Continuous reducibility for the real line*

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Abstract

We study Borel subsets of the real line up to continuous reducibility. We firstly show that every quasi-order of size ω_1 embeds into the quasi-order of Borel subsets of the real line up to continuous reducibility. We then prove that at least all the types of gaps in $\mathcal{P}(\omega)/\mathsf{fin}$ appear and determine several cardinal characteristics of this quasi-order. We also begin an analysis of the F_σ subsets of the real line by characterizing the sets reducible to $\mathbb{Q}$ and constructing the least non-trivial set below $\mathbb{Q}$.

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1 Introduction

Reducibility is a very useful concept in several contexts. The Wadge order for the Baire space and continuous (and Lipschitz) reducibility are important for describing the complexity of subsets of the Baire space.[1] Borel reducibility is a useful tool for studying Borel equivalence relations.

In this paper, we study Borel subsets of the real line up to continuous reducibility. Already the most basic structural results known for the Baire space fail in this setting: The Borel subsets of the real line up to continuous reducibility are ill-founded [3] and not semi-linear [12]. The aim of this paper is to describe the quasi-order of the Borel sets up to continuous reducibility. Roughly stated, our results show that continuous reducibility for the real line is much finer than for zero-dimensional Polish spaces. For zero-dimensional Polish spaces, the complexity of a Borel set is closely tied with continuous reducibility, but this is not true for the real line.

The first main result shows that $\mathcal{P}(\omega)/\mathsf{fin}$ can be embedded into this quasi-order. To prove this, we attach Borel sets to scattered countable linear orders so that a reduction between the sets attached to two linear orders induces an isomorphism of the first linear order onto a convex subset of the second linear order. The second main result shows that there is a least non-trivial set below $\mathbb{Q}$. The proof is based on a characterization of the F_σ sets reducible to $\mathbb{Q}$ and uses back-and-forth constructions of continuous maps.

In the next section, we collect basic facts about F_σ subsets of $\mathbb{R}$ up to continuous reducibility. In the third section, we construct an embedding of $\mathcal{P}(\omega)/\mathsf{fin}$ into the Borel subsets of $\mathbb{R}$ up to continuous reducibility, prove the existence of gaps, and determine several cardinal characteristics. In the last section, we characterize the F_σ sets reducible to $\mathbb{Q}$ and construct the least non-trivial set below $\mathbb{Q}$.

[1] Basic results on continuous reducibility for the Baire space can be found in [13].

2 Basic results

Let us quickly review our notation and the basic definitions. For the Borel hierarchy and terminology in descriptive set theory, see [6].

Definition 2.1. Let X be a topological space and $A, B \subseteq X$.

1. $\mathrm{cl}(A)$ denotes the closure of A in X.
2. A is *non-trivial* if $A \neq \emptyset$ and $A \neq X$.
3. A is *continuously reducible (Wadge reducible) to* B ($A \leq B$) if there is a continuous function $f\colon X \to X$ such that $f^{-1}(B) = A$.
4. A and B are *equivalent (Wadge equivalent)* ($A \equiv B$) if $A \leq B$ and $B \leq A$.
5. A is *strictly continuously reducible (strictly Wadge reducible) to* B ($A < B$) if $A \leq B$ but $B \nleq A$.
6. A is *comparable (Wadge comparable) to* B if $A \leq B$ or $B \leq A$.

Notice that $\leq$ is a quasi-order (i.e. a reflexive, transitive relation) and $\equiv$ is an equivalence relation on $\mathcal{P}(X)$. Moreover $\emptyset, X$ are the only $\leq$-minimal elements of $\mathcal{P}(X)$ and $\emptyset, X$ are incomparable if $X \neq \emptyset$. We will focus on continuous reducibility for the real line.

2.1 F_σ sets

In this section we collect facts about F_σ subsets of $\mathbb{R}$.

Lemma 2.2 (Selivanov [11])**.** Any two non-trivial open subsets of $\mathbb{R}$ are equivalent. The same holds for non-trivial closed sets.

Suppose that $A_0, ..., A_n \subseteq \mathbb{R}$. We say that A_n is *minimal above* $A_0, ..., A_{n-1}$ if there is no set $A \subseteq \mathbb{R}$ with $A_0, ..., A_{n-1} < A < A_n$. We say that A is *minimal* if it is minimal above $\emptyset, \mathbb{R}$. Non-trivial open or closed sets are examples for sets of this kind, and these two types of sets are incomparable. We now consider sets which are not comparable to sets of one of these types.

Definition 2.3. For a set $A \subseteq \mathbb{R}$, we consider the following conditions:

1. (I_1) Every point in A is an accumulation point in A from both sides, i.e. for any point x in A any open set U with $x \in U$, there are points y, z in A such that $y < x < z$.

2. (I_2) If A contains a bounded interval (a, b), then a, b belong to A.

We say that A satisfies (I) if satisfies both (I_1) and (I_2).

For example, any countable dense subset and its complement satisfy (I). Let us show that this does not happen for complexities strictly below F_σ.

Proposition 2.4. For any non-trivial set $A \subseteq \mathbb{R}$ the following are equivalent:

1. A satisfies (I_1).

2. $\mathbb{R} \setminus A$ satisfies (I_2).

3. No non-trivial closed set is continuously reducible to A.

Proof. The equivalence of (1) and (2) is immediate. Suppose that (2) holds for A but (3) fails. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous reduction of a non-trivial closed set B to A. We may assume that $\mathbb{R} \setminus B = (0, 1)$ by Lemma 2.2. Since f is continuous and $0, 1 \notin B$, $f((0, 1))$ is an interval contained in $\mathbb{R} \setminus A$ and at least one of its end points is not in A, contradicting (2). Suppose that (3) holds for A but (2) fails. Let $a \in A$ and $(a, b) \subseteq \mathbb{R} \setminus A$. We obtain a reduction of $[0, 1]$ to A which maps $[0, 1]$ to a, contradicting (3). $\square$

Hence a set A which satisfies (I) if and only if is incomparable with every non-trivial open set and every non-trivial closed set. Moreover, the complement and any continuous preimage of a set which satisfies (I) again satisfies (I).

Proposition 2.5. (I) fails for every non-trivial $\mathbf{\Delta}^0_2$ set $A \subseteq \mathbb{R}$.

Proof. Let A be a non-trivial set satisfying (I) and $B = \mathrm{cl}(A) \cap \mathrm{cl}(\mathbb{R} \setminus A)$, i.e. the boundary of A. By results of Kuratowski [7, p. 98, 99, 258, 417] it suffices to show that $B \cap A$ and $B \setminus A$ are dense in B. Since A and $\mathbb{R} \setminus A$ satisfy (I) by Proposition 2.4, it is sufficient to show that $B \setminus A$ is dense in B.

Suppose that $B \setminus A$ is not dense in B. Then there is an open interval (a, b) with $B \cap (a, b) \neq \emptyset$ and $B \cap (a, b) \subseteq A$. Then $A \cap (a, b) \neq \emptyset, (a, b)$, and $A \cap (a, b)$ is relatively closed in (a, b) because $B \cap (a, b) \subseteq A$ and $B \cap (a, b)$ is the boundary of $A \cap (a, b)$ in (a, b). Therefore $A \cap (a, b)$ does not satisfy (I_1) and so neither does A. $\square$

Let us first consider F_σ subsets of the Baire space.

Lemma 2.6. Every F_σ subset of the Baire space can be written as a disjoint union of countably many closed sets.

Proof. Suppose that $A = \bigcup_{n \in \omega} A_n$ where each A_n is closed. Let $A_n = [T_n]$ where each T_n is a tree on ω. Let $S_0 = T_0$. Let $(s_{n,i} \mid i < a_n)$ with $a_n \leq \omega$ enumerate the minimal nodes in $T_n \setminus \bigcup_{j<n} T_j$. Let $S_{n,i} = \{t \in T_n \mid t \subseteq s_{n,i}$ or $s_{n,i} \subseteq t\}$ for $i < a_n$. Then $\bigcup_{i \leq n} [T_i] = \bigsqcup_{i \leq n,\ j < a_i} [S_{i,j}]$. Hence $\bigcup_{n \in \omega} [T_n] = \bigsqcup_{n \in \omega,\ i < a_n} [S_{n,i}]$. $\square$

It is easy to see that a (nonempty bounded) open interval cannot be written as a disjoint union of countably many closed sets. However, such a decomposition can be done for any F_σ set satisfying condition (I_2).

Lemma 2.7. If a non-trivial F_σ set $A \subseteq \mathbb{R}$ satisfies (I_2), then it is a disjoint union of countably many closed sets.

Proof. We will construct a continuous reduction of A which collapses all non-degenerate subintervals of A (if they exist). Let $(B_n \mid n \in \omega)$ be a disjoint family of closed intervals or singletons with the following properties:

(a) $(B_n \mid n \in \omega)$ contains all the maximal closed non-degenerate subintervals in A,

(b) for every n, either $B_n \subseteq A$ or $B_n \subseteq \mathbb{R} \setminus A$,

(c) $\bigcup_{n \in \omega} B_n$ is dense, and

(d) for all m and n with $\max(B_m) < \min(B_n)$, there is a k such that $\max(B_m) < \min(B_k), \max(B_k) < \min(B_n)$ and $B_k \subseteq \mathbb{R} \setminus A$.

Notice that by condition (I_2) for A, all the end points of subintervals of A belong to A.

We define a sequence of weakly increasing continuous maps $f_n : \mathbb{R} \to \mathbb{R}$ by induction. Let $\{q_n \mid n \in \omega\}$ be an enumeration of $\mathbb{Q}$ without repetitions. Let f_0 map $\mathbb{R}$ to q_0.

To define f_{n+1}, choose the least k_{n+1} so that $q_{k_{n+1}}$ is not in the range of $f_n \upharpoonright \bigcup_{i \leq n} B_i$ and the union of $f_n \upharpoonright \bigcup_{i \leq n} B_i$ and the constant map on B_{n+1} with value $q_{k_{n+1}}$ is weakly increasing. Let us call this union to be f'_n. Let $C_{n+1} = \mathbb{R} \setminus \bigcup_{i \leq n+1} B_i$. We extend the map f'_n to a continuous map f_{n+1} on $\mathbb{R}$ which is affine on the bounded connected components of C_n and constant on the unbounded connected components of C_n (if they exist).

By the construction of $(f_n \mid n \in \omega)$, for every k there is some n with $q_k \in \mathrm{range}(f_n \upharpoonright \bigcup_{i \leq n} B_i)$. Since the union $\bigcup_{n \in \omega} B_n$ is dense and each f_n is weakly increasing, the sequence $(f_n \mid n \in \omega)$ converges uniformly on every bounded interval. Hence its pointwise limit f is continuous and weakly increasing. It follows from the construction of $(B_n \mid n \in \omega)$ that $f(x) < f(y)$ for all $x < y$ with $(x, y) \in (A \times (\mathbb{R} \setminus A)) \sqcup ((\mathbb{R} \setminus A) \times A)$. Hence $A = f^{-1}(f(A))$, i.e. f is a reduction from A to $f(A)$, and $\mathbb{R} \setminus f(A)$ is dense.

Since A is F_σ and $\mathbb{R}$ is σ-compact, $f(A)$ is F_σ. We write $f(A) = \bigcup_{n \in \omega} D_n$ where each D_n is closed. Let $D \subseteq \mathbb{R} \setminus f(A)$ be countable and dense in $\mathbb{R}$. Since $\mathbb{R} \setminus D$ is homeomorphic to a closed subset of the Baire space, we can write $f(A) = \bigsqcup_{n \in \omega} E_n$ where each E_n is closed by Lemma 2.6. Moreover the proof of Lemma 2.6 shows that we can choose each E_n so that $E_n \subseteq D_m$ for some m. Hence each E_n is closed in $\mathbb{R}$ and $A = f^{-1}(f(A)) = \bigsqcup_{n \in \omega} f^{-1}(E_n)$. $\square$

2.2 Semi-linearity

The Wadge order for zero-dimensional spaces has a quite simple structure: It is well-founded and semi-linear.

Definition 2.8. Let X be a nonempty set. A quasi-order $\leq$ on a class Π of subsets of X is *semi-linear* if $A \leq B$ or $B \leq X \setminus A$ for all $A, B \in \Pi$.

Lemma 2.9. Suppose that X is a zero-dimensional Polish space.

1. (Wadge, see [13, 6]) The quasi-order of continuous (and of Lipschitz) reducibility on the Borel subsets of X is semi-linear.
2. (Martin-Monk, see [6, Theorem 21.15]) The quasi-order of continuous (and of Lipschitz) reducibility on the Borel subsets of X is well-founded.

It is stated in [1, 12, 14] that the first part of this Lemma fails for the real line.

Example 2.10 ([12])**.** Let $A \subseteq \mathbb{R}$ be any non-trivial open set. Then $A \not\leq \mathbb{Q}$ and $\mathbb{Q} \not\leq \mathbb{R} \setminus A$.

Let Γ denote the class of sets $A \cap B$ where A is an open and B is a closed subset of $\mathbb{R}$. Let Δ denote the class of subsets A of $\mathbb{R}$ such that both A and its complement are in Γ. We will see below that the failure of Wadge's Lemma (the first item of Lemma 2.9) for the real line occurs already at the level of Δ sets. Note that the failure of the second part of Lemma 2.9 for Δ sets was proved in [3].

Let us now consider a small class of reductions on $\mathbb{R}$ which induce a well-founded semi-linear quasi-order on the Borel subsets of $\mathbb{R}$.

Definition 2.11. A function $f\colon \mathbb{R} \to \mathbb{R}$ is right (left) continuous if $f(x) = \lim_{n\to\infty} f(x_n)$ for every decreasing (increasing) sequence (x_n) with $\lim_{n\to\infty} x_n = x$.

Proposition 2.12. The Borel subsets of $\mathbb{R}$ up to right continuous (left continuous) reductions form a well-founded semi-linear quasi-order.

Proof. We will define an isomorphism from $({}^{\omega}\omega, <_{\mathrm{lex}})$ to $([0,1), <)$. Notice that any such isomorphism is continuous. We fix a bijection $g : \omega \to \mathbb{Z}$ and define a continuous bijection $f : {}^{\omega}\omega \to \mathbb{R}$ as follows.

Let L be the set of words $w = w_0 w_1 .. w_n 0^\infty$ in the alphabet ω. We equip L with the right lexicographical order, i.e. $v < w$ if $v(i) < w(i)$ for the least i with $v(i) \neq w(i)$. Let $L_n = \{w = w_0 w_1 ... w_m 0^\infty \in L \mid w_0 = n\}$. We choose an order isomorphism $f_n : L_n \to [g(n), g(n)+1) \cap \mathbb{Q}$ for each n. Let $f \colon {}^\omega\omega \to \mathbb{R}$ be the unique continuous map with $f(w_0 w_1 ... w_n 0^\infty) = f_{w_0}(w_1 w_2 ... w_n 0^\infty)$ for all $w = w_0 w_1 ... w_n 0^\infty \in {}^\omega\omega$. Then the restriction of f is an order isomorphism between $\{x \in {}^\omega\omega \mid x(0) = n\}$ and $[g(n), g(n)+1)$ for each n.

Notice that $f \circ h \circ f^{-1}$ is right continuous for every continuous $h : {}^\omega\omega \to {}^\omega\omega$. This shows that $f^{-1}(A) \leq f^{-1}(B)$ implies $A \leq B$ for all Borel sets $A, B \subseteq \mathbb{R}$. We obtain semi-linearity by combining this observation with the first item of Lemma 2.9.

Let us show that $A < B$ implies $f^{-1}(A) < f^{-1}(B)$ for all Borel sets $A, B \subseteq \mathbb{R}$. Since $B \not\leq A$, the previous argument shows $f^{-1}(B) \not\leq f^{-1}(A)$ and it is enough to see that $f^{-1}(A) \leq f^{-1}(B)$. If this fails then the first item of Lemma 2.9 implies that ${}^\omega\omega \setminus f^{-1}(B) \leq f^{-1}(A)$ and hence $\mathbb{R} \setminus B \leq A$. With $A < B$ we obtain $\mathbb{R} \setminus B \equiv B$ and $A \equiv B$, contradicting $A < B$.

The well-foundedness now follows from the second part of Lemma 2.9. The proof for left continuous reductions is analogous. $\square$

Let us again consider the function f in Proposition 2.12. Since $f^{-1} \restriction (\mathbb{R} \setminus \mathbb{Q})$ is continuous, it follows that f is a $(3,3)$-isomorphism.[2][3] Hence we obtain the statement of Proposition 2.12 for simultaneously right continuous, $(2,1)$, and $(3,3)$-functions. Note that this is optimal, since there are uncountable antichains in the quasi-order of the Borel subsets of $\mathbb{R}$ up to $(2,2)$-reducibility (see [8]). It is open whether this quasi-order is well-founded.

[2]A function $f \colon \mathbb{R} \to \mathbb{R}$ is a (m,n)-function if $f^{-1}(A) \in \mathbf{\Sigma}^0_m$ for every $A \in \mathbf{\Sigma}^0_n$. The Baire class 1 functions are the $(2,1)$-functions.

[3]The existence of a $(3,3)$-isomorphism between $\mathbb{R}$ and ${}^\omega\omega$ also follows from the result of Jayne-Rogers [5] that there are $(3,3)$-isomorphisms between any two uncountable finite-dimensional Polish spaces.

3 Embedding results and characteristics

Let us now describe some global features of the quasi-order $(\mathrm{Borel}(\mathbb{R}), \leq)$. We first show that $(\mathcal{P}(\omega), \subseteq^*)$ and hence every quasi-order of size ω_1 embeds into $(\mathrm{Borel}(\mathbb{R}), \leq)$. We then use variants of this result to prove the existence of gaps and to determine several cardinal characteristics of $(\mathrm{Borel}(\mathbb{R}), \leq)$.

3.1 Embeddings

Let $\subseteq^*$ denote inclusion up to finite error on $\mathcal{P}(\omega)$. It is known that every poset of size ω_1 embeds into $(\mathcal{P}(\omega), \subseteq^*)$ by Parovičenko's Theorem [9]. Let us write $\mathcal{P}(\omega)/\mathsf{fin}$ for the quotient of $\mathcal{P}(\omega)$ by the ideal fin of finite subsets of ω. We will construct an embedding of $(\mathcal{P}(\omega), \subseteq^*)$ into $(\mathrm{Borel}(\mathbb{R}), \leq)$, generalizing the construction in [4, Theorem 5.1.2].

We will embed a given scattered linear order L into $\mathbb{R}$ and attach a Borel set to this embedding such that every continuous reduction between two of these sets induces an isomorphism of L onto a convex[4] subset. This property will be used to encode $(\mathcal{P}(\omega), \subseteq^*)$ into continuous reducibility.

Let us consider embeddings of the following kind.

Definition 3.1 (Gaps and discrete embeddings)**.** Suppose that Q is a quasi-order and L is a linear order.

1. If $X, Y \subseteq Q$ are nonempty, then (X, Y) is a *gap* in Q if

    a. $a < c$ for all $a \in X$ and $c \in Y$ and

    b. there is no $b \in Q$ such that $a < b < c$ for all $a \in X$ and all $c \in Y$.

2. We call an embedding $f : L \to \mathbb{R}$ *discrete* if for all gaps (X, Y) in L, $\sup(f(X)) < \inf(f(Y))$ if and only if Y has a minimum.

Notice that every scattered[5] countable linear order can be discretely embedded into $\mathbb{R}$.

[4] A set $X \subseteq L$ is *convex* if $a, b \in X$ and $a \leq x \leq b$ imply that $x \in L$.

[5] A linear order is *scattered* if it has no subset isomorphic to $(\mathbb{Q}, <)$.

Definition 3.2. Suppose that L is a countable scattered linear order. We consider the lexicographical ordering on $L \times 4$. Suppose $r : L \times 4 \to \mathbb{R}$ is a discrete embedding. Let us write $\underline{ui} = r(u, i)$. Let

$$A_x = A_x^{L,r} = \bigsqcup_{u \in L} [\underline{u0}, \underline{u1}) \sqcup \bigsqcup_{u \in L \setminus x} [\underline{u2}, \underline{u3}] \sqcup \{\underline{u2} \mid u \in x\}$$

for $x \subseteq L$. Let us identify L with $L \times 1$ so that $\underline{u0} = \underline{u}$. Notice that $r \restriction (L \times 1)$ is discrete. If $\pi : L \to L$ is any function, let $r(\pi)$ denote the induced map $r \circ \pi \circ r^{-1}$ on $r(L)$. If $f : \mathbb{R} \to \mathbb{R}$ is a function with $f(r(L)) \subseteq r(L)$, let $r^{-1}(f)$ denote the induced map $r^{-1} \circ f \circ r$ on L.

The sets $A_x^{L,r}$ in Definition 3.2 are Γ subsets of $\mathbb{R}$. For this it is essential that r is discrete.

Lemma 3.3. Each set A_x is *rigid* in the following weak sense. Let I be a (possibly unbounded) open interval with $A_x \subseteq I$ such that $\sup(A_x) = \sup(I)$, and $\inf(A_x) \in I$ if and only if A_x has a minimum. Then for any continuous reduction f of A_x to a set $B \subseteq \mathbb{R}$, there is a (possibly unbounded) open interval J such that $(J, f(A_x)) = (J, B \cap J)$ is homeomorphic to (I, C), where C is the set A_x minus (possibly) some of the sets $[\underline{u2}, \underline{u3}]$ and $\{\underline{u2}\}$. Moreover, the reduction induces an isomorphism of L onto a convex subset of L.

Proof. If $f(\underline{u0}) < f(\underline{u1})$ for some $u \in L$, it follows that $[\underline{u0}, \underline{u1})$ is mapped onto $[f(\underline{u0}), f(\underline{u1}))$. If u has direct successor $v \in L$, then $f(\underline{v0})$ is the minimum of the least half-open interval in $f(A_x)$ above $f(\underline{u1})$, and if u has a direct predecessor $w \in L$, then $f(\underline{w1})$ is the supremum of the largest half-open interval below $f(\underline{u0})$. The situation is symmetric in the case $f(\underline{u0}) > f(\underline{u1})$. If follows by induction on the rank of L that f is either order-preserving or order-reversing on $r(L)$. Let us assume that f is order-preserving on $r(L)$. Since f is a continuous reduction, it follows that $f(A_x)$ comes from a discrete embedding. This implies that there is an open interval J so that $(J, f(A_x)) = (J, B \cap J)$ is homeomorphic to (I, A_x). $\square$

Lemma 3.4. Suppose that L is a scattered countable linear order and that $r : L \times 4 \to \mathbb{R}$ is a discrete embedding.

1. Suppose that $K \subseteq L$ is convex and $\pi : L \to K$ is an isomorphism. Moreover suppose that $x, y \subseteq L$ and $\pi(x) \subseteq y$. Then there is a weakly increasing continuous reduction $f \colon \mathbb{R} \to \mathbb{R}$ of A_x to A_y extending $r(\pi)$. If moreover $K = L$, then f can be chosen to be surjective.

2. Suppose that $f \colon \mathbb{R} \to \mathbb{R}$ is a continuous reduction of A_x to A_y. Then $f(r(L)) \subseteq r(L)$ and $r^{-1}(f)$ is an isomorphism of L onto a convex subset $K \subseteq L$. If f is surjective, then $K = L$.

Proof. 1. Suppose that $K \subseteq L$ is convex and $\pi : L \to K$ is an isomorphism. Moreover suppose that $x, y \subseteq L$ and $\pi(x) \subseteq y$. We define f in the following cases:

- Let f map $[\underline{u0}, \underline{u1})$ onto $[\underline{\pi u\ 0}, \underline{\pi u\ 1})$ and $[\underline{u1}, \underline{u2})$ onto $[\underline{\pi u\ 1}, \underline{\pi u\ 2})$ by strictly increasing maps.

- Suppose that $u \in L$. Let $s = \inf_{u<v} \underline{v}$ and $t = \inf_{u<v} \underline{\pi v}$. If $u \in x \subseteq \pi^{-1}(y)$, let f map $[\underline{u2}, \underline{u3}]$ onto $[\underline{\pi u\ 2}, \underline{\pi u\ 3}]$ and $[\underline{u3}, s]$ onto $[\underline{\pi u\ 3}, t]$ by strictly increasing maps. If $u \notin x$ and $\pi u \notin y$, let f map $[\underline{u2}, \underline{u3}]$ onto $[\underline{\pi u\ 2}, \underline{\pi u\ 3}]$ by strictly increasing maps. If $u \notin x$ and $\pi u \in y$, let f map $[\underline{u2}, \underline{u3}]$ to $\underline{\pi u\ 2}$ and $[\underline{u3}, s]$ onto $[\underline{\pi u\ 2}, t]$ by a strictly increasing map.

- If (A, B) is a gap in L, A has infinite cofinality, and B has a minimum, let $a = \sup(r(A))$, $b = \inf(r(B))$, $s = \sup(r(\pi(A)))$, and $t = \inf(r(\pi(B)))$. Then $a < b$ and $s < t$, since r is discrete. Let f map $[a, b]$ onto $[s, t]$ by a strictly increasing map.

- If (A, B) is a gap in L and B has infinite coinitiality, let $a = \sup(r(A))$ and $s = \sup(r(\pi(A)))$. Let f map a to s.

- If $r(L)$ has a lower bound, let $M = \{u \in L \colon \forall v \in K (u < v)\}$. Let $s = \sup(r(M))$, where $\sup(\emptyset) = -\infty$ (so $s = -\infty$ if K is coinitial in L) and $t = \inf(r(K))$. Let f map $(-\infty, \inf(r(L))]$ onto $(s, t]$ by a strictly increasing map.

- If $r(L)$ has an upper bound, let $M = \{u \in L\colon \forall v \in K(v < u)\}$. Let $s = \sup(r(K))$ and $t = \inf(r(M))$, where $\inf(\emptyset) = \infty$ (so $t = \infty$ if K is cofinal in L). Let f map $[\sup(r(L)), \infty)$ onto $[s, t)$ by a strictly increasing map.

Then f is a reduction of A_x to A_y with the desired properties.

2. To prove the second claim, let us suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous reduction of A_x to A_y. It follows from Lemma 3.3 that $r^{-1}(f)$ is an isomorphism of L onto a convex subset of L. □

Lemma 3.5. Let L be the set of ω-words $u = u_0u_1...$ in the alphabet $\mathbb{Z}$ such that u_n is eventually 0. Let $u \preceq v$ if $u(i) < v(i)$ for the largest i with $u(i) \neq v(i)$ for $u, v \in L$, or $u = v$. Then $(L, \preceq)$ is a scattered linear order. We fix a discrete embedding $r\colon L \to \mathbb{R}$ as in Lemma 3.4. Let $\bar{x} = \{u \in L \mid u \preceq 0^\infty \text{ or } u = 0^n10^\infty$[6] for some $n \in x\}$ for $x \subseteq \omega$. Let us consider the sets $A_{\bar{x}}$ defined in Lemma 3.4. Then for all $x, y \subseteq \omega$

1. If $x \subseteq^* y$, then there is a weakly increasing continuous surjective reduction $f\colon \mathbb{R} \to \mathbb{R}$ of $A_{\bar{x}}$ to $A_{\bar{y}}$.

2. If $A_{\bar{x}} \leq A_{\bar{y}}$, then $x \subseteq^* y$.

Proof. If $x \subseteq^* y$, then $\pi(u_0...u_n0^\infty) = u_0...u_{m-1}(u_m - 1)u_{m+1}...u_n0^\infty$ for $n > m = \max(x \setminus y)$ defines an automorphism of L with $\pi(\bar{x}) \subseteq \bar{y}$. The claim follows from the previous lemma.

If $A_{\bar{x}} \leq A_{\bar{y}}$, then there is an automorphism $\pi\colon L \to L$ with $\pi(\bar{x}) \subseteq \bar{y}$ by the previous lemma. If $\pi 0^\infty = u_0...u_m0^\infty$, then by induction on $n \geq m$, for all $u = u_0...u_n0^\infty$ there is some $v = v_0...v_n0^\infty$ with $\pi u = v$. Now $\pi(\bar{x}) = \bar{y}$ implies $x \subseteq^* y$. □

Notice that the linear order in Lemma 3.5 is the union of $\mathbb{Z}^n$ equipped with the lexicographical orders. Hence every isomorphism of L onto a convex subset is surjective. If the embedding r is chosen with image unbounded in both

[6] I.e. 0^n1 followed by an infinite string of 0.

directions and f is a continuous reduction of $A_{\bar{x}}$ to $A_{\bar{y}}$, then the proof of Lemma 3.5 shows that f is onto.

Note that as in the second part of Lemma 3.5, if f is a continuous reduction of $A_{\bar{x}}$ to any set, then $f(A_{\bar{x}})$ is equal to $A_{\bar{y}}$ defined relative to some discrete embedding of L for some $y \subseteq \omega$ with $x \subseteq^* y$. This follows from Lemma 3.3.

It follows from Lemma 3.5 that

Theorem 3.1. $(\mathcal{P}(\omega), \subseteq^*)$ embeds into $(\Gamma, \leq)$. Moreover:

1. Every quasi-order of size ω_1 embeds into $(\Gamma, \leq)$.

2. CH implies that $(\Gamma, \leq)$ is maximal with respect to embeddability among quasi-orders of size ω_1.

3. There is a model of ZFC in which 2^ω is arbitrarily large and $(\Gamma, \leq)$ is maximal with respect to embeddability among quasi-orders of size 2^ω.

Proof. The second claim follows from Parovičenko's Theorem [9]. The third claim follows from the analogous result for $\mathcal{P}(\omega)/\mathsf{fin}$ [2]. $\square$

Notice that we have not used the axiom of choice in the construction. Since every countable partial ordering on ω embeds into $(\mathcal{P}(\omega), \subseteq)$ by mapping n to the set of its predecessors, we obtain an embedding of any countable poset into $(\mathcal{P}(\omega), \subseteq^*)$ via a bijection from ω to $\omega \times \omega$ and hence an embedding into $(\Gamma, \leq)$ without the axiom of choice.

It is also possible to prove that $(\mathcal{P}(\omega), \subseteq)$ embeds into $(\Delta, \leq)$ by considering unions of order type ω of half-open intervals, closed intervals, and points unbounded upwards. However, we do not know whether $\mathcal{P}(\omega)/\mathsf{fin}$ embeds into $(\Delta, \leq)$.

Another natural question is whether the quasi-order of embedability of countable structures in a countable language embeds into $(\mathrm{Borel}(\mathbb{R}), \leq)$.

Remark 3.6. (Smaller classes of reductions) Do similar embeddings exist when we consider reducibility with respect to smaller classes of functions? Let us write $r_u = \inf_{u<v} \underline{v}$ for $u \in L$, where L and r are as in Lemma 3.5.

1. (Differentiable reductions) If the embedding r is chosen so that

$$\lim_{n\to\infty} \frac{\underline{u^n 3} - \underline{u^n 2}}{r_n - \underline{u^n 1}} = 0$$

for every increasing sequence (u^n) such that $(\underline{u}^n)_{n\in\omega}$ has an upper bound and for every decreasing sequence (u^n) such that $(\underline{u}^n)_{n\in\omega}$ has an lower bound, then it is easy to see that for all $x \subseteq^* y$ there is a weakly increasing differentiable Lipschitz reduction of $A_{\bar{x}}$ to $A_{\bar{y}}$.

2. (ε-Lipschitz reductions) Suppose r is chosen as we have just described and with bounded image. Let us choose a family of affine images $(A^i_{\bar{x}})_{i\in\mathbb{Z}}$ of $A_{\bar{x}}$ for each $x \subseteq \omega$ such that $\min(A^i_{\bar{x}}) = \min(A^i_{\bar{y}}) < \min(A^{i+1}_{\bar{x}}) = \min(A^{i+1}_{\bar{y}})$ for $x, y \subseteq \omega$ and $i \in \mathbb{Z}$ (and such that $A^i_{\bar{x}}$ has the same orientation as $A_{\bar{x}}$). Moreover, let us assume that the diameter of $A^i_{\bar{x}}$ is 2^i and the distance between $A^i_{\bar{x}}$ and $A^{i+1}_{\bar{x}}$ is 2^i for all $i \in \mathbb{Z}$. Let $C_{\bar{x}} = \bigcup_{i\in\mathbb{Z}} A^i_{\bar{x}}$. Then $C_{\bar{x}} \leq C_{\bar{y}}$ implies $x \subseteq^* y$ for all $x, y \subseteq \omega$. If $x \subseteq^* y$ then for every $\varepsilon \in \mathbb{R}^+$, there is a weakly increasing ε-Lipschitz reduction of $C_{\bar{x}}$ to $C_{\bar{y}}$. Note that $C_{\bar{x}} \in \Gamma$ for all $x \subseteq \omega$.

3. (C_1-reductions) We do not know if $(\mathcal{P}(\omega), \subseteq^*)$ embeds into the Borel subsets of $\mathbb{R}$ up to C_1-reducibility. Notice that it is not difficult to see by a variation of the previous arguments that $(\mathcal{P}(\omega), \subseteq)$ embeds into the Borel subsets of $\mathbb{R}$ up to C_∞-reducibility, but we do not know if $(\mathcal{P}(\omega), \subseteq)$ embeds into the Borel subsets of $[0,1]$ up to C_1-reducibility.

4. (Polynomial reductions) It is easy to see that ω^* (i.e. ω with the reverse order) embeds into Δ up to reducibility by polynomial functions by working with finite unions of half-open intervals, However, we do not know if ω^* embeds into the closed sets up to reducibility by polynomial functions, or if $(\mathcal{P}(\omega), \subseteq)$ embeds into the Borel subsets of $\mathbb{R}$ up to reducibility by polynomial functions or power series.

3.2 Gaps and cardinal characteristics

It is natural to ask how much $(\mathrm{Borel}, \leq)$ resembles $(\mathcal{P}(\omega), \subseteq^*)$ as a quasi-order. Let us show that there are gaps in $(\mathrm{Borel}, \leq)$, answering a question of Jörg Brendle. To this end, we again consider the family $(A_{\bar{x}})_{x \subseteq \omega}$ from Lemmas 3.4 and 3.5.

Proposition 3.7. Suppose that the range of the embedding $r \colon L \to \mathbb{R}$ in Lemma 3.4 is unbounded in both directions. Let $F(x) = A_{\bar{x}}$ for $x \subseteq \omega$. If (X, Y) is a gap in $\mathcal{P}(\omega)/\mathsf{fin}$, then $(F(X), F(Y))$ is a gap in $(\mathrm{Borel}(\mathbb{R}), \leq)$.

Proof. Suppose that $A_{\bar{x}}^{L,r} \leq B \leq A_{\bar{y}}^{L,r}$ for all $x \in X$ and all $y \in Y$. Let $f_x \colon \mathbb{R} \to \mathbb{R}$ be a continuous reduction of $A_{\bar{x}}^{L,r}$ to B for each $x \in X$. Then there are open intervals I, J and a set C as in Lemma 3.3 with $(J, f(A_x)) = (J, B \cap J)$ homeomorphic to (I, C).

Since $B \leq A_{\bar{y}}^{L,r}$ for some $y \subseteq \omega$, $f(A_{\bar{x}}^{L,r}) = B$ is unbounded in both directions. It follows that there are $z \subseteq \omega$ and a discrete embedding $s : L \times 4 \to \mathbb{R}$ with unbounded range in both directions such that $B = f(A_{\bar{x}}^{L,r}) = A_{\bar{z}}^{L,s}$. Since $A_{\bar{z}}^{L,r}$ and $A_{\bar{z}}^{L,s}$ differ only in the choice of the embedding and both r and s are discrete and their ranges are unbounded in both directions, there is a homeomorphism h of $\mathbb{R}$ with $h(A_{\bar{z}}^{L,s}) = A_{\bar{z}}^{L,r}$. Since $A_{\bar{x}}^{L,r} \leq A_{\bar{z}}^{L,r}$ for all $x \in X$, we have $x \subseteq^* z$ for all $x \in X$ by Lemma 3.5. Since $A_{\bar{z}}^{L,r} \leq A_{\bar{y}}^{L,r}$ for all $y \in Y$, we have $z \subseteq^* y$ for all $y \in Y$ by Lemma 3.5. However, there can be no such z since (X, Y) is a gap. $\square$

Hence there are gaps in $(\mathrm{Borel}(\mathbb{R}), \leq)$ of at least all the types of the gaps that appear in $\mathcal{P}(\omega)/\mathsf{fin}$. In particular, it follows from the existence of Hausdorff gaps and Rothberger gaps (see [10]) that

Corollary 3.8. There are gaps of the types (ω_1, ω_1^*), $(\omega, \mathfrak{b}^*)$, and $(\mathfrak{b}, \omega^*)$ in $(\mathrm{Borel}(\mathbb{R}), \leq)$, where $\mathfrak{b}$ denotes the bounding number.

We do not know if $(\mathrm{Borel}(\mathbb{R}), \leq)$ has exactly the same types of gaps (κ, λ) as $\mathcal{P}(\omega)/\mathsf{fin}$, for infinite cardinals κ, λ.

Some global features of the quasi-order $\mathcal{B} = (\mathrm{Borel}(\mathbb{R}), \leq)$ are described by the following cardinal characteristics.

Definition 3.9. Let $\min(\varphi) = \min(\{|X| \mid X \subseteq \mathrm{Borel}(\mathbb{R}),\ \varphi(X, \leq)\}$ where φ is any property, and similarly for sup. The *bounding number*, *dominating number*, *depth*, and *length* of $\mathcal{B}$ are defined as

1. $b_{\mathcal{B}} = \min(\text{unbounded})$,
2. $d_{\mathcal{B}} = \min(\text{dominating})$,
3. $dp_{\mathcal{B}} = \sup(\text{linearly ordered})$, and
4. $l_{\mathcal{B}} = \sup(\text{well-ordered})$.

The bounding and dominating numbers and the depth can be calculated in ZFC.

Proposition 3.10. 1. $b_{\mathcal{B}} = \omega_1$.

2. $d_{\mathcal{B}} = 2^{\omega}$.
3. $dp_{\mathcal{B}} = 2^{\omega}$.

Proof. 1. To see that $b_{\mathcal{B}} \geq \omega_1$, let us suppose that X is a countable family of Borel subsets of $\mathbb{R}$. We can choose homeomorphisms h_B between $\mathbb{R}$ and disjoint open intervals for $B \in X$. Then $\bigcup_{B \in X} h_B(B)$ is an upper bound for X.

To see that $b_{\mathcal{B}} \leq \omega_1$, let $r\colon \beta \times 2 \to \mathbb{R}$ be a discrete embedding for $\beta < \omega_1$. Let $\underline{\alpha i} = r(\alpha, i)$ for $(\alpha, i) \in \beta \times 2$ and let $A_\beta = \bigsqcup_{\alpha<\beta} [\underline{\alpha 0}, \underline{\alpha 1}]$. Then each A_β is rigid in the sense of Lemma 3.3 by the proof of Lemma 3.3.

Let us show that there is no upper bound for $(A_\beta \mid \beta < \omega_1)$. If $B \subseteq \mathbb{R}$ is an upper bound, let I_β, J_β be open intervals and let $h_\beta\colon I_\beta \to J_\beta$ be a homeomorphism between (I_β, A_β) and $(J_\beta, B \cap J_\beta)$ for each $\beta < \omega_1$. Let us assume that every h_β is increasing. Let $E \subseteq \omega_1$ be the set of additively closed countable ordinals. There is an interval H with rational end points and an unbounded set $F \subseteq E$ such that $H \subseteq J_\beta$ for all $\beta \in F$. Hence $J_\alpha \cap J_\beta \neq \emptyset$ for all $\alpha, \beta \in F$. Then $\sup(J_\alpha) < \sup(J_\beta)$ for all $\alpha, \beta \in F$ with $\alpha < \beta$ by the proof of Lemma 3.4. This is impossible since F is uncountable.

2. We will construct a family $(A_x \mid x \subseteq \omega)$ of sets in Γ such that for every $B \subseteq \mathbb{R}$ there are only countably many $x \subseteq \omega$ with $A_x \leq B$. For $x \subseteq \omega$ let $r_x \colon \omega^\omega \times 4 \to \mathbb{R}$ be order-preserving, continuous at ω^n for all $n \in x$, and discontinuous at ω^n for all $n \in \omega \setminus x$. Let $\bar{x} = \omega^\omega \setminus \{\omega^n \colon n \in x\}$ and

$$A_x = \bigsqcup_{\alpha \in \omega^\omega \setminus \bar{x}} [\underline{\alpha 0}, \underline{\alpha 1}) \sqcup \bigsqcup_{\alpha \in \bar{x}} (\underline{\alpha 0}, \underline{\alpha 1})$$

for $x \subseteq \omega$. It is easy to see that each A_x is rigid in the sense of Lemma 3.3.

Suppose that there is an uncountable set $E \subseteq \mathcal{P}(\omega)$ with $A_x \leq B$ for all $x \subseteq \omega$. Let us choose an uncountable set $F \subseteq E$ so that $x \triangle y = (x \setminus y) \cup (y \setminus x)$ is finite for all $x, y \in F$ with $x \neq y$. Let I_x be an open interval with $A_x \subseteq I_x$ and $\sup(I_x) = \sup(A_x)$ and let $h_x \colon I_x \to J_x$ be a homeomorphism between (I_x, A_x) and $(J_x, B \cap J_x)$ for all $x \in F$. Let us assume that every h_x is increasing. There is an interval H with rational end points and an uncountable set $G \subseteq F$ such that $H \subseteq J_x$ for all $x \in G$, so $J_x \cap J_y \cap B \neq \emptyset$ for all $x \in G$. A similar proof as in Lemmas 3.4 and 3.5 shows that $x \triangle y$ is finite for all $x, y \in G$ with $x \neq y$.

3. It is sufficient to embed the linear order $({}^\omega 2, \leq_{\text{lex}})$ into $(\mathcal{P}(\omega \times \omega), \subseteq^*)$ by Proposition 3.1. Let $\leq^*$ denote eventual domination on ${}^\omega\omega$, i.e. $f \leq^* g$ if there is an m so that $f(n) \leq g(n)$ for all $n \geq m$. Then $F \colon {}^\omega 2 \to {}^\omega\omega$, $F(x)(n) = \sum_{0 \leq i \leq n} 2^{n-i} x(i)$, is an order-preserving injection of $({}^\omega 2, \leq_{\text{lex}})$ into $({}^\omega\omega, \leq^*)$, and $G \colon {}^\omega\omega \to \mathcal{P}(\omega \times \omega)$, $G(x) = \{(n, m) \mid m < x(n)\}$, is an order-preserving injection of $({}^\omega\omega, \leq^*)$ into $(\mathcal{P}(\omega \times \omega), \subseteq^*)$. $\square$

Proposition 3.10 answers a question of Stefan Geschke. Note that the sets in the above proof are in Γ, so the characteristics for $(\Gamma, \leq)$ are the same as for $(\text{Borel}(\mathbb{R}), \leq)$. Hence there are no maximal elements in $(\mathbf{\Sigma}^0_\alpha, \leq)$ and in $(\mathbf{\Pi}^0_\alpha, \leq)$ for all countable ordinals $\alpha \geq 2$. This contrasts the situation for continuous reducibility for the Baire space where any proper $\mathbf{\Sigma}^0_\alpha$ set is maximal in $(\mathbf{\Sigma}^0_\alpha, \leq)$

and the same holds for $(\mathbf{\Pi}^0_\alpha, \leq)$.

Let us now consider the length $l_{\mathcal{B}}$. Notice that $\omega_1 \leq l_{\mathcal{B}} \leq 2^\omega$ by Proposition 3.1. We will need

Lemma 3.11. Suppose that κ is an infinite cardinal with $\alpha^\omega < \kappa$ for all $\alpha < \kappa$. Let $\mathbb{C}_\kappa$ denote the finite support product of κ many copies of Cohen forcing. Then in $V^{\mathbb{C}_\kappa}$, the length of any quasi-order on $\mathcal{P}(\omega)$ definable from a real and an ordinal is less than κ.

Proof. Suppose that $p \Vdash_{\mathbb{C}_\kappa} \varphi(x, y, \dot{x}, \check{\gamma})$ defines the strict part of a quasi-order on $\mathcal{P}(\omega)$, where $\dot{x}$ is a nice name for a subset of ω. Suppose that $(\dot{x}_\alpha \mid \alpha < \omega_1)$ is a sequence of nice $\mathbb{C}_\kappa$-names for subsets of ω with $p \Vdash \dot{x}_\alpha <_\varphi \dot{x}_\beta$ for all $\alpha < \beta < \omega_1$. We may assume that $p = 1$ and $\dot{x}$ is a name for an element of V by passing to an intermediate extension.

Let $s_\alpha = \mathrm{supp}(\dot{x}_\alpha) = \bigcup_{(\check{n}, p) \in \dot{x}_\alpha} \mathrm{supp}(p) \subseteq \kappa$ for each $\alpha < \kappa$. Note that each s_α is countable. We can assume that (s_α) forms a Δ-system by thinning out, and that the root is empty by passing to an intermediate extension. Thus we assume that (s_α) is a disjoint family.

Let $\mathbb{C}_\kappa(s) = \{p \in \mathbb{C}_\kappa \mid \mathrm{supp}(p) \subseteq s\}$ for $s \subseteq \kappa$. The function collapsing s_α to an ordinal induces an isomorphism between $\mathbb{C}_\kappa(s_\alpha)$ and $\mathbb{C}_{\beta_\alpha}$ for some $\beta_\alpha < \omega_1$. This maps $\dot{x}_\alpha$ to a nice $\mathbb{C}_{\omega_1}$-name $\pi(\dot{x}_\alpha)$ for a subset of ω. Since there are only $2^\omega < \kappa$ many such names and since κ is regular, there is an unbounded set $I \subseteq \kappa$ with $\pi(\dot{x}_\alpha) = \pi(\dot{x}_\beta)$ for all $\alpha, \beta \in I$.

Let us consider $\alpha, \beta \in I$ with $\alpha \neq \beta$. Since $s_\alpha \cap s_\beta \neq \emptyset$ and $\pi(\dot{x}_\alpha) = \pi(\dot{x}_\beta)$, there is an automorphism σ of $\mathbb{C}_\kappa$ with $\sigma(\dot{x}_\alpha) = \dot{x}_\beta$ and $\sigma(\dot{x}_\beta) = \dot{x}_\alpha$. Hence $1 \Vdash \dot{x}_\alpha <_\varphi \dot{x}_\beta$ if and only if $1 \Vdash \dot{x}_\beta <_\varphi \dot{x}_\alpha$, contradicting the assumption that φ defines a quasi-order. $\square$

Proposition 3.12. There are models of ZFC with

1. $l_{\mathcal{B}} = 2^\omega > \omega_1$,

2. $l_{\mathcal{B}} = \omega_1 < 2^\omega$, and

3. $\omega_1 < l_{\mathcal{B}} < 2^\omega$.

Proof. 1. Note that Martin's axiom (applied to almost disjoint forcing) implies that the tower number $\mathfrak{t}$ of $(\mathcal{P}(\omega), \subseteq^*)$ is 2^ω. Since there is a well-ordered sequence of length $\mathfrak{t}$ in $(\mathcal{P}(\omega), \subseteq^*)$, any model of $\mathsf{MA}+\neg\mathsf{CH}$ works by Proposition 3.1.

2. Let us force with $\mathbb{C}_{\omega_2}$ over a model of GCH. Then $l_{\mathcal{B}} < \omega_2$ and hence $l_{\mathcal{B}} = \omega_1$ in $V^{\mathbb{C}_{\omega_2}}$ by Lemma 3.11.

3. We force MA_{ω_1} over a model of GCH with the standard finite support c.c.c. iteration of length ω_2. In the extension $2^\omega = 2^{\omega_1} = \omega_2$, $2^{\omega_2} = \omega_3$, and $l_{\mathcal{B}} = \omega_2$ by MA_{ω_1}. We further force with $\mathbb{C}_{\omega_3}$ to obtain a model of $2^\omega = \omega_3$ and $l_{\mathcal{B}} = \omega_2$ by Lemma 3.11. $\square$

4 Below $\mathbb{Q}$

In this section we study $(F_\sigma, \leq)$. We first characterize the F_σ sets below $\mathbb{Q}$ and show that $\mathcal{P}(\omega)/\mathsf{fin}$ embeds into $(F_\sigma, \leq)$. We then construct the least non-trivial set below $\mathbb{Q}$.

4.1 A dichotomy for F_σ sets

We aim to prove

Theorem 4.1. The following conditions are equivalent for any F_σ set $A \subseteq \mathbb{R}$:

1. A satisfies (I).

2. There are no non-trivial closed or open sets B with $B \leq A$.

3. $A \leq \mathbb{Q}$.

This directly follows from Lemma 2.4 and

Lemma 4.1. A non-trivial F_σ subset of $\mathbb{R}$ reduces to $\mathbb{Q}$ if and only if it satisfies (I).

Proof. Notice that any continuous preimage of $\mathbb{Q}$ satisfies (I). Suppose that $A \subseteq \mathbb{R}$ is a non-trivial F_σ set satisfying (I). Then $A = \bigsqcup_{n\in s} A_n$, where each A_n is closed and $s \subseteq \omega$, by Lemma 2.7. Let $\mathbb{Q} = \{q_n \mid n \in \omega\}$. Let $\{p_n \mid n \in \omega\}$ be a dense subset of $\mathbb{R}\setminus\mathbb{Q}$. Let $(B_n \mid n \in t)$ enumerate all maximal closed intervals in $\mathbb{R} \setminus A$, where $t \subseteq \omega$. Note that any point in the interior of $\mathbb{R} \setminus B$ is contained in B_n for some n by property (I_1) for A.

We can assume that $s \cap t = \emptyset$ and that there are $a < b$ with

1. $A_n \subseteq (a, b)$,
2. $\bigcup_{j\in s,\ j<n} A_n \cap (a, b) = \emptyset$, and
3. $\bigcup_{j\in t,\ j<n} B_n \cap (a, b) = \emptyset$

by changing the indexing and partitioning the sets into finitely many closed pieces. Let $A = \bigcup_{j<n} A_n$, $B = \bigcup_{j<n} B_n$, and $f = \bigcup_{j<n} f_j$.

We will construct a sequence (f_n) of partial functions by induction. Let $f_n = f$ if $n \notin s \cup t$. Now we suppose that $n \in s \cup t$. Let $a_n = \min((A \cup B) \cap (-\infty, \min(A_n))$ and $b_n = \max((A \cup B) \cap (\max(A_n), \infty)$ where $\min(\emptyset) = -\infty$ and $\max(\emptyset) = \infty$. Let $c_n = f(a_n)$, if this is defined, and $c_n = -\infty$ if $a_n = -\infty$. Let $d_n = f(b_n)$, if this is defined, and $d_n = \infty$ if $b_n = \infty$. Let us extend f to f_n in the following cases:

1. Suppose that $c_n < d_n$. If $n \in s$ let k_n be minimal with $q_{k_n} \in (c_n, d_n)$. Then we extend f to f_n by mapping A_n to q_{k_n}. If $n \in t$ let k_n be minimal with $p_{k_n} \in (c_n, d_n)$. Then we extend f to f_n by mapping B_n to p_{k_n}.
2. Suppose that $d_n < c_n$. This is symmetric to the previous case.
3. Suppose that $c_n = d_n$. Let $I_n = (a_n, b_n)$. We choose $\bar{d}_n > c_n$ so that for $J_n = (c_n, \bar{d}_n)$ and $m < n$

   a. $\bar{d}_n - c_n < \frac{1}{2^n}.$,

b. $q_m \notin J_n$,

c. $q_{k_m} \notin J_n$, and

d. $\mathrm{cl}(J_n) \subseteq J_m$ if $c_n = d_n$ and $I_n \subseteq I_m$.

Notice that $c_n \neq c_m$ by the third condition in step m. Hence we can choose $\bar{d}_n$ such that J_n satisfies the fourth condition.

If $n \in s$ let k be minimal with $q_{k_n} \in (c_n, \bar{d}_n)$. Then we extend f to f_n by mapping A_n to q_{k_n}. If $n \in t$ let k_n be minimal with $p_{k_n} \in (c_n, \bar{d}_n)$. Then we extend f to f_n by mapping B_n to p_{k_n}.

Let $f = \bigcup_{n \in \omega} f_n$.

Claim 4.2. f is continuous.

Proof. Let $B = \bigcup_{n \in t} B_n$. It suffices to show that f preserves the limit of any strictly increasing (decreasing) sequence (x_n) in $A \cup B$ converging to $x \in A \cup B$. Let us consider the following cases:

1. If there is m with $x_n \in A_m$ for all n or with $x_n \in B_m$ for all n, then we are done.

2. If $x_n \in A_{m_n}$ for all m and $\sup(A_{m_n}) < \inf(A_{m_{n+1}})$ for all n, then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} q_{k_{m_n}} = f(x)$, since in the construction we always choose q_k and p_l with the least possible index as the next value: Assuming that $q_k, p_l \in (\lim_{n\to\infty} q_{k_{m_n}}, f(x))$ for some k and l leads to a contradiction.

3. If $x_n \in B_{m_n}$ for all m and $\sup(B_{m_n}) < \inf(B_{m_{n+1}})$ for all n, then again $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} p_{k_{m_n}} = f(x)$, since in the construction we always choose q_k and p_l with the least possible index as the next value.

All other cases reduce to one of these. The argument for decreasing sequences is symmetric. □

Claim 4.3. There is a unique continuous extension $g \colon \mathbb{R} \to \mathbb{R}$ of f.

Proof. Suppose (x_n) and (y_n) are strictly increasing (or decreasing) sequences in $A \cup B$ converging to $x \in \mathbb{R} \setminus (A \cup B)$. We need to see that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$.

Let us consider the following cases:

1. Suppose that $\sup(A_n) < x$ or $x < \inf(A_n)$. for all n. In this case the argument is similar as for the previous claim: Assuming that there are $q_k, p_l \in (\lim_{n\to\infty} f(x_n), \lim_{n\to\infty} f(y_n))$ or $q_k, p_l \in (\lim_{n\to\infty} f(y_n), \lim_{n\to\infty} f(x_n))$ leads to a contradiction, since in the construction we always choose q_k and p_l with the least possible index as the next value.

2. Suppose that there is an infinite sequence (n_i) such that $x \in I_{n_i}$ and $I_{n_j} \subseteq I_{n_i}$ for all $i < j$. Then $\mathrm{cl}(J_{n_j}) \subseteq J_{n_i}$ for all $i < j$. It follows from the choice of J_{n_i} that the diameter of J_{n_i} converges to 0. Hence $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$ is the unique element of $\bigcap_{i\in\omega} J_{n_i}$.

The other cases reduce to one of these. Note that g is unique since $A \cup B$ is dense in $\mathbb{R}$. □

Claim 4.4. $A = g^{-1}(\mathbb{Q})$.

Proof. We have $g(A) \subseteq \mathbb{Q}$ and $g(B) \subseteq \mathbb{R} \setminus \mathbb{Q}$ by the construction. It remains to show that $g(\mathbb{R} \setminus (A \cup B)) \subseteq \mathbb{R} \setminus \mathbb{Q}$. Suppose that $x \in \mathbb{R} \setminus (A \cup B))$.

Let us consider the two cases of the previous claim. In the first case, some A_n is mapped to q_k. If $x > \sup(A_n)$ then $g(x) > q_k$ and if $x < \inf(A_n)$ then $g(x) < q_k$ by the construction. In the second case $f(x) \in \bigcap_{n\in\omega} J_{n_i}$. Hence $g(x) \notin \mathbb{Q}$ by the choice of the sets J_n. □

This completes the proof of Lemma 4.1. □

Note that any nontrivial countable set $A \subseteq \mathbb{R}$ which satisfies (I) is continuously reducible to $\mathbb{Q}$ by Proposition 4.1. If A is not equivalent to $\mathbb{Q}$, then its closure is nowhere dense, and any two such sets are equivalent by a back-and-forth construction.

We will now study (Borel$(\mathbb{R}, \leq)$ below $\mathbb{Q}$. It is well known that

Lemma 4.5. If $A \subseteq \mathbb{R}$ is countable and dense in $\mathbb{R}$, then there is a homeomorphism h of $\mathbb{R}$ with $h(A) = \mathbb{Q}$. Hence $A \equiv \mathbb{Q}$.

Proof. There is an order isomorphism f between $(A, <)$ and $(\mathbb{Q}, <)$ by a standard back-and-forth argument. We define $g : \mathbb{R} \to \mathbb{R}$ by

$$h(r) = \sup\{f(a) \mid a \in A \text{ and } a < r\}.$$

This is a well-defined homeomorphism, since the topology of the real line is the order topology of $(\mathbb{R}, <)$. Clearly $h(A) = \mathbb{Q}$. $\square$

Lemma 4.6. $\mathbb{Q}$ is not minimal.

Proof. We will define a continuous function $f_{\mathcal{C}} \colon \mathbb{R} \to \mathbb{R}$ such that $\mathbb{R} \setminus A$ is nowhere dense for $A = f_{\mathcal{C}}^{-1}(\mathbb{Q})$. It is easy to see that this implies that $\mathbb{Q} \not\leq A$ and hence $A < \mathbb{Q}$. Let f denote the Cantor function, i.e. $f \colon [0, 1] \to [0, 1]$ is the unique continuous function such that

$$f(\sum_{n \geq 1} \frac{2a_n}{3^n}) = \sum_{n \geq 1} \frac{a_n}{2^n}$$

on the Cantor set (see [6, Exercise 3.4]) for $(a_n) \in {}^{\omega}2$ and f is constant on each open interval disjoint from the Cantor set. Note that the preimage of the irrationals under f is a subset of the Cantor set and hence is nowhere dense. Let the continuous extension $f_{\mathcal{C}} \colon \mathbb{R} \to \mathbb{R}$ of f obtained by translation

$$f_{\mathcal{C}}(x) = f(x - n) + n \quad \text{if } n \leq x < n + 1 \text{ for some integer } n$$

is as desired. $\square$

Let us further define $g_{\mathcal{C}} = f_{\mathcal{C}} + \sqrt{2}$. To see that $A = f_{\mathcal{C}}^{-1}(\mathbb{Q})$ and $B = g_{\mathcal{C}}^{-1}(\mathbb{Q})$ are incomparable, let us suppose that h reduces B to A. Then h is not constant and hence its range contains an open interval. In particular, it contains an open subset of A, and hence B would contain an open subset. But B is nowhere dense. The argument for $A \not\leq B$ is symmetric.

To see that the quasi-order (Borel$(\mathbb{R}, \leq)$ is quite complex below $\mathbb{Q}$, we will now embed $\mathcal{P}(\omega)/\mathsf{fin}$.

Proposition 4.7. There is an embedding of $(\mathcal{P}(\omega), \subseteq^*)$ to (Borel$(\mathbb{R}, \leq)$ below $\mathbb{Q}$.

Proof. We will construct a family $(A_x)_{x \subseteq \omega}$ of F_σ subsets of $\mathbb{R}$ by attaching homeomorphic images of the sets A and B in the previous paragraph. Let $r : \omega^\omega \times 2 \to \mathbb{R}$ be a continuous order-preserving embedding, where the ordinal ω^ω carries the order topology and $\omega^\omega \times 2$ carries the lexicographic ordering. we write $u^+ = u + 1$ and $\underline{ui} = r(u, i)$.

Let $I_u = (\underline{u0}, \underline{u1})$ and $J_u = (\underline{u1}, \underline{u^+0})$. Let $g_u \colon \mathbb{R} \to \mathbb{R}$ denote a homeomorphism with $g_u(0) = \underline{u0}$ and $g_u(1) = \underline{u1}$. Let $h_u \colon \mathbb{R} \to \mathbb{R}$ denote a homeomorphism with $h_u(0) = \underline{u1}$ and $g_u(1) = \underline{u^+0}$. Let $C_u = g_u(A \cap (0,1)) \cup h_u(B \cap (0,1))$. Let

$$A_x = \bigsqcup_{u \in \omega^\omega} C_u \sqcup \{\underline{u1} \colon u \in \omega^\omega\} \sqcup \{\underline{u0} \colon u = \omega^n + 1 \text{ for some } n \in \omega \setminus x\}.$$

These sets satisfy (I) and hence they are reducible to $\mathbb{Q}$ by Proposition 4.1. If $m = \max(x \setminus y) + 1$, then there is a continuous reduction f of A_x to A_y with $f(0) = \underline{\omega^m\ 0}$. The intervals $(\underline{u0}, \underline{u1})$, $(\underline{u1}, \underline{u^+0})$, and $(\underline{u^+1}, \underline{u^{++}0})$ are folded into $(\underline{u0}, \underline{u1})$ for all $n \in y \setminus x$ with $n > m$. If $x \not\subseteq^* y$, then there is no continuous reduction of A_x to A_y by a similar argument as in Lemma 3.4. $\square$

Problem 4.8. Are the sets reducible to $\mathbb{Q}$ up to surjective continuous reducibility (or up to weakly increasing continuous reducibility) well-founded?

4.2 The least set below $\mathbb{Q}$

In this section, we will construct a minimal set below $\mathbb{Q}$ and show that it is unique.

An *end point* of a set $A \subseteq \mathbb{R}$ is an $x \in A$ such that there is a $y < x$ with $(y, x) \cap A = \emptyset$ or a $y > x$ with $(x, y) \cap A = \emptyset$. Let $\mathcal{C} \subseteq [0, 1]$ denote the Cantor set (see [6, Exercise 3.4]). Let D_0 be the set of end points of $\mathcal{C}$. Let D_1 be a countable dense subset of $\mathcal{C} \setminus D_0$. Let D_2 be a countable dense subset of $[0, 1] \setminus \mathcal{C}$. Let $D = D_0 \sqcup D_1 \sqcup D_2$.

For any countable dense set $D \subseteq [0,1]$ with $0, 1 \in D$, let us fix an increasing homeomorphism $h_D\colon [0,1] \to [0,1]$ with $D = h_D[\{\frac{m}{2^n} \in [0,1] \mid m, n \in \omega\}]$. Let $f_{\mathcal{C}}\colon [0,1] \to [0,1]$ denote the Cantor function. Let $A = (h_D \circ f_{\mathcal{C}})^{-1}(\mathcal{C})$. Then A is obtained from $\mathcal{C}$ by blowing up each $d \in D$ to a closed interval. Note that it is not necessary to blow up the points in D_2 for our purposes, but we do this in order to be able to work with the Cantor function.

Lemma 4.9. Suppose $C \subseteq \mathbb{R}$ is a nonempty compact set. Then there is a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = \min(C)$, $f(1) = \max(C)$, and $A = f^{-1}(C)$.

Proof. Let us assume that $\min(C) = 0$ and $\max(C) = 1$. We enumerate by

- $(A_n)_{n<\omega}$ all maximal closed intervals in A,
- $(B_n)_{n<\omega}$ all maximal open intervals in $(0,1) \setminus A$,
- $(C_n)_{n<M}$ all maximal closed intervals in C and all end points of C which are not contained in a maximal closed interval in C, and
- $(D_n)_{n<N}$ all maximal open intervals in $(0,1) \setminus C$,

where $M, N \leq \omega$. Since there are infinitely many sets A_n whose end points are not end points of A, by choice of D_1, we can choose an enumeration such that $m < n$ whenever A_n and $cl(B_m)$ share an end point.

In step n we will define a partial map $f_n\colon \mathbb{R} \to \mathbb{R}$ with the properties

- $\mathrm{dom}(f_n)$ is a union of finitely intervals, including all A_m and B_m with $m < n$,
- $A_m \subseteq \mathrm{dom}(f_n)$ or $A_m \cap \mathrm{dom}(f_n) = \emptyset$ for all m,
- $B_m \subseteq \mathrm{dom}(f_n)$ or $B_m \cap \mathrm{dom}(f_n) = \emptyset$ for all m,
- $\mathrm{dom}(f_n)$ is closed,
- the end points of $\mathrm{dom}(f_n)$ are in A,
- $a, b \notin \mathrm{dom}(f_n)$ for all a, b with $(a,b) \cap \mathrm{dom}(f_n) = \emptyset$,

- range(f_n) is a union of finitely many sets of the form C_m or D_m, including all C_m and D_m with $m < n$,

- range(f_n) is closed,

- the end points of range(f_n) are in C,

- f_n is increasing on the end points of dom(f_n),

- f_n maps the end points of dom(f_n) onto the end points of range(f_n), and

- $f_n \restriction A_m$ and $f_n \restriction B_m$ are affine for all $A_m \subseteq \mathrm{dom}(f_n)$ and all $B_m \subseteq \mathrm{dom}(f_n)$.

Moreover, f_{n+1} will be an extension of f_n for each n, and $\bigcup_{n<\omega} \mathrm{dom}(f_n)$ and $\bigcup_{n<\omega} \mathrm{range}(f_n)$ will be dense in $\mathbb{R}$.

To define f_0, let a be maximal with $[0, a] \subseteq A$ and let b be minimal with $[b, 1] \subseteq A$. Moreover, let $\bar{a}$ be maximal with $[0, \bar{a}] \subseteq C$ and let $\bar{b}$ be minimal with $[\bar{b}, 1] \subseteq C$. We define f_0 by mapping $[-\infty, a]$ onto $[-\infty, \bar{a}]$ by a translation and $[b, \infty]$ onto $[\bar{b}, \infty]$ by a translation.

Let us assume that f_n satisfies the above properties. We will successively define extensions g_i of f_n for $i < 4$ such that

- $A_n \subseteq \mathrm{dom}(g_0)$,

- $B_n \subseteq \mathrm{dom}(g_1)$,

- $C_n \subseteq \mathrm{range}(g_2)$,

- $D_n \subseteq \mathrm{range}(g_3)$,

and g_i satisfies the above properties of f_n for each $i < 4$, except the conditions stating that certain A_m, B_m, C_m, D_m are contained in dom(f_n) or range(f_n).

We will then define $f_{n+1} = g_3$.

Step 4.9.1. Let $g_0 = f_n$ if $A_n \subseteq \mathrm{dom}(f_n)$. Note that if A_n contains an end point of A, then there is some $m < n$ such that A_n and $cl(B_m)$ have a common end point, by choice of the enumeration of the sets A_n. In this case, $A_n \subseteq \mathrm{dom}(f_n)$ by Step 4.9.2.

Otherwise, we define g_0 by extending $\mathrm{dom}(f_n)$ to A_n. Let $b = \min(A_n)$ and $c = \max(A_n)$. Let $a < b$ be maximal with $a \in \mathrm{dom}(f_n)$. Let $d > c$ be minimal with $d \in \mathrm{dom}(f_n)$.

We consider the following cases:

1. Suppose that $(f_n(a), f_n(d)) \cap C \neq \emptyset$. Let $[\bar{c}, \bar{d}]$ be a maximal closed subinterval of $(f_n(a), f_n(d)) \cap C$. We define g_0 by mapping $[c, d]$ onto $[\bar{c}, \bar{d}]$ by an increasing affine map.

2. Suppose that $(f_n(a), f_n(d)) \cap C = \emptyset$. Let (e, h) be a maximal open subinterval of $(a, d) \setminus A$. We define g_0 by mapping $[e, h]$ onto $[f_n(a), f_n(b)]$ by an increasing affine map. In addition, let $\epsilon = 2^{-n}|f_n(a) - f_n(d)|$, $g_0(x) = f_n(a) + \epsilon \min_{y \in A} |x - y|$ for $x \in [a, e]$, and $g_0(x) = f_n(d) - \epsilon \min_{y \in A} |x - y|$ for $x \in [h, d]$.

Step 4.9.2. Let $g_1 = g_0$ if $B_n \subseteq \mathrm{dom}(g_0)$. Otherwise, we define g_1 by extending $\mathrm{dom}(g_0)$ to B_n. Let $c = \min(B_n)$ and $d = \max(B_n)$. There are $b < c$ and $e > d$ with $[b, c] \subseteq A$ and $[d, e] \subseteq A$ by the choice of D_0. Let b be minimal and e maximal with these properties. Note that $[b, c] \cap \mathrm{dom}(g_0) = \emptyset$ and $[d, e] \cap \mathrm{dom}(g_0) = \emptyset$ by the choice of the enumeration of the sets A_m. Let $a < b$ be maximal with $a \in \mathrm{dom}(g_0)$ and let $h > e$ be minimal with $h \in \mathrm{dom}(g_0)$.

We consider the following cases:

1. Suppose $(g_0(a), g_0(h)) \cap C$ has at least 2 connected components. Then there is some m such that $[\bar{c}, \bar{d}] \subseteq (g_0(a), g_0(d))$ for $\bar{c} = \inf(D_m)$ and $\bar{d} = \sup(D_m)$. Let $\bar{b} \leq \bar{c}$ be minimal with $[\bar{b}, \bar{c}] \subseteq C$ and let $\bar{e} \geq \bar{d}$ be maximal with $[\bar{d}, \bar{e}] \subseteq C$. Notice that $g_0(a) < \bar{b}$ and $\bar{c} < g_0(e)$, since $\mathrm{range}(g_0)$ is a union of sets of the form C_m and D_m. We define g_1 by mapping $[b, c]$ onto $[\bar{b}, \bar{c}]$, $[c, d]$ onto $[\bar{c}, \bar{d}]$, and $[d, e]$ onto $[\bar{d}, \bar{e}]$ by increasing affine maps.

2. Suppose $(g_0(a), g_0(h)) \cap C$ has exactly one connected component C_m. Let $\bar{b} = \min(C_m)$ and $\bar{c} = \max(C_m)$. We define g_1 by mapping $[b, c]$ onto

$[\bar{b}, \bar{c}]$ and $[c, d]$ onto $[\bar{c}, g_0(h)]$ by increasing affine maps. Moreover, let $\epsilon = 2^{-n}|g_0(a) - g_0(h)|$ and let $g_1(x) = g_0(h) - \epsilon \min_{y \in A} |x - y|$ for $x \in [d, h]$.

3. Suppose $(g_0(a), g_0(h)) \cap C = \emptyset$. We define g_1 by mapping $[c, d]$ onto $[g_0(a), g_0(d)]$ by an increasing affine map. Moreover, let $\epsilon = 2^{-n}|g_0(a) - g_0(h)|$, $g_1(x) = g_0(a) + \epsilon \min_{y \in A} |x - y|$ for $x \in [a, c]$, and $g_1(x) = g_0(h) - \epsilon \min_{y \in A} |x - y|$ for $x \in [d, h]$

Step 4.9.3. Let $g_2 = g_1$ if $C_n \subseteq \mathrm{range}(g_1)$ or $n \geq M$. Otherwise, we define g_2 by extending $\mathrm{range}(g_1)$ to C_n. Let $\bar{b} = \min(C_n)$ and $\bar{c} = \max(C_n)$. Let $\bar{a} < \bar{b}$ be maximal with $\bar{a} \in \mathrm{range}(g_1)$ and $\bar{d} > \bar{c}$ minimal with $\bar{c} \in \mathrm{range}(g_1)$. Let a be the unique end point of $\mathrm{dom}(g_1)$ with $g_1(a) = \bar{a}$ and d the unique end point of $\mathrm{dom}(g_1)$ with $g_1(d) = \bar{d}$. We choose m such that $A_m \subseteq (a, d)$. We define g_2 by mapping A_m onto C_n by an increasing affine map.

Step 4.9.4. Let $g_3 = g_2$ if $D_n \subseteq \mathrm{range}(g_2)$ or $n \geq N$. Otherwise, we define g_3 by extending $\mathrm{range}(g_2)$ to D_n. Let $\bar{c} = \inf(D_n)$ and $\bar{d} = \sup(D_n)$. There are l, m with $\bar{c} = \max(C_l)$ and $\bar{d} = \min(C_m)$. Let $\bar{b} = \min(C_l)$ and $\bar{c} = \max(C_m)$.

We consider the following cases:

1. Suppose that $\mathrm{range}(g_2) \cap C_l = \emptyset$ and $\mathrm{range}(g_2) \cap C_m = \emptyset$. Let $\bar{a} < \bar{b}$ be maximal with $\bar{a} \in \mathrm{range}(g_2)$ and $\bar{h} > \bar{e}$ minimal with $\bar{h} \in \mathrm{range}(g_3)$. Let a and h be the unique end points of $\mathrm{dom}(g_2)$ with $g_2(a) = \bar{a}$ and $g_2(h) = \bar{h}$. There is i with $B_i \subseteq (a, h)$. Let $c = \inf(B_i)$ and $d = \sup(B_i)$. We choose j and k with $\max(A_j) = c$ and $\min(A_k) = d$. The conditions on g_2 imply $\mathrm{dom}(g_2) \cap A_j = \emptyset$ and $\mathrm{dom}(g_2) \cap A_k = \emptyset$. We define g_3 by mapping A_j onto C_l, B_i onto D_n, and A_k onto C_m by increasing affine maps.

2. Suppose that $\mathrm{range}(g_2) \cap C_l = \emptyset$ and $C_m \subseteq \mathrm{range}(g_2)$. Let $\bar{a} < \bar{b}$ be maximal with $\bar{a} \in \mathrm{range}(g_2)$. Let a and d be the unique end points of $\mathrm{dom}(g_2)$ with $g_2(a) = \bar{a}$ and $g_2(d) = \bar{d}$. There is i with $B_i \subseteq (a, d)$. Let $b = \inf(B_i)$ and $c = \sup(B_i)$. We choose j with $\max(A_j) = c$. Then $\mathrm{dom}(g_2) \cap A_j = \emptyset$ by the conditions on g_2. We define g_3 by mapping A_j onto C_l by an increasing affine map, if C_l is non-degenerate, and by

by a constant map if C_l consists of a point. We map B_i onto D_n by an increasing affine map. Moreover, let $\epsilon = 2^{-n}|\bar{c} - \bar{d}|$ and let $g_3(x) = g_2(d) - \epsilon \min_{y \in A} |x - y|$ for $x \in [c, d]$.

3. Suppose that $C_l \subseteq \mathrm{range}(g_2)$ and $\mathrm{range}(g_2) \cap C_m = \emptyset$. This is symmetric to the previous case.

4. Suppose that $C_l \subseteq \mathrm{range}(g_2)$ and $C_l \subseteq \mathrm{range}(g_2)$. Let c and d be the unique end points of $\mathrm{dom}(g_2)$ with $g_2(c) = \bar{c}$ and $g_2(d) = \bar{d}$. There is i with $B_i \subseteq (c, d)$. Let $a = \inf(B_i)$ and $b = \sup(B_i)$. Then $a, b \notin \mathrm{dom}(g_2)$ by the conditions on g_2. We define g_3 by mapping B_i onto D_n by an increasing affine map. Moreover let $\epsilon = 2^{-n}|\bar{c} - \bar{d}|$, $g_3(x) = g_2(c) + \epsilon \min_{y \in A} |x - y|$ for $x \in [c, a]$, and $g_3(x) = g_2(d) - \epsilon \min_{y \in A} |x - y|$ for $x \in [b, d]$.

Let $f_\infty = \bigcup_{n \in \omega} f_n$. An argument similar to the proof of Claim 4.4 shows that f has a unique continuous extension $f : \mathbb{R} \to \mathbb{R}$ and that $A = f^{-1}(C)$. □

Remark 4.10. Suppose $a, b \notin A$ and $(a, b) \cap A \neq \emptyset$.Then $(\mathbb{R}, A \cap (a, b))$ is homeomorphic to $(\mathbb{R}, A)$.

Proof. Let $f_{\mathcal{C}} \colon [0, 1] \to [0, 1]$ denote the Cantor function. Notice that $(\mathbb{R}, \mathcal{C} \cap (f_{\mathcal{C}}(a), f_{\mathcal{C}}(b)))$ is homeomorphic to $(\mathbb{R}, \mathcal{C})$ since $f_{\mathcal{C}}(a) \notin \mathcal{C}$. A back-and-forth constructionshows that there is such a homeomorphism which preserves $D_i \cap (f_{\mathcal{C}}(a), f_{\mathcal{C}}(b))$ for all $i < 3$. Since the Cantor map is weakly increasing, this implies the claim. □

Let

$$\mathrm{d}(C) = \sup_{x, y \in C} |x - y|$$

denote the *diameter* of a set C. Let $(E_n)_{n \in \omega}$ be a sequence of disjoint subsets of $\mathbb{R}$ with

- $(\mathbb{R}, A)$ is homeomorphic to $(\mathbb{R}, E_n)$ for each n,

- $\mathrm{d}(E_n) < 1$ for each n,

- $\lim_{n \to \infty} \mathrm{d}(E_n) = 0$, and

- $E = \bigsqcup_{n\in\omega} E_n$ is dense in $\mathbb{R}$,

where A is the set defined in the beginning of this section.

Lemma 4.11. Suppose that a_0, a_1 are in the interior of E and $E \cap (a_0, a_1)$ has infinitely many connected components. Moreover, suppose that $F \subseteq [b_0, b_1]$ is a non-trivial F_σ set with $b_0 < b_1$, $b_i \in F$ for $i < 2$, and F satisfies (I). Then there is a continuous function $f\colon [a_0, a_1] \to \mathbb{R}$ with $f(a_i) = b_i$ for $i < 2$ and $E \cap [a_0, a_1] = f^{-1}[F]$.

Proof. We proved in Lemma 2.7 from (I_2) that there is a family $(F_n)_{n\in\omega}$ of disjoint closed sets F_n with $F \cap [b_0, b_1] = \bigsqcup_{n\in\omega} F_n$. Let us partition the sets $E_n \cap [a_0, a_1]$ into finitely many pieces to obtain a sequence $(C_n\colon n < \omega)$ of disjoint closed sets with

- $E \cap [a_0, a_1] = \bigsqcup_{n\in\omega} C_n$,

- $\lim_{n\to\infty} \mathrm{d}(C_n) = 0$, and

- $(\mathbb{R}, C_n)$ is homeomorphic to $(\mathbb{R}, A)$ for each n.

We will define a sequence of partial functions $f_n\colon \mathbb{R} \to \mathbb{R}$ such that

- $\mathrm{dom}(f_n)$ and $\mathrm{range}(f_n)$ are closed,

- $C_m \subseteq \mathrm{dom}(f_n)$ for all $m < n$, and

- $F_m \subseteq \mathrm{range}(f_n)$ for all $m < n$.

Moreover, f_{n+1} will be an extension of f_n for all n and $\bigcup_{n\in\omega} \mathrm{dom}(f_n)$ will be dense in $[a_0, a_1]$. We will then define $f_\infty = \bigcup_{n\in\omega} f_n$ and show that there is a unique continuous extension $f\colon [a_0, a_1] \to \mathbb{R}$ and that $E \cap [a_0, a_1] = f^{-1}(F)$.

To define f_0, let $a > a_0$ be maximal with $[a_0, a] \subseteq E$ and let $b < a_1$ be minimal with $[b, a_1] \subseteq E$. This is where we use that a_0 and a_1 are in the interior of E. Let further $\bar{a} \geq b_0$ be maximal with $[b_0, \bar{a}] \subseteq F$ and $\bar{b} \leq b_1$ minimal with $[\bar{b}, b_1] \subseteq F$. We define f_0 by mapping $[a_0, a]$ to $[b_0, \bar{a}]$ by an increasing affine map, if $b_0 < \bar{a}$, and a constant map otherwise. We further map $[b, a_1]$ to $[\bar{b}, b_1]$ by an increasing affine map, if $\bar{b} < b_1$, and by a constant map otherwise.

We will extend f_n to f_{n+1} in two steps.

Step 4.11.1. Let $g_0 = f_n$ if $E_n \subseteq \mathrm{dom}(f_n)$. Otherwise, we will define g_0 by extending $\mathrm{dom}(f_n)$ to C_n.

Since $\mathrm{dom}(f_n)$ is closed, we can partition C_n into finitely many nonempty closed pieces so that for each piece C, there are $a, b \in \mathrm{dom}(f_n)$ with $C \subsetneq (a, b)$. Notice that each piece has infinitely many connected components. The partial map g_0 is defined by extending $\mathrm{dom}(f_n)$ to all such sets C in the following cases.

1. Suppose $f_n(a) < f_n(b)$. Let m be least with $F_m \subsetneq (f_n(a), f_n(b))$. We map C onto F_m as in Lemma 4.9.

2. Suppose $f_n(b) < f_n(a)$. Let m be least with $F_m \subsetneq (f_n(b), f_n(a))$. We map C onto F_m as in Lemma 4.9, but in reversed order.

3. Suppose $f_n(a) = f_n(b)$. Let us choose $\epsilon \leq 2^{-n}$ with $\mathrm{range}(f_n) \cap (f_n(a), f_n(a) + \epsilon) = \emptyset$. Let m be least with $F_m \subsetneq (f_n(a), f_n(a) + \epsilon)$. We can partition C into closed sets B_0 and B_1 such that $(\mathbb{R}, B_i)$ is homeomorphic to $(\mathbb{R}, A)$ for each i and $\max(B_0) < \min(B_1)$. We map B_0 onto F_m as in Lemma 4.9. We further map B_1 onto F_m as in Lemma 4.9, but in reverse order.

Step 4.11.2. We will extend g_0 to g_1 by mapping certain sets to F_n.

Since $\mathrm{range}(g_0)$ is closed, we can partition F_n into finitely many nonempty closed pieces so that for each piece G, there are $r, s \in \mathrm{range}(g_0)$ with $G \subsetneq (r, s)$. Let us assume that each piece G is a maximal subsets of F_n with this property. We will extend g_0 for each such set G.

Let us now consider such a set G and all pairs $a < b$ such that

- $G \subsetneq (g_0(a), g_0(b))$ or $G \subsetneq (g_0(b), g_0(a))$, and

- (a, b) is minimal with this property.

Such pairs exist since G is closed, and there are only finitely many pairs by the continuity of g_0.

Let us extend g_0 in the following cases.

1. Suppose that $G \subseteq (g_0(a), g_0(b))$. Let m be least with $C_m \subseteq (a, b)$. We map C_m onto G as in Lemma 4.9.

2. Suppose that $G \subseteq (g_0(b), g_0(a))$. Let m be least with $C_m \subseteq (a, b)$. We map C_m onto G as in Lemma 4.9, but with reversed orientation.

The extensions of $\mathrm{dom}(g_0)$ for different sets G are compatible by the choice of the sets G. This defines a partial function g_1. We complete Step 4.11.2 by defining $f_{n+1} = g_1$.

Let $f_\infty = \bigcup_{n\in\omega} f_n$. By a similar argument as in Claim 4.3, f_∞ has a unique continuous extension $f\colon \mathbb{R} \to \mathbb{R}$. By a similar argument as in Claim 4.4, $E \cap [a_0, a_1] = f^{-1}(F)$.

□

Theorem 4.2. If a non-trivial F_σ set $F \subseteq \mathbb{R}$ satisfies (I), then there is a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with $E = f^{-1}[F]$. Hence E is least among the sets below $\mathbb{Q}$.

Proof. Let $g : \mathbb{Z} \to \mathbb{R}$ be an order-preserving map such that range(g) is unbounded in both directions and $g(z)$ is in the interior of E for all $z \in \mathbb{Z}$. Since F is non-trivial and satisfies (I), there is an order-preserving map $h : \mathbb{Z} \to \mathbb{R}$ such that $h(z) \in F$ and $[h(z), h(z+1)] \not\subseteq F$ for all $z \in \mathbb{Z}$. We define f separately on the intervals $[g(z), g(z+1)]$ for $z \in \mathbb{Z}$ using the previous lemma. □

Towards a further analysis of Borel subsets of $\mathbb{R}$ up to continuous reducibility, the following problems appear.

Problem 4.12. 1. Is there a minimal F_σ set A with $A \not\leq \mathbb{Q}$?

2. Is there a minimal Σ^0_3 set?

Continuous reductions on $\mathbb{R}$ are close to continuous surjective reductions, leading to the following problem.

Problem 4.13. Describe the structure of the Borel subsets of ${}^\omega\omega$ and $\mathbb{R}$ up to continuous surjective reducibility.

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