$, since the remaining proof is analogous for arbitrary r.

We further choose a $\mathbb{Q}$-name $\dot{x}_{\mathbb{Q}}$ with $\Vdash_{\mathbb{R}} \dot{x}_{\mathbb{Q}} = \dot{x}$ and an $\mathrm{Add}(\kappa,1)$-name $\dot{y}$ for the $\mathrm{Add}(\kappa,1)$-generic real, so that $\Vdash_{\mathbb{Q}}\ \iota(\dot{y}) = \dot{x}_{\mathbb{Q}}$ by the definition of ι.

By Lemma 3.19, there is a perfect limit-closed level subset S of $\mathrm{Add}(\kappa,1)^2$ such that π_S is a projection and

$$\Vdash_{\mathrm{Add}(\kappa,1)} \mathbb{R}/\mathbb{Q}^{(\iota)} \simeq S/\mathrm{Add}(\kappa,1)^{\pi_S}. \tag{3.5}$$

It follows from the properties of $\dot{x}$ stated above that

$$\Vdash_{\mathbb{Q}} \Vdash_{\mathbb{R}/\mathbb{Q}\times\mathrm{Add}(\kappa,1)} \dot{x}_{\mathbb{Q}} \in A^{\kappa}_{\varphi,z}.$$

Since ι is a sub-isomorphism and by the properties of $\dot{y}$ and $\dot{x}_{\mathbb{Q}}$, this implies

$$\Vdash_{\mathrm{Add}(\kappa,1)} \Vdash_{\mathbb{R}/\mathbb{Q}^{(\iota)}\times\mathrm{Add}(\kappa,1)} \dot{y} \in A^{\kappa}_{\varphi,z}$$

and by equation 3.5,

$$\Vdash_{\mathrm{Add}(\kappa,1)} \Vdash_{S/\mathrm{Add}(\kappa,1)^{\pi_S}\times\mathrm{Add}(\kappa,1)} \dot{y} \in A^{\kappa}_{\varphi,z}. \tag{3.6}$$

Now suppose that $\dot{T}$ is a $\mathbb{P}_S$-name for the tree added by the $\mathbb{P}_S$-generic filter. In the next claim, we will identify $\dot{T}$ with the induced $\mathbb{P}_S \times \mathrm{Add}(\kappa,1)$-name.

Claim. $\mathbb{P}_S \times \mathrm{Add}(\kappa,1)$ *forces that* $\mathrm{p}[\dot{T}] \subseteq A^{\kappa}_{\varphi,z}$.

Proof. Suppose that $\dot{b}$ is a $\mathbb{P}_S$-name with $\mathbb{1} \Vdash_{\mathbb{P}_S} \dot{b} \in \mathrm{p}[\dot{T}]$. We can then find an adequate pair $(\dot{f},\dot{g})$ with $\mathbb{1}_{\mathbb{P}_S} \Vdash \bigcup \mathrm{ran}(\dot{f}) = \dot{b}$ and let $\mathbb{P}^* = \mathbb{P}^*_{\dot{f},\dot{g}}$.

Now let $\mathbb{T}$ be the dense subforcing of $\mathbb{P}^*$ that is introduced before Lemma 3.26. Moreover, suppose that G is $\mathbb{T}$-generic over V and $r \in G$. Since $\mathbb{T}$ is dense in $\mathbb{P}_S$, we can assume that $\dot{b}$ is a $\mathbb{T}$-name. Then there is an $([S_{(s_r,t_r)}/\mathrm{Add}(\kappa,1)_{s_r}]^{\pi_S})^{\dot{b}^G} \times \mathrm{Add}(\kappa,1)$-generic filter h over $W = V[\dot{b}^G]$ with $W[h] = V[G]$ by Lemma 3.26.

Since $\mathrm{Add}(\kappa,1)^2 \simeq \mathrm{Add}(\kappa,1)$ and since r forces that $[S_{(s_r,t_r)}/\mathrm{Add}(\kappa,1)_{s_r}]^{\pi_S}$ is a complete subforcing of $[S/\mathrm{Add}(\kappa,1)]^{\pi_S}$, the claim now follows from equation 3.6. □

Lemma 3.16 implies that $\mathbb{P}_S \times \mathrm{Add}(\kappa, 1)$ forces that player I has a winning strategy in $G_\kappa(\mathrm{p}[\dot{T}])$. Since $\mathbb{P}_S \times \mathrm{Add}(\kappa, 1)$ is sub-equivalent to $\mathrm{Add}(\kappa, 1)^2$, the statement now follows from the previous claim. □

In the next proof, we will use the notation $\mathrm{Col}(\lambda, X)$ for collapse forcings that was introduced before Theorem 2.19. We will further use the analogous notation $\mathrm{Add}(\lambda, X)$ to denote the subforcing of $\mathrm{Add}(\lambda, \nu)$ with support $X \subseteq \nu$ and let $G_X = G \cap \mathrm{Add}(\lambda, X)$ for any $\mathrm{Add}(\lambda, \nu)$-generic filter G.

Theorem 3.28. *Suppose that λ is an uncountable regular cardinal, $\mu > \lambda$ is inaccessible and ν is any cardinal. Then* $\mathrm{Col}(\lambda, {<}\mu) \times \mathrm{Add}(\kappa, \nu)$ *forces that $G_\lambda(A)$ is determined for every subset A of ${}^\lambda\lambda$ that is definable from an element of ${}^\lambda V$.*

Proof. We work in an extension of V by a fixed $\mathrm{Col}(\lambda, {<}\mu) \times \mathrm{Add}(\kappa, \nu)$-generic filter $G \times H$. First note that every $x \in {}^\lambda\lambda$ is an element of $V[G_\xi \times H_X]$ for some $\xi < \mu$ and some subset X of ν of size strictly less than μ, since $\mathrm{Col}(\lambda, {<}\mu) \times \mathrm{Add}(\kappa, \nu)$ has the μ-cc by the Δ-system lemma. In this situation, we will say that x is *absorbed by G_ξ, H_X*.

Now assume that $\varphi(x, y)$ is a formula with two free variables and $z \in {}^\lambda V$. We let

$$A^M = (A^\lambda_{\varphi,z})^{V[G\times H]} \cap M$$

for any transitive subclass M of $V[G \times H]$, where $A^\lambda_{\varphi,z}$ is given in Definition 1.10.

Since $\mathrm{Add}(\lambda, 1)$ is ${<}\lambda$-closed and $P(\mathrm{Add}(\lambda, 1))^V$ has size λ, the set of $\mathrm{Add}(\lambda, 1)$-generic elements of ${}^\lambda\lambda$ over V is comeager. Therefore, if there is no $\mathrm{Add}(\lambda, 1)$-generic element of ${}^\lambda\lambda$ over V in $A^\lambda_{\varphi,z}$, then by Lemma 3.8, player II has a winning strategy in $G_\lambda(A^\lambda_{\varphi,z})$. We can hence assume that there is an $\mathrm{Add}(\lambda, 1)$-generic element x of $A^\lambda_{\varphi,z}$ over V.

We will rearrange the generic extension to apply Lemma 3.27. To this end, we assume that x is absorbed by G_ξ, H_X as above. It follows from Lemma 1.21 that we can find a $\mathrm{Col}(\lambda, [\xi, \mu)) \times \mathrm{Add}(\lambda, 1)$-generic filter $g \times h$ with $V[G_{[\xi,\mu)}] = V[g \times h]$ and hence the generic extension can be written as

$$V[G \times H] = V[g \times H_{\mu \setminus X} \times h \times G_\xi \times H_X].$$

Since the filters $g \times H_{\nu \setminus X} \times h$ and $G_\xi \times H_X$ are mutually generic, it follows that x is also $\mathrm{Add}(\lambda, 1)$-generic over $V[g \times H_{\nu \setminus X} \times h]$.

Now let $W = V[g \times H_{\nu \setminus X}]$. Since the forcing $\mathrm{Col}(\lambda, {<}\xi) \times \mathrm{Add}(\lambda, X)$ has size λ in W, is ${<}\lambda$-closed and non-atomic, there is an $\mathrm{Add}(\lambda, 1)$-generic filter k over W with $W[k] = W[G_\xi \times H_X]$ by Lemma 1.21. Then $V[G \times H] = W[h \times k]$ is an $\mathrm{Add}(\lambda, 1)^2$-generic extension of W and

$$x \in (A^\lambda_{\varphi,z})^{W[h\times k]} \cap {}^\lambda\lambda \cap W[h \times k].$$

By Lemma 3.27, player I has a winning strategy in $G_\lambda(A^\lambda_{\varphi,z})$. □

By Lemma 3.6, the previous result implies that the almost Baire property for the class of definable sets considered there is consistent with arbitrary values of 2^λ. Moreover, as in the proof of Theorem 2.20, we immediately obtain the following result.

Theorem 3.29. *Suppose there is a proper class of inaccessible cardinals. Then there is a class generic extension $V[G]$ of V in which for every regular cardinal λ and for every subset A of ${}^\lambda\lambda$ that is definable from an element of ${}^\lambda V$, $G_\lambda(A)$ is determined.*

Since the almost Baire property immediately implies the Bernstein property, we obtain the following result as in the proof of Lemma 2.21.

Lemma 3.30. *Suppose that λ is an uncountable regular cardinal and all subsets of ${}^\lambda\lambda$ that are definable from elements of* ${}^\lambda\mathrm{Ord}$ *have the almost Baire property. Then the following statements hold.*

1. *All subsets of ${}^\lambda\lambda$ that are definable from elements of* ${}^\lambda\mathrm{Ord}$ *have the Bernstein property.*
2. *There is no well-order on ${}^\lambda\lambda$ that is definable from an element of* ${}^\lambda\mathrm{Ord}$.

It is further possible to obtain results for homogeneous sets for definable colorings for which player I has a winning strategy in G_κ, which extend Theorem 2.22 and will appear in a later paper.

4. Implications of resurrection axioms

In this section, we obtain versions of the main theorems from a variant of the resurrection axiom introduced by Hamkins and Johnstone [HJ14]. As above, we assume that λ is an uncountable regular cardinal. Moreover, we will use the sets $A_{\varphi,z}$ and A_φ given in Definition 1.10. Our result is motivated by the following sufficient condition for the existence of a perfect subset of a given $\mathbf{\Sigma}^1_1$ subset of ${}^\lambda\lambda$.

Lemma 4.1. (1) *Suppose that $\varphi(x,y)$ is a Σ^1_1-formula and $z \in {}^\lambda\mathrm{Ord}$ is a parameter. If $|A^\lambda_{\varphi,z}| > \lambda$ holds in every $\mathrm{Col}(\lambda, 2^\lambda)$-generic extension of V, then $A^\lambda_{\varphi,z}$ has a perfect subset.*

(2) *Suppose that $V = L$. Then there is a Π^1_1 formula $\varphi(x)$ such that $|A^\lambda_\varphi| > \lambda$ holds in every generic extension of V, but A^λ_φ does not have a perfect subset.*

Proof. For the first claim, it follows by standard arguments that there is a level subset S of $({}^{<\lambda}\lambda)^2$ with the property that $A^\lambda_{\varphi,z}$ is the projection of S in every outer model with the same V_λ as V. By the assumption, there are $\mathrm{Col}(\lambda, 2^\lambda)$-names σ, τ such that $\mathrm{Col}(\lambda, 2^\lambda)$ forces that (σ, τ) is a new element of $[S]$. Using these names, we can construct sequences $\langle p_u \mid u \in {}^{<\lambda}2\rangle$ of conditions in $\mathrm{Col}(\kappa, 2^\lambda)$ and $\langle (s_u, t_u) \mid u \in {}^{<\lambda}2\rangle$ of nodes in S such that the following conditions hold for all $u \subsetneq v$ in ${}^{<\lambda}2$.

(a) $p_s \Vdash t_s \subsetneq \sigma \;\&\; u_s \subsetneq \tau$.
(b) $p_u \subseteq p_v$, $s_u \subsetneq s_v$ and $t_u \subsetneq t_v$.
(c) $t_{u^\frown\langle 0\rangle} \neq t_{u^\frown\langle 1\rangle}$.

Let T denote the level subset of $({}^{<\lambda}\lambda)^2$ that is obtained as the downwards closure of the set of pairs (s_u, t_u) for $u \in {}^{<\lambda}2$. By the above conditions, its projection $\mathrm{proj}(T) = \{x \in {}^\lambda\lambda \mid \exists y \in {}^\lambda\lambda\ (x,y) \in [T]\}$ is a perfect subset of $A^\lambda_{\varphi,z}$.

For the second claim, we have a subtree T of ${}^{<\lambda}\lambda$ with $|[T]| > \lambda$ and no perfect subtrees by [LMRS16, Proposition 7.2]. We claim that the formula $\varphi(x)$ stating that $x \in [T]$ or $x \notin L$ satisfies the requirement. It follows from the choice of T that $[T]$ does not have a perfect subset. To show the remaining condition, we work in a generic extension $V[G]$ of V. If $(\lambda^+)^L = \lambda^+$, then $|A^\lambda_\varphi| \geq |[T]| \geq |[T]^L| > \lambda$. If $(\lambda^+)^L = \lambda$, then $|({}^\lambda\lambda)^L| \leq \lambda$ and hence $|A^\lambda_\varphi| > \lambda$ by the choice of φ. □

We now formulate the resurrection axiom at λ for a given class of forcings. By a *definable class of forcings* we will mean a class $\Gamma_{\varphi,z} = \{x \mid \varphi(x,z)\}$, where $\varphi(x,y)$ is a formula with two free variables with the property that it is provable in ZFC^- that x is a forcing for all sets x, y with $\psi(x,y)$, and z is a set parameter.

Definition 4.2. Assuming that Γ is a definable class of forcings, we define the *resurrection axiom* $\mathrm{RA}^\lambda(\Gamma)$ to hold if for all $\mathbb{P} \in \Gamma$, there is a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ such that $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \Gamma$ and $H_{\lambda^+} \prec^+ H_{(\lambda^+)^{V[G]}}$ holds for every $\mathbb{P} * \dot{\mathbb{Q}}$-generic filter G over V.

If λ is a regular cardinal, we say that ν is *λ-inaccessible* if $\nu > \lambda$ is regular and $\mu^{<\lambda} < \nu$ holds for all cardinals $\mu < \nu$. It can then be shown as in [HJ14, Theorem 18] that the axiom $\mathrm{RA}^\lambda(\Gamma)$ for the class of forcings $\mathrm{Col}(\lambda, {<}\nu)$, where ν is λ-inaccessible, is consistent from an uplifting cardinal $\mu > \lambda$ (see [HJ14, Definition 10]).

Lemma 4.3. *Suppose that ν is λ-inaccessible, $\varphi(x,y)$ is a formula and z is a set parameter. Then $\mathrm{Col}(\lambda, {<}\nu)$ forces the following statements.*

(1) *If $|A^\lambda_{\varphi,z}| > \lambda$, then $A^\lambda_{\varphi,z}$ has a perfect subset.*
(2) *If there is an $\mathrm{Add}(\lambda, 1)$-generic element of ${}^\lambda\lambda$ in $A^\lambda_{\varphi,z}$, then player I has a winning strategy in $G_\lambda(A^\lambda_{\varphi,z})$.*

Proof. Since ν is λ-inaccessible, it follows from a standard argument using the Δ-system lemma that $\mathrm{Col}(\lambda, {<}\nu)$ is ν-cc.

For the first claim, it follows from the assumption that there is a $\mathrm{Col}(\lambda, {<}\nu)$-name σ for a new element of $A^\lambda_{\varphi,z}$. By the ν-cc, we can assume that σ is a $\mathrm{Col}(\lambda, {<}\mu)$-name for some ordinal $\mu < \nu$. Since ν is λ-inaccessible, it is easy to see that there are unboundedly many cardinals $\mu \in \mathrm{Card} \cap \nu$ with $\mu^{<\lambda} = \mu$. To prove the claim, we work in a $\mathrm{Col}(\lambda, {<}\nu)$-generic extension of V. We can now show as in the proof of Theorem 2.19 (for $\mathrm{Col}(\lambda, {<}\nu)$ instead of $\mathrm{Col}(\kappa, {<}\lambda)$) that $A^{\lambda}_{\varphi,z}$ has a perfect subset.

For the second claim, it follows from the assumption that there is a $\mathrm{Col}(\lambda, {<}\nu)$-name σ for an $\mathrm{Add}(\kappa, 1)$-generic element of ${}^{\kappa}\kappa$ in $A^{\lambda}_{\varphi,z}$. We can again argue as in the proof of Theorem 3.28 (for $\mathrm{Col}(\lambda, {<}\nu)$ instead of $\mathrm{Col}(\kappa, {<}\lambda)$). □

Our last result follows immediately from Lemma 4.3 and the definition of the resurrection axiom.

Theorem 4.4. *Suppose that Γ is the class of forcings $\mathrm{Col}(\lambda, {<}\nu)$, where ν is λ-inaccessible. Assuming that $\mathrm{RA}^{\lambda}(\Gamma)$ holds, the following statements hold for every subset A of ${}^{\lambda}\lambda$ that is definable over $(H_{\lambda^+}, \in)$ with parameters in H_{λ^+}.*
1. *A has the perfect set property.*
2. *The game $G_{\lambda}(A)$ is determined.*

5. Questions

We conclude with some open questions. We first note that by standard arguments, an inaccessible cardinal is necessary to obtain the perfect set property for λ-Borel subsets of ${}^{\lambda}\lambda$. The most striking question is whether the conclusion of Theorem 3.28 can be achieved without an inaccessible cardinal as in [She84].

Question 5.1. *Can the almost Baire property for all subsets of ${}^{\lambda}\lambda$ definable from an element of ${}^{\lambda}\mathrm{Ord}$, for some uncountable regular cardinal λ, be forced over any model of ZFC?*

Moreover, we ask whether the conclusions of our results hold in the following other well-known models.

Question 5.2. *Do the conclusions of the main results, Theorem 2.19 and Theorem 3.28, hold in the Silver collapse* [Cum10, Definition 20.1] *and in the Kunen collapse* [Cum10, Section 20] *of an inaccessible cardinal μ to λ^+, where λ is any uncountable regular cardinal?*

Since the existence of winning strategies implies the existence ot winning tactics for the Banach-Mazur game of length ω, it is natural to consider the same problem in the present context.

Question 5.3. *Is it consistent that for some uncountable regular cardinal λ and for all subsets A of ${}^{\lambda}\lambda$ that are definable from elements of ${}^{\lambda}\mathrm{Ord}$, either player I or player II has a winning tactic in $G_{\lambda}(A)$?*

Moreover, in analogy to the Baire property, it is natural to ask the following question, which arose in a discussion with Philipp Lücke.

Question 5.4. *Does the almost Baire property for all subsets of ${}^{\lambda}\lambda$ definable from elements of ${}^{\lambda}\mathrm{Ord}$ imply a version of the Kuratowski-Ulam theorem?*

Finally, the similarities to other regularity properties suggest that our results can be extended as follows.

Question 5.5. *Can we prove results analogous to the main results for games associated to other regularity properties such as the Hurewicz dichotomy?*

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