The recognizability strength of infinite time Turing machines with ordinal parameters

Merlin Carl and Philipp Schlicht

Fachbereich Mathematik und Statistik, Universität Konstanz and
Lehrstuhl für theoretische Informatik, Universität Passau
Mathematisches Institut, Universität Bonn

Abstract. We study infinite time Turing machines that attain a special state at a given class of ordinals during the computation. We prove results about sets that can be recognized by these machines. For instance, the recognizable sets of natural numbers with respect to the cardinal-detecting infinite time Turing machines introduced in [Hab13] are contained in a countable level of the constructible hierarchy, and the recognizable sets of natural numbers with respect to finitely many ordinal parameters are constructible.

1 Introduction

Since the introduction of Infinite Time Turing Machines in [HL00], a variety of machine models of infinitary computability has been introduced: Turing machines that work with time and space bounded by an ordinal α, or with tape of length α but no time limit, or with tape and time both of length On; register machines that work in transfinite time and can store natural numbers or arbitrary ordinals in their registers; and so on.

A common feature to all these machine models of infinitary computations is that they are strongly linked to Gödel's constructible universe L. Since their operations are absolute between V and L, all objects that are writable by such machines are constructible. However, as was shown in [CSW], these machines can in a sense deal with objects far beyond L when one considers recognizability instead of computability, i.e. the ability of the machine to identify some real number x given in the oracle instead of producing x on the empty input. The model considered in [CSW] were Koepke's ordinal Turing machines (OTMs) with ordinal parameters.

A natural next step is then to determine how strong a machine type has to be to allow the recognizability of non-constructible real numbers.This motivates the question whether other of these models also have such strong properties.

It is relatively easy to deduce from Shoenfield's absoluteness theorem that, without ordinal parameters, recognizability for all machine types is restricted to a certain countable level L_σ of Gödel's constructible hierarchy L. The question must hence be what happens when we equip the other models with ordinal parameters. For *OTMs*, an ordinal parameter α is given to the machine by simply marking the αth cell of the working tape.

For *ITTMs* with the tape length ω, ordinal parameters cannot be introduced in this way. However, there is a rather natural way to make *ITTMs* work with ordinal parameters, first introduced in [Hab13]. Namely, we introduce a new inner machine state that is assumed whenever the current time is an element of a given class X of ordinals, and one hence marks one or several points of time instead of tape cells. In this way, an *ITTM* can be made to work relative to an arbitrary class of ordinals, where singletons correspond to single ordinal parameters. In this case, we will speak of X*-ITTMs* or *ordinal-detecting ITTMs*. For the class of cardinals X, these are the *cardinal-detecting ITTMs* studied in [Hab13].

In this paper, we study the recognizability strength of cardinal detecting *ITTMs* and more generally of ordinal-detecting *ITTMs*. We show that every recognizable set of natural numbers with respect to cardinal-detecting *ITTMs* is an element of L_σ, the first level of the constructible hierarchy where every Σ_1-statement that is true in L is already true (Theorem 9). However, these machines can recognize more real numbers than mere *ITTMs* (Lemma 12). Moreover, we show that every recognizable set of natural numbers with respect to *ITTMs* with finitely many ordinal parameters is constructible (Theorem 21). However, these machines recognize some sets of natural numbers outside of L_σ for certain ordinals α (Lemma 19). We conclude that even with ordinal parameters, *ITTM*-recognizability does not lead out of L.

2 Basic Notions and Results

Infinite Time Turing Machines, introduced by Hamkins and Kidder (see [HL00]), generalize Turing computability to transfinite working time. Their computations work like ordinary Turing computations at successor times, while the tape content at limit times is obtained as a cell-wise inferior limit of the sequence of earlier contents and the inner state at limit times is a special limit state. For details, we refer to [HL00].

There are various notions of computability associated with *ITTMs.*

Definition 1. Suppose that x and y are subsets of ω.

1. x is *writable* in the oracle y if and only if there is an *ITTM*-program P such that $P^y \downarrow= x$, i.e. P, run in the oracle y, halts with x on the output tape.
2. x is *eventually writable* in the oracle y if and only there is an *ITTM*-program P such that P, when run in the oracle y on the empty input, eventually has x on its output tape and never changes it again.
3. x is *accidentally writable* in the oracle y if and only there is an *ITTM*-program P such that P, when run in the oracle y on the empty input, has x on its output tape at some point, but may overwrite it later on.

The *ITTM*-recognizable sets are defined as follows.

Definition 2. Suppose that x and y are subsets of ω.

1. x is *ITTM-recognizable* or simply *recognizable* relative to y if and only if there is an *ITTM*-program P such that, for all subsets z of ω, $P^{z\oplus y} \downarrow= \delta_{z,x}$, where δ is the Kronecker symbol.
2. x is *non-deterministically ITTM-recognizable* if and only if there is a subset y of ω such that $x \oplus y$ is *ITTM*-recognizable.
3. The *recognizable closure*, denoted by $\mathcal{R}$, is the closure of the empty set under relativized recognizability.

We will call sets of natural numbers *reals.* The following alternative characterization of the recognizable closure works rather generally for models of infinite computation.

Lemma 3. The non-deterministically *ITTM*-recognizable reals are exactly those in $\mathcal{R}$.

Proof. If $x \oplus y$ is *ITTM*-recognizable, then clearly $x \in \mathcal{R}$. Suppose on the other hand that $x \in \mathcal{R}$. Then there is a sequence $\langle x_0, \ldots, x_n\rangle$ with $x = x_0$ such that x_n is *ITTM*-recognizable and x_i is *ITTM*-recognizable relative to x_{i+1} for all $i < n$. It is easy to see that the join $\bigoplus_{i\leq n} x_i$ is *ITTM*-recognizable by first identifying the last component and then successively the previous components.

It was observed in [CSW, Lemma 3.2] that $\mathcal{R} = L_\sigma$, where σ is least with the property that $L_\sigma \prec_{\Sigma_1} L$ or equivalently least with the property that every Σ_1-statement that it true in L is already true in L_σ.

We will need the following results.

Theorem 4. Suppose that y is a subset of ω.

1. [Wel09, Fact 2.4 & Fact 2.5 & Fact 2.6] There are countable ordinals $\lambda^y, \zeta^y, \Sigma^y$ with the following properties for all subsets x of ω.
   (a) x is writable if and only if $x \in L_{\lambda^y}[y]$.
   (b) x is eventually writable if and only if $x \in L_{\zeta^y}[y]$.
   (c) x is accidentally writable if and only if $x \in L_{\Sigma^y}[y]$.
2. [Wel14, p.11-12] An ITTM-program in the oracle y will either halt in strictly less than λ^y many steps, or it will run into an ever-repeating loop, repeating the sequence of configurations between ζ^y and Σ^y, that is of order-type Σ^y, from Σ^y on.
3. [Wel09, Theorem 1 & Corollary 2] [Wel14, Theorem 3] The triple $\langle \lambda^y, \zeta^y, \Sigma^y \rangle$ is the lexically least triple $\langle \alpha, \beta, \gamma \rangle$ of distinct ordinals with $L_\alpha \prec_{\Sigma_1} L_\beta \prec_{\Sigma_2} L_\gamma$.

Remark 5. An important result for *ITTMs*, and many other models of infinite computation, is the existence of *lost melodies*, i.e. real numbers that are writable, but not *ITTM*-recognizable. The existence of lost melodies for *ITTMs* was proved in [HL00]. For more on lost melodies for other machine types, see [Car14a,Car14b,Car15,CSW].

We now define how an infinite time Turing machine works relative to a class of ordinals.

Definition 6. For classes X of ordinals, an *X-ITTM* works like an *ITTM* with the modification that the machine state is a special reserved state if and only if the running time is an element of X. If an *ITTM*-program P is run relative to a class X, we will write X-P instead of P. When X consist of the ordinals in $\boldsymbol{\alpha} = \langle \alpha_0, \dots, \alpha_n \rangle$, we will write $\boldsymbol{\alpha}$-*ITTM* for the machine with special states at times in $\boldsymbol{\alpha}$, and for $\boldsymbol{\alpha} = \langle \alpha \rangle$ simply α-*ITTM*.

We thus obtain the *cardinal-detecting ITTMs* of [Hab13], called *cardinal-recognizing ITTMs* there, for the class of all cardinals. Note that there are several variants of *ITTMs*, for instance the original definition in [HL00] with three tapes of input, output and scratch, that we use here, its variant with only one tape, and the variant where at limit times, the head is set to the inferior limit of the previous head positions, instead of moving to the first tape cell. All proofs in this paper can be easily modified to work for each of these variants.

We take the opportunity to answer in the negative [Hab13, Question 10], which asked whether every real number accidentally writable by a cardinal-recognizing *ITTM* is also accidentally writable by a plain *ITTM*.

Theorem 7. There is a cardinal-recognizing *ITTM*-program that writes a code for Σ.

Proof. Let U denote an *ITTM*program that simulates al *ITTM*-programs simultaneously. The configuration c of U at time ω_1, when the special state is assumed for the first time, is the same as at time Σ and at time ζ [Wel14, p.11-12]. If c would occur prior to ζ, then U would start looping before time ζ, contradiction the assumption that U simulates all *ITTM*-programs. Hence c is accidentally writable, but not eventually writable. The L-least code for Σ is *ITTM*-writable from every real that is accidentally writable, but not eventually writable by the proof of [CH11, Proposition 4.6], Hence there is an *ITTM*-program P that computes a code for Σ from c. We now run P on the tape content of U when the special state is assumed for the first time. This program will halt with a code for Σ on the output tape, as required.

3 Recognizable reals relative to cardinals

In this section, we will determine the recognizable closure for cardinal-detecting *ITTMs*. It is easy to see that the parameter ω does not add recognizability strength. We begin by considering *ITTMs* with uncountable parameters.

Theorem 8. Every subset x of ω that is $\omega_1\alpha$-recognized by P for some ordinal α is an element of L_σ.

Proof. Suppose that P is a program that recognizes x and P^x halts with the final state s. Suppose that y is a subset of ω. The state of P^y at time ω_1 will be the same as at time Σ^y by Theorem 4. Moreover, between Σ^y and ω_1, the computation repeats a loop of length Σ^y by Theorem 4. Thus the state of P^y at time ω_1 is the same as at time Σ_y. Consequently, the computation will continue exactly the same whether the new inner state s is assumed at time $\omega_1\alpha$ or at time Σ^y. Since P recognizes x, ω_1-P^y and Σ^y-P^y both halt with the same output and the same final state s.

Let $c_{y,\alpha}$ denote the $L[y]$-least code for α, if α is countable in $L[y]$. We argue that the halting time of Σ^y-P^y is strictly less than $\lambda^{y\oplus c}$ for $c = c_{y,\Sigma^y}$. The tape content z of P^y at time Σ^y is accidentally writable in y and hence z is an element of $L_{\Sigma^y}[y]$. Therefore z is writable from $y \oplus c$ and $\lambda^z \leq \lambda^{y\oplus c}$. Then the halting time of Σ^y-P^y is strictly less than $\lambda^{y\oplus c}$.

We can thus characterize x as the unique real y with $\phi(x)$, where $\phi(y)$ is the statement that Σ^y-$P^y \downarrow = 1$ holds in $L_{\lambda^{y\oplus c_{y,\Sigma^y}}}[y]$. To see that $\phi(y)$ is a Σ_1-statement, we call a triple $\boldsymbol{\alpha} = \langle \alpha_0, \alpha_1, \alpha_2 \rangle$ a *y-triple* if $\alpha_0 < \alpha_1 < \alpha_2$ and $L_{\alpha_0}[y] \prec_{\Sigma_1} L_{\alpha_1}[y] \prec_{\Sigma_2} L_{\alpha_2}[y]$. Then $\phi(y)$ is equivalent to the Σ_1-statement that there is some γ such that in L_γ, the lexically least y-triple $\boldsymbol{\alpha} = \langle \alpha_0, \alpha_1, \alpha_2 \rangle$ and the lexically least $y \oplus c_{y,\alpha_2}$-triple $\boldsymbol{\beta} = \langle \beta_0, \beta_1, \beta_2 \rangle$ exist and α_2-$P^y \downarrow = 1$ holds in $L_{\beta_1}[y]$.

Since $\phi(x)$ holds, there is some $y \in L$ such that $\phi(y)$ holds in L by Shoenfield absoluteness. Then there is some z in L_σ such that $\phi(z)$ holds. This implies that $\omega_1 - P^z \downarrow = 1$, so $z = x$ and $x \in L_\sigma$.

Theorem 9. If x is a subset of ω that is recognized by an X-*ITTM*, where X is a proper class of ordinals of the form $\omega_1\alpha$, then $x \in L_\sigma$. In particular, this holds for subsets of ω recognized by a cardinal-detecting *ITTM*.

Proof. The proof is a variation of the proof of Theorem 8. Suppose that X-P recognizes x. A computation by P in the oracle x will assume its special state at the times $\langle \alpha_i \mid i < \gamma \rangle$ for some ordinal γ. Let x_i denote the tape contents at time α_i. Between the times α_i and α_{i+1}, we have an ordinary *ITTM*-computation in the oracle x with input x_ι on the tape. Such a computation will either halt or cycle from the time $\Sigma^{x\oplus x_i}$ with a loop of length $\Sigma^{x\oplus x_i}$. Now any α_i for $i \geq 1$ is a multiple of Σ^y for all reals y. In particular, the tape contents at time $\alpha_i + \Sigma^{x\oplus x_i}$ will be the same as at time α_{i+1}.

We can hence characterize x as the unique real y with $\phi(y)$, where $\phi(y)$ is the following statement. There is an ordinal δ, a sequence $\langle \alpha_i \mid i < \delta \rangle$ of ordinals and a sequence $\langle y_i \mid i < \delta \rangle$ of real numbers such that for all i with $i + 1 < \delta$, y_{i+1} is the tape contents of the computation of P with oracle y and input y_i at time $\Sigma^{y\oplus y_i}$, and the computation P^y with special state at the elements of the sequence $\langle \alpha_i \mid i < \delta \rangle$ halts with output 1.

As in the proof of Theorem 8, $\phi(y)$ is a Σ_1-statement. Since $\phi(x)$ is valid, $\phi(y)$ holds for some real y in L_σ. Then $y = x$ and $x \in L_\sigma$.

Theorem 10. The recognizable closure both for $\omega_1\alpha$-*ITTMs* for any ordinal α and for cardinal-detecting *ITTMs* is L_σ.

Proof. We first argue that the recognizable closure is contained in L_σ. Suppose that y is an element of L_σ and x is $ITTM$-recognizable from y in either machine type. By a relativization of the proofs of Theorem 8 and Theorem 9, x has a Σ_1-characterization in the parameter y. Since y is an element of L_σ, it is Σ_1-definable in L without parameters and hence can be eliminated from the definition of x. It follows that there is a Σ_1-formula ϕ such that x is the only witness for ϕ in L. Hence $x \in L_\sigma$.

Moreover, L_σ is contained in the recognizable closure for plain $ITTMs$ by [CSW, Lemma 3.2] and hence recognizable closures for both machine types are equal to L_σ.

The recognizability strength of cardinal-detecting $ITTMs$ is strictly higher than that of $ITTMs$ by the next results. The next lemma shows that not every real in the recognizable closure for $ITTMs$is itself recognizable.

Lemma 11. If $x \in L_\Sigma \setminus L_\lambda$, then x is not $ITTM$-recognizable.

Proof. Suppose that $x \in L_\Sigma$ and x is $ITTM$-recognizable. We consider an $ITTM$-program P which writes every accidentally writable real at some time. If x is $ITTM$-recognizable by a program Q, we can write x by letting P run and checking in each step with Q whether the contents of the output tape is equal to x, and in this case stop. Then x is writable and hence $x \in L_\lambda$.

Lemma 12. There is a real number that is $ITTM$-recognizable by a cardinal-detecting $ITTM$ and by an α-$ITTM$ for every $\alpha \geq \lambda$, but not $ITTM$-recognizable.

Proof. Let $0^\triangledown = \{\varphi \mid \varphi(0)\downarrow\}$ denote the *halting problem* or *jump* for $ITTMs$. Since $0^\triangledown$ is Σ_1-definable over L_λ, but certainly not $ITTM$-writable, we have $0^\triangledown \in L_\Sigma \setminus L_\lambda$ and hence $0^\triangledown$ is not $ITTM$-recognizable. We now argue that $0^\triangledown$ is writable by a cardinal-detecting $ITTM$ and α-$ITTM$-writable, hence it is recognizable with respect to these machines. This was already observed in [Hab13] for cardinal-detecting machines. We can simulate all $ITTM$-programs simultaneously and write 1 in the n-th place of the output tape when the n-th program has stopped. As all halting times are countable, the output tape will contain $0^\triangledown$ at time α, and the special state at time α allows us to stop.

4 Recognizable reals relative to finitely many ordinals

In this section, we consider what happens when we allow the machine to enter a special state at an ordinal time α. We first determine the writability strength of such machines. The next result follows from [CH11, Proposition 4.6]. We give a short proof from the λ-ζ-Σ theorem for the reader.

Lemma 13. Let λ^α denote the supremum of the halting times of α-$ITTMs$.

1. $\lambda^x > \Sigma$ for every real x with $\lambda^x \geq \zeta$.
2. $\lambda^\alpha > \Sigma$ for every $\alpha \geq \zeta$.

Proof. To prove the first claim, we first suppose that $\Sigma = \Sigma^x$. Since $\zeta \leq \lambda^x < \zeta^x$, this implies $L_\zeta \prec_{\Sigma_2} L_{\zeta^x} \prec_{\Sigma_2} L_\Sigma$ and this contradicts the minimality of Σ. Second, suppose that $\Sigma < \Sigma^x$. Then there is a triple $\langle \alpha, \beta, \gamma \rangle$ in L_{Σ^x} with $L_\alpha \prec_{\Sigma_1} L_\beta \prec_{\Sigma_2} L_\gamma$, namely $\langle \lambda, \zeta, \Sigma \rangle$. Since $L_{\lambda^x}[x] \prec_{\Sigma_1} L_{\Sigma^x}[x]$, we have $L_{\lambda^x} \prec L_{\Sigma^x}$ and therefore, there is such a triple in L_{λ^x}. Since Σ is the least value for γ for such triples $\langle \alpha, \beta, \gamma \rangle$, we have $\Sigma \leq \gamma < \lambda^x$.

The second claim is clear if $\alpha \geq \Sigma$. If $\alpha < \Sigma$, we can write an accidentally writable real x with an α-$ITTM$ that is not eventually writable. Since we can search for an L-level containing x and halt, it follows that $\lambda^x \geq \zeta$. Hence $\lambda^\alpha \geq \lambda^x > \Sigma$ by the first claim.

We remark that the assumption of Lemma 13 cannot be weakened to $\lambda^x > \lambda$ by the following counterexample. Let x be the L-minimal code for λ. Then clearly $\lambda^x > \lambda$. On the other hand, x is eventually writable, the L-minimal code for λ^x is writable relative to x and the eventually writable reals are closed under writability. Hence the L-minimal code for λ^x is eventually writable and therefore $\lambda^x < \zeta < \Sigma$.

Lemma 14. The following statements are equivalent for a real x.

1. x is α-*ITTM*-writable for some ordinal α.
2. x is *ITTM*-writable from some accidentally writable real number.
3. x is *ITTM*-writable from every accidentally writable real number that is not eventually writable.
4. x is an element of L_{λ^z}, where z is the L-least code for ζ.

Proof. Suppose that x is α-*ITTM*-writable for some ordinal α by a computation of a program P. Up to time α, this computation is just an ordinary *ITTM*-computation and hence at time α, the tape will contain some accidentally writable real number y. The rest of the computation will again be an ordinary *ITTM*-computation with the input y and thus the output will be *ITTM*-writable from y.

Suppose that x is ITTM-writable by a program P from some accidentally writable real number y. Suppose that Q is a program that has y on its tape at time α. If we run Q up to time α and then run P, this will write x.

The L-least code z for ζ is accidentally writable and hence the remaining implications follow from Lemma 13.

We obtain the following generalization of Lemma 14.

Lemma 15. The following statements are equivalent for a real x.

1. x is $\boldsymbol{\alpha}$-*ITTM*-writable for some sequence $\boldsymbol{\alpha}$ of length n.
2. x is *ITTM*-writable from x_{n-1} for some sequence $\boldsymbol{x} = \langle x_0, \dots, x_{n-1} \rangle$, where x_j is accidentally writable from $\bigoplus_{i<j} x_i$ for all $j < n$.
3. x is an element of $L_{\lambda^{z_{n-1}}}$, where $z_0 = 0$ and z_{i+1} is the L-least code for ζ^{z_i} for all $i < n-1$.

Proof. The implications follow by iterated application of Lemma 14.

To show that the recognizability strength of *ITTMs* with arbitrary ordinal parameters is beyond L_σ, we need the next definition and two well-known results. For technical convenience, we work with Jensen's J-hierarchy instead of Gödel's L-hierarchy. Note that $J_\alpha = L_\alpha$ if α takes one of the values λ, ζ, Σ or σ.

Definition 16. An ordinal α is an *index* if there is a real in $J_{\alpha+1} \setminus J_\alpha$.

Lemma 17. (Jensen) If α is an index, then there is a surjection from ω onto J_α that is definable over J_α and hence there is a code for J_α in $J_{\alpha+1}$.

Proof. This follows from the fact that $\langle J_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ is acceptable by [Zem02, Lemma 1.10.1].

Lemma 18 (folklore)**.** There are unboundedly many admissible indices α below ω_1^L.

Proof. There are unboundedly many indices below ω_1^L, since there are ω_1^L many reals in L. Suppose that α is an index. Let c denote the L-least code for α. Since α is an index, $c \in J_{\alpha+1}$ by Lemma 17. Suppose that $\beta = \omega_1^c$ is the least c-admissible ordinal. Then β is admissible and it remains to show that β is an index. Since $c \in J_\beta$ and J_β is the Skolem hull of c in J_β, there is a surjection from ω onto J_β that is definable over J_β. There is a real in $J_{\beta+1}$ that codes this surjection. Since J_β is admissible, c cannot be an element of J_β and hence β is an index.

The next result shows that there are *ITTM*-recognizable reals with respect to ordinal parameters beyond L_σ.

Lemma 19. Suppose that $\alpha = \omega\beta$ is an index and c is the L-least code for J_α in $J_{\alpha+1} \setminus J_\alpha$. Then c is α-*ITTM*-recognizable.

Proof. The claim is easy to see for $\alpha = \omega$, so we assume that $\alpha > \omega$. Suppose that the input is x. We first check whether x codes a set with an extensional relation and otherwise reject x. We then count through the ordinals of the set coded by x, i.e. in each step we search for the least next ordinal, while simultaneously searching for infinite strictly decreasing sequences of ordinals above. This is possible by keeping markers at all previous ordinals in every step. If we have exhausted the ordinals or if we find an infinite strictly decreasing sequence of ordinals in the structure coded by x before time α, then we reject x. This algorithm is carried out up to time α. After time α, we check if the structure coded by x is well-founded, and reject x if this is not the case. If the structure is well-founded, we check whether it is isomorphic to some J_β. In this case, we write a code for $J_{\beta+1}$ and check whether it contains a code for J_β. If it does, we determine the least such code in $J_{\beta+1} \setminus J_\beta$ and check if it is equal to c. If it is equal to c, then we accept x, and otherwise reject x. There is a code for J_α in $J_{\alpha+1} \setminus J_\alpha$ by Lemma 17. Therefore this algorithm accepts a real x if and only if $x = c$.

To show that every real that is *ITTM*-recognizable relative to finitely many ordinal parameters is in L, we need the following result.

Lemma 20. Suppose that x is an *ITTM*-recognizable subset of ω from n ordinal parameters. Then x is *ITTM*-recognizable from finitely many ordinal parameters strictly below $\omega_1 \cdot (n+1)$.

Proof. Suppose that x is *ITTM*-recognizable from $\boldsymbol{\alpha} = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle$ and $\boldsymbol{\alpha}$ is strictly increasing. We can assume that α_{n-1} is uncountable. Suppose that α_i^* is the remainder of the division of α_i by ω_1 for $i < n$. Suppose that $\boldsymbol{k} = \langle k_0, \ldots, k_l \rangle$ is the unique sequence such that $k_0 \leq n$ is least such that α_{k_0} is uncountable and for all $i < l$, $k_{i+1} < n$ is least such that the unique ordinal α with $\alpha_{k_i} + \alpha = \alpha_{k_{i+1}}$ is uncountable. Let $\beta_i = \alpha_i$ for $i < k_0$, $\beta_i = \omega_1 j + \alpha_i^*$ for $k_j \leq i < k_{j+1}$ and $j < l$, and let $\beta_i = \omega_1 l + \alpha_i^*$ for $k_l \leq i < n$.

Suppose that a program P recognizes x from the parameters $\alpha_0 < \cdots < \alpha_n$. For every input $\bar{x}$, $P^{\bar{x}}$ will cycle from time $\Sigma^{\bar{x}}$ between multiples of $\Sigma^{\bar{x}}$ by [cite Welch]. This implies that for every input $\bar{x}$, $P^{\bar{x}}$ will halt with the same state for the parameters $\alpha_0, \ldots, \alpha_n$ and the parameters $\beta_0, \ldots, \beta_n$. Hence P recognizes x from the parameters $\beta_0, \ldots, \beta_n$.

Theorem 21. Every subset of ω that is *ITTM*-recognizable from finitely many ordinal parameters is an element of L.

Proof. Suppose that x is *ITTM*-recognizable from finitely many ordinal parameters. Then x is recognized by a program P from finitely many ordinal parameters strictly below $\omega_1 \cdot (n+1)$ by Lemma 20.

We can assume that x is recognized by a single ordinal parameter δ with $\omega_1 \leq \delta < \omega_1 2$. The proof of the general case is analogous. Suppose that P^x with the special state at time δ halts at time η. Let $\delta = \omega_1 + \delta^*$ and $\eta = \omega_1 + \eta^*$. We consider the Σ_1-statement $\psi(\bar{x})$ stating that $P^{\bar{x}}$ with the special state at time $\Sigma^{\bar{x}} + \delta^*$ halts at time $\Sigma^{\bar{x}} + \eta^*$ and accepts $\bar{x}$. Since the program will cycle from time $\Sigma^{\bar{x}}$ in intervals of length $\Sigma^{\bar{x}}$ by Theorem 4, $\psi(\bar{x})$ is equivalent to the statement that $P^{\bar{x}}$ with the special state at time $\Sigma^{\bar{x}} \cdot \alpha + \delta^*$ halts at time $\Sigma^{\bar{x}} \cdot \alpha + \eta^*$ for some $\alpha \geq 1$, or equivalently for all $\alpha \geq 1$.

The statement $\psi(x)$ holds in V and in every generic extension of V. In particular, $\psi(x)$ holds in every $\mathrm{Col}(\omega, \zeta)$-generic extension $V[G]$ of V, where ζ is a countable ordinal with $\delta^*, \eta^* \leq \zeta$. Moreover $\exists \bar{x} \psi(\bar{x})$ holds in $L[G]$ by Shoenfield absoluteness, since $\exists \bar{x} \psi(\bar{x})$ is a Σ_1-statement and the parameters δ^* and η^* are countable in $L[G]$.

Suppose that θ is an L-cardinal such that L_θ is sufficiently elementary in L. Suppose that $M \prec L_\theta$ is countable with $\zeta + 1 \subseteq M$ and $\bar{M}$ is the transitive collapse of M. Suppose that g, h are mutually $\mathrm{Col}(\omega, \zeta)$-generic over $\bar{M}$ in V. The statement $\exists \bar{x} \psi(\bar{x})$ is forced over L_θ and $\bar{M}$ and therefore holds in $\bar{M}[g]$ and in $\bar{M}[h]$, witnessed by some reals x_g and x_h. Since P recognizes x with the special state at time δ, the uniqueness of x implies that $x_g = x_h = x$. Since g and h are mutually generic over $\bar{M}$, we have $\bar{M}[g] \cap \bar{M}[h] = \bar{M}$ and hence $x \in \bar{M}$. Since $\bar{M}$ is a subset of L, this implies $x \in L$.

This allows us to determine the recognizable closure with respect to ordinal parameters.

Theorem 22. The recognizable closure for *ITTMs* with single ordinal parameters and for *ITTMs* with finitely many ordinal parameters is $P(\omega)^L$.

Proof. This follows from Lemma 18, Lemma 19 and Theorem 21.

Note that Theorem 21 cannot be extended to countable sets of ordinal parameters, since it is easy to see that every real is writable from a countable set of ordinal parameters. The previous results suggest the question whether the number of ordinal parameters is relevant for the recognizability strength. The next result shows that this is the case.

Theorem 23. For every n, there is a subset x of ω that is *ITTM*-recognizable from $n+1$ ordinals, but not from n ordinals.

Proof. We define $\boldsymbol{x}_n = \langle x_0, \dots, x_n \rangle$, $\boldsymbol{\lambda}_n = \langle \lambda_0, \dots \lambda_n \rangle$, $\boldsymbol{\zeta}_n = \langle \zeta_0, \dots \zeta_n \rangle$, and $\boldsymbol{\Sigma}_n = \langle \Sigma_0, \dots \Sigma_n \rangle$ as follows for all n. Let $\zeta_0 = \zeta$ and $\zeta_{i+1} = \zeta^{x_i}$, where x_0 is the L-least code for ζ and x_{i+1} is the $L[\boldsymbol{x}_i]$-least code for ζ_{i+1}. Moreover, let $\lambda_0 = \lambda$, $\lambda_{i+1} = \lambda^{x_i}$, $\Sigma_0 = \Sigma$ and $\Sigma_{i+1} = \Sigma^{x_i}$. Then $\lambda^{x_i} > \Sigma_{i+1}$ for all i by the relativized version of Lemma 13. Moreover, let $\boldsymbol{\lambda}_n^y$, $\boldsymbol{\zeta}_n^y$ and $\boldsymbol{\Sigma}_n^y$ denote the relatived versions of $\boldsymbol{\lambda}_n$, $\boldsymbol{\zeta}_n$ and $\boldsymbol{\Sigma}_n$ for any real y.

Claim 24. A Cohen real x over L_{Σ_n+1} is not *ITTM*-recognizable from n ordinals.

Proof. Suppose that x is recognized by a program P in the parameter $\boldsymbol{\gamma} = \langle \gamma_0, \dots, \gamma_{n-1} \rangle$, where $\boldsymbol{\gamma}$ is strictly increasing. We define $\boldsymbol{\gamma}^* = \langle \gamma_0^*, \dots, \gamma_{n-1}^* \rangle$ as follows. Let $\gamma_0^* = \gamma_0$ if $\gamma_0 < \Sigma$ and $\gamma_0^* = \zeta_0 + \delta_0$ if $\gamma_0 \geq \Sigma$, where δ_0 is the remainder of the division of γ_0 by Σ. For all i with $i+1 < n$, let $\gamma_{i+1}^* = \gamma_{i+1}$ if $\gamma_{i+1} < \Sigma_{i+1}$ and $\gamma_{i+1}^* = \zeta_{i+1} + \delta_{i+1}$ if $\gamma_{i+1} \geq \Sigma_{i+1}$, where δ_{i+1} is the remainder of the division of γ_{i+1} by Σ_{i+1}. A computation with input y cycles from ζ^y in intervals of length Σ^y by Theorem 4. Since P^x with special states at $\boldsymbol{\gamma}$ accepts x, this implies that P^x with special states at $\boldsymbol{\gamma}^*$ accepts x as well.

For every Cohen real y over L_{Σ_n+1}, $\boldsymbol{\lambda}_n^y = \boldsymbol{\lambda}_n$, $\boldsymbol{\zeta}_n^y = \boldsymbol{\zeta}_n$ and $\boldsymbol{\Sigma}_n^y = \boldsymbol{\Sigma}_n$ by the variant of [CS, Lemma 3.12] for Cohen forcing (see also [Wel99, p.11]).

Since $L_{\Sigma_n}[y]$ is a union of admissible sets by the variant of [CS, Lemma 2.11] for Cohen forcing, the run of P^y is an element of $L_{\Sigma_n}[y]$.

Let σ be a name in $L_{\Sigma_{n+1}}$ for the run of P^x with special states at $\boldsymbol{\gamma}^*$. The statement that P^x with special states at $\boldsymbol{\gamma}^*$ accepts x is forced for σ by a condition p in Cohen forcing over L_{Σ_n} by the variant of [CS, Lemma 2.7] for Cohen forcing. Suppose that y is a Cohen generic over L_{Σ_n+1} with $x \neq y$ that extends the condition p. Then P^y accepts y by the truth lemma in the variant of [CS, Lemma 2.8] for Cohen forcing. This contradicts the uniqueness of x.

Claim 25. The L-least Cohen real over L_{Σ_n+1} is writable from $\boldsymbol{\zeta}_n$.

Proof. By the relativized version of Lemma 13 applied to $x_0, \ldots, x_n$, we can successively compute $x_0, \ldots, x_n$ from $\boldsymbol{\zeta}_n$. We can then compute codes for Σ_n, L_{Σ_n}, L_{Σ_n+1} and hence the L-least Cohen real over over L_{Σ_n} in L_{Σ_n+1} from x_n.

Remark 26. For finitely many parameters, the writability and recognizability strength does not change if we allow more than one special state, since such a program can be simulated with a single special state by coding the special states into tape cells.

5 Conclusion and open questions

We have seen that equipping *ITTMs* with the power to recognize one particular or all uncountable cardinals increases the set of *ITTM*-recognizable real numbers, but not the recognizable closure, which remains L_σ. Moreover, certain ordinals parameters enable an *ITTM* to recognize real numbers outside of L_σ, but *ITTM*-recognizability with finitely many ordinal parameters does not lead out of the constructible universe.

We conclude with the following open questions. Theorem 9 suggests the question whether the claim holds for all nonempty class of multiples of ω_1.

Question 27. If x is a subset of ω that is recognized by an X-*ITTM*, where X is any class of ordinals of the form $\omega_1\alpha$, then is $x \in L_\sigma$?

It is open whether the ordinals in Lemma 20 can be chosen to be countable.

Question 28. Is every real x that is *ITTM*-recognizable from an ordinal already *ITTM*-recognizable from a countable ordinal?

Let $\mathcal{R}_\alpha$ denote the recognizable closure with respect to *ITTMs* with the parameter α and let $\sigma(\alpha)$ denote the last ordinal $\gamma > \alpha$ with $L_\gamma \prec_{\Sigma_1} L$. It is open what is $\mathcal{R}_\alpha$ and whether there is a relationship between $\mathcal{R}_\alpha$ and $L_{\sigma(\alpha)}$.

Question 29. What is $\mathcal{R}_\alpha$ for arbitrary ordinals α?

The notion of *semi-recognizable reals* is defined by asking that the program halts for some input x and diverges for all other inputs. The notion of *anti-recognizable reals* is defined by asking the the program diverges for some input x and halts for all other inputs. The following question seems fundamental.

Question 30. Are there semi-recognizable reals and anti-recognizable reals that are not recognizable?

References

[Car14a] Merlin Carl. The distribution of *ITRM*-recognizable reals. *Ann. Pure Appl. Logic*, 165(9):1403–1417, 2014.

[Car14b] Merlin Carl. The lost melody phenomenon. In *Infinity, computability, and metamathematics*, volume 23 of *Tributes*, pages 49–70. Coll. Publ., London, 2014.

[Car15] Merlin Carl. Optimal results on recognizability for infinite time register machines. *J. Symb. Log.*, 80(4):1116–1130, 2015.

[CH11] Samuel Coskey and Joel David Hamkins. Infinite time decidable equivalence relation theory. *Notre Dame Journal of Formal Logic*, 52(2):203–228, 2011.

[CS] Merlin Carl and Philipp Schlicht. Randomness via infinite time machines and effective descriptive set theory. Submitted, 2016.

[CSW] Merlin Carl, Philipp Schlicht, and Philip Welch. Recognizable sets and woodin cardinals. Submitted, 2016.

[Hab13] Miha E. Habič. Cardinal-recognizing infinite time Turing machines. In *The nature of computation*, volume 7921 of *Lecture Notes in Comput. Sci.*, pages 231–240. Springer, Heidelberg, 2013.

[HL00] Joel David Hamkins and Andy Lewis. Infinite time Turing machines. *J. Symbolic Logic*, 65(2):567–604, 2000.

[Wel99] Philip D. Welch. Minimality arguments for infinite time Turing degrees. In *Sets and proofs (Leeds, 1997)*, volume 258 of *London Math. Soc. Lecture Note Ser.*, pages 425–436. Cambridge Univ. Press, Cambridge, 1999.

[Wel09] P. D. Welch. Characteristics of discrete transfinite time Turing machine models: halting times, stabilization times, and normal form theorems. *Theoret. Comput. Sci.*, 410(4-5):426–442, 2009.

[Wel14] P. D. Welch. Transfinite machine models. In *Turing's legacy: developments from Turing's ideas in logic*, volume 42 of *Lect. Notes Log.*, pages 493–529. Assoc. Symbol. Logic, La Jolla, CA, 2014.

[Zem02] Martin Zeman. *Inner models and large cardinals*, volume 5 of *De Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 2002.