Philipp Schlicht

Thin Equivalence Relations in $L(\mathbb{R})$ and Inner Models

2008

Mathematik

Thin Equivalence Relations in $L(\mathbb{R})$ and Inner Models

Inauguraldissertation zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften
durch den Fachbereich Mathematik und Informatik
der Westfälischen Wilhelms-Universität Münster

vorgelegt von Philipp Schlicht

– 2008 –

Dekan:

Prof. Dr. Dr. h.c. Joachim Cuntz

Gutachter:

Prof. Dr. Ralf Schindler

Prof. Dr. Gregory Hjorth, University of Melbourne

Tag der mündlichen Prüfung: 28.10.2008

Tag der Promotion: 28.10.2008

Contents

Introduction **v**

Overview **ix**

Acknowledgments **xi**

1 The framework **1**

1.1 Prewellorders and scales 1

1.1.1 Basic definitions and facts 1

1.1.2 Prewellorders under determinacy 5

1.2 Mice with Woodin cardinals 8

1.2.1 Premice, comparison, and $M_n^\#$ 9

1.2.2 Genericity iteration 12

1.2.3 The $\mathcal{Q}$-structure iteration strategy 14

1.2.4 Tools for ω_1-iterable premice 18

2 Lifting thin equivalence relations to forcing extensions **23**

2.1 Reasonable forcing 23

2.1.1 Absoluteness of $M_n^\#$ 24

2.1.2 Absoluteness of equivalence classes 27

2.2 Projective c.c.c. forcing 30

2.2.1 Absoluteness of equivalence classes 31

2.2.2 Prewellorders and generic absoluteness 34

3 The number of equivalence classes **37**

3.1 Co-κ-Suslin equivalence relations . . . 37

3.1.1 A few lemmas . . . 37

3.1.2 The theorem of Harrington and Shelah . . . 42

3.2 Projective equivalence relations . . . 47

3.2.1 $\mathbf{\Pi}^1_n$ and $\mathbf{\Sigma}^1_{2n+1}$ equivalence relations . . . 47

3.2.2 $\mathbf{\Sigma}^1_{2n}$ equivalence relations . . . 51

3.3 Equivalence relations in $L(\mathbb{R})$ . . . 55

3.3.1 Weak term condensation . . . 56

3.3.2 $\mathbf{\Pi}_1(J_\alpha(\mathbb{R}))$ and $\mathbf{\Sigma}_1(J_\alpha(\mathbb{R}))$ equivalence relations . . . 59

4 Inner models for thin equivalence relations **67**

4.1 Projective equivalence relations . . . 67

4.1.1 The main lemma . . . 69

4.1.2 The main theorem . . . 75

4.1.3 An inner model under CH . . . 81

4.2 Equivalence relations in $L(\mathbb{R})$ . . . 85

4.2.1 A direction in the main theorem . . . 85

4.2.2 $\mathbf{\Pi}_1(L(\mathbb{R}))$ equivalence relations and proper forcing . . . 88

5 Conclusion **89**

5.1 The context . . . 89

5.2 Directions for further work . . . 90

5.2.1 Extension of the main theorem . . . 90

5.2.2 The $\mathbf{\Sigma}^1_{2n+1}$ case . . . 90

5.2.3 Consistency strength . . . 90

5.2.4 Projective ordinals . . . 91

Bibliography **93**

Introduction

The answers to many important questions in mathematics are beyond the scope of the standard system of set theory ZFC. Such questions typically appear in descriptive set theory, cardinal arithmetic, topology, algebra, and other areas of mathematics. A particularly interesting example is the theory of definable equivalence relations, the topic of this dissertation.

Definable equivalence relations have become a focus of modern descriptive set theory. While current research centers around Borel equivalence relations, there has been a large amount of work on projective equivalence relations by Harrington and Sami [8], Hjorth [10, 12, 13], Hjorth and Kechris [14], Kechris [19], Louveau and Rosendal [25], Silver [44], and other researchers. Hjorth [11] and others have studied equivalence relations in the constructible universe $L(\mathbb{R})$ over the reals.

Iterable models with Woodin cardinals turned out to be extremely useful for analyzing definable equivalence relations, see Hjorth [12]. Mitchell and Steel [30] and Steel [46, 48] developed the theory of iterable mice with Woodin cardinals, which is one of the main tools in this dissertation. The original approach in the work of Harrington and Sami [8], Kechris [19], and others was to use the axiom of projective determinacy, which states that there are winning strategies for infinite two-player games with projective pay-off sets. This axiom allowed researchers to answer most of the interesting questions about projective sets. Meanwhile it is known from work of Martin, Steel, Woodin, and Neeman (see [23, 28, 32, 37]) that the two approaches are equivalent.

Thin projective equivalence relations, i.e. those with no perfect set of pairwise inequivalent reals, have been extensively studied, most notably by Harrington and Sami [8]. Research in this direction was in part motivated by the question of how many equivalence classes there are. For equivalence relations with a perfect set of pairwise inequivalent reals the number of equivalence classes is the size of the continuum $2^{\aleph_0}$.

A starting point in this field was Silver's famous theorem [44] that any thin $\mathbf{\Pi}^1_1$ equivalence relation has countably many equivalence classes. Subsequently this result had been generalized through the projective hierarchy by Harrington and Sami [8] assuming projective determinacy holds. In this case the number of equivalence classes can be calculated relative to the projective ordinals $\underset{\sim}{\delta}^1_n$, the suprema of the order types of $\mathbf{\Delta}^1_n$ prewellorders. The number of equivalence classes of thin $\mathbf{\Pi}^1_{2n+1}$ equivalence relations is strictly less than the projective ordinal $\underset{\sim}{\delta}^1_{2n+1}$, if $\underset{\sim}{\delta}^1_{2n+1}$ is a cardinal, and at most $Card(\underset{\sim}{\delta}^1_{2n+1})$ otherwise, see [8]. For the even levels the number of equivalence classes of thin $\mathbf{\Pi}^1_{2n+2}$ equivalence relations is at most $Card(\underset{\sim}{\delta}^1_{2n+1})$, see [8].

A quite different approach to determine the number of equivalence classes of thin equivalence relations comes from a question asked about thin equivalence relations which are co-κ-Suslin, i.e. equivalence relations whose complement is the projection of a tree T on $\omega \times \omega \times \kappa$ for a cardinal κ. Is the number of equivalence classes of such equivalence relations at most κ? Harrington and Shelah [9] answered this in the positive under the additional requirement that the complement of $p[T]$ is an equivalence relation in any Cohen generic extension. It turns out that their theorem is sufficient to determine the number of equivalence classes of thin $\mathbf{\Pi}^1_n$ equivalence relations, if the pointclasses $\mathbf{\Pi}^1_{2k+1}$ are scaled and all projective sets have the Baire property. The point is that if a set has a scale, then it is Suslin via the tree from the scale.

Since the number of equivalence classes of thin $\mathbf{\Pi}^1_n$ equivalence relations is bounded by a projective ordinal, it is natural to search for an inner model with fewer reals than V which has representatives in all equivalence classes of all thin $\mathbf{\Pi}^1_n$ equivalence relations defined from a parameter in the inner model. Hjorth [10] showed that as a consequence of Silver's theorem, every inner model has this property for $n = 1$. The candidates for such inner models for $n \geq 2$ are forcing extensions of fine structural inner models with Woodin cardinals. It is unclear, however, how to construct such an inner model for $n \geq 2$ without assuming the continuum hypothesis. Nevertheless, the inner models for $n = 2$ can be characterized. Hjorth [10] proved that if all reals have sharps, then the inner models with this property for $n = 2$ are exactly those which calculate ω_1 correctly and are correct about Σ^1_3 statements.

This research project was aimed at extending Hjorth's theorem to all even levels in the projective hierarchy. This is realized in the main theorem; the level of correctness is adapted and ω_1 is replaced by the tree T_{2n+1} from the canonical Π^1_{2n+1}-scale on the complete Π^1_{2n+1} set. Corresponding to the existence of sharps for reals in Hjorth's theorem, we assume the appropriate amount of projective determinacy, or equivalently the existence of certain ω_1-iterable premice with Woodin cardinals. The main theorem describes the inner models which have representatives in all equivalence classes of thin equivalence relations in a given projective pointclass of the form $\mathbf{\Pi}^1_{2n}$. Thus these inner models are characterized in a simple and beautiful way.

The proof of the main theorem, while a generalization of the proof of Hjorth's theorem, is substantially more complicated. Part of the proof is purely descriptive, whereas the more intricate direction hinges on a result which is proved separately as the main lemma. Let T^M_{2n+1} be the tree from the canonical Π^1_{2n+1}-scale computed in an inner model M with countably many reals. The idea of the main lemma is to reconstruct T^M_{2n+1} in an iterate of $M^\#_{2n}$. We apply Woodin's genericity iteration to build a stack of iteration trees in order to realize the reals of M as generic reals for the extender algebra over local Woodin cardinals in initial segments of iterates of $M^\#_{2n}$. We then form a forcing extension of the last model of the composition of the iteration trees. While this generic extension does not contain all reals of M, it does contain sufficiently many so that T^M_{2n+1} can be defined from the canonical Π^1_{2n+1}-scale.

The second ingredient for the proof of the main theorem is a result of Harrington and Shelah [9]. Suppose E is a thin equivalence relation which is co-κ-Suslin via the tree T. Harrington and Shelah proved that for any real x there is an infinitary formula simply definable from T which describes a neighborhood of x contained in the equivalence class of x. It is further known from Steel [46] that $M^\#_n$ is coded by a projective real. Combining this with the main lemma, the existence of a real satisfying the infinitary formula can be expressed in a projective way.

Let's look at the setting of the main theorem from a different perspective. Suppose V is a forcing extension of an inner model. Here the issue is whether a forcing introduces new equivalence classes to thin $\mathbf{\Pi}^1_{2n}$ equivalence relations. Foreman and Magidor [4] studied a related problem for thin κ-weakly homogeneously Suslin equivalence relations and the class of reasonable forcings, a large class of forcings which includes all proper forcings. They found out that reasonable forcing of size $\leq \kappa$ does not add new equivalence classes to such equivalence relations. Combined with Martin's and Steel's theorem [29], that every projective set is κ-

homogeneously Suslin if κ is the limit of ω many Woodin cardinals, it follows that reasonable forcing cannot add equivalence classes to thin projective equivalence relations.

We give a proof without large cardinals in V, but instead assuming that $M_n^\#(X)$ exists for every $X \in H_{\kappa^+}$, that reasonable forcing of size $\leq \kappa$ does not add new equivalence classes to thin provably $\boldsymbol{\Delta}^1_{n+2}$ equivalence relations. The proof relies on the fact that in this situation $M_n^\#(X)$ is absolute for forcing of size $\leq \kappa$.

Let's go back to the results about the number of equivalence classes mentioned on page 2. The results are applicable to thin $\boldsymbol{\Sigma}^1_{2n}$ equivalence relations, since these are $\boldsymbol{\Delta}^1_{2n}$ by a result of Harrington and Sami [8]. In fact Hjorth [10] has shown that every thin Σ^1_2 equivalence relation is Π^1_2 in any real coding $M_1^\#$. We generalize Hjorth's theorem to thin Σ^1_{2n} equivalence relations for all $n \geq 1$.

In order to extend this theorem to higher levels in $L(\mathbb{R})$, we consider the pointclass of Σ_1-definable sets of reals over $J_\alpha(\mathbb{R})$ for certain ordinals α beginning a Σ_1-gap. Woodin has constructed premice with sufficiently simple iteration strategies which can calculate whether a real is in a given set in this pointclass, assuming $\mathsf{AD}^{L(\mathbb{R})}$ (see Schindler and Steel [40]). Using this technique, we show that every thin equivalence relation which is Σ_1-definable over $J_\alpha(\mathbb{R})$ is Π_1-definable over $J_\alpha(\mathbb{R})$ in a real coding a suitable premouse.

These results reveal a general pattern in the structure of thin equivalence relations in $L(\mathbb{R})$ up to $(\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}$ under $\mathsf{AD}^{L(\mathbb{R})}$.

Overview

Chapter 1 first introduces the necessary definitions and facts about thin equivalence relations, prewellorders, and scales. We show that prewellorders induce thin equivalence relations under appropriate determinacy assumptions. Then important facts about premice due to Martin, Steel, and Woodin are presented. We discuss properties of $M_n^\#$, in particular its correctness and projective definability. Woodin's genericity iteration for making a real generic over an iterate of a premouse with a Woodin cardinal is described. Tools for $\omega_1 + 1$-iterable premice are then adapted to ω_1-iterable premice.

Chapter 2 studies liftings of thin projective equivalence relations to forcing extensions. We first work with the class of reasonable forcings, which comprises all proper forcings. Based on an idea of Foreman and Magidor [4], it is shown that reasonable forcings of size $\leq \kappa$ do not introduce new equivalence classes to thin provably $\mathbf{\Delta}^1_{n+3}$ equivalence relations if $M_n^\#(X)$ exists for every self-wellordered set $X \in H_{\kappa^+}$, where κ is an infinite cardinal. Adapting the argument to $\mathbf{\Sigma}^1_2$ c.c.c. forcings, we prove generic Σ^1_{n+3} absoluteness for such forcings from the assumption that $M_n^\#(x)$ exists for every real x. This generalizes the corresponding result of Woodin [52] for Cohen and random forcing. Moreover, in this situation no new equivalence classes are added to thin provably $\mathbf{\Delta}^1_{n+3}$ equivalence relations. The last result in this chapter states that if generic Σ^1_{n+1} Cohen absoluteness holds, then Cohen forcing does not add equivalence classes to prewellorders which are boolean combinations of $\mathbf{\Sigma}^1_n$ sets, for all $n \geq 1$. This lemma will be used in the next two chapters.

In *chapter 3* we first present a proof of the theorem of Harrington and Shelah [9] for counting the number of equivalence classes of any thin co-κ-Suslin equivalence relation $E = \mathbb{R}^2 - p[T]$, assuming that $\mathbb{R}^2 - p[T]$ is transitive in any Cohen generic extension of $L[T]$. The theorem is applied to calculate the number of equivalence classes for thin $\mathbf{\Pi}^1_n$ and $\mathbf{\Sigma}^1_{2n+1}$ equivalence relations, assuming the pointclasses

$\mathbf{\Pi}^1_{2k+1}$ are scaled and all projective sets have the Baire property. We further show that thin Σ^1_{2n} equivalence relations are Π^1_{2n} in any real coding $M^{\#}_{2n-1}$ for $n \geq 1$, generalizing the result for $n = 1$ from Hjorth [12]. Together with the previous theorems this determines the number of equivalence classes of thin $\mathbf{\Sigma}^1_{2n}$ equivalence relations. We then introduce suitable premice and discuss facts about them due to Woodin, which can be found in [40]. Using this technique, it is shown that thin $\Sigma_1(J_\alpha(\mathbb{R}))$ equivalence relations are $\Pi_1(J_\alpha(\mathbb{R}))$ in a real coding a suitable premouse, for appropriate ordinals α.

Chapter 4 is devoted to the proof of the main lemma and the main theorem. The main lemma shows that the tree T_{2n+1} from the canonical scale on the complete Π^1_{2n+1} set can be reconstructed in an iterate of $M^{\#}_{2n}$ for $n \geq 1$. In the main theorem we then characterize the inner models which have a representative in every equivalence class of every thin $\mathbf{\Pi}^1_{2n}$ equivalence relation defined from a parameter in the inner model, for $n \geq 1$. The conditions state that the inner model is Σ^1_{2n+1}-correct in V and that it calculates the tree T_{2n-1} correctly. We further build a transitive model with this property, assuming CH or merely $\underset{\sim}{\delta}^1_{2n+1} < \omega_2$. Then a version for $(\mathbf{\Pi}^2_1)^{L(\mathbb{R})}$ equivalence relations of one direction of the main theorem is derived. Finally, we show from the large cardinal assumption A_κ that proper forcing of size κ does not add equivalence classes to thin $(\mathbf{\Pi}^2_1)^{L(\mathbb{R})}$ equivalence relations, using a result of Neeman and Zapletal [36].

In the conclusion, in *Chapter 5,* the results are placed in context and related open problems are discussed.

Acknowledgments

First and foremost I would like to thank Prof. Ralf Schindler for supervising this project. Writing this dissertation would have been impossible without him. I am very grateful to Prof. Greg Hjorth for inspiring ideas and discussions and for his help during my stay at the University of California, Los Angeles in the academic year 2005/2006. I would further like to express my gratitude to the members of the Institut für Mathematische Logik und Grundlagenforschung at the University of Münster for discussions and the great work atmosphere.

My stay at UCLA was a great opportunity for my research. It was only made possible by financial support from UCLA from October 2005 through March 2006, thanks to Prof. Itay Neeman, and a doctoral fellowship from the German Academic Exchange Service DAAD from October 2005 through September 2006. I want to thank all people who made this possible.

I wish to thank everybody who helped me during the time this thesis was written. Finally, I would like to thank Flavia for her constant support and patience.

Chapter 1

The framework

This chapter presents standard definitions and facts which are used later. We work in the theory $\mathsf{ZF} + \mathsf{DC}$. For missing definitions in set theory, in particular descriptive set theory, we refer to Jech [16], Kanamori [17], Kechris [21], and Moschovakis [31].

1.1 Prewellorders and scales

In this section we discuss the basics about prewellorders and scales.

1.1.1 Basic definitions and facts

$\mathbb{R}$ as well as ${}^\omega\omega$ denotes Baire space, the set of sequences of natural numbers with the standard topology. The elements of $\mathbb{R}$ are called reals. A perfect set is a nonempty closed set of reals without isolated points. Clearly every perfect set has the size of the continuum.

Definition 1.1.1: *An equivalence relation $E \subseteq \mathbb{R} \times \mathbb{R}$ is called thin if there is no perfect set of pairwise inequivalent reals.*

The corresponding notion is also defined for prewellorders. Recall that a prewellorder is a wellfounded linear preorder.

Definition 1.1.2: *A prewellorder $\leq$ is called thin if there is no perfect set $P \subseteq \mathbb{R}$ such that $x < y$ or $y < x$ for any $x, y \in P$ with $x \neq y$.*

We will work with the projective pointclasses. By a pointclass we mean:

Definition 1.1.3: *A (lightface) pointclass* Γ *is a set* $\emptyset \neq \Gamma \subsetneqq \mathcal{P}(\mathbb{R})$ *which is closed under recursive preimages and finite intersections and unions. The dual of* Γ *is defined as* $\breve{\Gamma} = \{A \subseteq \mathbb{R} : \mathbb{R} - A\}$*. If* Γ *is a pointclass we write* $\Delta := \Gamma \cap \breve{\Gamma}$*.*

Of course for any pointclass Γ we have a corresponding pointclass of subsets of $\mathbb{R}^n$ via a recursive bijection $\mathbb{R} \to \mathbb{R}^n$.

Definition 1.1.4: *If* Γ *is a pointclass, then* $\utilde{\Gamma}$ *is defined as the pointclass of all preimages of sets in* Γ *under continuous functions. A boldface pointclass is a pointclass with* $\Gamma = \utilde{\Gamma}$*.*

Definition 1.1.5: *If* Γ *is a pointclass, then* $< \omega - \Gamma$ *denotes the pointclass of boolean combinations of sets in* Γ*, i.e. sets which are formed from sets in* Γ *by finite applications of union and complement.*

Some of the relevant structural properties of pointclasses are given by norms and scales.

Definition 1.1.6: *Suppose* Γ *is a pointclass and* $A \in \Gamma$*. A prewellorder* $\leq$ *with domain* $\mathbb{R}$ *is called a* Γ*-norm on* A *if* $x \leq y$ *and* $y \in A$ *imply* $x \in A$*, and* $\leq$ *is uniformly* Δ *in initial segments, i.e. there is a* Δ *set* $B \subseteq \mathbb{R}^2$ *with*

$$\{(x,y) \in \mathbb{R}^2 : x \leq y \wedge y \in A\} = B \cap \{(x,y) \in \mathbb{R}^2 : y \in A\}.$$

Let $\equiv$ *be the equivalence relation induced by* $\leq$ *and let*

$$rank(x) := otp(\{y : y < x\}/\equiv)$$

for $x \in A$ *and*

$$rank(x) := \infty$$

for $x \notin A$*.*

Definition 1.1.7: *Suppose* Γ *is a pointclass and* $A \in \Gamma$*. A sequence* $(\leq_n : n < \omega)$ *of* Γ*-norms on* A *with*

$$\{(x,y,n) \in \mathbb{R} \times \mathbb{R} \times \omega : x \leq_n y\} \in \Gamma$$

is called a Γ*-scale on* A *if there is a set* $B \in \Delta$ *with*

$$\{(x, y, n) \in \mathbb{R} \times \mathbb{R} \times \omega : x \leq_n y \wedge y \in A\} = B \cap \{(x, y, n) \in \mathbb{R} \times \mathbb{R} \times \omega : y \in A\},$$

and if $(x_k : k < \omega) \in {}^\omega\mathbb{R}$ *with* $x_k \to x$ *and* $rank_n(x_k) \to \alpha_n$ *(i.e.* $rank_n(x_k)$ *is eventually constant) for all* n*, then* $x \in A$ *and* $rank_n(x) \leq \alpha_n$*. Here* $rank_n$ *denotes the rank in* $\leq_n$*. A pointclass is scaled if there is a* Γ*-scale on every* $A \in \Gamma$*.*

With each scale one associates a tree, from which the scale can again be defined:

Definition 1.1.8: *Suppose* $(\leq_n : n \in \omega)$ *is a* Γ*-scale on* $A \in \Gamma$ *where* Γ *is a pointclass. The tree from the scale is defined as*

$$T = \{(x \upharpoonright n, (rank_0(x), ..., rank_{n-1}(x))) : x \in A \wedge n < \omega\}.$$

Note that $A = p[T]$ in the situation of the definition. Given $x \in p[T]$, there are ordinals α_n and reals x_k such that $x \upharpoonright k = x_k \upharpoonright k$ for all $k \in \omega$ and $rank_n(x_k) = \alpha_n$ for all $n \leq k$, so $x \in A$ by the semicontinuity of the scale.

The projective pointclasses Π^1_{2n+1} and Σ^1_{2n+2} and their boldface versions are scaled by the second periodicity theorem [21, theorem 39.8] if $Det(\boldsymbol{\Delta}^1_{2n})$ holds, where $\Delta^1_0 = \Delta^0_\omega$ denotes the pointclass of arithmetical sets. Let's fix the $\boldsymbol{\Pi}^1_{2n+1}$-complete Π^1_{2n+1} set and the $\boldsymbol{\Sigma}^1_{2n+2}$-complete Σ^1_{2n+2} set from the proof of the second periodicity theorem for each $n < \omega$. We will simply call these sets the complete Π^1_{2n+1} set and the complete Σ^1_{2n+2} set. Let's also fix the canonical scales on these sets from the proof of this theorem.

Definition 1.1.9: *Suppose* $Det(\boldsymbol{\Delta}^1_{2n})$ *holds. Then* T_{2n+1} *denotes the tree from the canonical* Π^1_{2n+1}*-scale on the complete* Π^1_{2n+1} *set.*

We will work with transitive models of a fragment of ZF between which wellfoundedness is absolute.

Definition 1.1.10: *A transitive set* $\mathbb{A}$ *is called admissible if* $(\mathbb{A}, \in) \models \mathsf{KP}$*. It is called* β*-admissible if* $(\mathbb{A}, \in) \models \mathsf{KP}$+*Axiom Beta.*

For a background on admissible sets see Barwise [3]. Axiom Beta asserts that every wellfounded relation can be collapsed to a transitive set, see [3, chapter I, section 9]. Note that the definition of β-admissible is not standard.

Lemma 1.1.11: *Every $\Sigma^1_{2n+2}(x)$ set is the projection of a tree which is uniformly defined from T_{2n+1} and x in every β-admissible set $\mathbb{A}$ with $T_{2n+1}, x \in \mathbb{A}$.*

PROOF: The tree T from the scale on the complete Σ^1_{2n+2} set is essentially T_{2n+1}, see [21, theorem 38.4]. Any $\Sigma^1_{2n+2}(x)$ set B for $x \in \mathbb{R}$ is the preimage of the complete Σ^1_{2n+2} set under some function $f : \mathbb{R} \to \mathbb{R}$ recursive in x. Then the tree

$$S := \{(s,h) \in (\omega \times Ord)^{<\omega} : \exists y = y_{s,h} \supset s \, \forall i < lh(s)(rank_i(f(y)) = h(i))\}$$

induces a $\Sigma^1_{2n+2}(x)$-scale on B. Now $rank_i(f(y))$ can be calculated from $f(y)$ and T in any β-admissible set $\mathbb{A}$ with $T, f(y) \in \mathbb{A}$. Since the existence of $y_{s,h}$ for given s, h is absolute between β-admissible sets, S is as required. ∎

We will further work with pointclasses of the form $\Sigma_n(J_\alpha(\mathbb{R}))$. Here $\Sigma_n(J_\alpha(\mathbb{R}))$ denotes the pointclass of sets of reals which are Σ_n-definable over $J_\alpha(\mathbb{R})$ without parameters. The next definition from [39] will be used in sections 3.3 and 4.2 only.

Definition 1.1.12: *A pointclass Γ is called a scaled Σ-pointclass if it is one of the following:*

1. $\Gamma = \Sigma^1_{2n+2}$ *for some* $n < \omega$,
2. $[\alpha, \beta]$ *is a* Σ_1*-gap with* $\alpha > 1$ *and* $\Gamma = \Sigma_1(J_\alpha(\mathbb{R}))$,
3. $[\alpha, \beta]$ *is a* Σ_1*-gap,* $\alpha = \beta > 1$, α *is* $\mathbb{R}$*-inadmissible (i.e.* $J_\alpha \not\models \mathsf{KP}$*), and* $\Gamma = \Sigma_{2i+1}(J_\alpha(\mathbb{R}))$ *for some* $i < \omega$, *or*
4. $[\alpha, \beta]$ *is a weak* Σ_1*-gap,* $\alpha < \beta$, *and* $\Gamma = \Sigma_{n+2i}(J_\beta(\mathbb{R}))$ *for some* $i < \omega$, *where* $n < \omega$ *is least such that* $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$.

Steel [45] proved from $\mathsf{AD}^{L(\mathbb{R})}$ that these pointclasses are actually scaled. Here, a Σ_1-gap $[\alpha, \beta]$ is a maximal interval so that the same Σ_1 statements with parameters in $\mathbb{R} \cup \{V_{\omega+1}\}$ are true in $J_\alpha(\mathbb{R})$ and $J_\beta(\mathbb{R})$. For the definition of weak gaps see [45, definition 3.2].

Since $J_1(\mathbb{R}) = V_{\omega+1}$, the pointclass $\Sigma^1_{2n+2} = \Sigma_{2n+2}(J_1(\mathbb{R}))$ corresponds to the case $\alpha = 1$.

1.1.2 Prewellorders under determinacy

Typical examples of thin equivalence relations are given by prewellorders. We will need the following facts to know that prewellorders induce thin equivalence relations under determinacy.

Lemma 1.1.13: *(Kechris [18]) Suppose* Γ *is a pointclass containing the* Π^0_1 *sets and* $Det(\Gamma)$ *holds. Then every* $\Game\Gamma$ *set has the Baire property and there is no* $\Game\Gamma$ *wellorder of the reals.*

PROOF: To prove that every $\Game\Gamma$ set has the Baire property, let $B \subseteq \mathbb{R}^2$ and

$$A = \Game B = \{x \in \mathbb{R} : \text{player 2 wins the game for } B_x\},$$

where $B_x := \{y \in \mathbb{R} : (x, y) \in B\}$. Basic open subsets of $\mathbb{R}$ and $\mathbb{R}^2$ are denoted by

$$U_s := \{x \in \mathbb{R} : x \restriction dom(s) = s\}$$

and

$$U_{s,t} := \{(x, y) \in \mathbb{R}^2 : x \restriction dom(s) = s \wedge y \restriction dom(t) = t\}$$

for $s, t \in \omega^{<\omega}$.

We first claim that the Banach-Mazur game for

$$A \cup (\mathbb{R} - U_s) = \Game[B \cup (\mathbb{R}^2 - U_{\emptyset,s})]$$

is determined for all $s \in \omega^{<\omega}$. In this game two players alternate playing finite sequences $s_0, s_1, ..$ and player 2 wins if $s_0 \frown s_1 \frown .. \in A \cup (\mathbb{R} - U_s)$. This game is equivalent to the Banach-Mazur game for $B \cup (\mathbb{R}^2 - U_t)$ by the game formula [18, theorem 3.3.1], and hence determined.

Now let S be the set of $s \in \omega^{<\omega}$ such that player 2 has a winning strategy in the Banach-Mazur game for $A \cup (\mathbb{R} - U_s)$. Then A is comeager in U_s for each $s \in S$ by the characterization of comeager sets in [21, theorem 8.33], so A is comeager in

$$U_1 := \bigcup_{s \in S} U_s.$$

The same theorem shows that for every $t \in \omega^{<\omega} - S$, the set A is meager in some nonempty open subset $U_{f(t)}$ of U_t, so A is meager in

$$U_2 := \bigcup_{t \in \omega^{<\omega} - S} U_{f(t)}.$$

Since $(\mathbb{R} - U_1) - U_2$ is nowhere dense, this implies that $A \triangle U_1$ is meager. Hence A has the Baire property.

Let's recall the proof that there is no wellorder of the reals with the Baire property. If $<$ were such a wellorder, we define

$$A := \{(x, y) \in \mathbb{R}^2 : x < y\},$$

$$B := \{(x, y) \in \mathbb{R}^2 : x > y\},$$

and

$$C := \{(x, y) \in \mathbb{R}^2 : x = y\}.$$

Then both A and B are not meager, since C is nowhere dense. Hence there is some $x \in \mathbb{R}$ such that

$$A_x := \{y \in \mathbb{R} : x < y\}$$

is not meager by the theorem of Kuratowski and Ulam. Choose z as $<$-minimal with this property. Again $A \cap (A_z \times A_z)$ and $B \cap (A_z \times A_z)$ are not meager. So there is some $x \in A_z$ with A_x not meager, contradicting the minimality of z. ∎

The proof of the previous lemma shows that there is no wellorder of the reals in the σ-algebra generated by $\Game\Gamma$.

Lemma 1.1.14: *(Kechris [18]) Suppose* Γ *is a boldface pointclass containing the* Π^0_1 *sets and* $Det(\Gamma)$ *holds. Then every prewellorder in* $\Game\Gamma$ *is thin.*

Proof: Let $\leq$ be a prewellorder in $\Game\Gamma$ and suppose $P \subseteq \mathbb{R}$ is a perfect set so that $x \not\leq y$ and $y \not\leq x$ for any two distinct $x, y \in P$. Then $\leq$ wellorders P. Now $\Game\Gamma$ is closed under continuous preimages since Γ is a boldface pointclass, so any continuous injective map $f : \mathbb{R} \to P$ induces a $\Game\Gamma$ wellorder of the reals, contradicting the previous lemma. ∎

The next two lemmas will be important for our purposes.

Lemma 1.1.15: *$Det(\mathbf{\Delta}^1_{2n})$ implies that every $\mathbf{\Pi}^1_{2n+1}$ norm is thin.*

PROOF: Suppose $P \subseteq \mathbb{R}$ is a perfect set whose elements have pairwise different norms $\neq \infty$ and let $f : \mathbb{R} \to P$ be a continuous injective map. Then f induces a $\boldsymbol{\Delta}^1_{2n+1}$ wellorder of the reals.

Now $Det(\boldsymbol{\Delta}^1_{2n})$ implies $Det(\boldsymbol{\Pi}^1_{2n})$ by [22, theorem 5.1] and further $\Game\boldsymbol{\Pi}^1_{2n} = \boldsymbol{\Sigma}^1_{2n+1}$ by [21, proposition 39.6]. So there is no $\boldsymbol{\Sigma}^1_{2n+1}$ wellorder of the reals by lemma 1.1.13. ∎

Note that the conclusion of the previous lemma follows from the Baire property or the Lebesgue measurability of all $\boldsymbol{\Delta}^1_{2n+1}$ sets alone.

Lemma 1.1.16: *$Det(\boldsymbol{\Pi}^1_{2n+1})$ implies that every $\boldsymbol{\Sigma}^1_{2n+2}$ norm is thin.*

PROOF: As the previous lemma; otherwise there is a $\boldsymbol{\Delta}^1_{2n+2}$ wellorder of the reals, contradicting lemma 1.1.13. ∎

It is sufficient to assume the Baire property or the Lebesgue measurability of all $\boldsymbol{\Delta}^1_{2n+2}$ sets for the previous lemma.

Lemma 1.1.17: *The following are equivalent:*

1. *every $\boldsymbol{\Delta}^1_2$ prewellorder of the reals is thin,*
2. *there is no $\boldsymbol{\Delta}^1_2$ wellorder of the reals, and*
3. *$L[x]$ does not contain $\mathbb{R}$ for any $x \in \mathbb{R}$.*

PROOF: Condition 1 clearly implies condition 2. To show that 2 implies 1, suppose $\leq$ is a $\boldsymbol{\Delta}^1_2$ prewellorder of the reals and $P \subseteq \mathbb{R}$ is perfect with $x < y$ or $x > y$ for any two distinct $x, y \in P$. Then any continuous injective map $f : \mathbb{R} \to P$ induces a $\boldsymbol{\Delta}^1_2$ wellorder of the reals.

Now condition 2 implies condition 3, since if $\mathbb{R} \subseteq L[x]$ for some $x \in \mathbb{R}$, then the order of constructibility of the reals is $\Delta^1_2(x)$. To show that 3 implies 2, note that $\mathbb{R} \subseteq L[x]$ by [16, theorem 25.39] if there is a $\Delta^1_2(x)$ wellorder of the reals. ∎

The projective ordinals are given by

Definition 1.1.18: *The n^{th} projective ordinal $\underset{\sim}{\delta}^1_n$ is the supremum of lengths of $\boldsymbol{\Delta}^1_n$ prewellorders for $n \geq 1$.*

We state some of their properties, since the projective ordinals play an essential role in calculating the number of equivalence classes of thin projective equivalence relations in section 3.2.

Lemma 1.1.19: *The following facts hold for the projective ordinals:*

1. *(Martin)* $\mathsf{ZF} + \mathsf{PD}$ *implies* $\underset{\sim}{\delta}^1_1 = \omega_1$ *and* $\underset{\sim}{\delta}^1_n \leq \omega_n$ *for* $n \leq 4$*,*
2. *(Kechris, Moschovakis)* $\mathsf{ZF} + \mathsf{PD}$ *implies* $\underset{\sim}{\delta}^1_n < \underset{\sim}{\delta}^1_{n+1}$ *for all* n*,*
3. *(Moschovakis)* $\mathsf{ZF} + \mathsf{AD}$ *implies that each* $\underset{\sim}{\delta}^1_n$ *is a cardinal, and*
4. *(Steel, Van Wesep [50])* $\mathsf{ZF} + \mathsf{AD}^{L(\mathbb{R})} + \underset{\sim}{\delta}^1_2 = \omega_2$ *is consistent relative to* $\mathsf{ZF} + \mathsf{AD} + \mathsf{AC}_{\mathbb{R}}$*.*

Proof: The proofs for parts 1 and 2 can be found in [20, theorem 9.1]. For part 3 see [20, theorem 2.2]. Note that Jackson [15] has computed all $\underset{\sim}{\delta}^1_n$ exactly under AD. For part 4 see [50]. Note that Woodin [53, theorem 3.17] proved that $\underset{\sim}{\delta}^1_2 = \omega_2$ holds if $\mathcal{P}(\omega_1)^\#$ exists and the nonstationary ideal on $\aleph_1$ is $\aleph_2$-saturated. ■

An important open question is how large the projective ordinal $\underset{\sim}{\delta}^1_n$ for $n \geq 3$ can be under $\mathsf{ZFC} + \mathsf{AD}^{L(\mathbb{R})}$. In fact, it is still open if $\mathsf{AD}^{L(\mathbb{R})}$ implies $\underset{\sim}{\delta}^1_n \leq \omega_n$ for all n, see [17, question 30.34].

Note that the consistency strength of $\underset{\sim}{\delta}^1_2 = \omega_2$ in the presence of sharps for reals is somewhere between a strong cardinal and a Woodin cardinal with a measurable cardinal above by work of Steel and Welch [51] and Woodin [53, theorem 3.25]. While we focus on the situation that $\mathsf{ZF} + \mathsf{PD}$ holds, one can consider the case that $0^\#$ does not exist. Note that MA and $\omega_1 = \omega_1^L$ already imply $\underset{\sim}{\delta}^1_2 = \omega_2$. This is because $\omega_1 = \omega_1^L$ implies that any subset of ω_1 can be coded as Δ_1^{HC} in a real by c.c.c. forcing so that ω_1 many dense subsets suffice to define the real. Moreover in this situation $\underset{\sim}{\delta}^1_3$ can be quite easily forced to be arbitrarily large with a forcing from Harrington [6].

1.2 Mice with Woodin cardinals

In this section tools for mice with Woodin cardinals are presented. The results are due to Martin, Steel, and Woodin. For missing definitions and proofs see

Martin and Steel [29], Mitchell and Steel [30], Schindler and Zeman [42], Steel [48], and Zeman [54]. Several facts about $\omega_1 + 1$-iterable premice are adapted to ω_1-iterable premice. The reason is that we only want to assume PD; all one can get from PD is the existence of ω_1-iterable premice with n Woodin cardinals for arbitrary $n < \omega$.

1.2.1 Premice, comparison, and $M_n^\#$

Definition 1.2.1: *A self-wellordered (swo) set is a set which codes a wellorder of itself. The height of a self-wellordered set X is*

$$ht(X) := \sup((Ord \cap tc(X)) \cup \omega).$$

Every self-wellordered set can be coded by a set $\sup(A) \cup A$, where A is a set of ordinals. Recall that the first level of the J-hierarchy built over a set X is defined as $J_0(X) = tc(\{X\})$.

Definition 1.2.2: *A potential X-premouse is a structure*

$$\mathcal{M} = (J_\beta^{\vec{F}}(X), \in, X, \vec{F} \restriction \beta, F_\beta)$$

where X is swo and $\vec{F}$ is a fine extender sequence relative to X. An X-premouse is a potential X-premouse all of whose proper initial segments are ω-sound; a premouse is simply a $\emptyset$-premouse. A boldface or relativized premouse is an X-premouse for some swo set X. $\mathcal{M}$ is called active if $F_\beta \neq \emptyset$, otherwise it is passive. We write $\vec{F}^{\mathcal{M}}$ for the extender sequence of $\mathcal{M}$.

For the definition of fine extender sequences see [48, definition 2.4] and for ω-sound [48, definition 2.17].

Definition 1.2.3: *Suppose*

$$\mathcal{M} = (J_\alpha^{\vec{F}}(X), \in, X, \vec{F} \restriction \alpha, F_\alpha)$$

and

$$\mathcal{N} = (J_\beta^{\vec{F}}(X), \in, X, \vec{F} \restriction \beta, F_\beta)$$

are X-premice where X is swo and $\alpha \leq \beta$ ($\alpha < \beta$). Then $\mathcal{M}$ is called a (proper) initial segment of $\mathcal{N}$ and we write $\mathcal{M} \trianglelefteq \mathcal{N}$ ($\mathcal{M} \triangleleft \mathcal{N}$). For notation write

$$\mathcal{N}||\alpha := (J_\alpha^{\vec{F}}(X), \in, X, \vec{F} \restriction \alpha, F_\alpha)$$

and

$$\mathcal{N}|\alpha := (J_\alpha^{\vec{F}}(X), \in, X, \vec{F} \restriction \alpha, \emptyset).$$

An ordinal δ is called a cutpoint of $\mathcal{M}$, if for no extender F on the $\mathcal{M}$-sequence do we have $crit(F) < \delta \leq lh(F)$. For the definition of iteration trees see [48, section 3.1]; for normal iteration trees see [54, section 4.2]. An iteration tree $\mathcal{T}$ on an X-premouse $\mathcal{M}$ is said to live on $\mathcal{M}|\alpha$ if all extenders in $\mathcal{T}$ have length less than α.

Definition 1.2.4: *Let $k \leq \omega$ and $\theta \in Ord$ and suppose X is swo. An X-premouse $\mathcal{M}$ is called (k, θ)-iterable if player 2 has a winning strategy in the iteration game $\mathcal{G}_k(\mathcal{M}, \theta)$ described in [48, section 3.1]. $\mathcal{M}$ is called normally (k, θ)-iterable if it is (k, θ)-iterable with respect to normal iteration trees. It is (normally) (k, θ)-iterable above δ if it is (normally) (k, θ)-iterable with respect to (normal) iteration trees all of whose extenders have critical points above δ. It is (normally) θ-iterable if it is (normally) (ω, θ)-iterable.*

We need the next two lemmas from [48, theorem 3.11] and [48, corollary 3.12].

Lemma 1.2.5: *(Comparison lemma) Let $\mathcal{M}$ and $\mathcal{N}$ be countable X-premice, where X is swo. Let δ be a cutpoint of both $\mathcal{M}$ and $\mathcal{N}$ and suppose $\mathcal{M}$ and $\mathcal{N}$ are normally $\omega_1 + 1$-iterable above δ and ω-sound above δ with $\mathcal{M}|\delta = \mathcal{N}|\delta$. Then there are countable iteration trees $\mathcal{S}$ on $\mathcal{M}$ and $\mathcal{T}$ on $\mathcal{N}$ with last models $\mathcal{M}_\alpha^\mathcal{S}$ and $\mathcal{M}_\beta^\mathcal{T}$ so that*

1. *$[0, \alpha]_\mathcal{S}$ does not drop in model or degree and $\mathcal{M}_\alpha^\mathcal{S} \trianglelefteq \mathcal{M}_\beta^\mathcal{T}$, or*
2. *$[0, \beta]_\mathcal{T}$ does not drop in model or degree and $\mathcal{M}_\beta^\mathcal{T} \trianglelefteq \mathcal{M}_\alpha^\mathcal{S}$.*

Lemma 1.2.6: *Let $\mathcal{M}$ and $\mathcal{N}$ be countable X-premice where X is swo. Let δ be a cutpoint of both $\mathcal{M}$ and $\mathcal{N}$ and suppose $\mathcal{M}$ and $\mathcal{N}$ are normally $\omega_1 + 1$-iterable above δ and ω-sound above δ with $\rho_\omega(\mathcal{M}), \rho_\omega(\mathcal{N}) \leq \delta$ and $\mathcal{M}|\delta = \mathcal{N}|\delta$. Then $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{M}$.*

Proof: Neither $\mathcal{M}$ nor $\mathcal{N}$ are moved in the coiteration, since they are ω-sound above δ and $\rho_\omega(\mathcal{M}), \rho_\omega(\mathcal{N}) \leq \delta$. ■

Let's recall the definition of Woodin cardinals:

Definition 1.2.7: *Suppose $A \subseteq V_\delta$. An ordinal $\kappa < \delta$ is A-reflecting in δ if for all $\alpha < \delta$ there is an extender F in V_δ with $crit(F) = \kappa$, $j_F(\kappa) > \alpha$, and*

$$j_F(A) \cap V_\alpha = A \cap V_\alpha,$$

where $j_F : V \to ult(V, F)$ is the ultrapower embedding.

Definition 1.2.8: *A cardinal δ is a Woodin cardinal if for every $A \subseteq V_\delta$ there is some $\kappa < \delta$ which is A-reflecting in δ.*

Now $M_n^\#$ can be defined:

Definition 1.2.9: *Let X be swo and $n \leq \omega$. An X-premouse $\mathcal{M}$ is n-small above δ if there is no extender F on the $\mathcal{M}$-sequence so that in $\mathcal{M}|crit(F)$ there are n Woodin cardinals above δ.*

Definition 1.2.10: *Let $\mathcal{M}$ be an active ω_1-iterable X-premouse, where X is swo, such that $\mathcal{M}$ is ω-sound above $ht(X)$ and $\rho_1(\mathcal{M}) \leq ht(X)$. Let F be the top extender of $\mathcal{M}$ and $n \leq \omega$. $\mathcal{M}$ is called $M_n^\#(X)$ if $\mathcal{M}|crit(F)$ is n-small and in $\mathcal{M}$ there are n Woodin cardinals below $crit(F)$. Moreover $M_n^\# := M_n^\#(\emptyset)$.*

Note that usually $M_n^\#$ is defined as an $\omega_1 + 1$-iterable premouse with the same first-order properties; thus $M_n^\#$ is unique by the comparison lemma. The $M_n^\#(X)$ defined here is also unique in the relevant case that $M_n^\#(x)$ exists for every $x \in \mathbb{R}$. Suppose we have two canditates for $M_n^\#(X)$ where X is swo. Let's consider the preimages of the candidates in the transitive collapse of a countable substructure of some large V_λ. Since these are ω_1-iterable, they can be compared in $M_n^\#(x)$ for some $x \in \mathbb{R}$ by the argument in lemma 1.2.20 below.

The standard definition of $M_n^\#(X)$ just states that it is countably iterable, i.e. all countable substructures are $\omega_1 + 1$-iterable, instead of being $\omega_1 + 1$-iterable itself. Everything would work if in the definition of $M_n(X)$ we only ask that countable substructures are ω_1-iterable.

Note that modulo Gödel numbers for first-order formulas, any x-premouse $\mathcal{M}$ with $x \in \mathbb{R}$ and $\rho_{k+1}(\mathcal{M}) = \omega$ comes with a code $z \in \mathbb{R}$ from the canonical $\Sigma_1^{(k)}$-definable surjection from ω onto $\mathcal{M}$, see [54, section 1.6] for the definition of $\Sigma_1^{(k)}$ formulas. Hence there is no need to distinguish between $M_n^\#(\mathcal{M})$ and $M_n^\#(z)$.

Definition 1.2.11: *Let $\mathcal{M}$ be an active ω_1-iterable X-premouse, where X is swo, such that $\mathcal{M}$ is ω-sound above $ht(X)$ and $\rho_1(\mathcal{M}) \leq ht(X)$. Let F be the top extender of $\mathcal{M}$ and suppose the topmost extender G below F is total. $\mathcal{M}$ is called $M_n^\dagger(X)$ if $\mathcal{M}|crit(G)$ is n-small and in $\mathcal{M}$ there are n Woodin cardinals below $crit(G)$. Moreover $M_n^\dagger := M_n^\dagger(\emptyset)$.*

Again $M_n^\dagger(X)$ is unique in the relevant situation that $M_n^\#(x)$ exists for every $x \in \mathbb{R}$. Since we would like to prove the main theorem from PD, we will use

Theorem 1.2.12: *(Harrington, Martin, Steel, Woodin, Neeman) The following are equivalent for $n < \omega$:*

1. *$Det(\mathbf{\Pi}^1_{n+1})$*
2. *there is an $M_n^\#(x)$ for every $x \in \mathbb{R}$*
3. *there is a unique $M_n^\#(x)$ for every $x \in \mathbb{R}$.*

Proof: See [37, theorem 5.3]. Harrington [7] proved the implication from 1 to 3 for $n = 0$. For arbitrary n see Koellner and Woodin [23]. Martin [26] proved that 2 implies 1 for $n = 0$, Neeman [32] has a proof for arbitrary n. The original proof for odd n is due to Woodin. Note that lemma 1.2.20 below can be used to show that 2 implies 3. ∎

1.2.2 Genericity iteration

We will use a theorem of Woodin to iterate an $\omega_1 + 1$-iterable premouse with a Woodin cardinal so that a given real is generic over the iterate for Woodin's extender algebra.

The extender algebra is built from a set of infinitary formulas. Let's state the necessary definitions. We let δ be an inaccessible cardinal and $\mathcal{L}$ a language which contains at least $\in$ and constants c for a real and $\dot{n}$ for each $n < \omega$. Let N be the set of atomic formulas $\dot{n} \in c$ for $n < \omega$. Now let $\mathcal{L}_{\delta,0,N}$ be the closure of N under negations and infinitary disjunctions and conjunctions of length less than δ. Note that one can equivalently work with the infinitary logic built over a language with propositional formulas p_n for $n < \omega$.

The infinitary proof calculus for this logic has the infinitary rule

$$\forall \alpha < \beta \ \vdash \varphi_\alpha \quad \Rightarrow \quad \vdash \bigwedge_{\alpha<\beta} \varphi_\alpha$$

in addition to the rules of first-order logic; for details see Barwise [3, chapter III, definition 5.1]. Let χ be the $\mathcal{L}_{\delta,0,N}$-sentence

$$\bigwedge_{n<\omega} (\forall x \in \dot{n} \bigvee_{m<n} x = \dot{m}) \wedge (\bigwedge_{m<n} \dot{m} \in \dot{n}) \wedge c \subseteq \omega,$$

which we add as an axiom. Hence in every model, c is interpreted as a subset of ω and each $\dot{n}$ is interpreted as n. Moreover, the infinitary disjunction of a sequence $\vec{\varphi} = (\varphi_\alpha : \alpha < \beta)$ of $\mathcal{L}_{\delta,0,N}$-formulas is denoted by $\bigvee_{\alpha<\beta} \varphi_\alpha$ or $\bigvee \vec{\varphi}$.

The following is Steel's version [48, section 7.2] of Woodin's extender algebra for fine structural mice.

Definition 1.2.13: *Let $\mathcal{M}$ be an X-premouse with Woodin cardinal δ, where X is swo. Let S be the set of all $\mathcal{L}_{\delta,0,N}$-formulas*

$$\bigvee \vec{\varphi} \leftrightarrow \bigvee j_E(\vec{\varphi}) \restriction \lambda$$

in $\mathcal{M}$, where

1. *$\vec{\varphi} = (\varphi_\alpha : \alpha < \kappa) \in \mathcal{M}$ is a sequence of $\mathcal{L}_{\delta,0,N}$-formulas with $\kappa < \delta$,*
2. *F is an extender on the $\mathcal{M}$-sequence with crit$(F) = \kappa \leq \lambda < \delta$,*
3. *$\nu(F)$ is a cardinal in $\mathcal{M}$, and*
4. *$j_F(\vec{\varphi}) \restriction \lambda \in J^{\mathcal{M}}_{\nu(F)}$.*

Here $\nu(F)$ is the natural length of F, see [48, definition 2.2]. Working in $\mathcal{M}$, the extender algebra $\mathbb{W}_\delta$ over δ is defined as the Lindenbaum algebra over $\mathcal{L}_{\delta,0,N}$ for provability from S; let

$$[\varphi] := \{\psi \in \mathcal{L}_{\delta,0,N} : S \vdash \varphi \leftrightarrow \psi\}$$

for $\varphi \in \mathcal{L}_{\delta,0,N}$ *and define*

$$\mathbb{W}_\delta := \{[\varphi] : \varphi \in \mathcal{L}_{\delta,0,N}\}$$

with partial ordering

$$[\varphi] \leq [\psi] :\Leftrightarrow S \vdash \varphi \to \psi.$$

The extender algebra $\mathbb{W}_\delta$ has size δ and the δ-c.c. in $\mathcal{M}$ by [48, theorem 7.14]. We will heavily use the next theorem of Woodin following Steel [48]:

Lemma 1.2.14: *(Genericity iteration) Let* $\mathcal{M}$ *be a countable* X*-premouse, where* X *is swo. Suppose* $\mathcal{M}$ *is normally* $\omega_1 + 1$*-iterable above* $\gamma < \delta$ *and* δ *is Woodin in* $\mathcal{M}$*. Then for each* $x \in \mathbb{R}$*, there is a countable iteration tree* $\mathcal{T}$ *on* $\mathcal{M}$ *with iteration map* π *and last model* $\mathcal{M}^{\mathcal{T}}_\alpha$ *such that* $[0, \alpha]_{\mathcal{T}}$ *does not drop in model and* x *is* $\mathbb{W}^{\mathcal{M}^{\mathcal{T}}_\alpha}_{\pi(\delta)}$*-generic over* $\mathcal{M}^{\mathcal{T}}_\alpha$*.*

Proof: See [48, theorem 7.14]. The idea is to iterate away the least extender which induces an axiom false for x. A reflection argument as in the proof of the comparison lemma shows that after countably many steps x is a model of $\pi(S)$, where π is the iteration map. It follows that x is $\mathbb{W}^{\mathcal{M}^{\mathcal{T}}_\alpha}_{\pi(\delta)}$-generic. ■

1.2.3 The $\mathcal{Q}$-structure iteration strategy

In this section we describe a partial iteration strategy based on so-called $\mathcal{Q}$-structures. In the relevant cases this is the unique iteration strategy.

Definition 1.2.15: *Suppose* $\mathcal{T}$ *is an iteration tree of limit length* θ *with models* $(\mathcal{M}_\alpha : \alpha < \theta)$ *and extenders* $(F_\alpha : \alpha < \theta)$*. Define*

$$\delta(\mathcal{T}) := \sup_{\alpha<\theta} lh(F_\alpha)$$

and

$$\mathcal{M}(\mathcal{T}) := \bigcup_{\alpha<\theta} \mathcal{M}_\alpha | lh(F_\alpha),$$

where $lh(F_\alpha)$ *denotes the length of* F_α*. The model* $\mathcal{M}(\mathcal{T})$ *is called the common part model of* $\mathcal{T}$*.*

$\mathcal{Q}$-structures for iteration trees are defined as follows.

Definition 1.2.16: *Let $\mathcal{T}$ be an iteration tree of limit length on an X-premouse $\mathcal{M}$, where X is swo. A $\mathcal{Q}$-structure for $\mathcal{T}$ is an X-premouse $\mathcal{Q}$ with*

1. *$\mathcal{M}(\mathcal{T}) \trianglelefteq \mathcal{Q}$ such that $\delta(\mathcal{T})$ is a cutpoint of $\mathcal{Q}$,*
2. *$\mathcal{Q}$ is ω_1-iterable above $\delta(\mathcal{T})$, and*
3. *the Woodin property of $\delta(\mathcal{T})$ is destroyed definably over $\mathcal{Q}$, i.e. there is a $k < \omega$ such that*
   - *(a) $\mathcal{Q}$ is $k+1$-sound and*
   - *(b) either $\rho_{k+1}(\mathcal{Q}) < \delta(\mathcal{T})$, or k is minimal such that there is a map $f : \delta(\mathcal{T}) \to \delta(\mathcal{T})$ which is $\Sigma_1^{(k)}$-definable over $\mathcal{Q}$ so that for no extender F on the $\mathcal{Q}$-sequence do we have $i_F(f)(crit(F)) \geq \nu(F)$.*

If in the previous definition $\mathcal{Q} = (J_\beta^{\vec{F}}(X), \in, X, \vec{F} \restriction \beta, F_\beta)$ is a proper initial segment of a premouse, then condition 3 simplifies to the statement that β is minimal with $J_{\beta+1}^{\vec{F}}(X) \vDash$ "$\delta(\mathcal{T})$ is not Woodin". Based on $\mathcal{Q}$-structures, one builds a partial iteration strategy:

Definition 1.2.17: *Let $\mathcal{T}$ be a normal iteration tree on an X-premouse $\mathcal{M}$, where X is swo. Let $\Sigma(\mathcal{T})$ be the unique cofinal branch $b \subseteq \mathcal{T}$ such that $\mathcal{M}_b$ is wellfounded and carries a $\mathcal{Q}$-structure $\mathcal{Q} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$, if such a branch exists. Let $\Sigma(\mathcal{T})$ be undefined if there is no such branch, or if there is one but it is not unique. This partial iteration strategy for normal iteration trees is called the $\mathcal{Q}$-structure iteration strategy.*

We have

Lemma 1.2.18: *If Σ is a θ-iteration strategy for normal iteration trees on an X-premouse $\mathcal{N}$, where X is swo, then Σ is a θ-iteration strategy for normal iteration trees on every initial segment $\mathcal{M} \trianglelefteq \mathcal{N}$.*

Let's consider the situation that $\mathcal{M}$ and $\mathcal{N}$ are X-premice which are θ-iterable via Σ, where X is swo. Suppose $\mathcal{M}|\delta = \mathcal{N}|\delta$ and δ is a cutpoint of both $\mathcal{M}$ and $\mathcal{N}$. Then every iteration tree $\mathcal{T}$ according to Σ on $\mathcal{M}$ living on $\mathcal{M}|\delta$ gives rise to an iteration tree on $\mathcal{N}$. In this case we say that $\mathcal{T}$ acts on $\mathcal{N}$.

In the relevant situation the $\mathcal{Q}$-structure iteration strategy Σ is the unique ω_1-iteration strategy for normal iteration trees:

Lemma 1.2.19: *Suppose $M_n^\#(x)$ exists for every $x \in \mathbb{R}$. Then Σ is the unique ω_1-iteration strategy for normal iteration trees on $M_n^\#(x)$.*

We prove a more general fact:

Lemma 1.2.20: *Suppose $M_n^\#(x)$ exists for every $x \in \mathbb{R}$. Let $\mathcal{N}$ be a countable X-premouse with n Woodin cardinals above δ and an extender above, where X is swo. Suppose $\mathcal{N}$ is normally ω_1-iterable above δ. Let $\mathcal{M} \in \mathcal{N}$ be a Y-premouse which is countable in $\mathcal{N}$ and normally ω_1-iterable above δ via an iteration strategy Σ', where Y is swo. Further suppose $\mathcal{M}$ and $\mathcal{N}$ are $(n+1)$-small above δ and ω-sound above δ with $\rho_\omega(\mathcal{M}) \leq \delta$ and $\rho_\omega(\mathcal{N}) \leq \delta$. Then $\mathcal{M}$ is normally ω_1+1-iterable above δ via Σ in $\mathcal{N}$. Moreover, Σ is the unique ω_1-iteration strategy for normal iteration trees on $\mathcal{M}$ in V.*

Proof: The proof is organized as an induction on n. We will show that

$$\Sigma'(\mathcal{T}) = \Sigma^{\mathcal{N}}(\mathcal{T}) = \Sigma(\mathcal{T})$$

for all normal iteration trees $\mathcal{T} \in \mathcal{N}$ on $\mathcal{M}$ above δ of limit length $\leq \omega_1^{\mathcal{N}}$.

Suppose this has been proved for all $k < n$ and let $\mathcal{T} \in \mathcal{N}$ be a normal iteration tree on $\mathcal{M}$ above δ of limit length $\leq \omega_1^{\mathcal{N}}$ according to Σ'. Let $b := \Sigma'(\mathcal{T})$. Then $\mathcal{M}_b^{\mathcal{T}}$ carries a $\mathcal{Q}$-structure $\mathcal{Q} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$, since $\mathcal{T}$ is a normal iteration tree and $\rho(\mathcal{M}) \leq \delta < \delta(\mathcal{T})$. We can inductively assume that $\mathcal{T}$ is according to Σ', Σ, and $\Sigma^{\mathcal{N}}$. It has to be shown that $\mathcal{Q} \in \mathcal{N}$. It can be assumed that $\mathcal{M}(\mathcal{T}) \lhd \mathcal{Q}$, so $\delta(\mathcal{T})$ is Woodin in $\mathcal{Q}$.

For $n = 0$ there are no extenders above $\delta(\mathcal{T})$ on the $\mathcal{Q}$-sequence, since $\mathcal{Q}$ is 1-small; in this case $\mathcal{Q} \in \mathcal{N}$ since $ht(\mathcal{Q}) < ht(\mathcal{N})$. Now suppose $n > 0$. Let κ be the critical point of an extender on the $\mathcal{N}$-sequence such that in $\mathcal{N}$ there are n Woodin cardinals between δ and κ. We do an $L[\vec{E}]$-construction over $\mathcal{M}(\mathcal{T})$ in $\mathcal{N}|\kappa$. $L[\vec{E}, \mathcal{M}(\mathcal{T})]^{\mathcal{N}|\kappa}$ inherits Woodin cardinals and iterability from $\mathcal{N}$, see [30, chapter 11].

We first prove $\mathcal{Q} \in \mathcal{N}$ in the special case $\mathcal{M} = M_n^\#(x)$. Otherwise it is not clear how to find a premouse $\mathcal{P}$ as in case 3 of the next claim.

Claim 1.2.21: *If $\mathcal{M} = M_n^\#(x)$ for some $x \in \mathbb{R}$, then $\mathcal{Q} \in \mathcal{N}$.*

Proof: We distinguish three cases.

Case 1: $\delta(\mathcal{T})$ is not Woodin in $L[\vec{E}, \mathcal{M}(\mathcal{T})]^{\mathcal{N}|\kappa}$.

Let $\alpha < \kappa$ be minimal such that $\delta(\mathcal{T})$ is not Woodin in $J_{\alpha+1}[\vec{E}, \mathcal{M}(\mathcal{T})]^{\mathcal{N}|\kappa}$. Let $\mathcal{P} := J_\alpha[\vec{E}, \mathcal{M}(\mathcal{T})]^{\mathcal{N}}$. Then $\mathcal{P}$ and $\mathcal{Q}$ are n-small above $\delta(\mathcal{T})$ and ω-sound above $\delta(\mathcal{T})$ with $\mathcal{P}|\delta(\mathcal{T}) = \mathcal{Q}|\delta(\mathcal{T})$ and $\rho_\omega(\mathcal{P}), \rho_\omega(\mathcal{Q}) \leq \delta(\mathcal{T})$. So $\mathcal{P}$ and $\mathcal{Q}$ can be coiterated in $M_{n-1}^\#(x)$ by the induction hypothesis, where $x \in \mathbb{R}$ codes $\mathcal{P}$ and $\mathcal{Q}$. Now $\mathcal{P}$ cannot be a proper initial segment of $\mathcal{Q}$ because $\mathcal{Q}$ is a $\mathcal{Q}$-structure. Thus $\mathcal{Q} \trianglelefteq \mathcal{P}$ and hence $\mathcal{Q} \in \mathcal{N}$.

Case 2: $\delta(\mathcal{T})$ is Woodin in $L[\vec{E}, \mathcal{M}(\mathcal{T})]^{\mathcal{N}|\kappa}$ and there is some $\mathcal{P}$ in the $L[\vec{E}]$-construction with $\rho_\omega(\mathcal{P}) \leq \delta(\mathcal{T})$.

Again $\mathcal{P}$ and $\mathcal{Q}$ can be compared in $M_n^\#(x)$, where $x \in \mathbb{R}$ codes $\mathcal{P}$ and $\mathcal{Q}$.

Case 3: $\delta(\mathcal{T})$ is Woodin in $L[\vec{E}, \mathcal{M}(\mathcal{T})]^{\mathcal{N}|\kappa}$ and the last projectum never falls below $\delta(\mathcal{T})$ in the $L[\vec{E}]$-construction.

Let $\mathcal{P}$ be the first model in the $L[\vec{E}]$-construction with $n-1$ Woodin cardinals above $\delta(\mathcal{T})$ and two extenders above. Then $\rho_1(\mathcal{P}) = \delta(\mathcal{T})$. $\mathcal{P}$ and $\mathcal{Q}$ can be coiterated in $M_n^\#(x)$ by the induction hypothesis, where $x \in \mathbb{R}$ codes $\mathcal{P}$ and $\mathcal{Q}$. But $\mathcal{Q}$ cannot win the coiteration, since $\mathcal{P}$ has more large cardinals. So $\mathcal{Q} \triangleleft \mathcal{P}$.

In fact, this case does not occur by the proof of the following claims. ∎

Claim 1.2.22: *If $\mathcal{N} = M_n^\#(x)$ for some $x \in \mathbb{R}$, then $\mathcal{Q} \in \mathcal{N}$.*

Proof: It suffices to show that case 3 cannot occur. Let $\mathcal{P}$ be the premouse obtained by adding the extender with critical point κ on top of $L[\vec{E}, \mathcal{M}(\mathcal{T})]^{\mathcal{N}|\kappa}$. Now $\mathcal{P}$ and $\mathcal{Q}$ can be compared, since $\mathcal{N}$ and hence $\mathcal{P}$ is ω_1+1-iterable in $M_n^\#(x)$ by the previous claim, where $x \in \mathbb{R}$ codes $\mathcal{N}$. But neither can iterate to an initial segment of the other. ∎

We finally conclude:

Claim 1.2.23: $\mathcal{Q} \in \mathcal{N}$.

Proof: Again it is sufficient that case 3 does not occur. This holds because $\mathcal{N}$ is $\omega_1 + 1$-iterable in $M_n^\#(x)$ by the previous claim, where $x \in \mathbb{R}$ codes $\mathcal{N}$. ∎

It remains to be shown that $b = \Sigma'(\mathcal{T}) \in \mathcal{N}$. Let g be $Col(\omega, ht(\mathcal{T}))$-generic over $\mathcal{N}$. Note that one can rearrange $\mathcal{N}[g]$ as a boldface premouse, see [41, lemma 1.4] and [49, section 3]. We can form a tree in $\mathcal{N}[g]$ searching for cofinal branches $b \subseteq \mathcal{T}$ with $\mathcal{Q} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$. The nodes consist of initial segments of b and partial finite $\in$-isomorphisms between $\mathcal{Q}$ and the corresponding model. Since wellfoundedness is absolute between $\mathcal{N}[g]$ and V, $\mathcal{N}[g]$ knows that there is a branch in this tree. But there is at most one cofinal branch in $\mathcal{T}$ with the required property by the argument in the proof of [48, corollary 6.14]. Hence $b \in \mathcal{N}$ by homogeneity of $Col(\omega, ht(\mathcal{T}))$. Thus $\Sigma(\mathcal{T}) = \Sigma^{\mathcal{N}}(\mathcal{T}) = b$.

To see that Σ is unique, consider a countable normal iteration tree $\mathcal{T}$ above δ on $\mathcal{M}$ of limit length according to an ω_1-iteration strategy Σ'. Let $x \in \mathbb{R}$ code $\mathcal{T}$. Then $\Sigma'(\mathcal{T}) = \Sigma^{M_n^\#(x)}(\mathcal{T}) = \Sigma(\mathcal{T})$. ■

1.2.4 Tools for ω_1-iterable premice

In this section we adapt the tools for ω_1+1-iterable premice from the previous sections to ω_1-iterable premice.

Lemma 1.2.24: *(Comparison lemma) Assume $M_n^\#(x)$ exists for every $x \in \mathbb{R}$. Let $\mathcal{M}$ and $\mathcal{N}$ be countable X-premice, where X is swo. Let δ be a cutpoint of $\mathcal{M}$ and $\mathcal{N}$ such that both are ω_1-iterable above δ and $\mathcal{M}|\delta = \mathcal{N}|\delta$. Further suppose $\mathcal{M}$ and $\mathcal{N}$ are $(n+1)$-small above δ and ω-sound above δ with $\rho_\omega(\mathcal{M}), \rho_\omega(\mathcal{N}) \leq \delta$. Then there are countable iteration trees $\mathcal{S}$ on $\mathcal{M}$ and $\mathcal{T}$ on $\mathcal{N}$ with last models $\mathcal{M}_\alpha^{\mathcal{S}}$ and $\mathcal{M}_\beta^{\mathcal{T}}$ so that*

1. *$[0,\alpha]_{\mathcal{S}}$ does not drop in model or degree and $\mathcal{M}_\alpha^{\mathcal{S}} \trianglelefteq \mathcal{M}_\beta^{\mathcal{T}}$, or*
2. *$[0,\beta]_{\mathcal{T}}$ does not drop in model or degree and $\mathcal{M}_\beta^{\mathcal{T}} \trianglelefteq \mathcal{M}_\alpha^{\mathcal{S}}$.*

Proof: $\mathcal{M}$ and $\mathcal{N}$ are ω_1+1-iterable in $M_n^\#(x)$ by lemma 1.2.20 where $x \in \mathbb{R}$ codes $\mathcal{M}$ and $\mathcal{N}$. So we can coiterate them in $M_n^\#(x)$ by lemma 1.2.5. ■

A consequence is

Lemma 1.2.25: *Suppose $M_n^\#(x)$ exists for every $x \in \mathbb{R}$. Let $\mathcal{M}$ and $\mathcal{N}$ be countable X-premice, where X is swo. Suppose δ is a cutpoint of $\mathcal{M}$ and $\mathcal{N}$ such that both are ω_1-iterable above δ and $\mathcal{M}|\delta = \mathcal{N}|\delta$. Further suppose that both $\mathcal{M}$ and $\mathcal{N}$ are $(n+1)$-small above δ and ω-sound above δ with $\rho_\omega(\mathcal{M}), \rho_\omega(\mathcal{N}) \leq \delta$. Then $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{M}$.*

We get a version of the genericity iteration for ω_1-iterable premice:

Lemma 1.2.26: *(Genericity iteration) Assume $M_n^\#(x)$ exists for every $x \in \mathbb{R}$. Let $\mathcal{M}$ be a countable X-premouse, where X is swo, such that δ is Woodin in $\mathcal{M}$ and $\mathcal{M}$ is ω_1-iterable above some $\gamma < \delta$. Further suppose $\mathcal{M}$ is $(n+1)$-small above γ and ω-sound above γ with $\rho_\omega(\mathcal{M}) \leq \gamma$. Then for each $x \in \mathbb{R}$, there is a countable iteration tree $\mathcal{T}$ on $\mathcal{M}$ with iteration map π and last model $\mathcal{M}_\alpha^\mathcal{T}$ such that $[0,\alpha]_\mathcal{T}$ does not drop in model and x is $\mathbb{W}_{\pi(\delta)}^{\mathcal{M}_\alpha^\mathcal{T}}$-generic over $\mathcal{M}_\alpha^\mathcal{T}$.*

Proof: Apply lemma 1.2.14 inside $M_n^\#(z)$ where $z \in \mathbb{R}$ codes $\mathcal{M}$ and x. ■

While forcing is usually applied to models of ZF, we would like to use the forcing theorem for small forcing over relativized premice in the next lemma. Let $\mathcal{M}$ be a relativized premouse and κ the critical point of an extender on the $\mathcal{M}$-sequence. Note that the forcing relation for any partial order $\mathbb{P} \in \mathcal{M}|\kappa$ is defined in $\mathcal{M}|\kappa$ and the forcing theorem holds for $\mathcal{M}|\kappa$, since this is a model of ZF. We are only interested in formulas whose quantifiers range over a bounded subset of $\mathcal{M}|\kappa$, especially projective formulas. The forcing theorem holds for such formulas since the relevant names are in $\mathcal{M}|\kappa$.

A key property of $M_n^\#$ is that it determines which Σ_{n+2}^1 statements about reals in $M_n^\#$ are true in V:

Lemma 1.2.27: *Let $n \leq k < \omega$ and assume $M_k^\#(x)$ exists for every $x \in \mathbb{R}$. Let $\mathcal{M}$ be a countable $(k+1)$-small X-premouse with $\rho_\omega(\mathcal{M}) \leq \gamma$ which is ω_1-iterable above γ, where X is swo. Suppose that in $\mathcal{M}$ there are n Woodin cardinals above γ and an extender above. Let δ be the least Woodin cardinal above γ in $\mathcal{M}$ if $n \geq 1$.*

1. *If n is even then $\mathcal{M} \prec_{\Sigma_{n+2}^1} V$, and*
2. *if n is odd then*

$$V \vDash \varphi(x) \Leftrightarrow \mathcal{M} \vDash \text{"}\exists p \in \mathbb{W}_\delta (p \Vdash_{\mathbb{W}_\delta} \varphi(\check{x}))\text{"}$$

for all Σ_{n+2}^1 formulas φ and all $x \in \mathbb{R} \cap \mathcal{M}$.

Proof: The proof is organized as an induction on n. For $n = 0$ we iterate $\mathcal{M}$ by an extender on the $\mathcal{M}$-sequence to a model of height $\geq \omega_1$. The conclusion follows from Shoenfield absoluteness, since this model has the same reals as $\mathcal{M}$.

Case 1: n is odd.

Suppose φ is a Π^1_{n+1} formula, $x \in \mathbb{R}$, and $y \in \mathbb{R} \cap \mathcal{M}$ so that $\varphi(x,y)$ holds. Do a genericity iteration on $\mathcal{M}$ for x and let $\pi : \mathcal{M} \to \mathcal{N}$ be the iteration map so that x is $\mathbb{W}_{\pi(\delta)}$-generic over $\mathcal{N}$. We get $\mathcal{N}[x] \models \varphi(x)$ from the induction hypothesis. Hence

$$\mathcal{N} \models \text{"}\exists p \in \mathbb{W}_{\pi(\delta)}\, (p \Vdash^{\mathcal{N}}_{\mathbb{W}_{\pi(\delta)}} \exists x \varphi(x, \check{y}))\text{"}$$

and the claim follows from elementarity of π.

For the other direction suppose there is a condition $p \in \mathbb{W}^{\mathcal{M}}_{\delta}$ which forces $\exists x \varphi(x, \check{y})$ over $\mathcal{M}$. Let x be $\mathbb{W}_{\delta} \restriction p$-generic over $\mathcal{M}$ in V. Then $\varphi(x,y)$ holds by the induction hypothesis.

Case 2: $n \geq 2$ is even.

Suppose φ is a Π^1_{n+1} formula and $y \in \mathbb{R} \cap \mathcal{M}$ so that $\exists x \varphi(x,y)$ holds. The assumptions imply Π^1_{n+1} uniformization via lemma 1.2.12. Let ψ be a Π^1_{n+1} formula and $x \in \mathbb{R}$ so that $\varphi(x,y)$ holds and x is unique with $\psi(x,y)$. We have to show that $x \in \mathcal{M}$.

Let $\pi : \mathcal{M} \to \mathcal{N}$ be an iteration map so that x is $\mathbb{W}_{\pi(\delta)}$-generic over $\mathcal{N}$. Let η be the least Woodin cardinal above δ in $\mathcal{M}$. We have

$$\exists p \in \mathbb{W}_{\pi(\eta)} (p \Vdash^{\mathcal{N}[x]}_{\mathbb{W}_{\pi(\eta)}} \psi(\check{x}, \check{y}))$$

by the induction hypothesis. So there is a condition $q \in \mathbb{W}_{\pi(\delta)}$ which forces

$$\exists p \in \mathbb{W}_{\pi(\eta)} (p \Vdash^{\mathcal{N}[\tau]}_{\mathbb{W}_{\pi(\eta)}} \psi(\tau, \check{y}))$$

such that x is $\mathbb{W}_{\pi(\delta)} \restriction q$-generic over $\mathcal{N}$, where τ is a name for the $\mathbb{W}_{\pi(\delta)}$-generic real. Let z be $\mathbb{W}^{\mathcal{N}}_{\pi(\delta)} \restriction q$-generic over $\mathcal{N}[x]$ in V. Then $\psi(z,y)$ holds and hence $z = x$. This implies that $\mathbb{W}_{\pi(\delta)} \restriction q$ is atomic and $x \in \mathcal{N}$. Then $x \in \mathcal{M}$ since the iteration does not add reals.

For the other direction suppose φ is a Π^1_{n+1} formula and y a real in $\mathcal{M}$ with

$$\mathcal{M} \models \text{"}\exists p \in \mathbb{W}_{\delta} (p \Vdash_{\mathbb{W}_{\delta}} \exists x \varphi(x, \check{y}))\text{"}.$$

Then $\varphi(x,y)$ holds by the induction hypothesis for any real x which witnesses this in a $\mathbb{W}_{\delta} \restriction p$-generic extension of $\mathcal{M}$.

The previous lemma is also true if $\mathcal{M}$ is uncountable. To show this one simply applies the lemma to a countable elementary substructure of $\mathcal{M}$. Note that the lemma also works for the forcing $Col(\omega, \delta)$. In fact this version of the lemma uses a weaker notion of iterability called n-iterability, see [32, definition 1.1] for a definition of n-iterability and [35, theorem 7.16] for the result.

If in the situation of the previous lemma there is an extra extender on top in $\mathcal{M}$, then $\mathcal{M} \prec_{\Sigma^1_{n+2}} V$ holds for odd n as well:

Lemma 1.2.28: *Suppose $n \leq k$ and $M_k^\#(x)$ exists for every $x \in \mathbb{R}$. Suppose $\mathcal{M}$ is a countable ω-sound $(k+1)$-small X-premouse which is ω_1-iterable above δ with $\rho_\omega(\mathcal{M}) \leq \delta$, where X is swo. Suppose there are n Woodin cardinals above δ in $\mathcal{M}$ and at least two total extenders above. Then $M_n^\#(x)$ is unique for every $x \in \mathbb{R} \cap \mathcal{M}$ and is calculated correctly by $\mathcal{M}$.*

PROOF: Do an $L[\vec{E}]$-construction over x in $\mathcal{M}$. Then the $M_n^\#(x)$ of both $\mathcal{M}$ and V occurs in the construction when one forms the core of an x-premouse with an extender above n Woodin cardinals for the first time. ■

Lemma 1.2.29: *Suppose $n \leq k$ and $M_k^\#(x)$ exists for every $x \in \mathbb{R}$. Suppose $\mathcal{M}$ is a countable ω-sound $(k+1)$-small X-premouse which is ω_1-iterable above δ with $\rho_\omega(\mathcal{M}) \leq \delta$, where X is swo. Suppose there are n Woodin cardinals above δ in $\mathcal{M}$ and at least two total extenders above. Then $\mathcal{M} \prec_{\Sigma^1_{n+2}} V$.*

PROOF: For n even this is true by lemma 1.2.27. Suppose n is odd and φ is a Σ^1_{n+2} formula. Let $x \in \mathbb{R}$ and let δ be the least Woodin cardinal in $M_n^\#(x)$. We know that $\mathcal{M}$ computes $M_n^\#(x)$ correctly by the previous lemma. So $\varphi(x)$ holds if and only if

$$\exists p \in \mathbb{W}_\delta^{M_n^\#(x)} \, (p \Vdash_{\mathbb{W}_\delta}^{M_n^\#(x)} \varphi(x))$$

holds if and only if $\mathcal{M} \models \varphi(x)$ by lemma 1.2.27. ■

The next two lemmas show that statements about $M_{2n}^\#(x)$ and $M_{2n}^\dagger(x)$ are projective.

Lemma 1.2.30: *Suppose $M_n^\#(x)$ exists for every $x \in \mathbb{R}$. Let $\mathcal{M}$ and $\mathcal{N}$ countable $(n+1)$-small X-premice with the same first order properties which are ω-sound above δ with $\rho_\omega(\mathcal{M}) \leq \delta$ and $\rho_\omega(\mathcal{N}) \leq \delta$, where X is swo. Suppose $\mathcal{M}$ is ω_1-iterable above δ and $\mathcal{N}$ is Π_{n+1}-iterable. Then $\mathcal{M} = \mathcal{N}$.*

Proof: The proof is organized as an induction. We sketch the proof for odd n following the proof of [46, lemma 2.2]. This proof has to be slightly modified since we don't have large cardinals in V. The case for even n can be similarly derived from the proof of [46, lemma 2.2]. The difference between the odd and even cases lies in the weak iteration game from [46].

Let $\mathcal{P} := M_n^\#(x)$ where $x \in \mathbb{R}$ codes $\mathcal{M}$ and $\mathcal{N}$. Let further g be a $Col(\omega, \omega_1^{\mathcal{P}})$-generic filter over $\mathcal{N}$ and define $\mathcal{R} := \mathcal{P}[g]$. Then $\mathcal{M}$ is $\omega_1 + 1$-iterable in $\mathcal{P}$ and in $\mathcal{R}$ by lemma 1.2.20. Note that Π_{n+1}-iterability is Π^1_{n+2} in the codes. Since there are n Woodin cardinals in $\mathcal{P}$ and in $\mathcal{R}$, it follows from lemma 1.2.29 that $\mathcal{N}$ is Π_{n+1}-iterable in $\mathcal{N}$ and in $\mathcal{P}$.

We define coiterations of $\mathcal{M}$ and $\mathcal{N}$ in both $\mathcal{P}$ and $\mathcal{R}$. For iteration trees on $\mathcal{M}$ of limit length choose the unique branch with a $\mathcal{Q}$-structure. For iteration trees $\mathcal{T}$ on $\mathcal{N}$ of limit length choose a cofinal branch with a wellfounded Π_n-iterable model. The winning position for player 2 in the weak iteration game $\mathcal{I}(\mathcal{N}, \delta, n{+}1)$ produces such a branch. A coiteration argument shows that the branch is unique. Note that the coiteration is possible by the induction hypothesis.

One can show that the same branches are chosen in the coiterations in $\mathcal{P}$ and $\mathcal{R}$ since the forcing $Col(\omega, \omega_1^{\mathcal{P}})$ is small. So the coiteration in $\mathcal{P}$ is an initial segment of the coiteration in $\mathcal{R}$. Hence there is at least one cofinal branch in the coiteration in $\mathcal{P}$. This is an element of $\mathcal{P}$ by homogeneity of $Col(\omega, \omega_1^{\mathcal{P}})$. Now the argument from the proof of the comparison lemma shows that the coiteration terminates after countably many steps. It follows that $\mathcal{M} = \mathcal{N}$. ∎

Lemma 1.2.31: *$M_n^\#(x)$ and $M_n^\dagger(x)$ are coded by $\Pi^1_{n+2}(x)$ singletons.*

Proof: Let $f : \omega \to M_n^\#(x)$ be the canonical Σ_1-definable surjection over $M_n^\#(x)$. Let

$$(k, m) \in z :\Leftrightarrow f(k) \in f(m)$$

for $k, m < \omega$, so that (ω, z) is isomorphic to $(M_n^\#(x), \in)$. Hence $z \subseteq \omega \times \omega$ is the unique set which codes $M_n^\#(x)$ and which computes itself via the canonical Σ_1-definable surjection computed in its transitive collapse. Now the set of $(x, y) \in \mathbb{R}^2$ so that x codes $M_n^\#(y)$ is a $\Pi^1_{n+2}(x, y)$ set by the previous lemma, since for sets of reals Π^{HC}_{n+1} is equivalent to Π^1_{n+2}. Thus z is a $\Pi^1_{n+2}(x)$ singleton. The same works for $M_n^\dagger$. ∎

Chapter 2

Lifting thin equivalence relations to forcing extensions

Suppose E is a provably $\mathbf{\Delta}^1_{n+1}$ equivalence relation and a forcing $\mathbb{P}$ preserves Σ^1_n truth. In this situation one can ask whether forcing with $\mathbb{P}$ introduces any new equivalence classes of E. In the first section of this chapter we show generic Σ^1_{n+3} absoluteness for reasonable forcing $\mathbb{P}$ of size κ, assuming that $M_n^\#(X)$ exists for every self-wellordered set $X \in H_{\kappa^+}$. We further show that in this situation $\mathbb{P}$ does not add equivalence classes to thin provably $\mathbf{\Delta}^1_{n+3}$ equivalence relations. The proof is based on an idea of Foreman and Magidor [4, section 3].

The second section utilizes the same method to derive analogous results for $\mathbf{\Sigma}^1_2$ c.c.c. forcing from projective determinacy. The generic absoluteness result generalizes the corresponding theorem for Cohen and random forcing from Woodin [52]. We further show that Cohen forcing does not add equivalence classes to $< \omega - \mathbf{\Pi}^1_n$ prewellorders for $n \geq 1$ if generic Σ^1_{n+1} Cohen absoluteness holds. In this chapter we work in $\mathsf{ZF} + \mathsf{DC}$.

2.1 Reasonable forcing

We work with a weaker version of the notion of proper forcing called reasonable forcing, introduced by Foreman and Magidor [4].

Definition 2.1.1: *Let $\mathbb{P}$ be a partial order and $p \in \mathbb{P}$.*

1. *Suppose N is a set with $p \in N$. Then p is called $(N, \mathbb{P})$-generic if for every maximal antichain $A \subseteq \mathbb{P}$ with $A \in N$ the set $A \cap N$ is predense below p.*

2. $\mathbb{P}$ *is called reasonable if for all* $q \in \mathbb{P}$ *and for some (for all) regular* $\lambda \geq (2^{2^{\overline{\overline{\mathbb{P}}}}})^+$ *there exist a countable elementary substructure* $N \prec H_\lambda$ *with* $q, \mathbb{P} \in N$ *and an* $(N, \mathbb{P})$*-generic condition* $r \leq q$.

Here H_λ can equivalently be replaced by V_λ. Let $\mathcal{P}_\kappa(\lambda) := \{X \subseteq \lambda : \overline{\overline{X}} < \kappa\}$ for $\kappa, \lambda \in Ord$. By standard proper forcing arguments we have

Lemma 2.1.2: *(Foreman and Magidor [4]) A forcing* $\mathbb{P}$ *is reasonable if and only if* $\mathcal{P}^V_{\omega_1}(\alpha)$ *is stationary in* $\mathcal{P}^{V^{\mathbb{P}}}_{\omega_1}(\alpha)$ *for every ordinal* α.

2.1.1 Absoluteness of $M_n^\#$

We will use

Lemma 2.1.3: *Suppose* $M_n^\#(X)$ *exists for all swo* $X \in H_\kappa$*, where* κ *is an uncountable cardinal. Then* $M_n^\#(X)$ *is* κ*-iterable for each swo* $X \in H_\kappa$.

Proof: We can assume that H_κ is closed under the relevant $\mathcal{Q}$-structures by the induction hypothesis. A reflection argument then shows that $M_n^\#(X)$ is κ-iterable. ■

The results in this section are based on the absoluteness of $M_n^\#(X)$:

Lemma 2.1.4: *Let* $\mathbb{P}$ *be a forcing of size* κ*, where* κ *is an infinite cardinal. Suppose* $M_n^\#(X)$ *exists for every swo* $X \in H_{\kappa^+}$. *Then for every* $\mathbb{P}$*-generic filter* G *over* V

1. $M_n^\#(X)$ *is normally* κ^+*-iterable in* $V[G]$ *via* Σ *for every swo* $X \in H_{\kappa^+}$,
2. $V[G] \models$ *"*$M_n^\#(X)$ *exists for every swo* $X \in H_{\kappa^+}$ *and is normally* κ^+*-iterable", and*
3. *suppose*
   (a) $H \prec V_\eta$ *is a countable substructure with* $\mathbb{P} \in H$ *where* η *is a large limit ordinal,*
   (b) $\bar{H}$ *is the transitive collapse of* H *with uncollapsing map* $\pi : \bar{H} \to H$ *and* $\pi(\bar{\mathbb{P}}) = \mathbb{P}$, $\pi(\bar{\kappa}) = \kappa$*, and*

(c) g is a $\bar{\mathbb{P}}$-generic filter over $\bar{H}$ in V,

then $M_n^\#(X)$ exists in $\bar{H}[g]$ for each swo $X \in H_{\bar{\kappa}^+}^{\bar{H}[g]}$ and is normally κ^+-iterable via Σ in both $\bar{H}[g]$ and V.

Proof: The proof works by induction on n. We get uniqueness of $M_n^\#(X)$ for $X \in H_{\kappa^+}$ by the argument in lemma 1.2.20.

1. In the case $n = 0$ the claim holds since all iterations are linear. Let $n \geq 1$ and suppose $X \in H_{\kappa^+}$ is swo. Let $\mathcal{M} := M_n^\#(X)$. We have to show that $\mathcal{M}$ is normally κ^+-iterable in $V[G]$ via Σ.

Suppose not. Then in $V[G]$ there is a normal iteration tree $\mathcal{T}$ on $\mathcal{M}$ of length $< \kappa^+$ which witnesses that $\mathcal{M}$ is not normally κ^+-iterable via Σ. I.e. $\mathcal{T}$ is according to Σ and $\mathcal{M}_{\Sigma(\mathcal{T})}$ is ill-founded or $\Sigma(\mathcal{T})$ is undefined. Let $\dot{\mathcal{T}}$ be a $\mathbb{P}$-name and $p \in \mathbb{P}$ a condition with

$$p \Vdash \text{"}\dot{\mathcal{T}} \text{ witnesses that } \check{\mathcal{M}} \text{ is not } \kappa^+\text{-iterable via } \Sigma\text{"}.$$

Now let $H \prec V_\eta$ be a countable substructure with $p, \mathbb{P}, \mathcal{M}, \dot{\mathcal{T}} \in H$ for some large limit ordinal η. Let $\bar{H}$ be the transitive collapse of H with uncollapsing map $\pi : \bar{H} \to H$ and $\pi(\bar{p}) = p$, $\pi(\bar{\mathbb{P}}) = \mathbb{P}$, $\pi(\bar{\mathcal{T}}) = \dot{\mathcal{T}}$, $\pi(\bar{\mathcal{M}}) = \mathcal{M}$. Then $\bar{\mathcal{M}}$ is κ^+-iterable in V since $\pi \restriction \bar{\mathcal{M}} : \bar{\mathcal{M}} \to \mathcal{M}$ is an elementary embedding.

Let g be a $\bar{\mathbb{P}}$-generic filter over $\bar{H}$ in V with $p \in g$. Then

$$\bar{H}[g] \models \text{"}\bar{\mathcal{T}}^g \text{ witnesses that } \mathcal{M} \text{ is not } \kappa^+\text{-iterable via } \Sigma\text{"}.$$

Let $\alpha < lh(\bar{\mathcal{T}}^g)$. $M_{n-1}^\#(\mathcal{M}(\bar{\mathcal{T}}^g \restriction \alpha))$ exists in in $\bar{H}[g]$ and is κ^+-iterable via Σ in both $\bar{H}[g]$ and V by the induction hypothesis 3.

Let $\mathcal{Q}(\bar{\mathcal{T}}^g \restriction \alpha)$ denote the $\mathcal{Q}$-structure for $\bar{\mathcal{T}}^g \restriction \alpha$ in $\bar{H}[g]$. We can compare $\mathcal{Q}(\bar{\mathcal{T}}^g \restriction \alpha)$ and $M_{n-1}^\#(\mathcal{M}(\bar{\mathcal{T}}^g \restriction \alpha))$ in $\bar{H}[g]$ by lemma 1.2.20. Hence

$$\mathcal{Q}(\bar{\mathcal{T}}^g \restriction \alpha) \trianglelefteq M_{n-1}^\#(\mathcal{M}(\bar{\mathcal{T}}^g \restriction \alpha)).$$

So $\bar{\mathcal{T}}^g$ is according to Σ in both $\bar{H}[g]$ and V.

Let $b := \Sigma^V(\bar{\mathcal{T}}^g)$. We have to show that $b \in \bar{H}[g]$. Let g' be $Col(\omega, lh(\bar{\mathcal{T}}^g))$-

generic over $\bar{H}[g]$ where $lh(\bar{\mathcal{T}}^g)$ is the length of $\bar{\mathcal{T}}^g$. Now $\mathcal{Q}(\bar{\mathcal{T}}^g) \in \bar{H}[g]$ since

$$\mathcal{Q}(\bar{\mathcal{T}}^g) \trianglelefteq M_{n-1}^{\#}(\mathcal{M}(\bar{\mathcal{T}}^g)).$$

One can build a tree in $\bar{H}[g][g']$ searching for a cofinal branch $b' \subseteq \bar{\mathcal{T}}^g$ with $\mathcal{Q}(\bar{\mathcal{T}}^g) \trianglelefteq \mathcal{M}_{b'}^{\bar{\mathcal{T}}^g}$. Since b is such a branch in V, there is a cofinal branch $b' \subseteq \bar{\mathcal{T}}$ in $\bar{H}[g][g']$ with $\mathcal{Q}(\bar{\mathcal{T}}^g) \trianglelefteq \mathcal{M}_{b'}^{\bar{\mathcal{T}}}$ by absoluteness of wellfoundedness. But there can be only one such branch by the argument in [48, corollary 6.14]. So $b = b'$ and $b' \in \bar{H}[g]$ by homogeneity of $Col(\omega, \bar{\mathcal{T}}^g)$. Hence

$$\Sigma^V(\bar{\mathcal{T}}^g) = b = \Sigma^{\bar{H}[g]}(\bar{\mathcal{T}}^g),$$

contradicting the assumption on $\bar{\mathcal{T}}^g$.

2. Suppose $X \in H_{\kappa^+}^{V[G]}$ is swo. Let's code X by a subset of κ and choose a nice $\mathbb{P}$-name for this set. So there is a $\mathbb{P}$-name $\tau \in H_{\kappa^+}$ with $\tau^G = X$. We can assume that $(\mathbb{P}, \tau)$ is swo; otherwise we work with a swo set in H_{κ^+} coding $\mathbb{P}$ and τ. Now $G' := G \cap M_n^{\#}(\mathbb{P}, \tau)$ is $\mathbb{P}$-generic over $M_n^{\#}(\mathbb{P}, \tau)$, since G is $\mathbb{P}$-generic over V. Moreover

$$x = \tau^{G'} \in M_n^{\#}(\mathbb{P}, \tau)[G'].$$

Then $M_n^{\#}(\mathbb{P}, \tau)$ is normally κ^+-iterable via Σ in $V[G]$ by 1. Since $\mathbb{P}$ is small compared to the critical points of the extenders on the $M_n^{\#}(\mathbb{P}, \tau)$ sequence, $M_n^{\#}(\mathbb{P}, \tau)[G']$ is normally κ^+-iterable via Σ in $V[G]$ as well. Let F be the top extender of $M_n^{\#}(\mathbb{P}, \tau)[G']$ and $\kappa := crit(F)$. Do an $L[\vec{E}]$ construction over X in $M_n^{\#}(\mathbb{P}, \tau)[G']|\kappa$. It follows from the argument of the commutativity lemma [5, lemma 3.2] that the premouse obtained by extending the $L[\vec{E}]$ model with the restriction of F is normally κ^+-iterable via Σ. Hence this is $M_n^{\#}(X)$ in $V[G]$.

3. Suppose $X \in H_{\bar{\kappa}^+}^{\bar{H}[g]}$ is swo. Let $\bar{\tau} \in H_{\bar{\kappa}^+}$ be a $\bar{\mathbb{P}}$-name for X and let $\tau := \pi(\bar{\tau})$. Let's assume that $(\mathbb{P}, \tau)$ is swo. Then $M_n^{\#}(\bar{\mathbb{P}}, \bar{\tau})^{\bar{H}}$ and $M_n^{\#}(\mathbb{P}, \tau)$ exist and are normally κ^+-iterable via Σ in $\bar{H}$ and V respectively and

$$\pi(M_n^{\#}(\bar{\mathbb{P}}, \bar{\tau})^{\bar{H}}) = M_n^{\#}(\mathbb{P}, \tau).$$

Then $M_n^{\#}(\bar{\mathbb{P}}, \bar{\tau})^{\bar{H}}$ is normally κ^+-iterable via Σ in $\bar{H}[g]$ by 1.

Let $g' := g \cap M_n^{\#}(\bar{\mathbb{P}}, \bar{\tau})^{\bar{H}}$. Then g' is a $\bar{\mathbb{P}}$-generic filter over $M_n^{\#}(\bar{\mathbb{P}}, \bar{\tau})^{\bar{H}}$ and $x = \bar{\tau}^{g'} \in M_n^{\#}(\bar{\mathbb{P}}, \bar{\tau})^{\bar{H}}[g']$. Now $M_n^{\#}(\bar{\mathbb{P}}, \bar{\tau})^{\bar{H}}[g']$ is normally κ^+-iterable via Σ in

$\bar{H}[g]$, since $\bar{\mathbb{P}}$ is small compared to the critical points of the extenders on the $M_n^\#(\bar{\mathbb{P}},\bar{\tau})^{\bar{H}}$ sequence. Moreover $M_n^\#(\bar{\mathbb{P}},\bar{\tau})^{\bar{H}}$ is normally κ^+-iterable via Σ in V, since

$$\pi \restriction M_n^\#(\bar{\mathbb{P}},\bar{\tau})^{\bar{H}} : M_n^\#(\bar{\mathbb{P}},\bar{\tau})^{\bar{H}} \to M_n^\#(\mathbb{P},\tau)$$

is elementary. Hence $M_n^\#(\bar{\mathbb{P}},\bar{\tau})^{\bar{H}}[g']$ is normally κ^+-iterable via Σ in V as well. As in part 2 we can build a model in $M_n^\#(\bar{\mathbb{P}},\bar{\tau})$ via an $L[\vec{E}]$-construction over X which is the $M_n^\#(X)$ of both $\bar{H}[g]$ and V. ■

Note that the lemma works for $M_n^\dagger$ with the same proof.

2.1.2 Absoluteness of equivalence classes

We will need a direct consequence of lemma 1.2.27:

Lemma 2.1.5: *Suppose M is a transitive model of* ZF *which computes $M_n^\#(x)$ correctly for every $x \in \mathbb{R} \cap M$. Then $M \prec_{\Sigma^1_{n+2}} V$.*

PROOF: Note that in this situation $M_n^\#(x)$ is unique by lemma 1.2.20. $M_n^\#(x)$ computes the truth value of Σ^1_{n+2} statements by lemma 1.2.27. ■

Definition 2.1.6: *Σ^1_n-absoluteness holds for a partial order $\mathbb{P}$ if $V \prec_{\Sigma^1_n} V[G]$ for any $\mathbb{P}$-generic filter G over V.*

Lemma 2.1.7: *(Martin, Solovay, Schindler) Suppose $M_n^\#(X)$ exists for every swo $X \in H_{\kappa^+}$, where κ is an infinite cardinal. Then Σ^1_{n+3}-absoluteness holds for every forcing of size κ.*

PROOF: We follow the proof of [38, theorem 1]. Suppose $\exists x \varphi(x,y)$ holds in some $\mathbb{P}$-generic extension of V, where φ is a Π^1_{n+2} formula and $y \in \mathbb{R}$. Let τ be a nice $\mathbb{P}$-name for a real and $p \in \mathbb{P}$ a condition with $p \Vdash_{\mathbb{P}} \varphi(\tau,\check{y})$. We can assume that $(\mathbb{P},\tau)$ is swo. Then $M_n^\#(\mathbb{P},\tau)$ exists since $\mathbb{P},\tau \in H_{\kappa^+}$.

Consider the tree T in V searching for a 5-tuple $(\mathcal{M},\pi,\bar{\mathbb{P}},g,x)$ such that

1. $\mathcal{M}$ is a countable premouse with $y \in \mathcal{M}$,
2. $\pi : \mathcal{M} \to M_n^\#(\mathbb{P},\tau)$ is elementary with $\pi(\bar{\mathbb{P}}) = \mathbb{P}$,
3. g is $\bar{\mathbb{P}}$-generic over $\mathcal{M}$, and

4. x is a real in $\mathcal{M}[g]$ such that $\mathcal{M}[g] \vDash \varphi(x, y)$ if n is even, and $\Vdash^{\mathcal{M}[g]}_{Col(\omega,\delta)} \varphi(\check{x}, \check{y})$ if n is odd, where δ is the least Woodin cardinal in $\mathcal{M}[g]$.

A branch in this tree defines a complete theory so that every existential statement in the theory is witnessed by a constant, giving rise to a model $\mathcal{M}[g]$, as well as a set of finite partial $\in$-isomorphisms whose union is an elementary map $\pi : \mathcal{M} \to M_n^\#(\mathbb{P}, \tau)$, witnessing that $\mathcal{M}$ is wellfounded.

Now let G be $\mathbb{P} \upharpoonright p$-generic over V. Then $g := G \cap M_n^\#(\mathbb{P}, \tau)$ is $\mathbb{P}$-generic over $M_n^\#(\mathbb{P}, \tau)$ and we have $x := \tau^g \in M_n^\#(\mathbb{P}, \tau)[g]$. Since $M_n^\#(\mathbb{P}, \tau)[g]$ is κ^+-iterable in $V[G]$ by lemma 2.1.4, the collapse of a countable elementary substructure of $M_n^\#(\mathbb{P}, \tau)[g]$ witnesses that T has a branch in $V[G]$ by lemma 1.2.27. Then T is also ill-founded in V and hence $V \vDash \exists x \varphi(x, y)$. ∎

For any set E with a fixed definition we always write E for the corresponding set in any forcing extension. If further $\mathbb{P}$ is a forcing and τ is a $\mathbb{P}$-name, then in any $\mathbb{P} \times \mathbb{P}$-generic extension τ defines two objects via the two $\mathbb{P}$-generic filters. We write τ and τ' for $\mathbb{P} \times \mathbb{P}$-names for these objects.

The idea for the next lemma and the next theorem comes from [4, theorem 3.4].

Lemma 2.1.8: *Let E be a thin $\mathbf{\Pi}^1_{n+3}$ equivalence relation. Suppose $\mathbb{P}$ is a forcing of size κ and $M_n^\#(X)$ exists for every swo $X \in H_{\kappa^+}$, where κ is an infinite cardinal. Let τ be a $\mathbb{P}$-name for a real. Then the set*

$$D := \{p \in \mathbb{P} : (p, p) \Vdash_{\mathbb{P} \times \mathbb{P}} \tau E \tau'\}$$

is dense.

Proof: Let $a \in \mathbb{R}$ so that E is $\Pi^1_{n+3}(a)$. Suppose D is not dense. Then there is a condition $p \in \mathbb{P}$ so that for every $q \leq p$ there are $r, s \leq q$ with

$$(r, s) \Vdash_{\mathbb{P} \times \mathbb{P}} \neg \tau E \tau'.$$

Let λ be a large limit ordinal and $H \prec V_\lambda$ a countable elementary substructure with $a, \mathbb{P}, p, \tau, \tau' \in H$. Let $\bar{H}$ be the transitive collapse with uncollapsing map $\pi : \bar{H} \to H$ and $\pi(\bar{\mathbb{P}}) = \mathbb{P}$, $\pi(\bar{p}) = p$, $\pi(\bar{\tau}) = \tau$, and $\pi(\bar{\tau}') = \tau'$.

Let $(D_n : n \in \omega)$ enumerate the open dense subsets in $\bar{H}$ of $\bar{\mathbb{P}} \times \bar{\mathbb{P}}$. We construct a family of conditions $(p_s : s \in 2^{<\omega})$ in $\bar{\mathbb{P}}$ such that

1. $p_\emptyset = \bar{p}$,
2. $p_s \leq p_t$ if $t \subseteq s$,
3. $(p_{s\frown 0}, p_{s\frown 1}) \Vdash_{\bar{\mathbb{P}}\times\bar{\mathbb{P}}} \neg\bar{\tau}E\bar{\tau}'$,
4. p_s decides $\bar{\tau} \restriction lh(s)$, and
5. $(p_s, p_t) \in D_0 \cap D_1 \cap ... \cap D_i$ if $s, t \in {}^i2$ and $s \neq t$

for all $s, t \in 2^{<\omega}$. When p_s is defined we choose as candidates for $p_{s\frown 0}$ and $p_{s\frown 1}$ conditions $r, s \leq p_s$ with $(r, s) \Vdash_{\mathbb{P}\times\mathbb{P}} \neg\tau E\tau'$. Then one enumerates the pairs of these conditions for all s of fixed length and extends the conditions to satisfy properties 4 and 5.

Now let

$$g_x := \{q \in \bar{\mathbb{P}} : \exists n \in \omega\, p_{x|n} \leq q\}$$

for each $x \in 2^{<\omega}$. Then g_x and g_y are mutually $\bar{\mathbb{P}}$-generic over $\bar{H}$ for $x, y \in 2^{<\omega}$ with $x \neq y$, so

$$\bar{H}[g_x, g_y] \models \neg\bar{\tau}^{g_x}E\bar{\tau}^{g_y}$$

by property 3. Since $\bar{H}[g_x, g_y]$ computes $M_n^\#(z)$ correctly for each $z \in \mathbb{R}\cap\bar{H}[g_x, g_y]$ by 3 of lemma 2.1.4 we have $\bar{H}[g_x, g_y] \prec_{\Sigma^1_{n+2}} V$ by the previous lemma. Since E is $\Pi^1_{n+3}(a)$ this implies

$$V \models \neg\bar{\tau}^{g_x}E\bar{\tau}^{g_y}$$

for $x \neq y$. Since $\bar{\tau}^{g_x}$ depends continuously on x, we get a perfect set of pairwise inequivalent reals in V. This would contradict that E is thin. ∎

A set is called provably $\Delta^1_n(a)$ for $a \in \mathbb{R}$ if there are Σ^1_n and Π^1_n formulas φ and ψ such that both $\varphi(., a)$ and $\psi(., a)$ define the set, and ZFC proves $\forall x, y(\varphi(x, y) \leftrightarrow \psi(x, y))$. For our purposes it will be sufficient to know that $\forall x(\varphi(x, a) \leftrightarrow \psi(x, a))$ holds in all generic extensions of sufficiently elementary substructures of V containing a.

Theorem 2.1.9: *Let $\mathbb{P}$ be a reasonable forcing of size κ, where κ is an infinite cardinal. Suppose $M_n^\#(X)$ exists for every $X \in H_{\kappa^+}$. Then $\mathbb{P}$ does not add equivalence classes to thin provably $\boldsymbol{\Delta}^1_{n+3}$ equivalence relations.*

PROOF: Suppose E is a thin provably $\Delta^1_{n+3}(a)$ equivalence relation where $a \in \mathbb{R}$. We use E to denote the set given by the same $\Sigma^1_{n+3}(a)$ and $\Pi^1_{n+3}(a)$ formulas in any $\mathbb{P}$-generic extension. This is an equivalence relation by lemma 2.1.7.

Suppose τ is a $\mathbb{P}$-name for a real and $p \in \mathbb{P}$ is a condition such that for every $x \in \mathbb{R}$ we have $p \Vdash_{\mathbb{P}} \neg \check{x} E \tau$. Let $q \leq p$ be a condition with

$$(q, q) \Vdash_{\mathbb{P}\times\mathbb{P}} \tau E \tau'$$

by the previous lemma. Since $\mathbb{P}$ is reasonable there is a large regular λ and a countable substructure $H \prec V_\eta$ with $a, \mathbb{P}, q, \tau, \tau' \in H$ such that there is an $(H, \mathbb{P})$-generic condition $r \leq q$. Let $\bar{H}$ be the transitive collapse of H with uncollapsing map $\pi : \bar{H} \to H$ and $\pi(\bar{\mathbb{P}}) = \mathbb{P}$, $\pi(\bar{q}) = q$, $\pi(\bar{\tau}) = \tau$, and $\pi(\bar{\tau}') = \tau'$.

Let g_0 be $\bar{\mathbb{P}} \restriction \bar{q}$-generic over $\bar{H}$ in V. Further let G be $\mathbb{P}$-generic over V with $r \in G$ and define $g_1 := \pi^{-1}{}''G$. Then $\bar{q} \in g_1$. As in the proof of lemma 2.1.7 g_1 is $\bar{\mathbb{P}}$-generic over $\bar{H}$.

Now let h be $\bar{\mathbb{P}} \restriction \bar{q}$-generic over both $\bar{H}[g_0]$ and $\bar{H}[g_1]$ in V. Let $x_0 := \bar{\tau}^{g_0}$, $x_1 := \bar{\tau}^{g_1}$, and $y := \bar{\tau}^h$. Then $x_1 = \tau^G$. Since $(\bar{q}, \bar{q}) \Vdash^{\bar{H}}_{\bar{\mathbb{P}}\times\bar{\mathbb{P}}} \bar{\tau} E \bar{\tau}'$ we have

$$\bar{H}[g_0, h] \vDash x_0 E y$$

and

$$\bar{H}[g_1, h] \vDash x_1 E y.$$

As in the proof of lemma 2.1.7 $\bar{H}[g_i, h]$ computes $M_n^\#(x)$ correctly for every $x \in \mathbb{R} \cap \bar{H}[g_i, h]$. Hence $\bar{H}[g_i, h] \prec_{\Sigma^1_{n+2}} V$ by lemma 2.1.5.

Since E is provably $\Delta^1_{n+3}(a)$, this shows that x_0, x_1, and y are equivalent with respect to E. But $x_0 \in V$ and on the other hand we assumed that x_1 is in a new equivalence class in $V[G]$, which is contradictory. ∎

2.2 Projective c.c.c. forcing

In this section we present versions of the results in the previous section for $\mathbf{\Sigma}^1_2$ c.c.c. forcing.

2.2.1 Absoluteness of equivalence classes

We use the notion of projective forcing from [1].

Definition 2.2.1: *Let $a \in \mathbb{R}$. A partially ordered set $\mathbb{P} \subseteq \mathbb{R}$ is called a $\Sigma^1_n(a)$ forcing if the partial order $\leq$ and the incompatibility relation $\perp$ are $\Sigma^1_n(a)$ subsets of $\mathbb{R}^2$.*

For example Cohen forcing, random forcing, and Amoeba forcing are Σ^1_1 c.c.c. forcings.

Lemma 2.2.2: *Let $\mathbb{P}$ be a $\Sigma^1_2(a)$ c.c.c. forcing where $a \in \mathbb{R}$. Suppose $M_n^\#(x)$ exists for every $x \in \mathbb{R}$. Then for every $\mathbb{P}$-generic filter G over V*

1. *$M_n^\#(x)$ is normally ω_1-iterable in $V[G]$ via Σ for every $x \in \mathbb{R}$,*
2. *$V[G] \models$ "$M_n^\#(x)$ exists for every $x \in \mathbb{R}$", and*
3. *suppose*
    (a) *$H \prec V_\eta$ is a countable substructure with $a \in H$ where η is a large limit ordinal,*
    (b) *$\bar{H}$ is the transitive collapse of H with uncollapsing map $\pi : \bar{H} \to H$, and*
    (c) *g is a $\mathbb{P}^{\bar{H}}$-generic filter over $\bar{H}$ in V,*

    then $M_n^\#(x)$ exists in $\bar{H}[g]$ for each $x \in \mathbb{R}^{\bar{H}[g]}$ and is normally ω_1-iterable via Σ in both $\bar{H}[g]$ and V.

Proof: The proof works by induction on n as in lemma 2.1.4. We get uniqueness of $M_n^\#(x)$ for $x \in \mathbb{R}$ by lemma 1.2.20.

1. This works just as in the proof of lemma 2.1.4.

2. Let $x \in \mathbb{R}^{V[G]}$ and let τ be a nice $\mathbb{P}$-name with $\tau^G = x$. Since $\mathbb{P} \subseteq \mathbb{R}$ is c.c.c. τ can be coded by a real, so $M_n^\#(\tau, a)$ exists. We can avoid working with $M_n^\#(\mathbb{P}, \tau)$ since $M_n^\#(\tau, a)$ has its own version of the forcing $\mathbb{P}$ and this is absolute between $M_n^\#(\tau, a)$ and V. We get

$$\forall y, y' \in \mathbb{P}(y \perp y' \Leftrightarrow \neg \exists z \in \mathbb{P}(z \leq y, y'))$$

in $M_n^\#(\tau, a)$ by Π^1_3 downwards absoluteness, where $\mathbb{P}, \leq, \perp$ are given by their $\Sigma^1_2(a)$ definition. Now for $y \in \mathbb{R}$ the statement

$$\text{"}y \text{ codes a countable subset of } \mathbb{P}\text{"}$$

is $\Sigma^1_2(a)$. Since

$$\text{"}y \text{ codes a countable predense subset of } \mathbb{P}\text{"}$$

holds if and only if y codes a subset $\{y_n : n < \omega\}$ of y and $\forall z \in \mathbb{P}\, \exists n (y_n \not\perp z)$, this is a combination of a $\Sigma^1_2(a)$ and a $\Pi^1_2(a)$ statement. So it is absolute between $M_n^\#(\tau, a)$ and V. Hence

$$G' := G \cap M_n^\#(\tau, a)$$

is $\mathbb{P}$-generic over $M_n^\#(\tau, a)$. Moreover

$$x = \tau^{G'} \in M_n^\#(\tau, a)[G'].$$

Now $M_n^\#(\tau, a)$ is normally ω_1-iterable via Σ in $V[G]$ by 1. Since $\mathbb{P}$ is small compared to the critical points of the extenders on the $M_n^\#(\tau, a)$ sequence, $M_n^\#(\tau, a)[G']$ is ω_1-iterable via Σ in $V[G]$ as well. We can construct $M_n^\#(x)$ in $M_n^\#(\tau, a)[G']$ via an $L[\vec{E}]$-construction as in lemma 2.1.4.

3. Let $x \in \mathbb{R}^{\bar{H}[g]}$. Let $\bar{\tau}$ be a nice $\mathbb{P}^{\bar{H}[g]}$-name with $\bar{\tau}^g = x$ and $\tau := \pi(\bar{\tau})$. Then $M_n^\#(\bar{\tau}, a)^{\bar{H}}$ and $M_n^\#(\tau, a)$ exist and are normally ω_1-iterable via Σ in $\bar{H}$ and V respectively and

$$\pi(M_n^\#(\bar{\tau}, a)^{\bar{H}}) = M_n^\#(\tau, a).$$

So $M_n^\#(\bar{\tau}, a)^{\bar{H}}$ is normally ω_1-iterable via Σ in $\bar{H}[g]$ by 1.

Let $g' := g \cap M_n^\#(\bar{\tau}, a)^{\bar{H}}$. Since the statement

$$\text{"}y \text{ codes a countable predense subset of } \mathbb{P}\text{"}$$

is absolute between $M_n^\#(\bar{\tau}, a)$ and V, we can conclude that g' is $\bar{\mathbb{P}}$-generic over $M_n^\#(\bar{\tau}, a)^{\bar{H}}$. Moreover $x = \bar{\tau}^{g'} \in M_n^\#(\bar{\tau}, a)^{\bar{H}}[g']$. Now $M_n^\#(\bar{\tau}, a)^{\bar{H}}[g']$ is normally ω_1-iterable via Σ in $\bar{H}[g]$, since $\bar{\mathbb{P}}$ is small compared to the critical points of the extenders on the $M_n^\#(\bar{\tau}, a)^{\bar{H}}$ sequence. Moreover $M_n^\#(\bar{\tau}, a)^{\bar{H}}$ is normally ω_1-

iterable via Σ in V since

$$\pi \upharpoonright M_n^\#(\bar{\tau}, a)^{\bar{H}} : M_n^\#(\bar{\tau}, a)^{\bar{H}} \to M_n^\#(\tau, a)$$

is elementary. Hence $M_n^\#(\bar{\tau}, a)^{\bar{H}}[g']$ is normally ω_1-iterable via Σ in V as well. Finally we can construct the $M_n^\#(x)$ of $\bar{H}[g]$ and V in $M_n^\#(\bar{\tau}, a)^{\bar{H}}[g']$ as in lemma 2.1.4. ■

Note that the existence of $M_n^\#(x)$ for every $x \in \mathbb{R}$ is equivalent to $Det(\mathbf{\Pi}^1_{n+1})$ by theorem 1.2.12. The previous lemma can be applied to generalize

Lemma 2.2.3: *(Woodin [52]) $Det(\mathbf{\Pi}^1_n)$ implies Σ^1_{n+2} Cohen (random) absoluteness. In fact for odd n it is sufficient to assume*

1. *$\mathbf{\Pi}^1_n$ is scaled and*
2. *all $\mathbf{\Delta}^1_{n+1}$ sets have the Baire property (are Lebesgue measurable).*

PROOF: See [52, lemma 2]. ■

Lemma 2.2.4: *Let $\mathbb{P}$ be a $\mathbf{\Sigma}^1_2$ c.c.c. forcing and suppose $M_n^\#(x)$ exists for every $x \in \mathbb{R}$. Then Σ^1_{n+3}-absoluteness holds for $\mathbb{P}$.*

PROOF: We follow the proof of lemma 2.1.7. Let $\mathbb{P}$ be a $\Sigma^1_2(a)$ forcing. Suppose $\exists x \varphi(x, y)$ holds in some $\mathbb{P}$-generic extension of V, where φ is a Π^1_{n+2} formula and $y \in \mathbb{R}$. Let τ be a nice name and $p \in \mathbb{P}$ a condition with $p \Vdash_{\mathbb{P}} \varphi(\tau, \check{y})$. Let further $\mathcal{M} := M_n^\#(a, y, \tau)$.

Consider the tree T in V searching for g and x such that

1. g is $\mathbb{P}^{\mathcal{M}}$-generic over $\mathcal{M}$ and
2. x is a real in $\mathcal{M}[g]$ such that $\mathcal{M}[g] \models \varphi(x, y)$ if n is even, and $\Vdash^{\mathcal{M}[g]}_{Col(\omega,\delta)} \varphi(\check{x}, \check{y})$ if n is odd, where δ is the least Woodin cardinal in $\mathcal{M}$.

Now let G be $\mathbb{P} \upharpoonright p$-generic over V. Then $g := G \cap \mathcal{M}$ is $\mathbb{P}^{\mathcal{M}}$-generic over $\mathcal{M}$ since $\mathbb{P}$ is $\Sigma^1_2(a)$. Since $\mathcal{M}[g]$ is normally ω_1-iterable in $V[G]$ by lemma 2.2.2, T has a branch in $V[G]$. Then T has a branch in V and hence $V \models \exists x \varphi(x, y)$. ■

Note that one cannot prove this from n Woodin cardinals:

Lemma 2.2.5: Σ^1_{n+3} *Cohen absoluteness does not hold in* $M_n^\#$ *for even* $n < \omega$.

PROOF: The set of reals $\mathbb{R} \cap M_n^\#$ is Σ^1_{n+1} for even n by [46, theorem 3.4]. So the statement that there is a Cohen real over $M_n^\#$ is Σ^1_{n+3}. Since $M_n^\#$ and the Cohen generic extension $M_n^\#[g]$ are Σ^1_{n+2}-correct in V by lemma 1.2.27, this statement holds true in $M_n^\#[g]$ but not in $M_n^\#$. ∎

We get analogues of lemma 2.1.8 and theorem 2.1.9:

Lemma 2.2.6: *Let* E *be a thin* $\mathbf{\Pi}^1_{n+3}$ *equivalence relation. Suppose* $\mathbb{P}$ *is a* $\mathbf{\Sigma}^1_2$ *c.c.c. forcing and* $M_n^\#(x)$ *exists for every* $x \in \mathbb{R}$. *Let* τ *be a* $\mathbb{P}$*-name for a real. Then the set*

$$D := \{p \in \mathbb{P} : (p, p) \Vdash_{\mathbb{P}\times\mathbb{P}} \tau E \tau'\}$$

is dense.

Theorem 2.2.7: *Let* $\mathbb{P}$ *be a* $\mathbf{\Sigma}^1_2$ *c.c.c. forcing and suppose* $M_n^\#(x)$ *exists for every* $x \in \mathbb{R}$. *Then* $\mathbb{P}$ *does not add equivalence classes to thin provably* $\mathbf{\Delta}^1_{n+3}$ *equivalence relations.*

2.2.2 Prewellorders and generic absoluteness

Let $N_{\mathbb{P}}$ denote the set of nice names $\tau = \{(p, \check{n}) : p \in A_n\}$ for reals, where each A_n is an antichain in a forcing $\mathbb{P}$. In case $\mathbb{P} \subseteq \mathbb{R}$ has the c.c.c. every nice name can be coded by a real.

Lemma 2.2.8: *(Bagaria, Bosch [1]) Suppose* $\mathbb{P}$ *is a c.c.c.* $\Sigma^1_n(x)$ *forcing and* φ *is a* Σ^1_k *(*Π^1_k*) formula where* $n \geq 1$ *and* $k \geq 2$. *Then*

$$R := \{(p, \tau) : \tau \in N_{\mathbb{P}} \wedge p \Vdash_{\mathbb{P}} \varphi(\tau)\}$$

is a $\Sigma^1_{n+k-1}(x)$ *(*$\Pi^1_{n+k-1}(x)$*) set.*

PROOF: The set $N_{\mathbb{P}}$ of nice $\mathbb{P}$-names for reals is a Π^1_n subset of $\mathbb{R}$ by [1, fact 2.6]. We sketch the proof of the lemma from [1] for Σ^1_k formulas. For $k = 2$ and a Π^1_1 formula ψ one can express

$$p \Vdash_{\mathbb{P}} \psi(\sigma, \tau)$$

by the Δ^1_2 statement that for every (for some) countable transitive model M containing p, σ, τ of a fixed finite fragment of ZFC such that the inclusion $\mathbb{P}^M \to \mathbb{P}$ is a complete embedding, we have that $p \Vdash^M_{\mathbb{P}^M} \psi(\sigma, \tau)$. Now by the forcing theorem

$$p \Vdash_{\mathbb{P}} \exists y \psi(\sigma, y) \;\Leftrightarrow\; \exists \tau \in N_{\mathbb{P}}\, (p \Vdash_{\mathbb{P}} \psi(\sigma, \tau)),$$

so R is $\Sigma^1_{n+1}(x)$. The rest is a straightforward induction on k. The proof for Π^1_k formulas is analogous. ■

We use the previous lemma to show

Lemma 2.2.9: Σ^1_{n+1} *Cohen absoluteness implies that Cohen forcing does not add any equivalence classes to* $< \omega - \mathbf{\Pi}^1_n$ *prewellorders.*

Proof: Suppose $\leq$ is a $< \omega - \mathbf{\Pi}^1_n$ prewellorder. Let $\mathbb{P}$ denote Cohen forcing and suppose G is $\mathbb{P}$-generic over V. We denote the relation given by the same definition in $V[G]$ by $\leq$ as well. This is a prewellorder in $V[G]$ since the statement that $\leq$ is a prewellorder is Π^1_{n+1}. Moreover

$$\mathbb{1} \Vdash_{\mathbb{P}} \text{"} \leq \text{ is a prewellorder"}$$

is Π^1_{n+1} by lemma 2.2.8. Hence this holds in $V[G]$ by Σ^1_{n+1}-absoluteness, so we get

$$\mathbb{1} \Vdash_{\mathbb{P} * \mathbb{P}} \text{"} \leq \text{ is a prewellorder"}.$$

Since the two-step iteration $\mathbb{P} * \mathbb{P}$ of Cohen forcing is equal to the product, the same is forced by $\mathbb{P} \times \mathbb{P}$.

Suppose $p \in \mathbb{P}$ and $\tau \in N_{\mathbb{P}}$ such that $p \Vdash_{\mathbb{P}} \tau \not\leq \check{x} \vee \check{x} \not\leq \tau$ for all $x \in \mathbb{R}$.

Case 1: There is a real $x \in \mathbb{R}$ and a condition $q \leq p$ with $q \Vdash_{\mathbb{P}} \tau \leq \check{x}$. In this case choose $x \in \mathbb{R}$ which is $\leq$-minimal with this property. Then

$$\forall y < x (p \Vdash_{\mathbb{P}} \tau \not\leq \check{y}),$$

which is a Π^1_{n+1} statement by lemma 2.2.8, since the map $y \mapsto \check{y}$ is Borel.

Let G and H be mutually $\mathbb{P} \restriction p$-generic filters over V. Then $\forall y < x (p \Vdash_{\mathbb{P}} \tau \not\leq \check{y})$ holds in $V[G]$ by Σ^1_{n+1}-absoluteness. In particular we have

$$p \Vdash^{V[G]}_{\mathbb{P}} \tau \not\leq \tau^G.$$

So $V[G \times H] \models \tau^H \not\leq \tau^G$ and by the same argument $V[G \times H] \models \tau^G \not\leq \tau^H$. This is contradictory, since $\leq$ is linear in $V[G \times H]$.

Case 2: There is a condition $q \leq p$ such that for every $x \in \mathbb{R}$ we have $q \Vdash_{\mathbb{P}} \check{x} < \tau$. Then

$$\forall x \in \mathbb{R}(q \Vdash_{\mathbb{P}} \check{x} < \tau).$$

Let G and H be mutually $\mathbb{P} \restriction q$-generic over V. Again we get $\forall x \in \mathbb{R}(q \Vdash_{\mathbb{P}} \check{x} < \tau)$ in $V[G]$. In particular

$$q \Vdash_{\mathbb{P}}^{V[G]} \tau^G < \tau$$

so that $V[G \times H] \models \tau^G < \tau^H$. By the same argument $V[G \times H] \models \tau^H < \tau^G$, which is impossible.

We conclude that

Corollary 2.2.10: *Cohen forcing does not add equivalence classes to $< \omega - \mathbf{\Pi}^1_n$ prewellorders if and only if Σ^1_{n+1} Cohen absoluteness holds, for $n \geq 1$.*

Proof: One direction is the previous lemma. For the other direction suppose we have proved Σ^1_k Cohen absoluteness for some $k \leq n$. Let G be Cohen generic over V. Suppose $V[G] \models \exists x \varphi(x, \vec{a})$ where $\varphi \in \Pi^1_k$ and $\vec{a} \in \mathbb{R}^{<\omega}$.

We define a prewellorder $\leq$ by letting $x \leq y$ if and only if $\varphi(x, \vec{a}) \vee \neg\varphi(y, \vec{a})$. Then one of the equivalence classes of the prewellorder is

$$\{x \in \mathbb{R} : \varphi(x, \vec{a})\}.$$

Since Cohen forcing does not add any equivalence classes, there is a real $x \in \mathbb{R} \cap V$ with $V \models \varphi(x, \vec{a})$. ■

Chapter 3

The number of equivalence classes

In this chapter we calculate the number of equivalence classes of thin $\mathbf{\Pi}^1_n$ and $\mathbf{\Sigma}^1_n$ equivalence relations under PD, based on Harrington's and Shelah's theorem [9] for counting the number of equivalence classes of thin co-κ-Suslin equivalence relations. For thin $\mathbf{\Pi}^1_n$ and $\mathbf{\Sigma}^1_{2n+1}$ equivalence relations, it is sufficient to assume that the pointclasses $\mathbf{\Pi}^1_{2k+1}$ are scaled for all $k < \omega$ and that all projective sets have the Baire property.

In the first section, we present a proof of the theorem of Harrington and Shelah [9]. We then apply this to compute the number equivalence classes of thin $\mathbf{\Pi}^1_n$ and $\mathbf{\Sigma}^1_{2n+1}$ equivalence relations in the next section. The third section shows that thin Σ^1_{2n} equivalence relations are Π^1_{2n} in any real coding $M^{\#}_{2n-1}$, for $n \geq 1$. In the last section, this result is extended to show that any $\Sigma_1(J_\alpha(\mathbb{R}))$ equivalence relation is $\Pi_1(J_\alpha(\mathbb{R}))$ in a real coding a suitable premouse for the appropriate ordinals α, assuming that AD holds in $L(\mathbb{R})$. In this chapter we work in the base theory $\mathsf{ZF} + \mathsf{DC}$.

3.1 Co-κ-Suslin equivalence relations

This section presents a proof of Harrington's and Shelah's theorem.

3.1.1 A few lemmas

Definition 3.1.1: *Suppose $\kappa \in Ord$. A set $A \subseteq \mathbb{R}^n$ is called κ-Suslin if $A = p[T]$ for some tree T on $\omega^n \times \kappa$. It is co-κ-Suslin if $\mathbb{R}^n - A$ is κ-Suslin.*

Note that if AD holds in $L(\mathbb{R})$, then the sets of reals which are κ-Suslin in $L(\mathbb{R})$ for some ordinal κ are exactly the $(\mathbf{\Sigma}^2_1)^{L(\mathbb{R})}$ sets of reals, as shown by Martin and Steel [27].

In this section, E denotes an equivalence relation which is co-κ-Suslin via T. Note that $\mathbb{R}^2 - p[T]$ is not necessarily an equivalence relation in generic extensions. For example, if $\mathbb{R}$ is wellorderable and

$$T := \{(s, (x, .., x)) \in (\omega \times \mathbb{R})^{<\omega} : s \subseteq x \wedge x \in \mathbb{R}^2 - E\},$$

then $p[T] = \mathbb{R}^2 - E$ is the same set in every generic extension, so whenever $\mathbb{P}$ adds reals $\mathbb{R}^2 - p[T]$ is not an equivalence relation in $V^{\mathbb{P}}$.

For the application to projective equivalence relations we note the following consequence of the second periodicity theorem [17, see 30.12]:

Lemma 3.1.2: *Assume* $\mathsf{ZF} + \mathsf{DC}$ *and* $Det(\mathbf{\Delta}^1_{2n})$. *Then every* $\mathbf{\Pi}^1_{2n+2}$ *set is co-*$\underset{\sim}{\delta}^1_{2n+1}$*-Suslin via the tree from a* $\mathbf{\Sigma}^1_{2n+2}$*-scale.*

The existence of a perfect set of pairwise inequivalent reals for $E = \mathbb{R}^2 - p[T]$ is not absolute between V and generic extensions. We work with a stronger and absolute version, which is called strongly thick in [9].

Definition 3.1.3: *Suppose* T *is a tree and* $E = \mathbb{R}^2 - p[T]$ *is an equivalence relation and* $S \subseteq T$. *We say* S *witnesses that* E *is not thin if there is a perfect set* $P \subseteq \mathbb{R}$ *such that* $(x, y) \in p[S]$ *for all* $x, y \in P$ *with* $x \neq y$.

For a tree T and a node $r \in T$ one says that r splits in T if there are $s, t \in T$ with $r \subseteq s, t$ and $s \perp t$.

Lemma 3.1.4: *The existence of a countable set* $S \subseteq T$ *which witnesses that* $E = \mathbb{R}^2 - p[T]$ *is not thin is absolute between transitive models of* ZF.

Proof: We define a partial order $(X, <)$ such that a countable set S with this property exists if and only if $(X, <)$ is ill-founded. Let X be the set of all triples (r, s, F) such that

1. r is a finite tree on ω,

2. $s \subseteq T$ is finite,

3. F is a finite set of finite functions,

4. for any two $u, v \in r$ with $u \neq v$ there is a function $f \in F$ with $(u, v, f) \in s$,

and let $(r, s, F) < (p, q, G)$ if

1. $p \subseteq r$,

2. $q \subseteq s$,

3. every node in p splits in r, and

4. for any $u, v \in p$ with $u \neq v$ and any $g \in G$ with $(w, x, g) \in q$ there are $y, z \in r$ and a function $f \in F$ with $u \subseteq y$, $v \subseteq z$, $g \subseteq f$, and $(y, z, f) \in s$.

If a countable set $S \subseteq T$ and a perfect set $P \subseteq \mathbb{R}$ witness that E is not thin, one can define $(r_n, s_n, F_n) \in X$ such that $(r_n, s_n, F_n) < (r_k, s_k, F_k)$ for all $k < n < \omega$ and for any two distinct $u, v \in r_n$ of the same length there are reals $x, y \in P$ and a function $f : \omega \to Ord$ with

1. $u \subseteq x$,

2. $v \subseteq y$,

3. $(u, v, f \restriction lh(u)) \in s_n$, and

4. $(x, y, f) \in [S]$.

If on the other hand $(X, <)$ is illfounded, let $((r_n, s_n, F_n) : n < \omega)$ be a strictly decreasing sequence in $(X, <)$. We can set $S := \{f \restriction k : \exists n < \omega\, (f \in F_n \wedge k < \omega)\}$ and $P := [U]$ where $U := \bigcup_{n \in \omega} r_n$.

This works without choice since X can be wellordered. ■

We will work with the infinitary logic $\mathcal{L}_{\infty,\omega}$ over a language $\mathcal{L}$ [3, chapter III, definition 1.5]. $\mathcal{L}_{\infty,\omega}$-formulas are distinguished from finitary formulas by the fact that disjunctions and conjunctions of arbitrary ordinal length are possible. Let $\mathcal{L}$ be a language which contains at least $\in$ and the following constants: c, d for reals, $\dot{f}$ for a function $f : \omega \to \kappa$, and $\dot{s}$ for each $s \in tc(\{T\})$. Let N be the set of atomic formulas $\dot{n} \in c$ with $n \in \omega$. We build $\mathcal{L}_{\infty,0,N}$ by starting with N and closing under negations and wellordered infinitary disjunctions and conjunctions. We will also write φ_d for the formula obtained from a formula $\varphi \in \mathcal{L}_{\infty,0,N}$ by replacing c with d. Note that instead of $\mathcal{L}_{\infty,0,N}$ one can equivalently work with the infinitary logic $\mathcal{L}_{\infty,0}$ built over a language with a set of propositional formulas $\{p_n : n < \omega\}$, as is done in [9] and [13].

Whether a statement $\varphi \in \mathcal{L}_{\infty,0,N}$ is true depends only on the truth value of each individual atomic statement $\dot{n} \in c$.

Definition 3.1.5: *Suppose $\varphi \in \mathcal{L}_{\infty,0,N}$ and $x \in \mathbb{R}$. Define the truth value of $\dot{n} \in c$ as true if and only if $n \in x$. This induces a truth value for φ by induction on the formula complexity. If this value is true we say that $\varphi(x)$ holds and x is a model of φ.*

We refer to the infinitary proof calculus from [3] which has the rule

$$\forall \alpha < \beta \vdash \varphi_\alpha \quad \Rightarrow \quad \vdash \bigwedge_{\alpha<\beta} \varphi_\alpha$$

in addition to the rules of first-order logic [3, chapter III, definition 5.1]. In the following let χ be the $\mathcal{L}_{\infty,\omega}$-formula

$$c \subseteq \omega \wedge d \subseteq \omega \wedge \bigwedge_{t \in tc(\{T\})} (\forall x \in \dot{t} \bigvee_{s \in t} x = \dot{s}) \wedge (\bigwedge_{s \in t} \dot{s} \in \dot{t}).$$

Then c, d are interpreted as reals and $\dot{s}$ takes the value s for each $s \in tc(\{T\})$ in any transitive model of χ. Note that for any admissible set $\mathbb{A}$ with $T \in \mathbb{A}$ we have $\chi \in \mathbb{A}$ by Δ_0-replacement, since we can assume that s and $\dot{s}$ are Δ_0-definable from each other for each $s \in tc(\{T\})$. We write $\vdash_\chi$ for the provability relation when χ is used as an axiom.

A theory in $\mathcal{L}_{\infty,\omega}$ is consistent if it is not contradictory in terms of infinitary proofs. Note that for hereditarily countable theories, this definition coincides with several other definitions of consistency:

Lemma 3.1.6: *The following are equivalent for any hereditarily countable theory $\Sigma \subseteq \mathcal{L}_{\infty,0,N}$:*

1. *Σ is consistent*
2. *Σ is consistent in any admissible set $\mathbb{A}$ with $\Sigma \in \mathbb{A}$*

3. *Σ has a model*

4. *there is a model of Σ in some generic extension*

5. *player 1 wins the closed game G_Σ from [13, section 2.2].*

PROOF: We sketch the relevant part that Σ has a model if it is consistent. Suppose Σ is consistent. We can assume that negations occur only at the atomic level in formulas in Σ, since every $\mathcal{L}_{\infty,0,N}$-formula is equivalent to a formula of this form. Suppose two players play a game G_Σ with the rules:

1. if player 1 plays $\bigvee_{\alpha<\beta} \chi_\alpha$, then player 2 has to play χ_α for some $\alpha < \beta$, and

2. both players can only play formulas which are consistent with Σ and the previously played moves,

where player 2 wins if the game does not stop after finitely many moves. Then player 2 has a winning strategy in G_Σ, since Σ is consistent. By letting player 1 play each disjunction in $tc(\Sigma)$ consistent with Σ and the previous moves at some point in the game, the play determines a real x so that $n \in x$ if and only if the formula $\dot{n} \in c$ was played during this run of the game. One shows by induction on the formula complexity that x models Σ.

The equivalence of 1 and 2 follows from the Barwise completeness theorem [3, part III, theorem 5.5]. ∎

Lemma 3.1.7: *A Cohen real adds a perfect set of mutually generic Cohen reals*

PROOF: Let $\mathbb{Q}$ be the forcing which consists of finite trees on ω, where $s \leq t$ if and only if $t \subseteq s$ and every node of t splits in s. Then $\mathbb{Q}$ is equivalent to Cohen forcing $\mathbb{P}$ since it is countable and has no atoms. Also any two branches of the tree added by $\mathbb{Q}$ are mutually $\mathbb{P}$-generic, since for every dense open set $D \subseteq \mathbb{P} \times \mathbb{P}$ the set

$$D' := \{t \in \mathbb{P} : \forall r, s \in t(r \neq s \Rightarrow \exists r', s' \in t(r \subseteq r' \wedge s \subseteq s' \wedge (r', s') \in D))\}$$

is dense in $\mathbb{Q}$. ∎

3.1.2 The theorem of Harrington and Shelah

In this section we give a proof of

Theorem 3.1.8: *(Harrington, Shelah [9]) Assume* ZF*. Suppose κ is an infinite cardinal and T is a tree on $\omega \times \omega \times \kappa$. Let $\mathbb{A}$ be an admissible set with $T \in \mathbb{A}$. Suppose $E = \mathbb{R}^2 - p[T]$ is a thin equivalence relation such that*

1. $\Vdash^{L[T]}_{Cohen}$ *"$\mathbb{R}^2 - p[T]$ is transitive" or*
2. *there is a Cohen real over $L[T]$ in V.*

Then for every $x \in \mathbb{R}$ there is a formula $\varphi \in \mathcal{L}_{\infty,0,N} \cap \mathbb{A}$ with

1. *$\varphi(x)$ and*
2. $\vdash^{\mathbb{A}}_{\chi} (\varphi \wedge \varphi_d) \rightarrow (c,d) \notin p[\dot{T}]$.

Proof: If there is a Cohen real x over $L[T]$ in V, then $\mathbb{R}^2 - p[T]$ is an equivalence relation in $L[T,x]$ by absoluteness of $p[T]$, since $E = \mathbb{R}^2 - p[T]$ is an equivalence relation in V. So we assume the first condition holds. Note that this condition is also true if $\mathbb{R}^2 - p[T]$ is transitive in a Cohen generic extension of V.

We work in $L[T]$. If the theory

$$\Sigma := \{\neg\varphi : \varphi \in \mathcal{L}_{\infty,0,N} \cap \mathbb{A} : \vdash_{\chi} (\varphi \wedge \varphi_d) \rightarrow (c,d) \notin p[\dot{T}]\}$$

is inconsistent, then no real can satisfy every statement in Σ. Here $\vdash_{\chi}$ can be equivalently replaced by $\vdash^{\mathbb{A}}_{\chi}$ by the Barwise completeness theorem [3, chapter III, theorem 5.5]. Then for every $x \in \mathbb{R}$ there is a formula φ satisfying the conditions and we are done.

Assume Σ is consistent and let $A \prec \mathbb{A}$ be a countable substructure with $\Sigma, T \in A$ and $\bar{A}$ its transitive collapse with uncollapsing map $\pi : \bar{A} \rightarrow A$, $\pi(\bar{T}) = T$, $\pi(\bar{\chi}) = \chi$, $\pi(\bar{\kappa}) = \kappa$, and $\pi(p) = \Sigma$. Further suppose $\dot{T}, \dot{n} \in A$, $\pi(\dot{T}) = \dot{T}$, $\pi(\dot{f}) = \dot{f}$, and $\pi(\dot{n}) = \dot{n}$ for all $n \in \omega$. Also assume that s and $\dot{s}$ are Δ_0-definable from each other for each $s \in tc(\{T\})$ to ensure that $\dot{T}$ is interpreted as $\bar{T}$ in every model of $\bar{\chi}$. We will refer to provability and consistency as provability from $\bar{\chi}$ and consistency with $\bar{\chi}$ and denote $\vdash_{\bar{\chi}}$ simply by $\vdash$.

Claim 3.1.9: *Suppose* $p \subseteq q \subseteq \mathcal{L}_{\infty,0,N} \cap \bar{A}$*,* q *is consistent,* q *is* $\underset{\sim}{\Sigma}_1$ *over* $\bar{A}$*. Then* $q \cup q_d \cup \{(c,d) \in p[\dot{T}]\}$ *is consistent.*

PROOF: Assume the theory is inconsistent, so it does not have a model. Now Barwise compactness [3, chapter III, theorem 5.6] implies that there is a theory $s \in \bar{A}$ with $\vdash q \rightarrow s$ such that $s \cup s_d \cup \{(c,d) \in p[\dot{T}]\}$ does not have a model. Hence this theory is inconsistent by lemma 3.1.6. We can replace s by a the conjunction $\varphi \in \mathcal{L}_{\infty,0,N} \cap \bar{A}$ of all formulas in s and get $\vdash (\varphi \wedge \varphi_d) \rightarrow (c,d) \notin p[\dot{T}]$. Then $\neg\varphi \in p \subseteq q$ by definition of Σ. But this cannot happen, since $\vdash q \rightarrow \varphi$ and q is consistent. ■

Using the claim, one would like to build a perfect tree of consistent theories starting with p such that every branch defines a complete theory. Then every branch defines a unique real. At the same time one would have to ensure that $(x,y) \in p[T]$ for reals x,y from distinct branches and this would contradict that E is thin.

However, this cannot work directly, since the assumption on Cohen forcing is necessary by remark 3.1.14. Instead one can construct a countable set $S \subseteq T$ and a perfect set in a Cohen generic extension witnessing that $\mathbb{R}^2 - p[T]$ is not thin and then apply lemma 3.1.4.

Claim 3.1.10: *Every consistent* $\mathcal{L}_{\infty,0,N}$*-theory* q *with* $p \subseteq q$ *which is* $\underset{\sim}{\Sigma}_1$ *over* $\bar{A}$ *is incomplete.*

PROOF: Suppose q is complete and consistent. Then the theory $q \cup q_d \cup \{(c,d) \in p[\dot{T}]\}$ is consistent by claim 3.1.6. So

$$q \cup q_d \cup \{(c,d,\dot{f}) \in [\dot{T}]\}$$

is consistent as well. This theory has a model $(x,y,f) \in L[T]$ since one can also apply lemma 3.1.6 if N is replaced by the set N' which additionally contains each formula $\dot{f}(\dot{n}) = \dot{\alpha}$ for $\alpha < \bar{\kappa}$ and $n < \omega$. Then $x = y$ since x and y are both models of q and q is complete. Hence $(x,x,f) \in [\bar{T}]$. Note that $p[\bar{T}] \subseteq p[T]$ since $\pi''\bar{T}$ is obtained from T by omitting all ordinals not in A. But this would imply $(x,x) \in p[T]$ and hence $(x,x) \notin E$. ■

Let $(\psi_n : n < \omega)$ enumerate the formulas in $\mathcal{L}_{\infty,0,N} \cap \bar{A}$. We can assume that negations only occur on the atomic level in all formulas.

We build a tree $\mathbb{C}$ in $L[T]$ whose nodes are consistent $\mathcal{L}_{\infty,0,N}$-theories $q \supseteq p$ which are $\underset{\sim}{\Sigma}_1$ over $\bar{A}$, ordered by inclusion. The root of the tree is p. We can construct the tree level by level and ensure that $\mathbb{C}$ is isomorphic to $2^{<\omega}$ and

1. $\psi_n \in q$ or $\neg\psi_n \notin q$ for all q on level n and
2. if $\psi_n \equiv \bigvee_{\alpha<\beta} \chi_\alpha$, $\psi_n \in q$, and q is on level $\langle n, k\rangle$ for some $k < \omega$, then for every $r \supseteq q$ on level $\langle n, k\rangle + 1$ there is some $\alpha < \beta$ with $\chi_\alpha \in r$.

Then every branch in $\mathbb{C}$ defines a consistent theory and a real which is a model of the theory. Note that there are no end nodes in $\mathbb{C}$ by claim 3.1.10. We will force with $(\mathbb{C}, \leq)$, where $\leq$ denotes reverse inclusion. If G is $\mathbb{C}$-generic, then $\bigcup G$ is a complete theory and defines a real x. The generic filter can also be recovered from the real x as

$$G = \{q \in \mathbb{C} : \forall\varphi \in q\, \varphi(x)\}.$$

Let N' be the set of atomic statements about c, d, and $\dot{f}$. Let $\mathbb{P}$ be a tree of consistent $\mathcal{L}_{\infty,0,N'}$-theories containing $p \cup p_d \cup \{c, d, \dot{f}) \in [\dot{T}]\}$ which are $\underset{\sim}{\Sigma}_1$ over $\bar{A}$. We build $\mathbb{P}$ in the same way as $\mathbb{C}$ such that additionally the value of $(c|n, d|n, \dot{f}|n)$ is decided on the n^{th} level and $\mathbb{P}$ is isomorphic to $2^{<\omega}$. Then every branch in $\mathbb{P}$ defines a consistent theory containing $p \cup p_d \cup \{(c, d, \dot{f}) \in [\dot{T}]\}$ and a triple (x, y, f) which is a model if this theory. We will force with $(\mathbb{P}, \leq)$ where $\leq$ denotes reverse inclusion.

Claim 3.1.11: *Suppose G is $\mathbb{P}$-generic and (x, y, f) is a model of the corresponding theory. Then both x and y are $\mathbb{C}$-generic.*

PROOF: Suppose $D \subseteq \mathbb{C}$ is open dense. It suffices to find $r \in D$ such that x models r. For $q \in \mathbb{P}$ let

$$q(c) := \{\varphi \in \mathcal{L}_{\infty,0,N} \cap \bar{A} : q \vdash \varphi\}$$

be the set of statements about c which are provable from q.

We claim that

$$D' := \{q \in \mathbb{P} : q(c) \in D\}$$

is dense in $\mathbb{P}$. So suppose $q \in \mathbb{P}$. There is a condition $r \in D$ with $q(c) \subseteq r$ since D is dense. If $q' := q \cup r$ was inconsistent, then by Barwise compactness and lemma 3.1.6 there would be a set $s \subseteq r$ with $s \in \bar{A}$ such that $q \vdash \neg \bigwedge s$. Hence $\neg \bigwedge s \in q(c)$. But $\bigwedge s$ is consistent with $q(c)$ since $s \subseteq r$. So q' is consistent. Since $r \subseteq q'(c)$ we have $q'(c) \in D$ and hence $q' \in D'$.

Choose a condition $q \in G \cap D'$ and let $r := q(c) \in D$. Then x models r, since $q \in G$ and (x, y, f) models $\bigcup G$. ■

Claim 3.1.12: *If (x, y) is $\mathbb{C} \times \mathbb{C}$-generic over $L[T]$, then $(x, y) \in p[T]$.*

Proof: Suppose there are conditions $q, q' \in \mathbb{C}$ with $(q, q') \Vdash_{\mathbb{C}\times\mathbb{C}} (\dot{x}_0, \dot{x}_1) \notin p[\check{T}]$, where $\dot{x}_0$ and $\dot{x}_1$ are names for the left and right generic reals. Then

$$s := q \cup q_d \cup \{(c, d) \in p[\dot{T}]\}$$

is consistent by claim 3.1.9. Let $s' \in \mathbb{P}$ be a condition with $s' \supseteq s$. Suppose G is Cohen-generic over $L[T]$. There is a $\mathbb{C} \upharpoonright q' \times \mathbb{P} \upharpoonright s$-generic filter over $L[T]$ in $L[T, G]$ by lemma 3.1.7. Let x be the $\mathbb{C}$-generic real and y and z the reals from the $\mathbb{P}$-generic filter as in claim 3.1.11. Then both y and z are $\mathbb{C} \upharpoonright q$-generic over $L[T]$. Since $p[T]$ is absolute, we have

$$L[T, G] \vDash (x, z) \notin p[T]$$

and

$$L[T, G] \vDash (y, z) \notin p[T]$$

since this is forced by (q, q') and

$$L[T, G] \vDash (x, y) \in p[\bar{T}].$$

But this cannot happen, since $p[\bar{T}] \subseteq p[T]$ and $\mathbb{R}^2 - p[T]$ is transitive in any Cohen generic extension of $L[T]$. ■

Hence Cohen forcing adds a perfect set of pairwise inequivalent reals by lemma 3.1.7 and claim 3.1.12.

Let τ be a $\mathbb{C}$-name for a sequence of ordinals with $\Vdash_{\mathbb{C}\times\mathbb{C}} (\dot{x}_0, \dot{x}_1, \tau) \in [\check{T}]$ by the forcing theorem, where $\dot{x}_0$ and $\dot{x}_1$ are names for the left and right generic reals. Since $\mathbb{C}$ is proper, there is in fact a countable set $S \subseteq T$ such that $\Vdash_{\mathbb{C}\times\mathbb{C}}$ $(\dot{x}_0, \dot{x}_1, \tau) \in [\check{S}]$. But then there would also be a countable subset of T witnessing that $\mathbb{R}^2 - p[T]$ is not thin in $L[T]$ by the absoluteness proved in lemma 3.1.4. This would imply that E is not thin in V. ∎

It is not clear whether the previous proof can be generalized to Sacks forcing, or other forcings whose conditions are trees on ω, instead of Cohen forcing.

Note that if φ satisfies the conditions in the previous theorem, then the set $\{y \in \mathbb{R} : \varphi(y)\}$ is contained in the equivalence class of x. The aim of the theorem was to show

Corollary 3.1.13: *(Harrington, Shelah [9]) Assume* ZF*. Suppose T is a tree on $\omega \times \omega \times \kappa$ and $E = \mathbb{R}^2 - p[T]$ is a thin equivalence relation such that*

1. $\Vdash^{L[T]}_{Cohen}$ *"$\mathbb{R}^2 - p[T]$ is transitive" or*
2. *there is a Cohen real over $L[T]$ in V.*

Then the equivalence classes of E can be wellordered with order type $\leq \kappa$.

Proof: Let $\mathbb{A}$ be the least admissible set with $T \in \mathbb{A}$. Then $\mathbb{A} = L_\alpha[T]$ for some ordinal α. Then $\overline{\overline{\mathbb{A}}} = \kappa$ since $\mathbb{A} = L_\alpha[T] = h^{\mathbb{A}}(\kappa \cup \{T\})$ by minimality of $\mathbb{A}$. Let Φ be the set of $\mathcal{L}_{\infty,0,N}$-formulas $\varphi \in \mathbb{A}$ satisfying the conditions in theorem 3.1.8 and let $(\varphi_\beta : \beta < \gamma)$ enumerate Φ for some $\gamma \leq \kappa$. Then every equivalence class is the union of sets of the form $\{x \in \mathbb{R} : \varphi_\beta(x)\}$ with $\beta < \gamma$. Hence

$$f([x]_E) := \min\{\beta < \gamma : \exists y \in [x]_E \ \varphi_\beta(xy)\}$$

is a rank function for the equivalence classes. ∎

Remark 3.1.14: *(Shelah [43]) The assumption that $\mathbb{R}^2 - p[T]$ is transitive in a Cohen generic extension cannot be eliminated from the previous corollary.*

Proof: Shelah [43] defines a finite support iteration of c.c.c. forcings of length ω_1 assuming $2^{\aleph_0} = \aleph_2$, so that in the generic extension there is a thin co-$\aleph_1$-Suslin equivalence relation with $2^{\aleph_0}$ equivalence classes. ∎

Note that if κ is a successor cardinal and T a tree on $\omega \times \omega \times \kappa$, then $L[T]$ is not a counterexample to the conclusion of corollary 3.1.13, since $L[T]$ has at most κ many reals by a standard argument.

3.2 Projective equivalence relations

In this section we use the theorem of Harrington and Shelah to determine the number of equivalence classes of thin projective equivalence relations relative to the projective ordinals, assuming PD.

3.2.1 $\mathbf{\Pi}^1_n$ and $\mathbf{\Sigma}^1_{2n+1}$ equivalence relations

Silver [44] proved

Lemma 3.2.1: *Assume* ZF*. Then every thin* $\mathbf{\Pi}^1_1$ *equivalence relation has countably many equivalence classes.*

Harrington's simpler proof of this result can be found in Jech [16, theorem 32.1]. The lemma follows from corollary 3.1.13, since $\mathbf{\Sigma}^1_1$ sets are $\aleph_0$-Suslin and the statement that a Π^1_1 formula defines an equivalence relation is Π^1_2 and hence absolute. Burgess proved

Lemma 3.2.2: *Assume* ZF*. Then every thin* $\mathbf{\Sigma}^1_1$ *equivalence relation has at most* $\aleph_1$ *many equivalence classes.*

For a proof see Jech [16, theorem 32.9]. This result is a consequence of corollary 3.1.13, since $\mathbf{\Sigma}^1_2$ sets are $\aleph_1$-Suslin and the Shoenfield tree projects to the complete Σ^1_2 set in any Cohen generic extension. For the same reason one has

Lemma 3.2.3: *Assume* ZF *and*

1. *there is a Cohen real over* $L[x]$ *or*
2. *there is an inner model which satisfies generic* Σ^1_3 *Cohen absoluteness.*

Then any thin $\Pi^1_2(x)$ *equivalence relation has at most* $\aleph_1$ *many equivalence classes.*

The conclusion from condition 1 is shown in in Harrington and Shelah [9]. Note that 2 implies 1 by Bartoszynski and Judah [2, theorems 9.2.12 and 9.2.1].

The previous facts generalize through the projective hierarchy:

Theorem 3.2.4: *Assume* ZF *and*

1. $\boldsymbol{\Pi}^1_{2n+1}$ *is scaled and*

2. *all* $\boldsymbol{\Delta}^1_{2n+2}$ *sets have the Baire property.*

Then the equivalence classes of any thin $\boldsymbol{\Pi}^1_{2n+2}$ *equivalence relation can be wellordered with order type* $\leq \underset{\sim}{\delta}^1_{2n+1}$. *Moreover, there is a thin* Σ^1_{2n+1} *equivalence relation whose equivalence classes can be wellordered with order type* $\underset{\sim}{\delta}^1_{2n+1}$.

Proof: Let E be a thin $\boldsymbol{\Pi}^1_{2n+2}$ equivalence relation and fix some $\boldsymbol{\Sigma}^1_{2n+2}$-scale $(\leq_k\colon k < \omega)$ of length $\underset{\sim}{\delta}^1_{2n+1}$ on $\mathbb{R}^2 - E$. In fact [31, 4C.14] and [21, theorem 38.4] imply that there is a scale of this length. We further have Σ^1_{2n+3} Cohen absoluteness by lemma 2.2.3. Then in every Cohen generic extension

1. E is an equivalence relation and

2. $(\leq_k\colon k < \omega)$ is a scale on $\mathbb{R}^2 - E$,

since both are Π^1_{2n+3} statements. Here E and $(\leq_k\colon k < \omega)$ are understood as the corresponding sets in the generic extension with the same definition as in the ground model. Now Cohen forcing does not change the tree T from the scale since no new equivalence classes are added to the relevant prewellorders by lemma 2.2.9. Hence $\mathbb{R}^2 - p[T]$ is an equivalence relation in any Cohen generic extension. Thus the equivalence classes of E can be wellordered with order type $\leq \underset{\sim}{\delta}^1_{2n+1}$ by corollary 3.1.13.

Let $\leq$ be a Π^1_{2n+1} norm on the complete Π^1_{2n+1} set $A \subseteq \mathbb{R}$, so $\leq$ has length $\underset{\sim}{\delta}^1_{2n+1}$. The norm induces a Σ^1_{2n+1} equivalence relation E defined by $(x, y) \in E$ if and only if $(x \leq y \wedge y \leq x) \vee x, y \notin A$. Moreover, E is thin by the argument in lemma 1.1.15. ■

The previous theorem implies that there is no difference in the possible number of equivalence classes of thin $\boldsymbol{\Sigma}^1_{2n+1}$ equivalence relations and of thin $\boldsymbol{\Pi}^1_{2n+2}$ equivalence relations. On the other hand, we will see in theorem 4.1.31 that there are inner models which have representatives in every equivalence class of every thin $\boldsymbol{\Sigma}^1_{2n+1}$ equivalence relation defined from a parameter in the inner model, but do not fulfill the same condition for thin $\boldsymbol{\Pi}^1_{2n+2}$ equivalence relations.

For the odd levels one has

Theorem 3.2.5: *Assume* ZF *and*

1. $\mathbf{\Pi}^1_{2n+1}$ *is scaled and*
2. *all* $\mathbf{\Delta}^1_{2n+2}$ *sets have the Baire property.*

Then the equivalence classes of any thin $\mathbf{\Pi}^1_{2n+1}$ *equivalence relation can be wellordered with order type* $< \utilde{\delta}^1_{2n+1}$. *Moreover, for every* $\alpha < \utilde{\delta}^1_{2n+1}$ *there is a thin* $\mathbf{\Delta}^1_{2n+1}$ *equivalence relation whose equivalence classes can be wellordered with order type at least* α.

Proof: Let E be a thin $\mathbf{\Pi}^1_{2n+1}$ equivalence relation. Let further $A \subseteq \mathbb{R}^3$ be a $\mathbf{\Pi}^1_{2n}$ set with $\mathbb{R}^2 - E = p[A]$ and fix a $\mathbf{\Pi}^1_{2n+1}$-scale $(\leq_k \colon k < \omega)$ on A. Then the prewellorders are actually $\mathbf{\Delta}^1_{2n+1}$, since A is $\mathbf{\Pi}^1_{2n}$. Since $cf(\utilde{\delta}^1_{2n+1}) > \omega$, this implies that the length α of the scale is less than $\utilde{\delta}^1_{2n+1}$. Now Σ^1_{2n+3} Cohen absoluteness holds by lemma 2.2.3. So the Π^1_{2n+2} statements

1. E is an equivalence relation,
2. $\mathbb{R}^2 - E = p[A]$, and
3. $(\leq_n \colon n < \omega)$ is a scale on A,

hold in every Cohen generic extension. Cohen forcing does not change the tree T from the scale, since no new equivalence classes are added to the relevant prewellorders by lemma 2.2.9. Hence $\mathbb{R}^2 - p[T]$ is an equivalence relation in any Cohen generic extension. Thus the equivalence classes of E can be wellordered with order type at most α by corollary 3.1.13.

Clearly for every $\alpha < \utilde{\delta}^1_{2n+1}$ there is a $\mathbf{\Delta}^1_{2n+1}$ prewellorder with order type at least α, by the definition of $\utilde{\delta}^1_{2n+1}$. ■

The extra assumptions in the previous theorems cannot be eliminated:

Lemma 3.2.6: *Let* $n < \omega$ *and assume there are* $2n$ *Woodin cardinals and a measurable above if* $n > 0$. *Then under* $\mathsf{ZFC} + Det(\mathbf{\Pi}^1_{2n})$ *there is no upper bound for the number of equivalence classes of thin* $\mathbf{\Sigma}^1_{2n+2}$ *equivalence relations.*

PROOF: M_{2n} denotes the inner class model defined by iterating the top extender of $M_{2n}^{\#}$ out of the universe. In particular $M_0 = L$. Then $M_{2n} \vDash Det(\mathbf{\Pi}^1_{2n})$ by lemma 1.2.12.

Let κ be an uncountable cardinal in M_{2n} and G a generic filter over M_{2n} for the finite support product of $(\kappa^+)^{M_{2n}}$ many Cohen forcings. Working in $M_{2n}[G]$, fix a set $A \subseteq \mathbb{R}$ of size κ. We claim that there is a c.c.c. forcing $\mathbb{P}$ in $M_{2n}[G]$ such that A is $\mathbf{\Pi}^1_{2n+2}$ in any $\mathbb{P}$-generic extension of $M_{2n}[G]$. For $n = 0$ this is Harrington's forcing from [6, §1]. The forcing has to be adapted if $n > 0$; in this case we have to find a sequence $(d_{\alpha,n} : \alpha < \omega_1, n < \omega)$ of distinct reals in $M_{2n}[G]$ which is Δ^{HC}_{2n+1} over $M_{2n}[G]$.

As for the case $n = 0$ we work with the sequence of all reals of M_{2n} in the order of constructibility. The canonical wellorder is shown to be Δ^{HC}_{2n+1} over M_{2n} in [46, theorem 4.5] by comparing reals in Π_{2n}-iterable, $2n$-small, ω-sound premice. The point is that $\mathcal{M} \trianglelefteq M_{2n}^{\#}$ for such premice $\mathcal{M}$ by [46, lemma 3.3]. Since Π_{2n}-iterability is Π^1_{2n+1} in the codes, it is absolute between M_{2n} and $M_{2n}[G]$. It follows that the sequence of reals of M_{2n} in the order of constructibility is Δ^{HC}_{2n+1} over $M_{2n}[G]$. ■

The number of equivalence classes of thin $\mathbf{\Pi}^1_n$ and $\mathbf{\Sigma}^1_{2n+1}$ equivalence relations can be calculated under PD by the two theorems above. On the other hand, the exact bounds for thin equivalence relations in a given lightface projective pointclass are still open. The next example defines a Δ^1_3 equivalence relation with exactly $Card(\underset{\sim}{\delta}^1_2)$ many equivalence classes from a prewellorder. Hence it is consistent that there is a Δ^1_3 equivalence relation with $\aleph_2$ many equivalence classes by lemma 1.1.19.

Example 3.2.7: *Assume $x^{\#}$ exists for every $x \in \mathbb{R}$. Let $(\iota^x_\alpha : \alpha \in Ord)$ enumerate the x-indiscernibles and define $u^x_2 := \iota^x_{\omega_1+1}$ for $x \in \mathbb{R}$. The prewellorder given by*

$$x \leq y :\Leftrightarrow u^x_2 \leq u^y_2$$

is Δ^1_3 and its length is $\underset{\sim}{\delta}^1_2$.

PROOF: Note that the class of x-indiscernibles and the theory of $L[x]$ are definable from $x^{\#}$ since

$$x^{\#} = \{\ulcorner \varphi(v_0, ..., v_n) \urcorner : L[x] \vDash \varphi(x, \omega_1^V, ..., \omega_n^V)\}.$$

Thus $u_2^x \leq u_2^y$ holds if and only if

$$L[x^{\#}, y^{\#}] \models \iota^x_{\omega_1^V+1} \leq \iota^y_{\omega_1^V+1}.$$

But this can be calculated from $(x^{\#}, y^{\#})^{\#}$ since ω_1^V is an $(x^{\#}, y^{\#})$-indiscernible. Now sharps for reals are defined by a Π^1_2 formula. Hence $u_2^x \leq u_2^y$ is Δ^1_3 in x, y.

The length of the prewellorder is $\underset{\sim}{\delta}^1_2$ since $\underset{\sim}{\delta}^1_2 = u_2 = \sup\{u_2^x : x \in \mathbb{R}\}$ in the presence of sharps for reals. ■

A similar example for Δ^1_5 is not known. It could be possible to realize this by comparing the heights of the transitive direct limit of iterates of $M_2^{\#}(x)$ and $M_2^{\#}(y)$ via iteration trees living below the respective least Woodin cardinal.

3.2.2 $\mathbf{\Sigma}^1_{2n}$ equivalence relations

Hjorth [12, lemma 2.5] showed that every thin $\Sigma^1_2(x)$ equivalence relation is Π^1_2 in any real coding $M_1^{\#}$, assuming $M_1^{\#}(x)$ exists and is $\omega_1 + 1$-iterable. This also works assuming $M_1^{\#}(x)$ exists for every $x \in \mathbb{R}$ via lemma 1.2.20. In this section this result and its proof are extended to the even levels of the projective hierarchy. The main ingredient is the next lemma, based on [12, lemma 2.2].

If $\mathbb{P}$ is a forcing and τ is a $\mathbb{P}$-name for a real, then in any $\mathbb{P} \times \mathbb{P}$-generic extension there are two corresponding reals from the $\mathbb{P}$-generic filters. We write τ and τ' for $\mathbb{P} \times \mathbb{P}$-name for these reals.

Lemma 3.2.8: *Let n be even and $k \geq n$, Suppose $M_k^{\#}(x)$ exists for every $x \in \mathbb{R}$. Let $E \subseteq \mathbb{R}^2$ be a thin $\Pi^1_{n+3}(x)$ equivalence relation where $x \in \mathbb{R}$. Let $\mathcal{M}$ be a countable $(k+1)$-small X-premouse which is ω_1-iterable above δ and ω-sound above δ with $\rho_\omega(\mathcal{M}) \leq \delta$, where X is swo. Suppose there are n Woodin cardinals above δ and an extender above them in $\mathcal{M}$. Let further $\mathbb{P}$ be a forcing of size $\leq \delta$ in $\mathcal{M}$. Then for every $\mathbb{P}$-name $\tau \in \mathcal{M}$ for a real the set D of conditions $p \in \mathbb{P}$ with*

$$(p, p) \Vdash^{\mathcal{M}}_{\mathbb{P}\times\mathbb{P}} \tau E \tau'$$

is dense in $\mathbb{P}$.

Proof: Suppose D is not dense. In this case let $p_\emptyset \in \mathbb{P}$ be a condition such that for every $q \leq p$ there are conditions $r, u \leq q$ with

$$(r,u) \Vdash^{\mathcal{M}}_{\mathbb{P}\times\mathbb{P}} \neg\tau E\tau'.$$

Let $(D_i : i < \omega)$ enumerate the dense open subsets of $\mathbb{P}\times\mathbb{P}$ in $\mathcal{M}$. One can inductively define a family $(p_s : s \in 2^{<\omega})$ of conditions in $\mathbb{P}$ such that for all $s, t \in 2^{<\omega}$

1. $p_s \leq p_t$ if $t \subseteq s$,
2. $(p_{s\frown 0}, p_{s\frown 1}) \Vdash^{\mathcal{M}}_{\mathbb{P}\times\mathbb{P}} \neg\tau E\tau'$,
3. p_s decides $\tau \restriction lh(s)$, and
4. $(p_s, p_t) \in D_0 \cap .. \cap D_i$ if $s, t \in {}^i 2$ and $s \neq t$.

Moreover let

$$g_y := \{p \in \mathbb{P} : \exists n < \omega(p_{y\restriction n} \leq p\}$$

for each $y \in \mathbb{R}$. Then $g_y \times g_z$ is $\mathbb{P}\times\mathbb{P}$-generic over $\mathcal{M}$ for any $y, z \in \mathbb{R}$ with $y \neq z$ by condition 4. Then

$$\mathcal{M}[g_y, g_z] \models \neg\tau^{g_y} E\tau^{g_z}$$

by condition 2. We have $\mathcal{M}[g_y, g_z] \prec_{\Sigma^1_{n+2}} V$ by lemma 1.2.27. Hence $\neg\tau^{g_y}E\tau^{g_z}$ as $\mathbb{R}^2 - E$ is $\Sigma^1_{n+3}(x)$. On the other hand the set $P := \{\tau^{g_y} : y \in \mathbb{R}\}$ is perfect since τ^{g_y} depends continuously on y by condition 2. This is a contradiction, since E is thin. ■

We will need

Lemma 3.2.9: *Let $n \leq k$ with $k \geq 1$ and suppose $M^{\#}_k(x)$ exists for every $x \in \mathbb{R}$. Let $\mathcal{N}$ be a countable active ω-sound ω_1-iterable $(k+1)$-small X-premouse with $\rho_\omega(\mathcal{N}) \leq \beta$, where X is swo. Let κ be the critical point of the top extender of $\mathcal{N}$ and $\mathcal{M} := \mathcal{N}|\kappa$. Let $\delta > \beta$ be the least Woodin cardinal in $\mathcal{N}$. Let $m < \omega$ be sufficiently large. Then there is a club $C \subseteq \delta$ which is uniformly definable in $\mathcal{M}$, so that for every $\gamma \in C$ we have for*

$$Y_\gamma := h^{\mathcal{M}}_{\Sigma_m}(V^{\mathcal{M}}_\gamma)$$

and X_γ its transitive collapse that

1. $X_\gamma \models$ "γ *is the least Woodin cardinal*" *and*

2. $X_\beta \lhd X_\gamma \lhd \mathcal{M}$ *for all* $\beta \in C \cap \gamma$.

Proof: We define a sequence $(\gamma_\alpha : \alpha < \delta)$ by induction and then set

$$C := \{\gamma_\alpha : \alpha < \delta\}.$$

Note that $\Sigma_1^{(m-1)}$ coincides with Σ_m over $\mathcal{M}$ since $\mathcal{M}$ is ω-sound and is a model of ZF. To define γ_0 let $Y^0 := h^{\mathcal{M}}_{\Sigma_1^{(m-1)}}(\emptyset)$ via the canonical Skolem functions. Let

$$Y^{i+1} := h^{\mathcal{M}}_{\Sigma_1^{(m-1)}}(V^{\mathcal{M}}_{\sup(Y^i \cap \delta)+1})$$

for $i < \omega$ and define

$$\gamma_0 := \sup(\bigcup_{i<\omega} Y^i \cap \delta).$$

We have $\gamma_0 < \delta$ since δ is inaccessible in $\mathcal{M}$ and further

$$Y_{\gamma_0} = \bigcup_{i<\omega} Y^i = h^{\mathcal{M}}_{\Sigma_1^{(m-1)}}(V^{\mathcal{M}}_{\gamma_0}).$$

To define $\gamma_{\alpha+1}$ in the successor step start with $Y^0 := h^{\mathcal{M}}_{\Sigma_1^{(m-1)}}(V^{\mathcal{M}}_{\sup(Y_{\gamma_\alpha} \cap \delta)+1})$. Let

$$Y^{i+1} := h^{\mathcal{M}}_{\Sigma_1^{(m-1)}}(V^{\mathcal{M}}_{\sup(Y^i \cap \delta)+1})$$

for $i < \omega$ and let

$$\gamma_{\alpha+1} := \sup(\bigcup_{i<\omega} Y^i \cap \delta).$$

Again we have

$$Y_{\gamma_{\alpha+1}} = h^{\mathcal{M}}_{\Sigma_1^{(m-1)}}(V^{\mathcal{M}}_{\gamma_{\alpha+1}}).$$

For limits $\mu < \delta$ define $\gamma_\mu := \sup_{\alpha<\mu} \gamma_\alpha$, so that $Y_{\gamma_\mu} = h^{\mathcal{M}}_{\Sigma_1^{(m-1)}}(V^{\mathcal{M}}_{\gamma_\mu})$ as well. Since δ in inaccessible in $\mathcal{M}$, we have $\gamma_\alpha < \delta$ for each $\alpha < \delta$.

Let X_{γ_α} be the transitive collapse of Y_{γ_α} for $\alpha < \delta$ and let $\sigma_\alpha : X_{\gamma_\alpha} \to Y_{\gamma_\alpha}$ be the uncollapsing map. Then γ_α is the least Woodin cardinal in X_{γ_α}, since $\pi_\alpha(\gamma_\alpha) = \delta$. The construction ensures that $J^{\vec{F}}_{\gamma_\alpha} = V^{\mathcal{M}}_{\gamma_\alpha}$ for each $\alpha < \delta$ where $\vec{F} = \vec{F}^{\mathcal{M}}$.

Moreover X_{γ_α} is m-sound above γ_α, since Y_{γ_α} is the $\Sigma_1^{(m-1)}$-hull of γ_α in $\mathcal{M}$. It is clear that $crit(\sigma_\alpha) = \gamma_\alpha$ and $\rho_m(X_{\gamma_\alpha}) = \gamma_\alpha$. Hence the condensation lemma can be applied, see [54, theorem 5.5.1] and [30, theorem 8.2]. We have $F^{\mathcal{M}}_{\gamma_\alpha} = \emptyset$ since γ_α is a cardinal. So the case that X_{γ_α} is an ultrapower of an initial segment of $\mathcal{M}$ by F_{γ_α} can be ruled out. Hence $X_{\gamma_\alpha} \trianglelefteq \mathcal{M}$. One can now conclude that $X_{\gamma_\alpha} \triangleleft X_{\gamma_\beta} \triangleleft \mathcal{M}$ for all $\alpha < \beta < \delta$. ■

The previous lemma will also be used in the proof of the main lemma in section 4.1. For the next theorem we actually only need a single element of the club in the lemma.

Hjorth [12, lemma 2.5] proved the next theorem for $n = 1$:

Theorem 3.2.10: *Let $n \geq 1$ and suppose $M^{\#}_{2n-1}(x)$ exists for every $x \in \mathbb{R}$. Then every thin $\Sigma^1_{2n}(r)$ equivalence relation is Π^1_{2n} in any real coding $M^{\#}_{2n-1}(r)$, for $r \in \mathbb{R}$.*

Proof: Let E be a thin $\Sigma^1_{2n}(r)$ equivalence relation. Define $\mathcal{M} := M^{\#}_{2n-1}(r)$ and let δ be the least Woodin cardinal in $\mathcal{M}$. Let η and τ be $\mathbb{W}_\delta$-names in $\mathcal{M}$ such that $\Vdash^{\mathcal{M}}_{\mathbb{W}_\delta} \dot{x} = \eta \oplus \tau$, where $\dot{x}$ is a name for the $\mathbb{W}_\delta$-generic real. Then the set D of conditions $p \in \mathbb{W}^{\mathcal{M}}_\delta$ with

$$(p,p) \Vdash^{\mathcal{M}}_{\mathbb{W}_\delta \times \mathbb{W}_\delta} \eta E \eta' \wedge \tau E \tau'$$

is dense in $\mathbb{W}^{\mathcal{M}}_\delta$ by lemma 3.2.8. Let κ be the critical point of the top extender of $\mathcal{M}$. Let's choose some $\gamma \in C$ where $C \subseteq \delta$ is the club from the previous lemma. Let X_γ be the corresponding initial segment of $\mathcal{M}$ with uncollapsing map $\sigma : X_\gamma \to Y_\gamma$ and $\sigma(\bar{D}) = D$.

We claim that any two reals x and y are E-inequivalent if and only if there are

1. reals x' and y' and
2. an iteration tree on $\mathcal{M}$ living on $\mathcal{M}|\gamma$ according to Σ with iteration map $\pi : \mathcal{M} \to \mathcal{N}$

such that

1. $x' \oplus y'$ is $\mathbb{W}_{\pi(\gamma)}$-generic over $\mathcal{N}$,

2. $\mathcal{N}[x', y'] \models \neg x' E y'$, and

3. xEx' and yEy'.

Condition 2 is equivalent to $\neg x'Ey'$ since $\mathcal{N}[x', y'] \prec_{\Sigma^1_{2n}} V$ by lemma 1.2.29. So these conditions imply that $\neg xEy$.

On the other hand suppose $\neg xEy$. Let $\mathcal{T}$ be a countable iteration tree on $M^{\#}_{2n-1}(x)$ living below γ with iteration map $\pi : \mathcal{M} \to \mathcal{N}$ such that $x \oplus y$ is $\mathbb{W}_{\pi(\gamma)}$-generic over $\pi(X)$ by lemma 1.2.26. Since $\bar{D}$ is dense in $\mathbb{W}^{X_\gamma}_{\gamma}$, there is a condition $p \in \pi(\bar{D})$ such that $x \oplus y$ is $\mathbb{W}_{\pi(\gamma)} \restriction p$-generic over $\pi(X)$.

Now let $x' \oplus y'$ be $\mathbb{W}_{\pi(\gamma)} \restriction p$-generic over both $\pi(X)[x, y]$ and over $\mathcal{N}$. We have

$$\pi(X)[x, x'] \models xEx'$$

and

$$\pi(X)[y, y'] \models yEy',$$

since this is forced by (p, p). Then lemma 1.2.27 shows that xEx' and yEy' hold, since there are $2n-2$ Woodin cardinals in $\pi(X)[x, x']$ and in $\pi(X)[y, y']$. Thus $\neg x'Ey'$ and hence y, y', and π satisfy conditions 1, 2, and 3.

It remains to show that the existence of y, y', and π satisfying conditions 1, 2, and 3 is a Σ^1_{2n} statement in any real coding $\mathcal{M}$. It therefore suffices to know that the statement that $\mathcal{T}$ is an iteration tree living on $\mathcal{M}|\gamma$ according to Σ is Σ^1_{2n} in $\mathcal{M}$. The point is that $\mathcal{T}$ is such an iteration tree if and only if the corresponding iteration tree on X_γ is according to Σ, since both strategies choose the unique branch with a $\mathcal{Q}$-structure. Since X_γ is $(2n-1)$-small, the $\mathcal{Q}$-structures for X_γ are $(2n-2)$-small. So they can be identified by a Π^1_{2n-1} formula by lemma 1.2.30. Hence the statement that $\mathcal{T}$ is according to Σ is Σ^1_{2n} in any real coding $\mathcal{M}$. ∎

Note that Harrington and Sami [8, theorem 5] proved that every thin $\boldsymbol{\Sigma}^1_{2n}$ equivalence relation is $\boldsymbol{\Delta}^1_{2n}$ and every thin $\boldsymbol{\Pi}^1_{2n+1}$ equivalence relation is $\boldsymbol{\Delta}^1_{2n+1}$ from PD, without identifying the parameters.

3.3 Equivalence relations in $L(\mathbb{R})$

In this section the result in the previous section is extended to equivalence relations which are Σ_1-definable over certain initial segments of $L(\mathbb{R})$. Woodin has constructed higher level analogues of $M_n^\#$ which can be used to analyze such sets. Assume $\mathsf{AD}^{L(\mathbb{R})}$ throughout this section, except in the last lemma.

We fix ordinals α and β with

$$\alpha = \sup(\{\gamma < \beta : \gamma \text{ is critical}\}) < \beta$$

such that either $[\alpha, \beta]$ is a weak Σ_1-gap, or $\beta - 1$ exists and $[\alpha, \beta - 1]$ is a strong Σ_1-gap. Here γ is critical if and only if there is some set $A \subseteq \mathbb{R}$ such that both A and $\mathbb{R} - A$ have a scale in $J_{\gamma+1}(\mathbb{R})$, but A does not have a scale in $J_\gamma(\mathbb{R})$. For the definition of weak and strong Σ_1-gaps see [45, definition 3.2]. We will consider the pointclass of $\Sigma_1(J_\alpha(\mathbb{R}))$ sets of reals. The largest such α is $(\underset{\sim}{\delta}_1^2)^{L(\mathbb{R})}$. In this case one has the pointclass of $\Sigma_1(L(\mathbb{R}))$ sets of reals by [45, lemma 1.12].

3.3.1 Weak term condensation

This section discusses results of Woodin about suitable premice. A more detailed account can be found in Schindler and Steel [40] and a slightly different approach in Steel [47]. Recall that the height of a self-wellordered set X is defined as $ht(X) := \sup((Ord \cap tc(X)) \cup \omega)$.

Definition 3.3.1: *Let X be a bounded subset of ω_1. The lower-part model $Lp^\alpha(X)$ is defined as the the union of all X-premice $\mathcal{M}$ with $\rho_\omega(\mathcal{M}) \leq \sup(ht(X))$ such that $J_\alpha(\mathbb{R}) \models$ "$\mathcal{M}$ is ω_1-iterable".*

Definition 3.3.2: *Let X be a bounded subset of ω_1. An X-premouse $\mathcal{M}$ is called suitable if*

1. *there is a unique Woodin cardinal $\delta = \delta^\mathcal{M}$ in $\mathcal{M}$,*
2. *$\mathcal{M} = \bigcup_{n<\omega} \mathcal{M}_n$ where $\mathcal{M}_0 := \mathcal{M}|\delta$ and $\mathcal{M}_{n+1} := Lp^\alpha(\mathcal{M}_n)$ for $n < \omega$, and*
3. *if $\gamma < \delta$ is a cardinal in $\mathcal{M}$, then $Lp^\alpha(\mathcal{M}|\gamma) \models$ "γ is not Woodin".*

Note that there are exactly ω many cardinals above $\delta^\mathcal{M}$ in $\mathcal{M}$ by condition 2.

Definition 3.3.3: *Let $\mathcal{M}$ be a suitable X-premouse, where X is a bounded subsetq of ω_1. An ω_1-iteration strategy Σ for $\mathcal{M}$ is called fullness-preserving if for every iteration tree $\mathcal{T}$ on $\mathcal{M}$ according to Σ which lives on $\mathcal{M}|\delta^\mathcal{M}$,*

1. $\mathcal{N}$ *is suitable if* $[0,\gamma]_\mathcal{T}$ *does not drop, and*

2. $J_\alpha(\mathbb{R}) \models$ *"$\mathcal{N}$ is ω_1-iterable" if* $[0,\gamma]_\mathcal{T}$ *drops,*

where $\mathcal{T}$ *has length* γ *and last model* $\mathcal{N}$*.*

Definition 3.3.4: *Let* Σ *be an* ω_1*-iteration strategy for a suitable* X*-premouse* $\mathcal{M}$*, where* X *is a bounded subset of* ω_1*. Let* $A \subseteq \mathbb{R}$ *and let* $\tau_A \in \mathcal{M}$ *be a* $\mathbb{P}$*-name where* $\mathbb{P} \in \mathcal{M}$ *is a forcing. Then* τ_A *captures* A *over* $\mathcal{M}$ *with respect to* $\mathbb{P}$ *and* Σ *if*

$$\pi(\tau_A)^g = A \cap \mathcal{N}[g]$$

for every non-dropping iteration tree according to Σ *with map* $\pi : \mathcal{M} \to \mathcal{N}$ *and for every* $\pi(\mathbb{P})$*-generic filter* g *over* $\mathcal{N}$*.* τ_A *is called a capturing term for* A*.*

Theorem 3.3.5: *(Woodin) Let* $x \in \mathbb{R}$ *and let* $A \subseteq \mathbb{R}$ *be* $\Sigma_1(J_\alpha(\mathbb{R}))$ *in the parameter* $x \in \mathbb{R}$*. Then there are*

1. *a suitable* X*-premouse* $\mathcal{M}$ *and*

2. *a fullness-preserving* ω_1*-iteration strategy* Σ *for* $\mathcal{M}$ *in* $L(\mathbb{R})$

such that for every cardinal $\mu \geq \delta^\mathcal{M}$ *in* $\mathcal{M}$ *and every forcing* $\mathbb{P}$ *of size* $< \mu$ *in* $\mathcal{M}$*, there is a* $Col(\omega,\mu) \times \mathbb{P}$*-name* $\tau \in \mathcal{M}$ *which captures* A *over* $\mathcal{M}$ *with respect to* Σ*.*

A weak capturing property is retained in substructures of the premouse in the previous theorem:

Lemma 3.3.6: *Let* $(\leq_n : n < \omega)$ *be a* $\Sigma_1(J_\alpha(\mathbb{R}))$*-scale on a* $\Sigma_1(J_\alpha(\mathbb{R}))$ *set* A *in the parameter* $x \in \mathbb{R}$*. Let* $\mathcal{N}$ *be a suitable* X*-premouse, where* X *is a bounded subset of* ω_1*. Let* $\mathbb{P}$ *be a homogeneous forcing of size* μ *in* $\mathcal{N}$ *and* $\mu^{+\mathcal{N}} \leq \lambda \leq Ord^\mathcal{N}$*. Suppose* τ_A *and* τ *are* $\mathbb{P}$*-names so that*

1. τ_A *captures* A *and*

2. τ *captures* $(\leq_n : n < \omega)$

over $\mathcal{N}$*. Suppose further that* $m < \omega$ *is sufficiently large and* $\pi : \mathcal{M} \to \mathcal{N}|\lambda$ *is* Σ_m*-elementary with* $\pi(\bar{\mathbb{P}}) = \mathbb{P}$*,* $\pi(\bar{\tau}_A) = \tau_A$*, and* $\tau \in rng(\pi)$*. Then* $\bar{\tau}_A^g \subseteq A$ *for every* $\bar{\mathbb{P}}$*-generic filter* g *over* $\mathcal{N}$*.*

Proof: For any $\mathbb{P}$-generic filter g over $\mathcal{N}$, let U_g be the tree from the scale $(\leq_n : n < \omega)$ calculated in $\mathcal{N}[g]$. Thus

$$U_g = \{(x|n, (rank_0^{\mathcal{N}[g]}(x), .., rank_{n-1}^{\mathcal{N}[g]}(x))) : x \in A \cap \mathcal{N}[g], n < \omega\}.$$

Then U_g can be defined in $\mathcal{N}$, since $\mathbb{P}$ is homogeneous and A and $(\leq_n : n < \omega)$ are captured by τ_A and τ over $\mathcal{N}$. Hence $U := U_g$ does not depend on g. In fact we have $U \in \mathcal{N}|\lambda$. Moreover U and the canonical $\mathbb{P}$-name $\check{U}$ are in $rng(\pi)$ since they are defined from $\mathbb{P}$, τ_A, and τ.

We have

$$\Vdash_{\mathbb{P}}^{\mathcal{N}} \tau_A \subseteq p[\check{U}]$$

since

$$\tau_A^g = A \cap \mathcal{N}[g] \subseteq p[U]$$

for any $\mathbb{P}$-generic filter g over $\mathcal{N}$. Hence

$$\Vdash_{Col(\omega,\gamma)}^{\mathcal{M}} \bar{\tau}_A \subseteq p[\pi^{-1}(\check{U})]$$

by elementarity of π. So $\bar{\tau}_A^g \subseteq p[\pi^{-1}(U)]$ for any $\mathbb{P}$-generic filter g over $\mathcal{M}$. We also have $p[\pi^{-1}(U)] \subseteq p[U]$ and moreover $p[U] \subseteq A$ by the semicontinuity of the scale. Hence

$$\bar{\tau}_A^g \subseteq p[\pi^{-1}(U)] \subseteq p[U].$$

■

One has a similar property for iterates:

Lemma 3.3.7: *(Weak term condensation) Let* $(\leq_n : n < \omega)$ *be a* $\Sigma_1(J_\alpha(\mathbb{R}))$*-scale on a* $\Sigma_1(J_\alpha(\mathbb{R}))$ *set* A *in the parameter* $x \in \mathbb{R}$*. Let* $\mathcal{N}$ *be a suitable* X*-premouse with* ω_1*-iteration strategy* Σ*, where* X *is a bounded subset of* ω_1*. Let* $\mathbb{P}$ *be a homogeneous forcing of size* μ *in* $\mathcal{N}$ *and* $\mu^{+\mathcal{N}} \leq \lambda \in Ord^{\mathcal{N}}$*. Suppose* τ_A *and* τ *are* $\mathbb{P}$*-names so that*

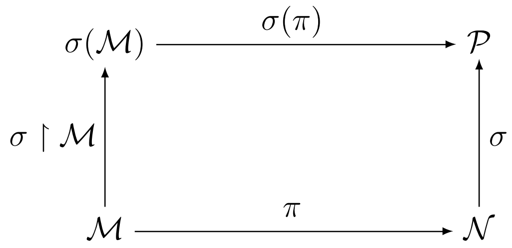
1. τ_A *captures* A *and*

2. τ *captures* $(\leq_n : n < \omega)$

over $\mathcal{N}$ *with respect to* Σ*. Let* $m < \omega$ *be sufficiently large and* $\pi : \mathcal{M} \to \mathcal{N}|\lambda$ *elementary with* $\pi(\bar{\mathbb{P}}) = \mathbb{P}$*,* $\pi(\bar{\tau}_A) = \tau_A$*, and* $\tau \in rng(\pi)$*. Let further* $\sigma : \mathcal{N} \to \mathcal{P}$

be the map from an iteration tree according to Σ along a branch which does not drop. Then $\sigma(\bar{\tau}_A)^g \subseteq A$ for every $\sigma(\bar{\mathbb{P}})$-generic filter g over $\sigma(\mathcal{M})$.

Proof: The diagram



commutes. Apply the previous lemma to $\sigma(\mathcal{M})$ and $\mathcal{P}$. ∎

3.3.2 $\mathbf{\Pi}_1(J_\alpha(\mathbb{R}))$ and $\mathbf{\Sigma}_1(J_\alpha(\mathbb{R}))$ equivalence relations

In this section we show that every thin $\Sigma_1(J_\alpha(\mathbb{R}))$ equivalence relation is $\Pi_1(J_\alpha(\mathbb{R}))$ in a real coding a suitable premouse. We will use

Lemma 3.3.8: *Assume $\mathsf{AD}^{L(\mathbb{R})}$ and suppose Σ is an ω_1-iteration strategy in $L(\mathbb{R})$ for an X-premouse $\mathcal{M}$, where $X \in HC$ is swo. Then Σ is an $\omega_1 + 1$-iteration strategy.*

Proof: See [48, lemma 7.11]. Let $\mathcal{T}$ be an iteration tree on $\mathcal{M}$ of length ω_1 in $L(\mathbb{R})$ according to Σ. Let j be the ultrapower map from the club ultrafilter on ω_1. Then $j \restriction L[\mathcal{T}]$ is elementary since $L[\mathcal{T}] \models \mathsf{ZFC}$. Hence $j(\mathcal{T})$ is an iteration tree of length $> \omega_1$ extending $\mathcal{T}$. ∎

Thus the comparison lemma, the genericity iteration, and the condensation lemma hold for suitable ω_1-iterable X-premice, where $X \in HC$ is swo.

Suppose $\mathbb{P}$ and $\mathbb{Q}$ are forcings and τ is a $\mathbb{P}$-name for a real. Then τ defines two reals in any $\mathbb{P} \times \mathbb{P} \times \mathbb{Q}$-generic extension via the $\mathbb{P}$-generic filters. We will write τ and τ' for $\mathbb{P} \times \mathbb{P} \times \mathbb{Q}$-names for these reals.

Lemma 3.3.9: *Let E be a thin equivalence relation which is $\Sigma_1(J_\alpha(\mathbb{R}))$ in the parameter $x \in \mathbb{R}$. Let $\mathcal{M}$ be a suitable X-premouse with $x \in \mathcal{M}$, where X is a bounded subset of ω_1. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcings in $\mathcal{M}$ and τ_E a $\mathbb{P} \times \mathbb{P} \times \mathbb{Q}$-name* *which captures E over $\mathcal{M}$. Then for every $\mathbb{P}$-name $\tau \in \mathcal{M}$ for a real, the set D of conditions $p \in \mathbb{P}$ with*

$$(p, p, \mathbb{1}) \Vdash^{\mathcal{M}}_{\mathbb{P}\times\mathbb{P}\times\mathbb{Q}} (\tau, \tau') \in \tau_E$$

is dense in $\mathbb{P}$.

Proof: Suppose D is not dense. In this case let $p_\emptyset$ be a condition such that for every $q \leq p_\emptyset$ there are conditions $r, u \leq q$ with

$$(r, u, \mathbb{1}) \Vdash^{\mathcal{M}}_{\mathbb{P}\times\mathbb{P}\times\mathbb{Q}} (\tau, \tau') \notin \tau_E.$$

Let $(D_i : i < \omega)$ enumerate the dense open subsets of $\mathbb{P} \times \mathbb{P}$ in $\mathcal{M}$. As in the proof of lemma 3.2.8 one can inductively construct a family $(p_s : s \in 2^{<\omega})$ of conditions in $\mathbb{P}$ so that for all $s, t \in 2^{<\omega}$ we have

1. $p_s \leq p_t$ if $t \subseteq s$,
2. $(p_{s\frown 0}, p_{s\frown 1}, \mathbb{1}) \Vdash^{\mathcal{M}}_{\mathbb{P}\times\mathbb{P}\times\mathbb{Q}} (\tau, \tau') \notin \tau_E$,
3. p_s decides $\tau \restriction lh(s)$, and
4. $(p_s, p_t) \in D_0 \cap .. \cap D_i$ if $s, t \in {}^i2$ and $s \neq t$.

Moreover let

$$g_y := \{p \in \mathbb{P} : \exists n < \omega (p_{y\restriction n} \leq p)\}$$

for each $y \in \mathbb{R}$. Then $g_y \times g_z$ is $\mathbb{P} \times \mathbb{P}$-generic over $\mathcal{M}$ for any $y, z \in \mathbb{R}$ with $y \neq z$ by condition 4. Now for any $\mathbb{Q}$-generic filter g over $\mathcal{M}[g_y \times g_z]$ we have

$$\tau_E^{g_y\times g_z\times g} = E \cap \mathcal{M}[g_y \times g_z \times g],$$

since τ_E is a capturing term for E. Moreover

$$\mathcal{M}[g_y \times g_z \times g] \models (\tau^{g_y}, \tau^{g_z}) \notin \tau_E^{g_y\times g_z\times g}$$

holds by condition 2 and hence $(\tau^{g_y}, \tau^{g_z}) \notin E$. But the set $\{\tau^{g_y} : y \in \mathbb{R}\}$ is perfect, since τ^{g_y} depends continuously on y by condition 3. This is a contradiction since E is thin. ∎

Lemma 3.3.10: *Let $\mathcal{M}$ be a suitable X-premouse, where X is a bounded subset of ω_1. Let $\mu > \delta^{\mathcal{M}}$ be a cardinal in $\mathcal{M}$ and let $a \in \mathcal{N}|\mu$. Let $k < \omega$ be sufficiently large. There is an ordinal $\bar{\delta} < \delta^{\mathcal{M}}$ such that for $Z := h^{\mathcal{M}|\mu}_{\Sigma_k}(V^{\mathcal{M}}_{\bar{\delta}} \cup \{a\})$ with uncollapsing map $\zeta : Y \to Z$ we have $\zeta(\bar{\delta}) = \delta^{\mathcal{M}}$ and $Y \lhd \mathcal{M}$.*

Proof: As in lemma 3.2.9. The point is that the construction yields $\bar{\delta} < \delta^{\mathcal{M}}$ since $\rho(\mathcal{M}|\mu) = \infty$ by acceptability. ■

With these two lemmas we can show

Theorem 3.3.11: *Let E be a thin equivalence relation which is $\Sigma_1(J_\alpha(\mathbb{R}))$ in some parameter $x \in \mathbb{R}$. Then there is a bounded set $X \subseteq \omega_1$ and a suitable X-premouse $\mathcal{M}$ so that E is $\Pi_1(J_\alpha(\mathbb{R}))$ in any real coding $\mathcal{M}$.*

Proof: Let $\mathcal{M}$ be a suitable X-premouse with ω_1-iteration strategy Σ and σ_E a $\mathbb{W}_\delta \times \mathbb{W}_\delta \times Col(\omega, \delta^{+\mathcal{M}})$ -name which captures E over $\mathcal{M}$ with respect to Σ. Let's also assume there is a capturing term in $\mathcal{M}$ for a scale on E. Define $\mu := \delta^{+\mathcal{M}}$. Let further η and τ be $\mathbb{W}_\delta$-names for reals so that $\Vdash^{\mathcal{M}}_{\mathbb{W}_\delta} \dot{x} = \eta \oplus \tau$, where $\dot{x}$ is a name for the $\mathbb{W}_\delta$-generic real. Then the set D of conditions $p \in \mathbb{W}_\delta$ with

$$(p, p, \mathbb{1}) \Vdash^{\mathcal{M}}_{\mathbb{W}_\delta \times \mathbb{W}_\delta \times Col(\omega,\mu)} (\eta, \eta') \in \sigma_E \wedge (\tau, \tau') \in \sigma_E$$

is dense by lemma 3.3.9.

Now let $k < \omega$ be sufficiently large. Let $\bar{\delta} < \delta^{\mathcal{M}}$ and $Z \prec_{\Sigma_k} \mathcal{N}$ as in the previous lemma, so that Z contains everything relevant. Let further $\zeta : Y \to Z$ be the uncollapsing map and $\zeta(\bar{\mu}) = \mu$, $\zeta(\bar{\sigma}_E) = \sigma_E$, $\zeta(\bar{\eta}) = \eta$, $\zeta(\bar{\tau}) = \tau$. There is a $\mathbb{W}_{\bar{\delta}} \times \mathbb{W}_{\bar{\delta}} \times Col(\omega, \mu)$-name τ_E which captures E over $\mathcal{M}$, since the forcings $\mathbb{W}_\delta \times \mathbb{W}_\delta \times Col(\omega, \mu)$ and $\mathbb{W}_{\bar{\delta}} \times \mathbb{W}_{\bar{\delta}} \times Col(\omega, \mu)$ are equivalent.

Let $x, y \in \mathbb{R}$. We claim that $\neg xEy$ if and only if there are

1. reals x' and y' and
2. a map $\pi : \mathcal{M} \to \mathcal{N}$ from an iteration tree on $\mathcal{M}$ living on $\mathcal{M}|\bar{\delta}$ according to Σ

such that

1. $x' \oplus y'$ is $\mathbb{W}_{\pi(\bar{\delta})}$-generic over $\mathcal{N}$,

2. $\mathbb{1} \Vdash^{\mathcal{N}[x',y']}_{Col(\omega,\pi(\mu))} (x',y') \notin \tau_E$, and

3. xEx' and yEy'.

Condition 2 is equivalent to $\neg x'Ey'$ since τ_E captures E over $\mathcal{N}$. So conditions 1, 2, and 3 imply $\neg xEy$.

On the other hand suppose $\neg xEy$. Let $\mathcal{T}$ be an iteration tree on $\mathcal{M}$ living on $\mathcal{M}|\bar{\delta}$ according to Σ with iteration map $\pi : \mathcal{M} \to \mathcal{N}$ such that $x \oplus y$ is $\mathbb{W}_{\pi(\delta)}$-generic over $\pi(X)$ by lemma 1.2.14. Let $p \in \bar{D}$ be a condition such that $x \oplus y$ is $\mathbb{W}_{\pi(\gamma)} \restriction \pi(p)$-generic over $\pi(X)$. Let $x' \oplus y'$ be $\mathbb{W}^{\mathcal{N}}_{\pi(\gamma)} \restriction \pi(p)$-generic over both $\pi(X)[x \oplus y]$ and $\mathcal{N}$. We have

$$(p, p, \mathbb{1}) \Vdash^{X}_{\mathbb{W}_{\bar{\delta}} \times \mathbb{W}_{\bar{\delta}} \times Col(\omega,\bar{\mu})} (\bar{\eta}, \bar{\eta}') \in \pi(\bar{\sigma}_E) \wedge (\bar{\tau}, \bar{\tau}') \in \pi(\bar{\sigma}_E)$$

by elementarity of ζ, and

1. $(\pi(p), \pi(p), \mathbb{1}) \Vdash^{\pi(X)}_{\mathbb{W}_{\pi(\bar{\delta})} \times \mathbb{W}_{\pi(\bar{\delta})} \times Col(\omega,\pi(\bar{\mu}))} (\pi(\bar{\eta}), \pi(\bar{\eta}')) \in \pi(\bar{\sigma}_E)$ and

2. $(\pi(p), \pi(p), \mathbb{1}) \Vdash^{\pi(X)}_{\mathbb{W}_{\pi(\bar{\delta})} \times \mathbb{W}_{\pi(\bar{\delta})} \times Col(\omega,\pi(\bar{\mu}))} (\pi(\bar{\tau}), \pi(\bar{\tau}')) \in \pi(\bar{\sigma}_E)$

by elementarity of π. Hence xEx' and yEy' hold by lemma 3.3.7, since $x \oplus x'$ and $y \oplus y'$ are $\mathbb{W}_{\pi(\gamma)} \restriction p$-generic over $\pi(X)$. So $\neg x'Ey'$ and thus the conditions 1, 2, and 3 hold.

Moreover, the ω_1-iteration strategy for iteration trees living on $\mathcal{M}|\bar{\delta}$ is given by $\mathcal{Q}$-structures which have ω_1-iteration strategies in $J_\alpha(\mathbb{R})$. Hence the existence of x', y', and π satisfying conditions 1, 2, and 3 is $\Sigma_1(J_\alpha(\mathbb{R}))$ in any real which codes $\mathcal{M}$. ■

In the rest of this section we give an alternative proof of theorem 3.3.11 for thin $\mathbf{\Sigma}_1(L(\mathbb{R}))$ equivalence relations. Note that $\mathbf{\Sigma}_1(L(\mathbb{R})) = (\mathbf{\Sigma}^2_1)^{L(\mathbb{R})}$ by [45, lemma 1.12]. Although this is entirely covered by theorem 3.3.11, the proof is simpler in the sense that it avoids Woodin's theory of suitable premice. We work with $M^\#_\omega$ instead.

Theorem 3.3.12: *(Steel) Let $\mathcal{M}$ be a countable $\omega_1 + 1$-iterable ω-sound X-premouse, where X is swo. Suppose λ is a limit of Woodin cardinals in $\mathcal{M}$. Then in any $Col(\omega, \mathbb{R})$-generic extension of V there is an iteration map $\pi : \mathcal{M} \to \mathcal{N}$ and a $Col(\omega, < \pi(\lambda))$-generic filter g over $\mathcal{N}$ with $\mathbb{R}^V = \mathbb{R}^{\mathcal{N}[g]}$.*

PROOF: See [48, theorem 7.19]. ■

Lemma 3.3.13: *If $M_\omega^\#(x)$ exists and is ω_1+1-iterable for some $x \in \mathbb{R}$, then* $\mathsf{AD}^{L(\mathbb{R})}$ *holds.*

PROOF: Let λ be the supremum of the Woodin cardinals in $M_\omega^\#(x)$. Then AD holds in $L(\mathbb{R}^*)$, where

$$\mathbb{R}^* := \bigcup_{\alpha<\lambda} \mathbb{R}^{M_\omega^\#(x)[g\cap Col(\omega,\alpha)]}$$

and g is $Col(\omega,<\lambda)$-generic over $M_\omega^\#(x)$ by a theorem of Woodin, see [32, theorem 3.1]. Together with lemma 3.3.12 this implies that AD holds in $L(\mathbb{R})$. ■

Lemma 3.3.14: *Suppose $M_\omega^\#(x)$ exists and is ω_1+1-iterable for some $x \in \mathbb{R}$. Let E be a thin equivalence relation definable from x in $L(\mathbb{R})$. Let λ be the supremum of the Woodin cardinals in $M_\omega^\#(x)$ and let $\dot{\mathbb{R}}$ be the canonical $Col(\omega,<\lambda)$-name for*

$$\mathbb{R}^* := \bigcup_{\alpha<\lambda} \mathbb{R}^{V[G\cap Col(\omega,\alpha)]},$$

where G is a $Col(\omega,<\lambda)$-generic filter over V. Suppose $\mathbb{P}$ is a forcing of size $<\lambda$ in $M_\omega^\#(x)$. Then for every $\mathbb{P}$-name $\tau \in M_\omega^\#(x)$ for a real, the set D of conditions $p \in \mathbb{P}$ with

$$(p,p,\mathbb{1}) \Vdash^{M_\omega^\#(x)}_{\mathbb{P}\times\mathbb{P}\times Col(\omega,<\lambda)} \text{"} L(\dot{\mathbb{R}}) \models \tau E \tau' \text{"}$$

is dense in $\mathbb{P}$.

PROOF: We follow the proof of lemma 3.3.9. Suppose D is not dense. Then one can construct a family $(p_s : s \in 2^{<\omega})$ of conditions in $\mathbb{P}$ so that for all $s,t \in 2^{<\omega}$ we have

1. $p_s \leq p_t$ if $t \subseteq s$,
2. $(p_{s\frown 0}, p_{s\frown 1}, \mathbb{1}) \Vdash^{M_\omega^\#(x)}_{\mathbb{P}\times\mathbb{P}\times Col(\omega,<\lambda)} \text{"} L(\dot{\mathbb{R}}) \models \neg\tau E \tau' \text{"}$,
3. p_s decides $\tau \restriction lh(s)$, and
4. $(p_s,p_t) \in D_0 \cap .. \cap D_i$ if $s,t \in {}^i2$ and $s \neq t$.

We define

$$g_y := \{q \in \mathbb{P} : \exists n < \omega(p_{y|n} \leq q)\}$$

for each $y \in \mathbb{R}$. Then the set of all τ^{g_y} is perfect. Moreover, $g_y \times g_z$ is $\mathbb{P} \times \mathbb{P}$-generic over $M_\omega^\#(x)$ for any $y, z \in \mathbb{R}$ with $y \neq z$ by condition 4. So for any $Col(\omega, < \lambda)$-generic filter g over $M_\omega^\#(x)[g_y \times g_z]$ and

$$\mathbb{R}^* := \bigcup_{\alpha<\lambda} \mathbb{R}^{M_\omega^\#(x)[g_y \times g_z][g \cap Col(\omega,\alpha)]}$$

we have

$$L(\mathbb{R}^*) \models \neg \tau^{g_y} E \tau^{g_z}$$

by condition 2. Hence $\neg \tau^{g_y} E \tau^{g_z}$ for any two $y, z \in \mathbb{R}$ with $y \neq z$ by theorem 3.3.12 applied to $M_\omega^\#(x)[g_y \times g_z]$. This is impossible since E is thin. ■

Lemma 3.3.15: *Suppose $M_\omega^\#(r)$ exists and is ω_1+1-iterable for some $r \in \mathbb{R}$. Let κ be the critical point of the top extender of $M_\omega^\#(r)$ and let $k < \omega$ be sufficiently large. Let δ be the least Woodin cardinal in $M_\omega^\#(r)$. Then there is $\bar\delta < \delta$ so that for $Y := h_{\Sigma_k}^{M_\omega^\#(r)|\kappa}(V_{\bar\delta}^{M_\omega^\#(r)})$ and the uncollapsing map $\zeta : X \to Y$ we have $\zeta(\bar\delta) = \delta$ and $X \lhd M_\omega^\#(r)$.*

Proof: As in lemma 3.2.9. ■

Theorem 3.3.16: *Suppose $M_\omega^\#(r)$ is ω_1+1-iterable for some $r \in \mathbb{R}$. Then every thin equivalence relation which is $\Sigma_1(L(\mathbb{R}))$ in r is $\Pi_1(L(\mathbb{R}))$ in any real coding $M_\omega^\#(r)$.*

Proof: We follow the proof of theorem 3.3.11. Let E be a thin equivalence relation which is $\Sigma_1(L(\mathbb{R}))$ in r. Let δ be the least Woodin cardinal in $M_\omega^\#(r)$ and λ the supremum of the Woodin cardinals in $M_\omega^\#(r)$. Let further η and τ be names such that $\Vdash_{\mathbb{W}_\delta}^{M_\omega^\#(r)} \dot{x} = \eta \oplus \tau$, where $\dot{x}$ is a name for the $\mathbb{W}_\delta$-generic real. Then the set D of conditions $p \in \mathbb{W}_\delta$ with

$$(p, p, \mathbb{1}) \Vdash_{\mathbb{W}_\delta \times \mathbb{W}_\delta \times Col(\omega,<\lambda)}^{M_\omega^\#(r)} \text{"} L(\dot{\mathbb{R}}) \models \eta E \eta' \wedge \tau E \tau' \text{"}$$

is dense in $\mathbb{W}_\delta$ by the previous lemma.

Let κ be the critical point of the top extender of $M_\omega^\#(r)$ and let $k < \omega$ be sufficiently large. Suppose that $\bar\delta < \delta$ and Y are chosen as in the previous lemma with uncollapsing map $\zeta : X \to Y$. Let further $\zeta(\bar\lambda) = \lambda$, $\zeta(\bar D) = D$, $\zeta(\bar\tau) = \tau$, and $\zeta(\bar\eta) = \eta$.

Let $x, y \in \mathbb{R}$. We claim that $\neg x E y$ if and only if there are

1. reals x' and y' and
2. a map $\pi : M_\omega^\#(r) \to \mathcal{N}$ from a countable iteration tree on $M_\omega^\#(r)$ living on $M_\omega^\#(r)|\bar{\delta}$

such that

1. $x' \oplus y'$ is $\mathbb{W}_{\pi(\gamma)}$-generic over $\mathcal{N}$,
2. $\mathbb{1} \Vdash^{M_\omega^\#(r)[x',y']}_{Col(\omega,<\pi(\lambda))}$ "$L(\dot{\mathbb{R}}) \models \neg x'Ey'$ ", and
3. xEx' and yEy'.

Condition 2 is equivalent to $\neg x'Ey'$ by lemma 3.3.12. Hence conditions 1, 2, and 3 imply that $\neg xEy$.

On the other hand suppose $\neg xEy$. Let $\pi : M_\omega^\#(r) \to \mathcal{N}$ be a map from an iteration tree on $M_\omega^\#(r)$ living on $M_\omega^\#(r)|\bar{\delta}$ such that $x \oplus y$ is $\mathbb{W}_{\pi(\bar{\delta})}$-generic over $\pi(X)$. Let $p \in \pi(\bar{D})$ such that $x \oplus y$ is $\mathbb{W}_{\pi(\bar{\delta})} \restriction p$-generic over $\pi(X)$. Moreover let $x' \oplus y'$ be $\mathbb{W}_{\pi(\bar{\delta})} \restriction p$-generic over both $\pi(X)[x, y]$ and $\mathcal{N}$. We have

1. $(p, p, \mathbb{1}) \Vdash^{\pi(X)}_{\mathbb{W}_{\pi(\bar{\delta})} \times \mathbb{W}_{\pi(\bar{\delta})} \times Col(\omega,<\pi(\bar{\lambda}))}$ "$L(\dot{\mathbb{R}}) \models \pi(\bar{\eta})E\pi(\bar{\eta}')$" and
2. $(p, p, \mathbb{1}) \Vdash^{\pi(X)}_{\mathbb{W}_{\pi(\bar{\delta})} \times \mathbb{W}_{\pi(\bar{\delta})} \times Col(\omega,<\pi(\bar{\lambda}))}$ "$L(\dot{\mathbb{R}}) \models \pi(\bar{\tau})E\pi(\bar{\tau}')$"

by elementarity of ζ and π.

Now in any $Col(\omega, \mathbb{R})$-generic extension of V there is an iteration map $\pi(X) \to \pi'(X)$ and a $Col(\omega, < \pi'(\bar{\lambda}))$-generic filter g over $\pi'(X)$ with $\mathbb{R}^V = \mathbb{R}^{\pi'(X)[g]}$ by lemma 3.3.7. Hence $L(\mathbb{R})^{\pi'(X)[g]} = L_\alpha(\mathbb{R})$ for some ordinal α. The previous formulas 1 and 2 imply that $L_\alpha(\mathbb{R}) \models xEx'$ and $L_\alpha(\mathbb{R}) \models yEy'$. Then $L(\mathbb{R}) \models xEx'$ and $L(\mathbb{R}) \models yEy'$ since E is $\Sigma_1(L(\mathbb{R}))$ in r.

It remains to show that E is $\Pi_1(L(\mathbb{R}))$ in any real coding $M_\omega^\#(r)$. Note that $\mathsf{AD}^{L(\mathbb{R})}$ holds by lemma 3.3.13. So the $\omega_1 + 1$-iteration strategy for $M_\omega^\#(r)$ restricted to iteration trees living on X is $\Sigma_1(L(\mathbb{R}))$ in the parameter $M_\omega^\#(r)$ by [48, lemma 7.9] and [48, theorem 7.10]. Hence the existence of an iteration map π which satisfies conditions 1, 2, and 3 is $\Sigma_1(L(\mathbb{R}))$ in any real coding $M_\omega^\#(r)$. ∎

Harrington's and Shelah's theorem 3.1.8 can be used to determine the number of equivalence classes of thin equivalence relations whose complement is in a scaled Σ-pointclass, see definition 1.1.12.

Lemma 3.3.17: *Let $\Gamma = \Sigma_n(J_\gamma(\mathbb{R}))$ be a scaled Σ-pointclass and suppose*

1. *$\Gamma(x)$ is scaled and*
2. *there is no ω_1-sequence of reals in $J_{\gamma^{+L(\mathbb{R})}}(\mathbb{R})$.*

Let E be a thin equivalence relation in $\breve{\Gamma}(x)$ where $x \in \mathbb{R}$ and let $(\leq_n : n < \omega)$ be a $\Gamma(x)$-scale on $\mathbb{R}^2 - E$ of length κ. Then the equivalence classes of E can be wellordered with order type $\leq \kappa$.

Proof: Let T be the tree from the scale $(\leq_n : n < \omega)$. Then T is in $J_{\gamma^{+L(\mathbb{R})}}(\mathbb{R})$ since $J_{\gamma^{+L(\mathbb{R})}} \models \mathsf{ZF}^-$. We further have

$$\mathbb{R} \cap L[T] = \mathbb{R} \cap J_{\gamma^{+L(\mathbb{R})}}[T]$$

by condensation since $\gamma^{+L(\mathbb{R})}$ is a cardinal in $L[T]$. Now $\mathbb{R} \cap L[T]$ is wellorderable in $J_{\gamma^{+L(\mathbb{R})}}(\mathbb{R})$ and hence countable. So there is a Cohen real over $L[T]$ in V. This implies that the equivalence classes of E can be wellordered in $L(\mathbb{R})$ with order type $\leq \kappa$ by corollary 3.1.13. ∎

Chapter 4

Inner models for thin equivalence relations

This chapter is devoted to inner models and thin equivalence relations. In the first section we study projective equivalence relations, while the second section is dedicated to equivalence relations in $L(\mathbb{R})$. We work in ZFC.

4.1 Projective equivalence relations

Let M be a transitive set or class. We consider the property that there are representatives in M for all equivalence classes of all thin $\mathbf{\Pi}^1_{2n}$ equivalence relations defined from a parameter in M. In this section we will characterize the inner models with this property. A first observation is that such a model contains at least $Card(\underset{\sim}{\delta}^1_{2n-1})$ many reals by theorem 3.2.4, assuming $\mathbf{\Pi}^1_{2n-1}$ determinacy. Among others we will show that such a model computes $\underset{\sim}{\delta}^1_{2n-1}$ correctly.

Hjorth [10, theorem 3.1] characterized the inner models with this property for thin $\mathbf{\Pi}^1_2$ equivalence relations:

Theorem 4.1.1: *(Hjorth [10]) Assume $x^{\#}$ exists for every $x \in \mathbb{R}$. The following are equivalent for an inner model M:*

1. *M has a representative in every equivalence class of every thin $\Pi^1_2(x)$ equivalence relation for any $x \in \mathbb{R} \cap M$,*
2. *(a) $M \prec_{\Sigma^1_3} V$ and*

   (b) $\omega_1^M = \omega_1^V$.

We extend this result to thin $\mathbf{\Pi}^1_{2n}$ equivalence relations for any $n \geq 1$. The level of correctness is adapted and ω_1 is replaced with the tree from the canonical Π^1_{2n+1} scale.

Main Theorem 4.1.2: *Suppose $n \geq 1$ and $M^{\#}_{2n-2}(x)$ exists for every $x \in \mathbb{R}$. Let $(\leq_k : k < \omega)$ denote the canonical scale on the complete Π^1_{2n-1} set C and let $\equiv_k$ denote the induced thin Σ^1_{2n-1} equivalence relations. The following are equivalent for any transitive model M of* ZF*:*

1. *every equivalence class of every thin $\Pi^1_{2n}(r)$ equivalence relation has a representative in M for all $r \in \mathbb{R} \cap M$,*
2. (a) *$M \prec_{\Sigma^1_{2n+1}} V$ and*
   (b) *$T^M_{2n-1} = T^V_{2n-1}$,*
3. (a) *$M \prec_{\Sigma^1_{2n+1}} V$ and*
   (b) *for some $k < \omega$ every equivalence class of $\equiv_k$ (except possibly the ∞ class $\mathbb{R} - C$) has a representative in M.*

Compared with the proof of theorem 4.1.1, the argument is substantially more involved and makes essential use of mice with Woodin cardinals. The proof of the implication from condition 3 to condition 1 hinges on the main lemma 4.1.15, which states that T_{2n+1} can be reconstructed in an iterate of $M^{\#}_{2n}$. To show this, one first produces local Woodin cardinals below the least Woodin cardinal in $M^{\#}_{2n}$ using lemma 3.2.9. Then an iteration tree is built by successively applying the genericity iteration to the local Woodin cardinals. A density argument shows that one can define T_{2n+1} in the last model of the iteration tree. To apply the main lemma 4.1.15 in the proof of the main theorem, we first find an infinitary formula by the theorem of Harrington and Shelah and then express the existence of a real satisfying this formula in a projective way via the main lemma.

Finally, we will construct a transitive model satisfying the properties of the main theorem 4.1.2, if CH holds or merely $\underset{\sim}{\delta}^1_{2n+1} < \omega_2$. The difference to the proof of the main lemma is that here we enumerate the reals with order type ω_1 and build a stack of countable iteration trees of height ω_1. In each step the next real in the enumeration is realized as a generic real over an initial segment of the iterate for the extender algebra over a local Woodin cardinal.

4.1.1 The main lemma

In this section we prove

Main Lemma 4.1.3: *Let $n \geq 1$ and assume $M_{2n}^{\#}(x)$ exists for every $x \in \mathbb{R}$. Suppose M is a transitive model of a sufficiently large finite fragment of* ZF*. Suppose $\mathbb{R} \cap M$ is countable and M calculates $M_{2n}^{\#}(x)$ correctly for each $x \in \mathbb{R} \cap M$. Let $r \in \mathbb{R} \cap M$ and let δ be the least Woodin cardinal in $M_{2n}^{\#}(r)$. There is a countable iteration tree on $M_{2n}^{\#}(r)$ with iteration map $\pi : M_{2n}^{\#}(r) \to \mathcal{N}$ so that T_{2n+1}^{M} is definable in $\mathcal{N}$ uniformly in the parameter $r \in \mathbb{R}$.*

We will build an iteration tree on $M_{2n}^{\#}(r)$ with last model $\mathcal{N}$ and reconstruct T_{2n+1}^{M} in a $Col(\omega, < \omega_1^M)$-generic extension of $\mathcal{N}$. Let M, r, δ, and n be as in the main lemma for the rest of this section.

Claim 4.1.4: $M \prec_{\Sigma_{2n+1}^1} V$.

Proof: Since every $x \in \mathbb{R} \cap M$ is generic over some iterate of $M_{2n}^{\#}(r)$ for the extender algebra at the least Woodin cardinal by lemma 1.2.26, one can construct $L[\vec{E}, x]$ in this generic extension up to the critical point of the top extender. We then construct the $M_{2n-1}^{\#}(x)$ of both M and V by attaching the top extender of the iterate of $M_{2n}^{\#}(r)$ on top of this model. Now the claim follows from lemma 2.1.5. ■

Let $C \subseteq \delta$ be the club from lemma 3.2.9 applied to $M_{2n}^{\#}(r)$. Now the set

$$S := \{\gamma \in C : M_{2n}^{\#}(r) \models \gamma \text{ is inaccessible}\}$$

is stationary in δ, since δ is Mahlo in $M_{2n}^{\#}(r)$. Let $\bar{S}$ be the set of limit points of S and

$$\lambda_r := \min(S \cap \bar{S}).$$

Let $(\gamma_k : k < \omega)$ be a sequence in M of ordinals in S with supremum λ_r. We define $X_k := X_{\gamma_k}$ for $k < \omega$. Note that each X_k is ω_1-iterable via the $\mathcal{Q}$-structure iteration strategy Σ.

Now let $(x_k : k < \omega)$ enumerate $\mathbb{R} \cap M$ and set $\mathcal{N}_0 := M_{2n}^{\#}(r)$. We construct premice $\mathcal{N}_k$ for $k \geq 1$ and countable iteration trees $\mathcal{T}_k$ on $\mathcal{N}_k$ in M for $k < \omega$ such that

1. the composition $\mathcal{T}_0 \frown .. \frown \mathcal{T}_k$ is an iteration tree according to Σ with map $\pi_{k+1} = \pi_{0,k+1} = \pi_{k,k+1} \circ .. \circ \pi_{0,1} : \mathcal{N}_0 \to \mathcal{N}_{k+1}$,
2. x_k is $\mathbb{W}_{\pi_{k+1}(\gamma_{k+1})}$-generic over $\pi_{k+1}(X_{k+1})$, and
3. $\mathcal{T}_k$ lives on $\mathcal{N}_k|\pi_k(\gamma_{k+1})$ and all extenders in $\mathcal{T}_k$ have critical points above $\pi_k(\gamma_k)$.

Suppose $\mathcal{N}_k$ and $\mathcal{T}_i$ have been defined for $i < k$. Note that $\pi_k(X_{k+1}) \lhd \mathcal{N}_k$ and $\pi_k(X_{k+1})|\pi_k(\gamma_{k+1}) = \mathcal{N}_k|\pi_k(\gamma_{k+1})$. There is a countable iteration tree $\mathcal{T}_k$ on $\mathcal{N}_k$ according to Σ with map $\pi_{k,k+1}$ so that x_k is $\mathbb{W}_{\pi_{k,k+1}(\pi_k(\gamma_{k+1}))}$-generic over $\pi_{k,k+1}(X_{k+1})$ by the genericity iteration. Here $\mathcal{T}_k$ lives on $\mathcal{N}_k|\pi_k(\gamma_{k+1})$ and all extenders have critical points above $\pi_k(\gamma_k)$. We define $\mathcal{N}_{k+1}$ as the last model of $\mathcal{T}_k$.

One can easily check that the composition $\mathcal{T}_0 \frown .. \frown \mathcal{T}_k$ is an iteration tree on $M_{2n}^\#(r)$, since it follows from condition 3 and the rules of the iteration game that $\mathcal{N}_k$ is the only model in $\mathcal{T}_0 \frown .. \frown \mathcal{T}_{k-1}$ to which an extender in $\mathcal{T}_k$ can be applied. Since the composition $\mathcal{T}$ of $(\mathcal{T}_k : k < \omega)$ is according to Σ, the direct limit $\mathcal{N}$ along the unique cofinal branch is wellfounded. Let $\pi_{k,\omega} : \mathcal{N}_k \to \mathcal{N}$ denote the direct limit maps.

Note that it follows from condition 3 that $\pi_k(\gamma_k) = \pi_{0,\omega}(\gamma_k)$ and

$$\pi_{k,\omega} \restriction V_{\pi_k(\gamma_k)+\omega}^{\mathcal{N}_k} = id.$$

This implies that $\mathbb{W}_{\pi_k(\gamma_k)}^{\mathcal{N}_k} = \mathbb{W}_{\pi_k(\gamma_k)}^{\pi_k(X_k)} = \mathbb{W}_{\pi_{0,\omega}(\gamma_k)}^{\pi_{0,\omega}(X_k)}$ and the forcing has the same subsets in $\pi_k(X_k)$ and $\pi_{0,\omega}(X_k)$. Hence x_k is $\mathbb{W}_{\pi_{0,\omega}(\gamma_{k+1})}^{\pi_{0,\omega}(X_{k+1})}$-generic over $\pi_{0,\omega}(X_{k+1})$.

Claim 4.1.5: $\sup_{k<\omega} \pi_k(\gamma_k) = \omega_1^M$.

Proof: Since initial segments of $\mathcal{T}$ are countable in M, we have $\pi_k(\gamma_k) < \omega_1^M$ for every $k < \omega$. Thus $\sup_{k<\omega} \pi_k(\gamma_k) \leq \omega_1^M$.

To show that $\sup_{k<\omega} \pi_k(\gamma_k) \geq \omega_1^M$, suppose $\alpha < \omega_1^M$ is given. Let x_k code α where $k < \omega$. Then x_k is $\mathbb{W}_{\pi_{k+1}(\gamma_{k+1})}$-generic over $\pi_{k+1}(X_{k+1})$. Now $\pi_{k+1}(\gamma_{k+2})$ is inaccessible in $\pi_{k+1}(X_{k+2})$ and hence in $\pi_{k+1}(X_{k+1}) \subseteq \pi_{k+1}(X_{k+2})$. Thus it is still inaccessible in $\pi_{k+1}(X_{k+1})[x_k]$. This implies

$$\alpha < \omega_1^{\pi_{k+1}(X_{k+1})[x_k]} < \pi_{k+1}(\gamma_{k+2}) \leq \pi_{k+2}(\gamma_{k+2}). \qquad \blacksquare$$

If $\mathbb{P}$ is a forcing and τ is a $\mathbb{P}$-name for a real, then in any $\mathbb{P}\times\mathbb{P}$-generic extension τ defines two reals via the $\mathbb{P}$-generic filters. Let τ and τ' be $\mathbb{P}\times\mathbb{P}$-names for these reals. Let $\leq_i$ denote the Π^1_{2n+1} prewellorders from the canonical Π^1_{2n+1} scale on the complete Π^1_{2n+1} set A for $i<\omega$. We write $\equiv_i$ for the induced thin Σ^1_{2n+1} equivalence relations, i.e. $x\equiv_i y$ if and only if $(x\leq_i y\wedge y\leq_i x)\vee x,y\notin A$.

Claim 4.1.6: *Let τ be a name for the $\mathbb{W}_{\pi_k(\gamma_k)}$-generic real. Then the set $D_{j,k}$ of conditions $p\in\mathbb{W}^{\pi_k(X_k)}_{\pi_k(\gamma_k)}$ with*

1. *p decides $\tau\restriction j$ and*
2. *$(p,p)\Vdash^{\pi_k(X_k)}_{\mathbb{W}_{\pi_k(\gamma_k)}\times\mathbb{W}_{\pi_k(\gamma_k)}}\tau\equiv_i\tau'$ for every $i\leq j$*

is dense for all $j,k<\omega$.

Proof: Let $j,k<\omega$ and let σ be a name for the $\mathbb{W}_\delta$-generic real. Then the set of conditions $p\in\mathbb{W}^{M^\#_{2n}(r)}_\delta$ with

$$(p,p)\Vdash^{M^\#_{2n}(r)}_{\mathbb{W}_\delta\times\mathbb{W}_\delta}\sigma\equiv_i\sigma'$$

is dense for every $i<j$ by lemma 3.2.8. Since these sets are dense open and the set of conditions which decide $\sigma\restriction j$ is dense open, we have that their intersection is dense open. The claim follows by elementarity. ∎

We use the notation

$$[\alpha,\beta):=\{\gamma<\kappa:\alpha\leq\gamma<\beta\}$$

and

$$Col(\omega,[\alpha,\beta)):=\{f:\omega\times[\alpha,\beta):\forall n<\omega\forall\gamma\in[\alpha,\beta)\,f(n,\gamma)<\gamma\}$$

ordered by reverse inclusion. Then

$$Col(\omega,<\beta)\cong Col(\omega,<\alpha)\times Col(\omega,[\alpha,\beta))$$

for all ordinals $\alpha<\beta$.

Claim 4.1.7: *There is a $Col(\omega,<\omega_1^M)$-generic filter g over $\mathcal{N}$ in V with*

$$\mathbb{R}^{\mathcal{N}[g]}\subseteq M.$$

PROOF: Let g_0 be a $Col(\omega, < \gamma_0)$-generic filter over $\mathcal{N}$ in M. Then $\pi_2(\gamma_2)$ is inaccessible in $\mathcal{N}[g_0]$ and hence $\mathcal{P}(\pi_1(\gamma_1))^{\mathcal{N}[g_0]}$ is countable in M. Now let g_1 be a $Col(\omega, < \pi_1(\gamma_1))$-generic filter over $\mathcal{N}$ in M with

$$g_1 \cap Col(\omega, < \gamma_0) = g_0.$$

Similarly we choose $Col(\omega, < \pi_k(\gamma_k))$-generic filters g_k over $\mathcal{N}$ with

$$g_{k+1} \cap Col(\omega, < \pi_k(\gamma_k))$$

for each $k < \omega$. Finally let $g := \bigcup_{k<\omega} g_k$.

To see that g is $Col(\omega, < \omega_1^M)$-generic over $\mathcal{N}$, note that the forcing $Col(\omega, < \omega_1^M)$ has the ω_1^M-c.c. in $\mathcal{N}$ since ω_1^M is regular in $\mathcal{N}$. So for any maximal antichain $A \subseteq Col(\omega, < \omega_1^M)$ there is some $k < \omega$ with $A \subseteq Col(\omega, < \pi_k(\gamma_k))$. Hence $g_k \cap A \neq \emptyset$.

By considering only the nice names for reals we get $\mathbb{R}^{\mathcal{N}[g]} = \bigcup_{k<\omega} \mathbb{R} \cap \mathcal{N}[g_k]$. ■

We fix a $Col(\omega, < \omega_1^M)$-generic filter g over $\mathcal{N}$ as in the previous claim and let $\mathbb{R}^* := \mathbb{R}^{\mathcal{N}[g]}$.

Claim 4.1.8: *For all $x \in \mathbb{R} \cap M$ and $j < \omega$ there is a real $y \in \mathbb{R}^*$ such that $x \upharpoonright j = y \upharpoonright j$ and $M \models x \equiv_i y$ for every $i \leq j$.*

PROOF: Let $x \in \mathbb{R} \cap M$ and find $k < \omega$ with $x = x_k$. Let

$$\mathbb{P} := \mathbb{W}^{\pi_{k+1}(X_{k+1})}_{\pi_{k+1}(\gamma_{k+1})}.$$

Then x is $\mathbb{P}$-generic over $\pi_{k+1}(X_{k+1})$. Let $D_{j,k+1}$ be the dense set from claim 4.1.6 and choose a condition $p \in D_{j,k+1}$ in the generic filter for x. Since the set $\mathcal{P}(\pi_{k+1}(\gamma_{k+1}))^{\pi_{k+1}(X_{k+1})}$ is countable in $\mathcal{N}[g]$, there is a $\mathbb{P} \upharpoonright p$-generic real y over $\pi_{k+1}(X_{k+1})$ in $\mathcal{N}[g]$. We directly get $x \upharpoonright j = y \upharpoonright j$ by choice of p.

We claim that $x \equiv_i y$ for every $i \leq j$. To see this, choose another real $z \in \mathbb{R} \cap M$ which is $\mathbb{P} \upharpoonright p$-generic over both $\pi_{k+1}(X_{k+1})[x]$ and $\pi_{k+1}(X_{k+1})[y]$. Since

$$(p, p) \Vdash^{\pi_{k+1}(X_{k+1})}_{\mathbb{P}\times\mathbb{P}} \tau \equiv_i \tau',$$

we have

$$\pi_{k+1}(X_{k+1})[x, z] \models x \equiv_i z$$

and

$$\pi_{k+1}(X_{k+1})[y, z] \models y \equiv_i z$$

for each $i \leq j$. Now $\pi_{k+1}(X_{k+1})[x, z]$ and $\pi_{k+1}(X_{k+1})[y, z]$ are $2n$-small boldface premice with $2n-1$ Woodin cardinals above $\pi_{k+1}(\gamma_{k+1})$ which are ω_1-iterable above $\pi_{k+1}(\gamma_{k+1})$ in M and project to $\pi_{k+1}(\gamma_{k+1})$ or below. Hence both are Σ^1_{2n}-correct in M by lemma 1.2.27. We can conclude that

$$M \vDash x \equiv_i z \equiv_i y$$

by Σ^1_{2n+1} upwards absoluteness. ■

Claim 4.1.9: *T^M_{2n+1} is definable from r in $\mathcal{N}[g]$.*

PROOF: We have $M \prec_{\Sigma^1_{2n+1}} V$ by claim 4.1.4. Now $\mathcal{N}[g]$ is a countable ω-sound $2n$-small boldface premouse with $2n-1$ Woodin cardinals above ω_1^M and $\rho_\omega(\mathcal{N}[g]) \leq \omega_1^M$ which is ω_1-iterable above ω_1^M in V. Moreover $\mathcal{N}[g]$ computes Σ^1_{2n+1} truth in V by lemma 1.2.27. So for any $x, y \in \mathbb{R} \cap \mathcal{N}[g]$ and $k < \omega$ we can calculate in $\mathcal{N}[g]$ whether $V \vDash x \leq_k y$ holds. Using the previous claim, we can define T^M_{2n+1} in $\mathcal{N}[g]$ in the parameter $\pi_{0,\omega}(\lambda_r)$, which was defined from r. ■

By homogeneity of $Col(\omega, < \omega_1^M)$ we get that T^M_{2n+1} is definable from r in $\mathcal{N}$ and hence an element of $\mathcal{N}$.

Remark 4.1.10: $\mathcal{N}[g] \not\prec_{\Sigma^1_{2n+3}} V$.

PROOF: $M^\#_{2n}(r)$ is a $\Pi^1_{2n+2}(r)$ singleton by 1.2.31. Supposing $\mathcal{N}[g] \prec_{\Sigma^1_{2n+3}} V$ we would have $M^\#_{2n}(r) \in \mathcal{N}[g]$. But this implies $M^\#_{2n}(r) \in \mathcal{N}$ by homogeneity of $Col(\omega, < \omega_1^M)$ and hence $M^\#_{2n}(r) \in M^\#_{2n}(r)$, a contradiction. ■

Remark 4.1.11: *If $M^\#_{2n}(X)$ exists for every $X \in H_{(2^{\aleph_0})^+}$, then the iterability of $M^\#_{2n}$ is not affected by forcing with $Col(\omega, \mathbb{R})$ by lemma 2.1.4. In this case one can construct the iteration tree in the proof of the main lemma for $M = V$ in $V^{Col(\omega,\mathbb{R})}$. The construction produces a forcing extension $\mathcal{N}[g]$ of an iterate of $M^\#_{2n}$ in $V^{Col(\omega,\mathbb{R})}$.*

We get a simpler version of the main lemma for $n = 0$ based on

Lemma 4.1.12: *let M be a transitive model of* ZF*. Then $T_1^M = T_1^V$ if and only if $\omega_1^M = \omega_1^V$.*

PROOF: Since $ht(T_1) = \omega_1$ we know that $T_1^M = T_1^V$ implies $\omega_1^M = \omega_1^V$. On the other hand the Shoenfield tree is absolute relative to ω_1. Moreover, the tree T_1 from the canonical Π^1_1 scale is absolute relative to the Shoenfield tree by the proof of [21, theorem 36.12]. ■

Lemma 4.1.13: *Let M be a transitive model of* ZF*. Suppose $r \in \mathbb{R} \cap M$ and $r^\#$ exists in M. Let κ be the critical point of the top extender of $M_0^\#(r)$ and $\pi : M_0^\#(r) \to \mathcal{N}$ the map from iterating the top extender in ω_1^M many steps. Then*

$$\pi(\kappa) = \omega_1^M$$

and

$$T_1^{\mathcal{N}[g]} = T_1^M$$

for every $Col(\omega, <\omega_1^M)$-generic filter g over $\mathcal{N}$.

PROOF: Since $\omega_1^{\mathcal{N}[g]} = \omega_1^M$ we have $T_1^{\mathcal{N}[g]} = T_1^M$ by the previous lemma. ■

One can derive a different version of the main lemma for $M_{2n}^\dagger(r)$ with essentially the same proof, based on the following version of lemma 3.2.8:

Lemma 4.1.14: *Suppose $m \leq k$ and $M_k^\#(x)$ exists for every $x \in \mathbb{R}$. Let $E \subseteq \mathbb{R}^2$ be a thin $\Pi^1_{m+3}(x)$ equivalence relation with $x \in \mathbb{R}$. Let $\mathcal{M}$ be a countable $(k+1)$-small X-premouse which is ω_1-iterable above δ and ω-sound above δ with $\rho_\omega(\mathcal{M}) \leq \delta$, where X is swo. Suppose there are m Woodin cardinals above δ and at least two extenders above them in $\mathcal{M}$. Let $\mathbb{P}$ be a forcing of size $\leq \delta$ in $\mathcal{M}$. Then for every $\mathbb{P}$-name τ in $\mathcal{M}$ for a real the set D of conditions $p \in \mathbb{P}$ with*

$$(p, p) \Vdash^{\mathcal{M}}_{\mathbb{P}\times\mathbb{P}} \tau E \tau'$$

is dense in $\mathbb{P}$.

PROOF: The proof is the same as for lemma 3.2.8. We need the extra extender on top to make sure that $\mathcal{M}[g_y, g_z] \prec_{\Sigma_{m+2}} V$ by lemma 1.2.29, where $g_y \times g_z$ is $\mathbb{P} \times \mathbb{P}$-generic over $\mathcal{M}$. ■

Lemma 4.1.15: *Let $n \geq 1$ and assume $M_{2n}^{\#}(x)$ exists for every $x \in \mathbb{R}$. Suppose M is a transitive model of* ZF *such that $\mathbb{R} \cap M$ is countable and M calculates $M_{2n}^{\#}(x)$ correctly for each $x \in \mathbb{R} \cap M$. Moreover suppose $r \in \mathbb{R} \cap M$ and $M_{2n}^{\dagger}(r)$ exists in M and is calculated correctly. Let δ be the least Woodin cardinal in $M_{2n}^{\dagger}(r)$. There are*

1. *a countable iteration tree on $M_{2n}^{\dagger}(r)$ with iteration map $\pi : M_{2n}^{\dagger}(r) \to \mathcal{N}$ and*
2. *an ordinal $\lambda_r < \delta$ definable in $M_{2n}^{\dagger}(r)$ uniformly in the parameter $r \in \mathbb{R}$*

such that

$$\pi(\lambda_r) = \omega_1^M$$

and

$$T_{2n+1}^{\mathcal{N}[g]} = T_{2n+1}^{M},$$

where g is any $Col(\omega, < \omega_1^M)$-generic filter over $\mathcal{N}$.

4.1.2 The main theorem

In this section we show

Main Theorem 4.1.16: *Suppose $n \geq 1$ and $M_{2n-2}^{\#}(x)$ exists for every $x \in \mathbb{R}$. Let $(\leq_k : k < \omega)$ denote the canonical scale on the complete Π_{2n-1}^1 set C and let $\equiv_k$ denote the induced thin Σ_{2n-1}^1 equivalence relations. The following are equivalent for any transitive model M of* ZF*:*

1. *every equivalence class of every thin $\Pi_{2n}^1(r)$ equivalence relation has a representative in M for all $r \in \mathbb{R} \cap M$*
2. (a) *$M \prec_{\Sigma_{2n+1}^1} V$ and*
   (b) *$T_{2n-1}^M = T_{2n-1}^V$*
3. (a) *$M \prec_{\Sigma_{2n+1}^1} V$ and*
   (b) *for some $k < \omega$ every equivalence class of $\equiv_k$ (except possibly the ∞ class $\mathbb{R} - C$) has a representative in M.*

Note that theorem 4.1.1 is the special case of the equivalence of conditions 1 and 2 for $n = 0$ since $T_1^M = T_1^V$ if and only if $\omega_1^M = \omega_1^V$ by lemma 4.1.12.

The first part of the proof of the main theorem is purely descriptive. Note that the assumptions of the main theorem imply $Det(\mathbf{\Pi}^1_{2n-1})$ by theorem 1.2.12.

Claim 4.1.17: *Under the assumptions of the main theorem, condition 1 implies conditions 2 and 3.*

PROOF: *2 (a).* It suffices to show that $A \cap M \neq \emptyset$ for every nonempty $\Pi^1_{2n}(r)$ set A with $r \in \mathbb{R} \cap M$. Let $\leq$ be the $\Pi^1_{2n}(r)$ prewellorder from a $\Sigma^1_{2n}(r)$ norm on $\mathbb{R} - A$. We have $[x]_\leq = A$ for every $x \in A$, where

$$[x]_\leq := \{y \in \mathbb{R} : x \leq y \wedge y \leq x\}.$$

Then $A \cap M \neq \emptyset$, since the induced $\Pi^1_{2n}(r)$ equivalence relation is thin by lemma 1.1.16.

2 (b). Condition 1 implies

$$rank_k^M(x) = rank_k^V(x)$$

for all $x \in \mathbb{R} \cap M$ and $k < \omega$ since each norm in the Π^1_{2n-1}-scale induces a thin Σ^1_{2n-1} equivalence relation. Hence $T^M_{2n-1} \subseteq T_{2n-1}$. We have to show that $T_{2n-1} \subseteq T^M_{2n-1}$.

Suppose $(s, f) \in T_{2n-1}$ and $m = lh(s) = lh(f)$. Let A be the canonical complete Π^1_{2n-1} set. Choose $x_0 \in A$ with

$$(s, f) = (x_0 \restriction m, (rank_0(x_0), .., rank_{m-1}(x_0)).$$

We inductively define reals $x_k \in A \cap M$ for $1 \leq k \leq m$ with

$$(s \restriction k, f \restriction k) = (x_k \restriction k, (rank_0(x_k), .., rank_{k-1}(x_k)))$$

so x_m witnesses that $(s, f) \in T^M_{\Pi^1_{2n-1}}$. Let $\leq_k$ denote the k^{th} Σ^1_{2n-1} prewellorder from the canonical Π^1_{2n-1}-scale on A and $\equiv_k$ the induced equivalence relation. Moreover let

$$U_t := \{x \in \mathbb{R} : x \restriction lh(t) = t\}$$

for $t \in \omega^{<\omega}$.

Case 1: $k = 1$. Define xE_1y if and only if

$$x, y \notin U_{s\restriction 1} \vee (x, y \in U_{s\restriction 1} \wedge x \equiv_0 y).$$

Then E_1 is a Σ^1_{2n-1} equivalence relation which is thin by lemma 1.1.14, since it is induced by a Σ^1_{2n-1} prewellorder. There is a real $x_1 \in \mathbb{R} \cap M$ with $x_1(0) = s(0)$ and $x_1 \equiv_0 x_0$ by condition 1 applied to E_1.

Case 2: $2 \leq k \leq m$. Suppose $x_i \in \mathbb{R} \cap M$ is defined for $1 \leq i < k$. Then the set

$$U := \{x \in \mathbb{R} : \forall i < k - 1(x \equiv_i x_{k-1})\}$$

is $\Delta^1_{2n-1}(x_{k-1})$ since $x_{k-1} \in A$. Now define xE_ky if and only if

$$x, y \notin U_{s\restriction k} \cap U \vee (x, y \in U_{s\restriction k} \cap U \wedge x \equiv_{k-1} y).$$

Then E_k is a $\Sigma^1_{2n-1}(x_{k-1})$ equivalence relation. It is thin since it is induced by a $\Sigma^1_{2n-1}(x_{k-1})$ prewellorder. Moreover we have $x_0 \in U_{s\restriction k} \cap U$. Hence there is a real $x_k \in \mathbb{R} \cap M$ with $x_k \restriction k = s \restriction k$ and $x_k \equiv_i x_0$ for all $1 \leq i < k$ by condition 1 for E_k.

3 (b). Since $\equiv_k$ is a thin Σ^1_{2n-1} equivalence relation for each $k < \omega$. ■

Remark 4.1.18: *In condition 1 one can equivalently replace thin $\mathbf{\Pi}^1_{2n}$ equivalence relations by $\mathbf{\Pi}^1_{2n}$ prewellorders, thin $\mathbf{\Pi}^1_{2n}$ linear preorders, or $\mathbf{\Sigma}^1_{2n}$ norms.*

For the other implications will use

Lemma 4.1.19: *Suppose T is the tree from a scale on a set containing $x \in \mathbb{R}$ and $\mathbb{A}$ is β-admissible with $x, T \in \mathbb{A}$. Then $rank_k(x)$ is definable from x and T in $\mathbb{A}$ for every $k < \omega$.*

Proof: $\mathbb{A} \models \mathsf{KP} + Axiom\, Beta$ since $\mathbb{A}$ is β-admissible, see definition 4.1.19. Let $\alpha < \underset{\sim}{\delta}^1_{2n-1}$ be least with

$$\mathbb{A} \models \exists f \in {}^{\omega}(\underset{\sim}{\delta}^1_{2n-1})\,((x, f) \in [T] \wedge f(k) = \alpha).$$

We show that $rank_k(x) = \alpha$.

For $rank_k(x) \leq \alpha$ suppose $f \in {}^\omega(\underset{\sim}{\delta}^1_{2n-1})$ and $(x, f) \in [T]$. Let $(x_i : i < \omega)$ be a sequence of reals with

$$(x_i|i, (rank_0(x_i), ..., rank_{i-1}(x_i))) = (x|i, (f(0), ..., f(i-1))) \in [T]$$

for every $i < \omega$. Hence $x_i \to x$ and the sequence $(rank_k(x_i) : i < \omega)$ is eventually constant with value $f(k)$ for each $k < \omega$. The semicontinuity of the scale implies $rank_k(x) \leq f(k)$.

For $\alpha \leq rank_k(x)$ we have to find a function $f \in {}^\omega(\underset{\sim}{\delta}^1_{2n-1})$ in $\mathbb{A}$ with $(x, f) \in [T]$ and $f(k) = \alpha$. Such functions are exactly the branches of the tree

$$S := \{s \in {}^{<\omega}(\underset{\sim}{\delta}^1_{2n-1}) : (x|lh(s), s) \in T \wedge s(k) = \alpha\}.$$

In V the sequence $(rank_k(x) : k < \omega)$ is the pointwise minimal branch of S by the semicontinuity of the scale. Now Axiom Beta implies that every wellfounded relation can be collapsed to a transitive set, so wellfoundedness becomes a Δ_1 predicate. Hence S has a branch f in $\mathbb{A}$ as well by Δ_1 absoluteness. ■

The previous lemma shows that condition 2 in the main theorem implies condition 3. We shall now prove condition 1 from condition 3. Let $r \in \mathbb{R} \cap M$ and let E be a thin $\Pi^1_{2n}(r)$ equivalence relation. We fix a real $x \in \mathbb{R}$. The goal is to find a real $\bar{x} \in \mathbb{R} \cap M$ with $xE\bar{x}$.

The idea of the proof is as follows. Let $\mathbb{A}$ be the least β-admissible set containing T_{2n-1} as an element and choose a formula $\varphi \in \mathcal{L}_{\infty,0,N} \cap \mathbb{A}$ with $\varphi(x)$ as in theorem 3.1.8. One can find a real $t \in \mathbb{R} \cap M$ so that φ can be defined from t and T_{2n-1}. One can now use the main lemma to reconstruct T_{2n-1} in an iterate of $M^\#_{2n-2}(t \oplus u)$ for arbitrary reals $u \in \mathbb{R}$. This allows you to express the existence of a real $\bar{x}$ with $\varphi(\bar{x})$ by a $\Sigma^1_{2n+1}(x, r)$ statement. Since M is sufficiently correct, there is such a real in M. Finally the choice of φ implies that $xE\bar{x}$.

Let T be a tree defined from T_{2n-1} and r as in lemma 1.1.11 with $E = \mathbb{R}^2 - p[T]$. In order to apply the theorem of Harrington and Shelah we need to know that

Claim 4.1.20: *$\mathbb{R}^2 - p[T]$ is an equivalence relation in any Cohen generic extension of V.*

Proof: Let G be Cohen generic over V. We get Σ^1_{2n+1} Cohen forcing absoluteness from $Det(\mathbf{\Pi}^1_{2n-1})$ by lemma 2.2.3. Then $T^V_{2n-1} = T^{V[G]}_{2n-1}$ since Cohen forcing does not add equivalence classes to the relevant prewellorders by lemma 2.2.9. So $\mathbb{R}^2 - p[T]$ is defined by the same Π^1_{2n} formula in $V[G]$. Hence this is an equivalence relation in $V[G]$ by Σ^1_{2n+1} Cohen absoluteness. ∎

Suppose $x \in \mathbb{R}$ and $\mathbb{A}$ is the least admissible set with $T_{2n-1} \in \mathbb{A}$. Let $\varphi \in \mathcal{L}_{\infty,0,N} \cap \mathbb{A}$ be a formula with

1. $\varphi(x)$ and
2. $\varphi(y) \Rightarrow xEy$ for every $y \in \mathbb{R}$

by theorem 3.1.8.

Claim 4.1.21: *There is a real $t \in \mathbb{R} \cap M$ and a formula ψ which defines φ from T_{2n-1} and t in every β-admissible set A with $T_{2n-1}, t \in A$.*

Proof: Let A be a minimal β-admissible set containing T_{2n-1} and t as elements. Then A is the Σ_2 Skolem hull of $\underset{\sim}{\delta}^1_{2n-1} \cup \{T_{2n-1}\}$ in itself by minimality of A, since Axiom Beta is a Π_2 statement. Now there is a Σ_2 Skolem function for A which is uniformly Σ_3 over A by [42, theorem 1.15] and the following paragraph. So φ is Σ_3-definable in A from T_{2n-1} and some $\vec{\alpha} = (\alpha_0, .., \alpha_j) \in (\underset{\sim}{\delta}^1_{2n-1})^{<\omega}$. Since the length of each Π^1_{2n-1} norm in the scale is $\underset{\sim}{\delta}^1_{2n-1}$, we can choose reals $t_i \in \mathbb{R} \cap M$ with $rank_k(t_i) = \alpha_i$ for $i \leq j$ by condition 3 (b). Then the join $t := t_0 \oplus .. \oplus t_j$ works by the previous lemma. ∎

Let $t \in \mathbb{R} \cap M$ and ψ be as in the previous claim. Let $\varphi_{s,S}$ denote the formula defined by ψ from a real s and a tree S. Let T_u be the term from the main lemma 4.1.15 which defines T_{2n-1} from $u \in \mathbb{R}$ in an iterate of $M^{\#}_{2n-2}(u)$. Note that T_u does not depend on the model M in the main lemma.

Claim 4.1.22: *For every real $u \in \mathbb{R}$ we have that $\varphi(u)$ holds if and only if $M^{\#}_{2n-2}(t \oplus u) \models \varphi_{t,T_u}(u)$.*

Proof: Let $N \prec V_\eta$ be countable and contain all relevant parameters, where η is a large limit ordinal. Let $\bar{N}$ be its transitive collapse with uncollapsing map $\pi : \bar{N} \to N$ and $\pi(\bar{\varphi}) = \varphi$. There is an iteration $M^{\#}_{2n-2}(t \oplus u) \to \mathcal{N}$ with

$$T_u^{\mathcal{N}} = T_{2n-1}^{\bar{N}}$$

by the main lemma 4.1.15 applied to $\bar{N}$. Since claim 4.1.21 is true in $\bar{N}$ we have

$$\varphi^{\mathcal{N}}_{t,T_u} = \varphi^{\mathcal{N}}_{t,T^{\bar{N}}_{2n-1}} = \bar{\varphi}.$$

Hence

$$\varphi(u) \Leftrightarrow \bar{\varphi}(u) \Leftrightarrow \mathcal{N} \models \varphi_{t,T_u}(u) \Leftrightarrow M^{\#}_{2n-2}(t \oplus u) \models \varphi_{t,T_u}(u).$$

The last equivalence holds by elementarity of the iteration map. ∎

Now the previous claim expresses the existence of a real $\bar{x}$ with $\varphi(\bar{x})$ by a Σ^1_{2n+1} formula, since $M^{\#}_{2n-2}(t \oplus u)$ is a $\Pi^1_{2n}(t \oplus u)$ singleton uniformly in t and u by lemma 1.2.31. Since $M \prec_{\Sigma^1_{2n+1}} V$, there is a real $\bar{x} \in \mathbb{R} \cap M$ with $\varphi(\bar{x})$.

Note that Harrington's proof [16, theorem 32.1] of Silver's theorem shows

Lemma 4.1.23: *(Harrington) For every equivalence class $[x]$ of every thin Π^1_1 equivalence relation, there is a Δ^1_1 set $X \neq \emptyset$ with $X \subseteq [x]$.*

The technique from the proof of the main theorem can be used to show a similar fact. Let $rank_k$ denote the k^{th} rank in the canonical Π^1_{2n-1}-scale for $k < \omega$.

Lemma 4.1.24: *Let $n \geq 1$ and suppose $Det(\mathbf{\Pi}^1_{2n-1})$ holds. Let $A \subseteq \mathbb{R}$ be closed under finite join $\oplus$. Suppose that for every $\alpha < \underset{\sim}{\delta}^1_{2n-1}$ there are $r \in A$ and $k < \omega$ with $rank_k(r) = \alpha$. Then for every thin Π^1_{2n} equivalence relation E and every real x, there exist a real $r \in A$ and a nonempty $\Delta^1_{2n+1}(r)$ set X such that $x \in X \subseteq [x]_E$.*

Proof: Given a formula φ with $\varphi(x)$ as in the proof of the main theorem, we can choose a real $t \in A$ satisfying claim 4.1.21. It follows from claim 4.1.22 that the set $\{u \in \mathbb{R} : \varphi(u)\}$ is $\Delta^1_{2n+1}(t)$. ∎

We conclude this section with two remarks about the proof of the main theorem.

Remark 4.1.25: *It is unclear whether condition 2 (b) in the main theorem can be replaced by $\underset{\sim}{\delta}^1_{2n-1} = (\underset{\sim}{\delta}^1_{2n-1})^M$.*

Note that if there is model satisfying the assumptions of the main theorem for which forcing with $Col(\omega, \omega_1)$ does not change $\underset{\sim}{\delta}^1_3$ and at the same time $V \prec_{\Sigma^1_5} V^{Col(\omega,\omega_1)}$ holds, then the condition $T^M_3 = T^V_3$ cannot be replaced by $(\underset{\sim}{\delta}^1_3)^M = \underset{\sim}{\delta}^1_3$ in the main theorem for $n = 2$. However, it is not clear how one could obtain such a model.

Remark 4.1.26: *Claim 4.1.22 can be used to prove from* PD *that every thin projective equivalence relation is induced by a projective prewellorder.*

Note that this fact is proved from in [8, theorem 5] from a determinacy assumption which is locally weaker than in this case. Suppose E is a thin equivalence relation and T is a tree as in the main theorem. Let x and y be reals. A prewellorder which induces E can be defined by comparing infinitary formulas φ with $\varphi(x)$ and ψ with $\psi(y)$ as above in the constructibility order of the least admissible set containing T.

4.1.3 An inner model under CH

In this section we construct an inner model which fulfills the conditions in the main theorem 4.1.16, assuming that the continuum hypothesis holds or just $\underset{\sim}{\delta}^1_{2n+1} < \omega_2$.

Theorem 4.1.27: *Assume* CH*. Let $n \geq 1$ and suppose $M_n^\#(x)$ exists for every $x \in \mathbb{R}$. Let's further suppose that $M_n^\dagger$ exists if n is odd. Then there is a transitive model M of* ZFC *of size $\aleph_1$ so that M has representatives in all equivalence classes of all thin provably $\mathbf{\Delta}^1_{n+2}$ equivalence relations defined from a parameter in M. Moreover, $m = n + 2$ is maximal with $M \prec_{\Sigma^1_m} V$.*

Let's first suppose that $n \geq 2$ is even. In this case we inductively build a stack of ω_1 many iteration trees on an iterate of $M_n^\#$ with direct limit model $\mathcal{N}$ such that every real is generic over an initial segment of $\mathcal{N}$. The model required for the theorem will be a generic extension $\mathcal{N}[g]$, where g is a $Col(\omega, < \omega_1^V)$-generic filter over $\mathcal{N}$ in V. The difference to the proof of the main lemma 4.1.15 is that the initial segments of the generic filter have to be defined in the course of the induction to ensure the required property for $\mathcal{N}[g]$.

As the first model in the stack of iteration trees we construct an iterate of $M_n^\#$ which contains a sequence of local Woodin cardinals of order type ω_1. Let δ be the least Woodin cardinal in $M_n^\#$ and let $\bar{C}$ be the set of limit points $< \delta$ of the club $C \subseteq \delta$ from lemma 3.2.9. Note that the set of critical points of extenders on the $M_n^\#$-sequence below is stationary in δ, since δ is Woodin in $M_n^\#$. Choose such a critical point $\gamma \in \bar{C}$. We define $\mathcal{N}_0$ as the iterate of $M_n^\#$ obtained by iterating the extender with critical point γ on the $M_n^\#$-sequence with least index ω_1 many times.

Then $\mathcal{N}_0$ is ω_1-iterable with respect to iteration trees living on $\mathcal{N}_0|\omega_1$, since the relevant iteration maps commute by the argument in the commutativity lemma [5, lemma 3.2]. Moreover the image D of $C \cap \gamma$ is a club in ω_1. We enumerate D by $(\gamma_\alpha : \alpha < \omega_1)$ and let $(X_{\gamma_\alpha} : \alpha < \omega_1)$ be the corresponding initial segments of $\mathcal{N}_0$ from lemma 3.2.9 such that γ_α is Woodin in X_{γ_α} for all $\alpha < \omega_1$.

As a bookkeeping device we fix a bijective map $f : \omega_1 \to \omega_1 \times \omega_1$ with $\eta \leq \alpha$ if $f(\alpha) = (\zeta, \eta)$. Let's inductively construct

1. a premouse $\mathcal{N}_\alpha$,
2. a countable iteration tree $\mathcal{T}_\alpha$ on $\mathcal{N}_\alpha$,
3. a filter g_α, and
4. a set $R_\alpha = \{x_{\eta,\alpha} : \eta < \omega_1\} \subseteq \mathbb{R}$

for each $\alpha < \omega_1$, such that for all $\beta < \omega_1$

1. the composition of $(\mathcal{T}_\alpha : \alpha < \beta)$ is an iteration tree on $\mathcal{N}_0$ according to Σ with map $\pi_\beta = \pi_{0,\beta} : \mathcal{N}_0 \to \mathcal{N}_\beta$,
2. g_β is $Col(\omega, < \pi_\beta(\gamma_\beta))$-generic over $\mathcal{N}_\beta$,
3. $g_\alpha = g_\beta \cap Col(\omega, < \pi_\alpha(\gamma_\alpha))$ for $\alpha < \beta$,
4. $x_{f(\beta)}$ is $\mathbb{W}_{\pi_{\beta+1}(\gamma_{\beta+1})}$-generic over $\pi_{\beta+1}(X_{\gamma_{\beta+1}})$,
5. $\mathcal{T}_\beta$ lives on $\mathcal{N}_\beta|\pi_\beta(\gamma_{\beta+1})$ and all extenders in $\mathcal{T}_\beta$ have critical points above $\pi_\beta(\gamma_\beta)$, and
6. there is a representative in R_β for every equivalence class of every thin provably $\Delta^1_{n+2}(r)$ equivalence relation with $r \in \mathbb{R} \cap \mathcal{N}_\beta[g_\beta]$.

In each successor step $\beta + 1 < \omega_1$ we fix a set $R_\beta \subseteq \mathbb{R}$ which fulfills condition 6. Let $(x_{\alpha,\beta} : \alpha < \omega_1)$ enumerate R_β. There is a countable iteration tree $\mathcal{T}_\beta$ on $\mathcal{N}_\beta$ living on $\mathcal{N}_\beta|\pi_\beta(\gamma_{\beta+1})$ with iteration map $\pi_{\beta,\beta+1}$ so that $x_{f(\beta)}$ is $\mathbb{W}_{\pi_{\beta+1}(\gamma_{\beta+1})}$-generic over $\pi_{\beta,\beta+1}(\pi_\beta(X_{\gamma_{\beta+1}}))$ by lemma 1.2.26. Let $\mathcal{N}_{\beta+1}$ be the last model of

$\mathcal{T}_\beta$. We further choose a $Col(\omega, < \pi_{\beta+1}(\gamma_{\beta+1}))$-generic filter $g_{\beta+1}$ over $\mathcal{N}_{\beta+1}[g_\beta]$ in V.

In each limit step $\beta \leq \omega_1$ let $\mathcal{N}_\beta$ be the direct limit of the unique cofinal branch in the composition of $(\mathcal{T}_\alpha : \alpha < \beta)$. We define $g_\beta := \bigcup_{\alpha<\beta} g_\alpha$. Then g_β is $Col(\omega, < \pi_\beta(\gamma_\beta))$-generic over $\mathcal{N}_\beta$. The reason is that $\pi_\beta(\gamma_\beta)$ is inaccessible in $\mathcal{N}_\beta$, so $Col(\omega, \pi_\beta(\gamma_\beta))$ has the $\pi_\beta(\gamma_\beta)$-c.c. in $\mathcal{N}_\beta$.

Finally let $\mathcal{N} := \mathcal{N}_{\omega_1}$, $g := g_{\omega_1}$, and

$$\mathbb{R}^* := \mathbb{R} \cap \mathcal{N}[g].$$

Claim 4.1.28: *Let $r \in \mathbb{R}^*$ and suppose E is a thin provably $\mathbf{\Delta}^1_{n+2}(r)$ equivalence relation. Then for every $x \in \mathbb{R}$ there is some $y \in \mathbb{R}^*$ with xEy.*

Proof: Let $\alpha < \omega_1$ be an ordinal with $r \in \mathbb{R} \cap \mathcal{N}[g_\alpha]$. We have

$$\mathbb{R} \cap \mathcal{N}[g_\alpha] = \mathbb{R} \cap \mathcal{N}_{\alpha+1}[g_\alpha],$$

since the set of nice $Col(\omega, < \pi_\alpha(\gamma_\alpha))$-names for reals in $\mathcal{N}$ is contained in $\mathcal{N}_{\alpha+1}$. We can assume that $x \in R_\alpha$. Now let $\beta, \eta < \omega_1$ be ordinals with $x = x_{\eta,\alpha}$ and $f(\beta) = (\eta, \alpha)$. Note that this implies $\beta \geq \alpha$. Then x is $\mathbb{P}$-generic over $\pi_{\beta+1}(X_{\beta+1})$ for

$$\mathbb{P} := \mathbb{W}^{\pi_{\beta+1}(X_{\beta+1})}_{\pi_{\beta+1}(\gamma_{\beta+1})}.$$

Let τ be a name for the $\mathbb{P}$-generic real. Then there is a condition p in the generic filter for x with

$$(p,p) \Vdash^{\pi_{\beta+1}(X_{\beta+1})}_{\mathbb{P}\times\mathbb{P}} \tau E \tau'$$

by lemma 3.2.8.

Let $y \in \mathbb{R}^*$ be $\mathbb{P} \restriction p$-generic over $\pi_{\beta+1}(X_{\gamma_{\beta+1}})$. Let further $z \in \mathbb{R}$ be $\mathbb{P} \restriction p$-generic over both $\pi_{\beta+1}(X_{\beta+1})[x]$ and $\pi_{\beta+1}(X_{\beta+1})[y]$. Then

$$\pi_{\beta+1}(X_{\beta+1})[x, z] \vDash xEz$$

and

$$\pi_{\beta+1}(X_{\beta+1})[y, z] \vDash yEz$$

hold, since this is forced by (p,p). Now $\pi_{\beta+1}(X_{\beta+1})[x, z]$ and $\pi_{\beta+1}(X_{\beta+1})[y, z]$ are both Σ^1_{n+1}-correct in V by lemma 1.2.29. Hence x, y, and z are E-equivalent in V. ■

Claim 4.1.29: $\mathcal{N}[g] \prec_{\Sigma^1_{n+2}} V$.

Proof: Let's assume $k \leq n+1$ and $\mathcal{N}[g] \prec_{\Sigma^1_k} V$. Suppose $\exists x \varphi(x,a)$ holds, where φ is a Π^1_k formula and $a \in \mathbb{R} \cap \mathcal{N}[g]$. We define an equivalence relation E by

$$xEy :\Leftrightarrow (\varphi(x) \wedge \varphi(y)) \vee (\neg\varphi(x) \wedge \neg\varphi(y)).$$

Then there is a real $x \in \mathbb{R} \cap \mathcal{N}[g]$ with $\mathcal{N}[g] \models \varphi(x,a)$ by the previous claim. ■

Claim 4.1.30: $\mathcal{N}[g] \not\prec_{\Sigma^1_{n+3}} V$.

Proof: Suppose $\mathcal{N}[g] \prec_{\Sigma^1_{n+3}} V$. Since $M^\#_n$ is a Π^1_{n+2} singleton by lemma 1.2.31, this implies $M^\#_n \in \mathcal{N}[g]$. Thus $M^\#_n \in \mathcal{N}$ by homogeneity. But $\mathcal{N}$ is an iterate of $M^\#_n$, so this is impossible. ■

Now $M := \mathcal{N}[g]|\kappa$ is the model required in theorem 4.1.27, where κ is the critical point of the top extender of $\mathcal{N}$. For odd n the proof is the analogous, with $M^\#_n$ replaced by $M^\dagger_n$ and an application of lemma 4.1.14 instead of lemma 3.2.8.

A similar result which does not use the continuum hypothesis can be shown with the same technique.

Theorem 4.1.31: *Let $n \geq 1$ and suppose $M^\#_{2n}(x)$ exists for every $x \in \mathbb{R}$. Suppose that $\underset{\sim}{\delta}^1_{2n+1} < \omega_2$. Then there is a transitive model M of* ZFC *of size $\aleph_1$ which has a representative in every equivalence class of every thin provably $\boldsymbol{\Delta}^1_{2n+2}$ equivalence relation defined from a parameter in M. In particular M calculates T_{2n+1} and $\underset{\sim}{\delta}^1_{2n+1}$ correctly. Moreover, $m = 2n+2$ is maximal with $M \prec_{\Sigma^1_m} V$.*

Proof: The proof is analogous to theorem 4.1.27. The point is that in the course of the induction we can choose the required sets $R_\alpha \subseteq \mathbb{R}$ of size $\aleph_1$, since every thin provably $\boldsymbol{\Delta}^1_{2n+2}$ equivalence relation has at most $Card(\underset{\sim}{\delta}^1_{2n+1}) < \omega_2$ many equivalence classes by theorem 3.2.4. It follows as in the previous claim that $M \not\prec_{\Sigma^1_{2n+3}} V$. ■

Note that every model constructed with this technique has exactly $\aleph_1$ many reals. Hence the method does not work if $\underset{\sim}{\delta}^1_{2n-1} \geq \omega_2$ for some $n \geq 2$. In this situation, any inner model which satisfies the conditions of the main theorem for some $n \geq 2$ will calculate $\underset{\sim}{\delta}^1_{2n-1}$ correctly. Thus such a model has at least $\aleph_2$ many reals.

4.2 Equivalence relations in $L(\mathbb{R})$

In this section we show one direction of an analogue of the main theorem for scaled Σ-pointclasses.

4.2.1 A direction in the main theorem

Suppose Γ is a scaled Σ-pointclass, see definition 1.1.12. Let T_Γ denote the tree from the canonical Γ-scale on the complete $\underset{\sim}{\Gamma}$ set $A \subseteq \mathbb{R}$. Let further M be a transitive set or class. Then the n^{th} norm in the scale restricted to $\mathbb{R} \cap M$ induces a rank function

$$rank_n^{\mathbb{R}\cap M} : \mathbb{R} \cap M \to Ord.$$

We define

$$T_\Gamma^{\mathbb{R}\cap M} := \{(x|n, (rank_0^{\mathbb{R}\cap M}(x), .., rank_{n-1}^{\mathbb{R}\cap M}(x))) : x \in A \cap M \wedge n < \omega\}$$

as the tree from the scale restricted to the reals of M.

Theorem 4.2.1: *Let Γ be a scaled Σ-pointclass. Suppose $\underset{\sim}{\Gamma}$ determinacy holds and M is an inner model such that*

1. *for every thin $\breve{\Gamma}(x)$ equivalence relation with $x \in \mathbb{R} \cap M$ every equivalence class has a representative in M.*

Then

2. (a) *if $x \in \mathbb{R} \cap M$ and $A \in \Gamma(x)$ or $A \in \breve{\Gamma}(x)$ then*

$$A \neq \emptyset \Leftrightarrow A \cap M \neq \emptyset,$$

and

   (b) $T_\Gamma^{\mathbb{R}\cap M} = T_\Gamma$.

Proof: *2 (a).* Suppose $A \in \Gamma(x)$ and $A \neq \emptyset$. Since $\Gamma(x)$ is normed, there is a $\breve{\Gamma}(x)$ prewellorder $\leq$ such that

$$(y \leq z \wedge z \in A) \Rightarrow y \in A$$

for all $z \in \mathbb{R}$. The equivalence relation induced by $\leq$ is thin by lemma 1.1.14. So $[y]_{\leq} \cap M$ is nonempty for every $y \in \mathbb{R}$, where

$$[y]_{\leq} := \{z \in \mathbb{R} : y \leq z \wedge z \leq y\}.$$

For $y \in A$ this implies that there is $z \in \mathbb{R} \cap M$ with $z \in A$.

Suppose on the other hand that $A \in \breve{\Gamma}(x)$ and $A \neq \emptyset$. In this case let $\leq$ be a $\breve{\Gamma}(x)$ prewellorder on $\mathbb{R} - A$ with $[y]_{\leq} = A$ for every $y \in A$. Again we get $A \cap M \neq \emptyset$, since the induced $\breve{\Gamma}(x)$ equivalence relation is thin.

2 (b). We have
$$rank_n^{\mathbb{R}\cap M}(y) = rank_n(y)$$
for all $y \in \mathbb{R} \cap M$ and $n < \omega$ by condition 1, since each norm in the Γ-scale induces a thin $\breve{\Gamma}$ equivalence relation. Hence $T_\Gamma^{\mathbb{R}\cap M} \subseteq T_\Gamma$. We have to show that $T_\Gamma \subseteq T_\Gamma^{\mathbb{R}\cap M}$.

Suppose $(s, f) \in T_\Gamma$ and $n = lh(s) = lh(f)$. Let A be the canonical complete Γ set. Choose $x_0 \in A$ with

$$(s, f) = (x_0 \restriction n, (rank_0(x_0), .., rank_{n-1}(x_0)).$$

We inductively define reals $x_k \in A \cap M$ for $1 \leq k \leq n$ with

$$(s \restriction k, f \restriction k) = (x_k \restriction k, (rank_0(x_k), .., rank_{k-1}(x_k))),$$

so x_n witnesses that $(s, f) \in T_\Gamma^{\mathbb{R}\cap M}$. Let $\leq_k$ denote the k^{th} prewellorder in $\breve{\Gamma}$ from the Γ-scale and $\equiv_k$ the induced equivalence relation. Let

$$U_t := \{x \in \mathbb{R} : x \restriction lh(t) = t\}$$

for $t \in \omega^{<\omega}$.

Case 1: $k = 1$. Define xE_1y if and only if

$$x, y \notin U_{s\restriction 1} \vee (x, y \in U_{s\restriction 1} \wedge x \equiv_0 y).$$

Then E_1 is a thin $\breve{\Gamma}$ equivalence relation which is thin by lemma 1.1.14, since it is induced by a $\breve{\Gamma}$ prewellorder. There is a real $x_1 \in \mathbb{R} \cap M$ with $x_1(0) = s(0)$ and

$x_1 \equiv_0 x_0$ by condition 1 applied to E_1.

Case 2: $2 \leq k \leq n$. Suppose $x_i \in \mathbb{R} \cap M$ is defined for $1 \leq i < k$. Then the set

$$U := \{x \in \mathbb{R} : \forall i < k-1(x \equiv_i x_{k-1})\}$$

is $\Delta(x_{k-1})$ since $x_{k-1} \in A$. Define xE_ky if and only if

$$x, y \notin U_{s\restriction k} \cap U \vee (x, y \in U_{s\restriction k} \cap U \wedge x \equiv_{k-1} y).$$

Then E_k is a $\breve{\Gamma}(x_{k-1})$ equivalence relation. It is thin since it is induced by a $\breve{\Gamma}(x_{k-1})$ prewellorder. Moreover we have $x_0 \in U_{s\restriction k} \cap U$. Hence there is a real $x_k \in \mathbb{R} \cap M$ with $x_k \restriction k = s \restriction k$ and $x_k \equiv_i x_0$ for all $1 \leq i < k$ by condition 1 applied to E_k.

This implies $T_\Gamma = T_\Gamma^{\mathbb{R} \cap M}$. ■

Note that condition 2 (a) in the previous theorem corresponds to the absoluteness in the projective case, see theorem 4.1.16.

Remark 4.2.2: *Suppose there are ω Woodin cardinals and a measurable above. Then condition 2 (a) in the previous theorem does not imply $L(\mathbb{R})^M \prec_{\Sigma^2_1} L(\mathbb{R})^V$.*

Proof: We show that $M = M_\omega^\#$ is a counterexample. To check that $M_\omega^\#$ fulfills condition 2 (a) in the previous theorem, suppose $x \in \mathbb{R} \cap M_\omega^\#$ and $A \subseteq \mathbb{R}$ is a nonempty $\Sigma^2_1(x)^{L(\mathbb{R})}$ set. Then A contains a real a so that $\{a\}$ is $\Delta^2_1(x)$ by the argument of [48, theorem 7.20]. Let φ be a Σ^2_1 formula with

$$z = a \Leftrightarrow L(\mathbb{R}) \models \varphi(x, z)$$

for all $z \in \mathbb{R}$. Letting λ be the supremum of the Woodin cardinals of $M_\omega^\#$, we have

$$n \in a \Leftrightarrow \Vdash^{M_\omega^\#}_{Col(\omega,<\lambda)} \exists z(\varphi(x, z) \wedge n \in z)$$

by [48, theorem 7.19]. This implies $a \in M_\omega^\#$.

On the other hand $M_\omega^\#$ believes that there is a wellorder of the reals in its $L(\mathbb{R})$ by [48, corollary 7.21]. The existence of a wellorder of the reals is a Σ^2_1 statement. But there is no such wellorder in $L(\mathbb{R})^V$ since the large cardinal assumption implies $AD^{L(\mathbb{R})^V}$ by [35, theorem 8.24]. ■

4.2.2 $\boldsymbol{\Pi}_1(L(\mathbb{R}))$ equivalence relations and proper forcing

Suppose κ is an infinite cardinal. We use the large cardinal property A_κ from Neeman and Zapletal [36] which states the existence of a class inner model M and a countable ordinal λ with

1. $M = L(V_\lambda^M)$,
2. $M \models$ "λ is the supremum of ω Woodin cardinals", and
3. M is uniquely $\kappa^+ + 1$-iterable.

Neeman and Zapletal showed

Theorem 4.2.3: *(Neeman and Zapletal [36]) Suppose A_κ holds, where κ is an infinite cardinal. If $\mathbb{P}$ is a proper forcing of size $\leq \kappa$ and G is a $\mathbb{P}$-generic filter over V, then there is an elementary embedding $j : L(\mathbb{R})^V \to L(\mathbb{R})^{V[G]}$ which fixes the ordinals.*

Together with Harrington's and Shelah's theorem 3.1.8 this implies

Corollary 4.2.4: *Suppose A_κ holds, where κ is an infinite cardinal. Then proper forcing of size $\leq \kappa$ does not add equivalence classes to thin $(\boldsymbol{\Pi}^2_1)^{L(\mathbb{R})}$ equivalence relations.*

Proof: A_κ implies $\mathsf{AD}^{L(\mathbb{R})}$ by [48, lemma 7.15] and Woodin's theorem [32, theorem 3.1] that if λ is a limit of Woodin cardinals and $\mathbb{R}^*$ is the set of reals of a symmetric collapse below λ, then $L(\mathbb{R}^*) \models \mathsf{AD}$.

Suppose E is a thin $(\boldsymbol{\Pi}^2_1)^{L(\mathbb{R})}$ equivalence relation. Then E is co-λ-Suslin for $\lambda = (\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}$ via the tree T from a $(\boldsymbol{\Sigma}^2_1)^{L(\mathbb{R})}$ scale on the complement of E, since $\mathsf{AD}^{L(\mathbb{R})}$ holds.

If G is generic over V for a proper forcing of size $\leq \kappa$, then there is an elementary embedding

$$j : L(\mathbb{R}) \to L(\mathbb{R})^{V[G]}$$

which fixes the ordinals by the previous theorem. In particular $\mathbb{R}^2 - p[T]$ is an equivalence relation in every Cohen generic extension of V. So there is a wellorder of the equivalence classes of E in $L(\mathbb{R})$ by corollary 3.1.13. Since j fixes the ordinals, there are no new equivalence classes in $V[G]$. ∎

Chapter 5

Conclusion

In this chapter we place the results in context and state several related open problems.

5.1 The context

It is common practice to investigate how mathematical objects differ between models of set theory. For instance equivalence classes of an equivalence relation can vanish in an inner model. In the main theorem 4.1.16 we have described the inner models in which this does not happen to thin projective equivalence relations, and reduced the problem of identifying such an inner model to checking whether it is sufficiently correct and calculates the tree from a scale correctly. These inner models give us information about the structure of the equivalence relations.

At first glance it is quite surprising that large cardinals play a role in the proof of theorems about thin projective equivalence relations which do not mention large cardinals. For instance, the main theorem 4.1.16 can be stated in terms of determinacy without any reference to large cardinals. However, iterable models with large cardinals are essential for the proof of the theorem and it seems unlikely that it can be shown without these techniques.

A crucial point of the whole analysis is that the techniques can be extended beyond the projective sets via suitable premice which capture sets in higher pointclasses. The results in section 3.3 reveal a general pattern in the structure of thin equivalence relations in $L(\mathbb{R})$. Thus the current developement allows us to generalize the known techniques to a higher level.

5.2 Directions for further work

There are several open questions which should be further pursued. The most interesting of these is whether there are analogues of the main theorem for higher levels in $L(\mathbb{R})$.

5.2.1 Extension of the main theorem

The main theorem 4.1.16 leaves open whether a similar characterization is valid for higher pointclasses in $L(\mathbb{R})$. In fact, the proof of the main lemma 4.1.15 does not generalize to thin $\mathbf{\Pi}_1(J_\alpha(\mathbb{R}))$ equivalence relations for the appropriate ordinals α, since the corresponding version of claim 4.1.8 does not go through. We do not know if this works for some pointclasses of the form $\mathbf{\Pi}_n(J_\alpha(\mathbb{R}))$. It is possible that this can be show with a similar argument as for the main theorem by replacing $M_{2n-2}^{\#}$ with a suitable premouse.

5.2.2 The $\mathbf{\Sigma}_{2n+1}^1$ case

Hjorth [10, corollary 2.17] has characterized the inner models which have representatives in all equivalence classes of thin $\mathbf{\Sigma}_1^1$ equivalence relations defined from a parameter in the inner model. He showed that these are exactly the models which calculate ω_1 correctly, provided all reals have sharps. Hjorth's proof and the main theorem might give hints on the corresponding question for thin $\mathbf{\Sigma}_{2n+1}^1$ equivalence relations.

5.2.3 Consistency strength

Various questions related to consistency strength arise, for instance whether the conclusion of theorem 3.2.10 implies the existence of $M_{2n-1}^{\#}$. It would be of interest to know if the property that reasonable forcing does not add equivalence classes to thin projective equivalence relations has large cardinal strength, for example whether it implies the existence of $0^{\#}$. It is conceivable that this property holds in Woodin's model of projective absoluteness.

5.2.4 Projective ordinals

The method of theorem 4.1.31 for constructing an inner model which calculates projective ordinals correctly only works for projective ordinals below ω_2. An approach to build such an inner model for projective ordinals above ω_2 could be to construct a model from a directed system of iterates of $M_n^\#$. Regarding the overall problematic, it appears of great interest to construct canonical inner models of this kind.

The techniques provided in this dissertation might prove helpful tools for the solution of these problems. Future work will further expand our knowledge about equivalence relations in $L(\mathbb{R})$ beyond what is known today.

Bibliography

[1] Joan Bagaria and Roger Bosch. Projective forcing. *Ann. Pure Appl. Logic*, 86(3):237–266, 1997.

[2] Tomek Bartoszyński and Haim Judah. *Set theory. On the structure of the real line.* A K Peters Ltd., Wellesley, MA, 1995.

[3] Jon Barwise. *Admissible sets and structures.* Perspectives in Mathematical Logic. Springer, Berlin, 1975.

[4] Matthew Foreman and Menachem Magidor. Large cardinals and definable counterexamples to the continuum hypothesis. *Ann. Pure Appl. Logic*, 76(1):47–97, 1995.

[5] Gunter Fuchs, Itay Neeman, and Ralf Schindler. A criterion for coarse iterability. submitted. available at *http://wwwmath.uni-muenster.de/logik/Personen/fuchs*.

[6] Leo Harrington. Long projective wellorderings. *Ann. Math. Logic*, 12(1):1–24, 1977.

[7] Leo Harrington. Analytic determinacy and $0^\sharp$. *J. Symbolic Logic*, 43(4):685–693, 1978.

[8] Leo Harrington and Ramez Sami. Equivalence relations, projective and beyond. In *Logic Colloquium '78 (Mons, 1978)*, volume 97 of *Stud. Logic Foundations Math.*, pages 247–264. North-Holland, Amsterdam, 1979.

[9] Leo Harrington and Saharon Shelah. Counting equivalence classes for co-κ-Souslin equivalence relations. In *Logic Colloquium '80 (Prague, 1980)*, volume 108 of *Stud. Logic Foundations Math.*, pages 147–152. North-Holland, Amsterdam, 1982.

[10] Greg Hjorth. Thin equivalence relations and effective decompositions. *J. Symbolic Logic*, 58(4):1153–1164, 1993.

[11] Greg Hjorth. A dichotomy for the definable universe. *J. Symbolic Logic*, 60(4):1199–1207, 1995.

[12] Greg Hjorth. Some applications of coarse inner model theory. *J. Symbolic Logic*, 62(2):337–365, 1997.

[13] Greg Hjorth. Glimm-Effros for coanalytic equivalence relations. preprint, 2006. available at *http://www.math.ucla.edu/~greg/research.html*.

[14] Greg Hjorth and Alexander S. Kechris. Analytic equivalence relations and Ulm-type classifications. *J. Symbolic Logic*, 60(4):1273–1300, 1995.

[15] Steve Jackson. A computation of δ^1_5. *Mem. Amer. Math. Soc.*, 140(670), 1999.

[16] Thomas Jech. *Set theory. The third millennium edition, revised and expanded*. Springer Monographs in Mathematics. Springer, Berlin, 2003.

[17] Akihiro Kanamori. *The higher infinite. Large cardinals in set theory from their beginnings*. Perspectives in Mathematical Logic. Springer, Berlin, 1994.

[18] Alexander S. Kechris. Forcing in analysis. In *Higher set theory (Proc. Conf., Math. Forschungsinst., Oberwolfach, 1977)*, volume 669 of *Lecture Notes in Math.*, pages 277–302. Springer, Berlin, 1978.

[19] Alexander S. Kechris. On transfinite sequences of projective sets with an application to $\mathbf{\Sigma}^1_2$ equivalence relations. In *Logic Colloquium '77 (Proc. Conf., Wrocław, 1977)*, volume 96 of *Stud. Logic Foundations Math.*, pages 155–160. North-Holland, Amsterdam, 1978.

[20] Alexander S. Kechris. AD and projective ordinals. volume 689 of *Lecture Notes in Math.*, pages 91–132. Springer, Berlin, 1978.

[21] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer, New York, 1995.

[22] Alexander S. Kechris and Robert M. Solovay. On the relative consistency strength of determinacy hypotheses. *Trans. Amer. Math. Soc.*, 290(1):179–211, 1985.

[23] Peter Koellner and W. Hugh Woodin. Large cardinals from determinacy. In Matthew Foreman and Akihiro Kanamori, editors, *Handbook of Set Theory*. Springer, Berlin, to appear.

[24] Kenneth Kunen. *Set theory. An introduction to independence proofs*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1980.

[25] Alain Louveau and Christian Rosendal. Complete analytic equivalence relations. *Trans. Amer. Math. Soc.*, 357(12):4839–4866 (electronic), 2005.

[26] Donald A. Martin. Measurable cardinals and analytic games. *Fund. Math.*, 66:287–291, 1969/1970.

[27] Donald A. Martin and John R. Steel. The extent of scales in $L(\mathbb{R})$. In *Cabal seminar 79–81*, volume 1019 of *Lecture Notes in Math.*, pages 86–96. Springer, Berlin, 1983.

[28] Donald A. Martin and John R. Steel. A proof of projective determinacy. *J. Amer. Math. Soc.*, 2(1):71–125, 1989.

[29] Donald A. Martin and John R. Steel. Iteration trees. *J. Amer. Math. Soc.*, 7(1):1–73, 1994.

[30] William J. Mitchell and John R. Steel. *Fine structure and iteration trees*, volume 3 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, 1994.

[31] Yiannis N. Moschovakis. *Descriptive set theory*, volume 100 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1980.

[32] Itay Neeman. Optimal proofs of determinacy. *Bull. Symbolic Logic*, 1(3):327–339, 1995.

[33] Itay Neeman. Optimal proofs of determinacy. II. *J. Math. Log.*, 2(2):227–258, 2002.

[34] Itay Neeman. Inner models and ultrafilters in $L(\mathbb{R})$. *Bull. Symbolic Logic*, 13(1):31–53, 2007.

[35] Itay Neeman. Determinacy in $L(\mathbb{R})$. In Matthew Foreman and Akihiro Kanamori, editors, *Handbook of Set Theory*. Springer, Berlin, to appear. available at *http://www.math.ucla.edu/~ineeman*.

[36] Itay Neeman and Jindřich Zapletal. Proper forcing and $L(\mathbb{R})$. *J. Symbolic Logic*, 66(2):801–810, 2001.

[37] Ernest Schimmerling. A core model toolbox and guide. In Matthew Foreman and Akihiro Kanamori, editors, *Handbook of Set Theory*. Springer, Berlin, to appear. available at *http://www.math.cmu.edu/~eschimme/papers/papers.html*.

[38] Ralf Schindler. The role of absoluteness and correctness. slides from the Barcelona conference on set theory, Sept 16-20, 03, 1996. available at *http://wwwmath.uni-muenster.de/logik/Personen/rds*.

[39] Ralf Schindler and John R. Steel. The strength of AD. preprint, 2006. available at *http://wwwmath.uni-muenster.de/logik/Personen/rds*.

[40] Ralf Schindler and John R. Steel. *The core model induction*. in preparation. available at *http://wwwmath.uni-muenster.de/logik/Personen/rds*.

[41] Ralf Schindler and John R. Steel. The self-iterability of $L[E]$. submitted. available at *http://wwwmath.uni-muenster.de/logik/Personen/rds*.

[42] Ralf Schindler and Martin Zeman. Fine structure. In Matthew Foreman and Akihiro Kanamori, editors, *Handbook of Set Theory*. Springer, Berlin, to appear. available at *http://wwwmath.uni-muenster.de/logik/Personen/rds*.

[43] Saharon Shelah. On co-κ-Souslin relations. *Israel J. Math.*, 47(2-3):139–153, 1984.

[44] Jack H. Silver. Counting the number of equivalence classes of Borel and coanalytic equivalence relations. *Ann. Math. Logic*, 18(1):1–28, 1980.

[45] John R. Steel. Scales in $L(\mathbb{R})$. In *Cabal seminar 79–81*, volume 1019 of *Lecture Notes in Math.*, pages 107–156. Springer, Berlin, 1983.

[46] John R. Steel. Projectively well-ordered inner models. *Ann. Pure Appl. Logic*, 74(1):77–104, 1995.

[47] John R. Steel. Woodin's analysis of $HOD^{L(\mathbb{R})}$. preprint, 1996. available at *http://math.berkeley.edu/~steel/papers/Publications.html*.

[48] John R. Steel. An outline of inner model theory. In Matthew Foreman and Akihiro Kanamori, editors, *Handbook of Set Theory*. Springer, Berlin, to appear. available at *http://math.berkeley.edu/~steel/papers/Publications.html*.

[49] John R. Steel. Scales in $K(\mathbb{R})$. to appear in the reissue of the Cabal Seminar. available at *http://math.berkeley.edu/~steel/papers/Publications.html*.

[50] John R. Steel and Robert Van Wesep. Two consequences of determinacy consistent with choice. *Trans. Amer. Math. Soc.*, 272(1):67–85, 1982.

[51] John R. Steel and Philip D. Welch. Σ^1_3 absoluteness and the second uniform indiscernible. *Israel J. Math.*, 104:157–190, 1998.

[52] W. Hugh Woodin. On the consistency strength of projective uniformization. In *Proceedings of the Herbrand symposium (Marseilles, 1981)*, volume 107 of *Stud. Logic Found. Math.*, pages 365–384. North-Holland, Amsterdam, 1982.

[53] W. Hugh Woodin. *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, volume 1 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 1999.

[54] Martin Zeman. *Inner models and large cardinals*, volume 5 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 2002.