Continuous reducibility and dimension

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August 15, 2012

Abstract

Let us consider a positive-dimensional metric space, i.e. at some point there is no clopen local base. We construct a family of size continuum of Borel subsets of the metric space so that any two sets are incomparable with respect to continuous reducibility.

1 Introduction

In various contexts, instead of analyzing a given structure on a metric space, it is useful to describe how it relates to other structures via reducibility. The notion of reducibility is given by a class of functions, for example, Borel measurable, Lebesgue measurable, or ω-universally Baire measurable reducibility of equivalence relations on Polish spaces, continuous reducibility (Wadge reducibility) of subsets of the Baire space, Turing reducibility or polynomial time reducibility of sets of natural numbers.

Reducibility of Borel subsets of zero-dimensional Polish spaces is well understood for Lipschitz functions, continuous functions, $\mathbf{\Delta}^{\mathbf{0}}_{\alpha}$-functions, Borel measurable functions, and other interesting classes of Borel measurable functions [1, 6]. The results for Borel measurable functions hold for all uncountable Polish spaces via Borel isomorphisms. Let us consider continuous reductions on metric spaces.

Definition. *Suppose that X is a metric space and $A, B \subseteq X$.*

- *A is* continuously reducible *to B ($A \leq^X B$ or $A \leq B$) if there is a continuous map $f\colon X \to X$ with $A = f^{-1}(B)$.*
- *A and B are* equivalent *($A \equiv^X B$ or $A \equiv B$) if $A \leq B$ and $B \leq A$.*

- *A and B are* incomparable *($A \perp^X B$ or $A \perp B$) if $A \not\leq B$ and $B \not\leq A$.*

Let us consider the class Borel^X of Borel subsets of X up to continuous reducibility. A key result for zero-dimensional Polish spaces is

Lemma (Martin-Wadge)**.** *Suppose that X is a zero-dimensional Polish space.*

- *$A \leq B$ or $B \leq X \setminus A$ for all Borel sets $A, B \subseteq X$, i.e.* Borel^X *is* semi-linearly ordered.

- *There are no three pairwise incomparable Borel subsets of X.*

The first part follows from Borel determinacy, but notice that Louveau proved Wadge determinacy in second-order arithmetic. So we obtain a linear order by identifying equivalent sets and complements. This fails for the real line.

Lemma (Andretta [1])**.** *There are three pairwise incomparable Borel subsets of $\mathbb{R}$.*

In this paper, we extend this to all positive-dimensional metric spaces. Let Γ denote the class of intersections of an open set and a closed set. Let us call $A \subseteq \mathcal{P}(X)$ an *antichain* if its elements are pairwise incomparable.

Theorem 1.1. *(ZF) For every positive-dimensional metric space, there is an antichain in Γ of size continuum.*

We prove this in the next section. In the last section, we prove that for any locally compact metric space, there is an antichain of subsets of size the power set of the continuum.

2 The construction

To prove Theorem 1.1, let X denote a metric space of positive dimension and suppose that there is no clopen local base at $b \in X$. There is an open neighborhood U of b such that no clopen neighborhood of b is contained in U. We choose $r \in \mathbb{R}^+$ with $B_r(b) \subseteq U$ and strictly increasing sequences $(r_n)_{n\in\mathbb{N}}$ with $r_0 = 0$, $r_n < r_{n+1}$ for all n, and $\sup_n r_n = r$.

Definition 2.1.

- *Let $B_n = B_{r_{2n+1}}(b) \setminus B_{r_{2n}}(b)$.*

- *Let $C_n = \overline{B}_{r_{2n+1}}(b) \setminus \overline{B}_{r_{2n}}(b)$.*

- *If $s = (m_i)_i$ is a strictly increasing sequence of natural numbers, let*
  - $s_i = [m_i, m_{i+1}) \cap \mathbb{N}$ *for* $i \in \mathbb{N}$,
  - $s^{\uparrow} = \bigcup_{i \in 2\mathbb{N}} s_i$,
  - $s^{\downarrow} = \bigcup_{i \in 2\mathbb{N}+1} s_i$,
  - $D^s_n = B_n$ *if* $n \in s^{\uparrow}$,
  - $D^s_n = C_n$ *if* $n \in s^{\downarrow}$,
  - $U^s_n = B_{r_{2n+3}} \setminus (B_{r_{2n}} \cup D^s_n \cup D^s_{n+1})$,
  - $A_s = \bigsqcup_{n \in \mathbb{N}} D_n$.

The sets A_s and A_t will be incomparable for appropriately chosen sequences $s = (m_i)$ and $t = (n_i)$. To see when A_s is reducible to A_t, we will capture combinatorial information about potential reductions of A_s to A_t by sets of graph reductions.

Notation and Definition 2.2. *Suppose that G, H are graphs.*

- *The vertex and edge sets of G are denoted by V_G, E_G.*
- $\bar{G} = V_G \cup E_G$.
- *A graph homomorphism $f\colon G \to H$ is identified with the induced map on $\bar{G}$.*
- *A graph G together with a 2-coloring $c_G\colon \bar{G} \to 2$ is called a* colored graph.
- *A* reduction $f\colon G \to H$ of colored graphs *is a graph homomorphism with $c_G(t) = c_H(f(t))$ for all $t \in \bar{G}$.*
- *A colored graph G is* good *if*
  - $V_G \in \mathbb{N}$ *and*
  - $E_G = \{(n, n+1) \mid n+1 \in V_G\}$.
- *A colored graph G is* nice *if*
  - $V_G = \mathbb{N}$,
  - $E_G = \{(n, n+1) \mid n \in \mathbb{N}\}$,
  - $c_G((n, n+1)) = 1$ *if and only if n is even and $n+1 \in V_G$, and*
  - *if n is even, then either $c_G(n) = 1$ or $c_G(n+1) = 1$.*

- *If G is a colored graph, we define G^* by attaching an additional* 0*-colored loop at every vertex of G.*

- *Suppose that F, G are nice. An* unfolding *of F, G is a a pair $f\colon H \to F^*$, $g\colon H \to G^*$ of reductions of colored graphs, where H is good.*

Let us consider sequences $s = (m_i)_{i \in \mathbb{N}}$ and $t = (n_i)_{i \in \mathbb{N}}$ in $\mathbb{N}$ in the next definitions.

Definition 2.3.
- *Let G_s denote the unique nice graph with $c_G(2n) = 1$ if and only if $c_G(2n+1) = 0$ if and only if $n \in s^{\uparrow}$ (analogously for G_t).*

- *Let $D_n = D_n^s$, $E_n = E_n^t$, $U_n = U_n^s$, $V_n = U_n^t$.*

- *Let $d_n^s = \{2n, (2n, 2n+1)\}$ if $n \in s^{\uparrow}$ and $d_n^s = \{(2n, 2n+1), 2n+1\}$ if $n \in s^{\downarrow}$. Let $d_n = d_n^s$ and $e_n = d_n^t$. We define u_n^s, u_n, v_n analogously.*

Notice the exact correspondences for the pairs D_n, d_n and U_n, u_n. We assume from now on that $h\colon X \to X$ is a continuous reduction of A_s to A_t.

Definition 2.4. *(Matching points) Let $l \in \mathbb{N}$.*

- *if $f\colon H \to G_s^*$, $g\colon H \to G_t^*$ is an unfolding, let* $\mathrm{Match}_{f,g}^l$ *denote the set of $x \in X$ such that*
  - *there is n with $f(n) = 0$ and $g(n) = l$,*
  - *if $x \in D_m$, then there are $k \in \mathbb{N}$, $v \in \bar{H}$ with $h(x) \in E_k$, $f(v) \in d_m$, and $g(v) \in e_k$,*
  - *if $x \in U_m$, then there are $k \in \mathbb{N}$, $v \in \bar{H}$ with $h(x) \in V_k$, $f(v) \in u_m$, and $g(v) \in v_k$,*

  In this case we say that x, f, g, l match.

- *Let* Match^l *denote the union of* $\mathrm{Match}_{f,g}^l$ *where f, g is an unfolding of G_s^*, G_t^*.*

Lemma 2.5. Match^l *is relatively clopen in $B_r(b)$ for all $l \in \mathbb{N}$.*

Proof. It follows from the definition that Match^l is relatively open in $B_r(b)$.

To show that Match^l is relatively closed in $B_r(b)$, suppose that $x = \lim_{i \to \infty} x_i$, $x_i \in \mathrm{Match}^l$ for all i, and $x, x_i \in B_r(b)$ for all i. We can assume that there is n with $x_i \in D_n$ for all i or $x_i \in U_n$ for all i. There is an unfolding f_0, g_0 such that x_0, f_0, g_0, l match.

If $x_i \in D_n$ for all i, then it is easy to see that x, f_0, g_0, l match. Let us suppose that $x_i \in U_n$ for all i. If $x \in U_n$, then it is again easy to see that x, f_0, g_0, l match.

Suppose that $x \in D_n \cap S_{r_{2n+1}}$, $h(x) \in S_{r_{2k+1}}$, and $2k+1 \notin f_0(g_0^{-1}(2n+1))$. Then $2k+2 \in f_0(g_0^{-1}(2n+1))$. We define g from g_0 by unfolding a 0-colored loop at $2n+1$ into two edges. We define f from f_0 by mapping those edges to $(2k+2, 2k+1)$ and $(2k+1, 2k+2)$. It follows that x, f, g, l match.

The remaining cases are symmetric. $\square$

Let $\Delta n_i = n_{i+1} - n_i$ if $(n_i)_{i\in\mathbb{N}}$ is a sequence in $\mathbb{N}$. Let us now suppose that $(m_i)_i$, $(n_i)_i$, $(\Delta m_i)_i$, and $(\Delta n_i)_i$ are strictly increasing.

Definition 2.6. *(Unfolded pairs) Let uf_l denote the set of pairs (n, k) such that there is an unfolding f, g of G_s, G_t and m, i with $f(i) = 0$, $f(m) = n$, $g(i) = l$, and $g(m) = k$.*

Lemma 2.7. *If $l \in \mathbb{N}$, there are $j \in \mathbb{N}$, $m \in \mathbb{Z}$ so that for all $k \in \mathbb{N}$ and $n \geq j$, $(n, k) \in uf_l$ if and only if $k = n + m$.*

Proof. Let i be least with $l < n_i$. Since (Δm_i), (Δn_i) are strictly increasing, it is easy to see that there is j with $(j, n_i) \in uf_l$. Let j be the least such. It follows that $j = n_k$ for some k. Let $m = n_i - j$. Since the sequences and their differences are strictly increasing, it is easy to check that j, m have the desired property. $\square$

Let F_t denote the tail equivalence relation on ${}^\omega\omega$, i.e. $(x, y) \in F_t$ if there are m and n with $x(m+i) = x(n+i)$ for all i.

Lemma 2.8. *If $A_s \leq A_t$, then (Δm_i), (Δn_i) are tail equivalent.*

Proof. We consider a continuous reduction $h\colon X \to X$ of A_s to A_t as above. Suppose that $h(b) \in E_l$. Then $b \in \mathrm{Match}^l$. Recall that b has no clopen neighborhood $V \subseteq U$ by the choice of U and that Match^l is relatively clopen in $B_r(b)$ by Lemma 2.5. So we may choose $x_n \in \mathrm{Match}^l \cap S_{r_n}$ and witnesses f_n, g_n for every n. It follows from the previous lemma that (Δm_i), (Δn_i) are tail equivalent. $\square$

Notice that the set S of strictly increasing sequences $(n_i)_i$ is closed in ${}^\omega\omega$. Since the equivalence classes of F_t are countable, there is a perfect subset of S of F_t-inequivalent sequences. Let T denote the set of sequences of partial sums of sequences in the perfect set. Then $A_s \not\leq A_t$ for all $s \neq t$ in T. This completes the proof of Theorem 1.1.

We do not know if the result can be improved to sets in Γ with complements in Γ.

Remark 2.9. *1. There is a compact connected metric space $X \subseteq \mathbb{R}^3$ whose subsets $\neq \emptyset, X$ form an antichain [2, Theorem 11], so more assumptions are necessary to embed other posets (see [4] and Corollary 3.3).*

2. *The proof of Theorem 1.1 works for continuity instead of sequential continuity; instead of metrizability it is sufficient to assume that X is normal, but in this case* DC *is used to choose the sequence of neighborhoods of b.*

3. *There is an infinite-dimensional countable quasi-Polish space X such that* Borel^X *is almost well-ordered [7, Remark 5.33]. Hence Theorem 1.1 fails for some non-normal spaces.*

3 Corollaries

Let us conclude with three corollaries to the proof.

Corollary 3.1. *The following are equivalent for a Polish space X:*

1. *X is zero-dimensional.*
2. Borel^X *is semi-linearly ordered.*
3. Borel^X *contains no infinite antichains.*
4. Borel^X *contains no uncountable antichains.*
5. Borel^X *is well-quasi-ordered, i.e. it contains no infinite decreasing sequences and no infinite antichains.*

Proof. Continuous reducibility for Borel subsets of zero-dimensional Polish spaces is well-founded by a result of Martin-Monk [5, Theorem 21.15]. □

Corollary 3.2. *Suppose that X is a locally compact metric space of positive dimension. Then $\mathcal{P}(X)$ contains an antichain of size the power set of the continuum.*

Proof. We choose a positive-dimensional open set $V \subseteq X$ with compact closure. Let $z \subseteq X$ denote the set of points in X with no clopen local base and let $nz = X \setminus z$. Let z^V denote the set of points in V with a clopen neighborhood $W \subseteq V$ and $nz^V \setminus z^V$.

Notice that the closure $cl(nz) \cap V$ is uncountable; otherwise V is the union of a countable family of zero-dimensional disjoint closed subsets, but this cannot be the case since V has positive dimension (see [3, Theorem II.2]).

Let us choose $C \subseteq cl(nz) \cap V$ nowhere dense and homeomorphic to the Cantor space. We further choose $b_n \in nz \setminus C$ and $(r_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^+$ so that for $B = \{b_n \mid n \in \mathbb{N}\}$ and for all $m \neq n$

- $\overline{B}_{r_n}(b_n) \subseteq V$,
- $B_{r_m}(b_m) \cap B_{r_n}(b_n) = \emptyset$,
- $B_{r_n}(b_n) \cap C = \emptyset$,
- $b_n \in nz^{B_{r_n}(b_n)}$, and
- $cl(B) \supseteq C$.

Then $\lim_n r_n = 0$ since $cl(V)$ is compact. Let $(t_n)_{n \in \mathbb{N}}$ denote a sequence of distinct elements of the set T in the previous section. As in Definition 2.1, we now choose $r_{n,i}$ with $r_{n,i} < r_{n,i+1}$ and $\sup_{i \in \mathbb{N}} r_{n,i} = r_n$ and define $A_{t_n} \subseteq B_{r_n}(b_n)$. Suppose that $f\colon {}^\omega\omega \to C$ is injective and define for $x \subseteq {}^\omega\omega$

$$A_x = f[x] \sqcup \bigsqcup_{n \in \mathbb{N}} A_{t_n}.$$

Claim. *If $x \neq y$, then $A_x \not\leq A_y$.*

Proof. Suppose that $h\colon X \to X$ is a continuous reduction of A_x to A_y and $n \in \mathbb{N}$. Our aim is to show that $f(a) \in A_{t_m}$ for some m and some $a \in nz^{B_{r_n}(b_n)}$. Then $m = n$ by the proof of Theorem 1.1. Since $cl(B) \supseteq C$, this implies $h \restriction C = id \restriction C$ and hence $x = y$.

To show the existence of a, let $z^{B_r(b)} = \bigcup_{i \in \mathbb{N}} U_i$ with $U_i \subseteq B_{r_n}(b_n)$ clopen and the radii converging to 0.

Let us argue that there is no clopen in $cl(nz^{B_{r_n}(b_n)})$ neighborhood $U \subseteq nz^{B_{r_n}(b_n)}$ of b_n. Otherwise we can separate U and $nz^{B_{r_n}(b_n)} \setminus U$ by disjoint open sets $I, J \subseteq V$ and $cl(I) \subseteq V$ using compactness. Then $W = U \cup \bigcup_{U_i \cap I \neq \emptyset} U_i$ is closed in V by the choice of I. So $W \subseteq B_{r_n}(b_n)$ is a clopen neighborhood of

b_n, contradicting the assumption on b_n. Hence $nz^{B_{r_n}(b_n)}$ contains points in all distances $< r_n$ to b_n.

Towards a contradiction, assume that there is no $a \in nz^{B_{r_n}(b_n)}$ with $f(a) \in A_{t_m}$ for some m. Then $f(b_n) \in C$ and $f^{-1}(X) \cap S_{r_{n,1}}(b_n) \neq \emptyset$ for all clopen in C neighborhoods $X \subseteq C$ of $f(b_n)$ in C by the preceding argument. Since $S_{r_{n,1}}$ is compact, there is $a \in S_{r_{n,1}}$ with $f(a) = f(b_n)$. Since $b_n \in A_x$ if and only if $a \notin A_x$ by Definition 2.1, this contradicts our assumption that f is a reduction of A_x to A_y. □

This completes the proof of Corollary 3.2. □

We do not know if local compactness is necessary in Corollary 3.2. We refer to [4] for embedding results for continuous reducibility for the real line but note that we obtain

Corollary 3.3. *Suppose that $X = \bigsqcup_{n \in \mathbb{N}} X_n$ is metric and each X_n is clopen in X and positive-dimensional. Then $\subseteq_{\mathcal{P}(\omega)}$ embeds into Γ^X.*

Proof. We choose distinct $s_n \in T$ and define $A_{s_n} \subseteq X_n$ as in the previous section.

Let $A_x = \bigsqcup_{n \in x} A_{s_n}$ for $x \subseteq \omega$. Then $A_x \leq A_y$ implies $x \subseteq y$ as in the proof of Lemma 2.8. Moreover, it is easy to construct a continuous reduction for $A_x \leq A_y$ if $x \subseteq y$. □

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