CHOICELESS RAMSEY THEORY OF LINEAR ORDERS

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Abstract. We investigate symmetric and asymmetric partition relations for linear orders without choice, i.e. the existence of a subset in one of finitely many given order types which is homogeneous for a given colouring of the finite subsets of a linear order. More specifically, we consider linear orders of the form $\langle {}^{\gamma}2, <_{lex}\rangle$, where γ is an ordinal and $<_{lex}$ denotes the lexicographical order. We obtain stronger partition relations than what is possible with choice, for instance it is consistent that $\langle {}^{\kappa}2, <_{lex}\rangle \longrightarrow (\langle {}^{\kappa}2, <_{lex}\rangle)^2_2$ where $\kappa^{<\kappa} = \kappa > \omega$. Motivated by work of Erdős, Milner and Rado, we prove various negative partition relations with finite exponents for linear orders of the form $\langle {}^{\gamma}2, <_{lex}\rangle$. We use these results to determine which partition relations of the forms $\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (K, M)^n$ and $\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (\bigvee_{\nu<\lambda} K_\nu, \bigvee_{\nu<\mu} M_\nu)^n$ for $n \leq 4$ and linear orders K, M, K_ν, M_ν are consistent.

Contents

1. Introduction

In this paper, we study Ramsey theory of linear orders without the axiom of choice in the theory ZF. We work in this theory throughout the paper.

1.1. **Some Ramsey theory.** We begin with some definitions and facts from Ramsey theory. Structures with finitely many relations (usually linear orders) are denoted as K, L, M and a structure is identified with its underlying set. We use greek letters to denote ordinals, i.e. a cardinal ν is always assumed to be an ordinal.

Definition 1.1. *Suppose that L, M are structures in the same signature and ν is a cardinal.*

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(i) $[L]^M$ *denotes the set of substructures of* L *which are isomorphic to* M.
(ii) *Suppose that* $f\colon [L]^M \to \nu$ *is a colouring and* $i < \nu$. *A set* $H \subseteq L$ *is* (f,i)*-homogeneous if* $f(x) = i$ *for all* $x \in [H]^M$.
(iii) *Suppose that* $f\colon [L]^M \to \nu$ *is a colouring and* $i < \nu$. *A set* $H \subseteq L$ *is* f*-homogeneous if it is* (f,i)*-homogeneous for some* $i < \nu$.

We will consider the following partition relations.

Definition 1.2. *Suppose that* K, L, M *are structures and* ν *is a cardinal.*
(i) $L \longrightarrow (M)^K_\nu$ *states that for every colouring* $f\colon [L]^K \to \nu$, *there is some* f*-homogeneous* $H \in [L]^M$.
(ii) $L \longrightarrow [M]^K_\nu$ *states that for every* $f\colon [L]^K \to \nu$, *there is some* $H \in [L]^M$ *with* $\mathrm{range}(f \restriction [H]^K) \neq \nu$.
(iii) $L \longrightarrow (M_0, \ldots, M_{n-1})^K$ *states that for every* $f\colon [L]^K \to n$, *there are* $i < n$ *and* $H \in [L]^{M_i}$ *such that* H *is* (f,i)*-homogeneous.*
(iv) $L \longrightarrow (M_{0,0} \vee \ldots \vee M_{0,k_0}, \ldots, M_{n-1,0} \vee \ldots \vee M_{n-1,k_{n-1}})^K$ *states that for every* $f\colon [L]^K \to n$, *there are* $i < n$, $k \leqslant k_i$, *and* $H \in [L]^{M_{i,k}}$ *such that* H *is* (f,i)*-homogeneous.*

If L is a linear order and each $M_{i,j}$ is an ordinal $\alpha_{i,j}$, then Definition 1.2(iv) is equivalent to $L \longrightarrow (\alpha_0, \ldots, \alpha_{n-1})^K$ where $\alpha_i := \min_{j \leqslant k_i} \alpha_{i,j}$ for every $i < n$.

We consider partition relations with exponent at least 2, and Proposition 1.3 below motivates the focus on linear orders. Let us first mention the case of exponent 1.

A structure L is *indivisible* if it satisfies $L \longrightarrow (L)^1_2$. If L is an indivisible structure with only one unary relation, then the relation is trivial, i.e. either full or empty. If L is any non-scattered countable linear order, i.e. L contains a copy of $\mathbb{Q}$, then L is indivisible. There are many interesting indivisible structures, for instance some countable metric spaces [007D].

If on the other hand L is a structure with a single binary relation, $L \longrightarrow (L)^2_2$ holds, and the domain of L can be linearly ordered (by a linear order which may be unrelated to L), then L is necessarily a linear order or trivial, by the following result. We will identify a relation with its restriction to the set of tuples with pairwise different coordinates.

Proposition 1.3. *Suppose that* L *is an infinite structure with a single binary relation and* $L \longrightarrow (L)^2_2$.
(1) *If the domain of* L *can be linearly ordered (by a linear order which may be unrelated to* L*), then* L *is a linear order or trivial, i.e. either full or empty.*
(2) *If the domain of* L *can be wellordered, then* L *is a wellorder with order type* ω *or a weakly compact cardinal.*

Proof. Note that $L \longrightarrow (L)^2_2$ implies $L \longrightarrow (L)^2_n$ for all $n \in \omega$. For the first claim, suppose that R_L is the binary relation of L and R is a linear order on the domain of L. Let

$$f_0(x,y) = 0 \text{ if } [(x,y) \in R \Rightarrow (x,y) \in R_L] \text{ and } [(y,x) \in R \Rightarrow (y,x) \in R_L]$$
$$f_1(x,y) = 0 \text{ if } [(x,y) \in R \Rightarrow (y,x) \in R_L] \text{ and } [(y,x) \in R \Rightarrow (x,y) \in R_L]$$

and choose the value 1 otherwise in each case. Let $f(x,y) = 2f_0(x,y) + f_1(x,y)$. The remaining claims follow. □

This generalises to dimensions $n \geqslant 3$ as follows.

Definition 1.4. *Let* $P(S_n)$ *denote the power set of the symmetric group* S_n.
(i) *If* L *is a structure whose only relation is a linear order* $<_L$ *and* $t \in P(S_n)$, *let* $L^{(t)}$ *denote the structure whose only relation is the set of tuples* $(x_{\sigma(0)}, \ldots, x_{\sigma(n-1)})$ *with* $x_{\sigma(0)} <_L x_{\sigma(1)} <_L \cdots <_L x_{\sigma(n-1)}$ *and* $\sigma \in t$.
(ii) *If* M *is a structure whose only relation is* n*-ary, then* M *is* induced by a linear order *if there is a linear order* L *with the same domain as* M *and some* $t \subseteq P(n)$ *with* $M = L^{(t)}$.

Proposition 1.5. *(1) If* $\mathbb{N}$ *is the structure of the natural numbers with the standard order and* $t \in P(S_n)$, *then* $\mathbb{N}^{(t)} \longrightarrow (\mathbb{N}^{(t)})^m_k$ *for all* $k, m \in \omega$.
(2) Suppose that L *is a structure whose only relation is* n*-ary relation for some* $n \geqslant 2$, $L \longrightarrow (L)^n_2$, *and the domain of* L *can be linearly ordered. Then* L *is induced by a linear order.*

Proof. The first claim follows from Ramsey's theorem. The second claim is proved as in Proposition 1.3. □

In this paper, we consider the following problem.

Problem 1.6. *Suppose that $n \geqslant 1$. For which pairs (L, M) of linear orders is there a linear order K with $K \longrightarrow (L, M)^n$?*

Since the answer depends on whether the axiom of choice holds, we consider Problem 1.6 in the following contexts.

(1) For arbitrary linear orders, assuming the the axiom of choice.
(2) For linear orders on ${}^{\kappa}\kappa$, the set of functions $f\colon \kappa \to \kappa$, assuming that $\kappa^{<\kappa} = \kappa$, so in particular ${}^{\mu}\kappa$ is wellordered for all $\mu < \kappa$, but assuming that ${}^{\kappa}\kappa$ is not wellordered.
(3) For arbitrary linear orders without the axiom of choice, and more specifically for linear orders on ${}^{\kappa}\kappa$ assuming that ${}^{\mu}2$ is not wellordered for some $\mu < \kappa$.

For instance, the situation in 2 occurs in the model $L(P(\kappa))$ after forcing with $Col(\kappa, < \lambda)$, where $\lambda > \kappa$ is inaccessible, and 3 is fulfilled for linear orders of size at least $\aleph_1$ in models of the axiom of determinacy.

The *lexicographical order* $\langle {}^{\kappa}\kappa, <_{lex} \rangle$ is defined by $x <_{lex} y$ if $x \neq y$ and $x(\alpha) < y(\alpha)$ for the least $\alpha < \kappa$ with $x(\alpha) \neq y(\alpha)$.

Section 2 is concerned with partition relations for $\langle {}^{\gamma}2, <_{lex} \rangle$. Sections 3 and 4 is concerned with asymmetric negative partition relations without choice. The combined results of Section 2 and Sections 3 and 4 determine which partition relations of the form $\langle {}^{\omega}2, <_{lex} \rangle \longrightarrow (L, M)^n$ with $n \geqslant 2$ are consistent without choice.

The results in Sections 3 and 4 were proved by the last author together with the second author and extend results from [014We2]. We would like to thank Paul Larson for letting us include his Theorem 1.25.

1.2. **Partition relations assuming the axiom of choice.** We recall some known results on partition relations with choice. Partition relations for linear orders, in contrast to well-orders, were studied in [956ER, 963EH, 965Kr, 971E, 972EM, 974La].

Lemma 1.7. *Suppose that* ZFC *holds. Then $L \not\longrightarrow (\omega^*, \omega)^2$ for all linear orders L.*

Proof. The proof is similar to the proof of $\omega_1 \not\longrightarrow (\omega_1)^2_2$ in [933Si]. We consider a well-order on the domain of L and colour a pair depending on whether the well-order agrees with the natural order on this pair. □

This strongly limits the possibilities for positive partition relations under the axiom of choice. In particular, in any partition relation of the form $K \longrightarrow (L, M)^2$, we can assume that L, M are well-ordered, or that M is finite. Even for well-orders K, L, M, there are many difficult open questions for these relations (see [010HL, 979No, 993B, 008Jo, 010Sc, 014We]). Instead of considering these relations, we focus on linear orders L such that L, L^* are not well-ordered.

For partition relations with exponent at least 3, similar ideas as in Lemma 1.7 led to the following results.

Theorem 1.8. [965Kr, 971E] *Suppose that* ZFC *holds. For any linear order L*

(1) $L \not\longrightarrow (\omega^* + \omega, 4)^3$ and
(2) $L \not\longrightarrow (\omega + \omega^*, 4)^3$.

The linear orders on the right side of the arrows are optimal, since $\omega \longrightarrow (\omega)^m_n$ and $\omega^* \longrightarrow (\omega^*)^m_n$ hold by Ramsey's theorem.

A further problem is to determine the valid partition relations which allow finitely many order types linked by a disjunction, instead of a single order type. For example, in the context of choice, the occurence of $\omega^* \vee \omega$ in a partition relation for a linear order states that there is an infinite homogeneous set with arbitrary order type. The occurence of $\omega^* + \omega \vee \omega + \omega^*$ in a partition relation for a linear order states that there is an infinite homogeneous set L such that L and L^* are not well-ordered.

Theorem 1.9. [971E] *Suppose that* ZFC *holds. Then $L \not\longrightarrow (\omega^* + \omega \vee \omega + \omega^*, 5)^3$ for all linear orders L.*

Question 1.10. [971E] *Suppose that* ZFC *holds. Is there a linear order L with $L \longrightarrow (\omega^* + \omega \vee \omega + \omega^*, 4)^3$?*

Let us mention two negative relations for $\langle {}^\kappa 2, <_{lex} \rangle$ with choice. In the following proof, the topology on ${}^\kappa\kappa$ is given by the basic open sets $N_t = \{x \in {}^\kappa\kappa \mid t \subseteq x\}$ for $t \in {}^{<\kappa}\kappa$. A perfect subset of ${}^\kappa\kappa$ is a set of the form $[T] = \{x \in {}^\kappa\kappa \mid \forall \alpha < \kappa (x \upharpoonright \alpha \in T)\}$, where $T \subseteq {}^{<\kappa}\kappa$ is a perfect tree, i.e. a $< \kappa$-closed tree whose splitting nodes are cofinal in T.

Theorem 1.11. [908Be] *Suppose that* ZFC *holds and $\kappa^{<\kappa} = \kappa$. Then $\langle {}^\kappa 2, <_{lex} \rangle \not\longrightarrow (\langle {}^\kappa 2, <_{lex} \rangle)^1_2$.*

Proof. The counterexample is a κ-Bernstein set, i.e. a set $A \subseteq {}^\kappa\kappa$ such that A and its complement do not have perfect subsets. The set is constructed by diagonalization along an enumeration of the perfect subsets of ${}^\kappa\kappa$. □

A *meagre* subset of ${}^\kappa\kappa$ is a union of κ nowhere dense subsets of ${}^\kappa\kappa$, and a *comeagre* set is such that its complement is meagre.

Theorem 1.12. *Suppose that* ZFC *holds and $\kappa^{<\kappa} = \kappa$. Then $\langle {}^\kappa 2, <_{lex} \rangle \not\longrightarrow (\langle {}^\kappa 2, <_{lex} \rangle, 3)^2$.*

Proof. Suppose that $\langle C_\alpha \mid \alpha < 2^\kappa \rangle$ enumerates all perfect subsets of ${}^\kappa 2$. We choose an injective sequence $(x_\alpha, y_\alpha)_{\alpha < 2^\kappa}$ as follows. In step α, we find distinct $x_\alpha, y_\alpha \in C_\alpha$ with $x_\alpha \neq x_\beta$, $x_\alpha \neq y_\beta$, $y_\alpha \neq x_\beta$, and $y_\alpha \neq x_\beta$ for all $\beta < \alpha$. Let

$$G = \{(x_\alpha, y_\alpha) \mid \alpha < 2^\kappa\} \cup \{(y_\alpha, x_\alpha) \mid \alpha < 2^\kappa\}.$$

Let $f \colon [{}^\kappa 2]^2 \to 2$ denote the characteristic function of G, i.e. $f(x, y) = 1$ if $(x, y) \in G$ and $f(x, y) = 0$ otherwise. Note that every order preserving injection $f \colon \langle {}^\kappa 2, <_{lex} \rangle \hookrightarrow \langle {}^\kappa 2, <_{lex} \rangle$ is discontinuous in at most κ points for the following reason. Every point in which f is discontinuous defines a nontrivial interval in $\langle {}^\kappa\kappa, <_{lex} \rangle$, and the intervals from two distinct such points are are disjoint. It follows that f is continuous on a perfect set. This implies that for every isomorphism $f \colon \langle {}^\kappa 2, <_{lex} \rangle \to \langle {}^\kappa 2, <_{lex} \rangle$, there is a perfect set C such that $f \upharpoonright C$ is a homeomorphism (see e.g. [014L, Cor. 5.3]). Hence $\langle {}^\kappa 2, G \rangle$ contains no independent set isomorphic to $\langle {}^\kappa 2, <_{lex} \rangle$ and no complete subgraph of size 3. □

1.3. **Partition relations assuming $\kappa^{<\kappa} = \kappa$.** We consider the lexicographical order $\langle {}^\kappa 2, <_{lex} \rangle$ for cardinals κ such that $\kappa^{<\kappa} = \kappa$, but ${}^\kappa\kappa$ is not necessarily well-ordered. The following is our first main result.

Theorem 1.13. *Suppose that V is a model of* ZFC *and κ is regular. There is a symmetric extension of V by a $< \kappa$-closed κ^+-c.c. forcing in which $\langle {}^\kappa 2, <_{lex} \rangle \longrightarrow (\langle {}^\kappa 2, <_{lex} \rangle)^2_2$ holds.*

It follows from Theorem 3.3 and Theorem 3.1 that Theorem 1.13 cannot be extended to exponent 3. For instance, the colouring which maps a triple to its splitting type does not have a large homogeneous set. The splitting type is defined as follows.

Definition 1.14. *Suppose that $\gamma \in Ord$.*

(1) Let
$$\delta_{x,y} = \delta(x, y) = \min\{\alpha < \gamma \mid x(\alpha) \neq y(\alpha)\}$$
for $x, y \in {}^\gamma\gamma$.

(2) Let
$$\Delta_{x,y} = \Delta(x, y) = x \upharpoonright \delta(x, y)$$
for $x, y \in {}^\gamma\gamma$.

(3) Let
$$\beta^{(h)}_{x,y} = \beta_h(x, y) = h(\delta(x, y))$$
for $x, y \in {}^\gamma\gamma$ and $h \colon \gamma \hookrightarrow |\gamma|$ one-to one.

(4) Suppose that $A, B \in M_n(\mathbb{Z})$ and $\bar{A} = \{A_{i,j} \mid i, j < n\}$, $\bar{B} = \{B_{i,j} \mid i, j < n\}$. Then A, B are (order) isomorphic *if A can be converted into B by a composition of a permutation of the indices with an order isomorphism $\pi \colon \bar{A} \leftrightarrow \bar{B}$.*

(5) The branching type *or* splitting type *of a tuple $x_0 <_{lex} \ldots <_{lex} x_{n-1}$ in $\langle {}^\gamma 2, <_{lex} \rangle$ is the isomorphism type of the matrix $(\Delta_{x_i, x_j}, <)_{i,j<n}$.*

(6) $\langle {}^\gamma 2, <_{lex} \rangle \longrightarrow_t (\langle {}^\gamma 2, <_{lex} \rangle)^m_n$ holds if for every colouring $f \colon [{}^\gamma 2]^m \to n$, there is a set isomorphic to $\langle {}^\gamma 2, <_{lex} \rangle$ which is separately homogeneous for f in each branching type.

Therefore we consider sets which are separately homogeneous in each splitting type. Partition relations $\longrightarrow_t$ for the linear order $\langle {}^{\omega}2, <_{lex} \rangle$ were considered by Blass [981Bl].

Lemma 1.15. *Suppose that κ is a regular cardinal.*

(1) The linear orders $\langle {}^{\kappa}2, <_{lex} \rangle$ and $\langle {}^{\kappa}\kappa, <_{lex} \rangle$ are bi-embeddable.
(2) The linear orders $\langle {}^{\omega}2, <_{lex} \rangle$, $\langle {}^{\omega}\omega, <_{lex} \rangle$, and $\langle \mathbb{R}, < \rangle$ are bi-embeddable.

Proof. The linear order $\langle {}^{\kappa}\kappa, <_{lex} \rangle$ is embeddable into $\langle {}^{\kappa}2, <_{lex} \rangle$ by the map $f \colon {}^{\kappa}\kappa \to {}^{\kappa}2$, where $f((\alpha_i)_{i<\kappa})$ is the concatenation of $1^{(\alpha_i)}{}^\frown 0$ for all $i < \kappa$.

The linear order $\langle \mathbb{R}, < \rangle$ is isomorphic to $\langle {}^{\omega}\omega, <_{lex} \rangle \cdot \langle \mathbb{Z}, < \rangle$. □

Since these linear orders are bi-embeddable, they satisfy the same partition relations.

Theorem 1.16. [981Bl] $\langle \mathbb{R}, < \rangle \longrightarrow_t \langle \mathbb{R}, < \rangle^m_n$ *holds for continuous colourings for all m, n.*

Proof. This is proved in [981Bl] using the Halpern-Läuchli theorem. To see that this holds without choice, suppose that a real x codes the continuous colouring. We apply Blass' theorem in $L[x]$ and obtain a closed set coded by a tree T. The statement that $[T]$ is homogeneous up to the branching type for the colouring coded by x is a Π^1_1 statement in x and T, and hence this holds in V. □

For uncountable cardinals κ, the analogue of Blass' theorem is connected with large cardinal properties of κ.

Theorem 1.17. *If $\kappa > \omega$ and $\langle {}^{\kappa}2, <_{lex} \rangle \longrightarrow_t (\kappa^* \vee \kappa)^3_2$, then κ is weakly compact.*

Proof. If $f \colon [\kappa]^2 \to 2$ is a colouring, we define $g_f \colon [{}^{\kappa}2]^3 \to 2$ as follows. Suppose that $x, y, z \in {}^{\kappa}2$ are distinct and $A = \{x, y, z\}$. Let $\Delta_A := \{\Delta(x, y), \Delta(y, z), \Delta(z, x)\}$. Note that $|\Delta_A| = 2$. Let $g_f(A) := f(\Delta_A)$. Suppose that $H \subseteq {}^{\kappa}2$ is homogeneous for g_f up to the branching type and that H is isomorphic to κ^* or to κ. Then $I := \{\Delta(x, y) \mid x, y \in H\}$ has order type κ and is homogeneous for f. □

Note that $\langle {}^{\kappa}2, <_{lex} \rangle \longrightarrow_t (\kappa^* \vee \kappa)^2_2$ does not imply that κ is weakly compact, by Theorem 1.13.

Question 1.18. (1) *Is it consistent that $\kappa = \kappa^{<\kappa} > \omega$ and $\langle {}^{\kappa}2, <_{lex} \rangle \longrightarrow_t (\langle {}^{\kappa}2, <_{lex} \rangle)^m_n$ holds for all m, n?*
(2) *If $\kappa = \kappa^{<\kappa} > \omega$ and $\langle {}^{\kappa}2, <_{lex} \rangle \longrightarrow_t (\langle {}^{\kappa}2, <_{lex} \rangle)^3_2$, is κ measurable?*

1.4. **Partition relations in models of determinacy.** Partition relations for cardinals in models of determinacy have been intensively studied. Let us recall some results.

Definition 1.19. (1) *The* strong partition property *holds for a cardinal κ if $\kappa \longrightarrow (\kappa)^\kappa_\mu$ for all $\mu < \kappa$.*
(2) *Let θ denote the supremum of the ordinals α such that there is a surjection $f \colon P(\omega) \twoheadrightarrow \alpha$.*

Note that the strong partition property for ω is equivalent to the statement that all subsets of $[\omega]^\omega$ are Ramsey.

Theorem 1.20. *(1)* [976Pr] *The axiom of determinacy of games of reals $\mathsf{AD}_{\mathbb{R}}$ implies that ω has the strong partition property.*
(2) Martin [003Ka, Theorem 18.12]*,* [004JM, 990Ja, 981K] *The axiom of determinacy* AD *implies that ω_1 has the strong partition property.*
(3) [008KW, 983KW] *Suppose that $V = L(\mathbb{R})$. Then* AD *holds if and only if there are unboundedly many strong partition cardinals below θ.*

It is open whether the strong partition property for ω follows from AD [003Ka, Question 27.18] and what is its consistency strength [003Ka, Question 11.16]. The strong partition property for ω_1 has more consistency strength than the strong partition property for ω by the next result.

Theorem 1.21. *(1)* [977Ma] *It is consistent from an inaccessible cardinal that ω has the strong partition property.*
(2) [970Kl] *Every uncountable cardinal with the strong partition property is measurable.*

We ask which partition relations for linear orders hold if AD holds and $V = L(\mathbb{R})$. Note that the strong partition property for κ implies that $\langle {}^{\omega_1}2, <_{lex} \rangle$ is indivisible.

Theorem 1.22. *Suppose that κ has the strong partition property. Then $\langle {}^{\kappa}2, <_{lex}\rangle \longrightarrow (\langle {}^{\kappa}2, <_{lex}\rangle)^1_2$ holds.*

Proof. The claim follows from the strong partition property by identifying elements of $[\kappa]^\kappa$ with their characteristic functions in 2^κ. □

We ask if this generalises to exponent 2.

Question 1.23. *Suppose that the axiom of determinacy holds in $V = L(\mathbb{R})$. Does this imply $\langle {}^{\omega_1}2, <_{lex}\rangle \longrightarrow (\langle {}^{\omega_1}2, <_{lex}\rangle)^2_2$?*

1.5. **Embedding linear orders into $\langle {}^{\kappa}2, <_{lex}\rangle$.** Every linear order of size κ embeds into $\langle {}^{\kappa}2, <_{lex}\rangle$ by a result of Hausdorff (see [949Ha, Chapter 6, Section 8]). If $\langle \kappa, <_L\rangle$ is a linear order, we map each $\gamma < \kappa$ to the characteristic function in ${}^{\kappa}2$ of the set of predecessors of γ in $<_L$ with $\alpha < \gamma$.

The negative partition results for suborders of $\langle {}^{\kappa}2, <_{lex}\rangle$ in the following sections suggest the question whether every linear order embeds into $\langle {}^{\kappa}2, <_{lex}\rangle$ for some cardinal κ. In models such that every linear order embeds into $\langle {}^{\kappa}2, <_{lex}\rangle$ for some cardinal κ, Theorem 4.16 and Theorem 4.26 hold for all linear orders.

Let $\mathbb{P}$ denote the forcing $P(\omega)$ ordered by inclusion up to finite error. We asked whether in a $\mathbb{P}$-generic extension of $L(\mathbb{R})$, there is a linear order which does not embed into $\langle {}^{\kappa}2, <_{lex}\rangle$ for any cardinal κ, if $L(\mathbb{R})$ is a model of determinacy. This was solved by Paul Larson in unpublished work (see Theorem 1.25 below).

The following is stated in [011CK, Section 1.1] without a proof.

Lemma 1.24. *Suppose that there is a measurable cardinal above ω Woodin cardinals. Let $(x, y) \in E_0$ if $x(n) = y(n)$ for all but finitely many n, for $x, y \in {}^{\omega}\omega$. Then there is no linear order in $L(\mathbb{R})$ of the equivalence classes of E_0.*

Proof. Suppose that in $L(\mathbb{R})$, $\phi(x, y, z, \alpha)$ defines a linear order on the equivalence classes of E_0, where $z \in {}^{\omega}2$ and $\alpha \in Ord$. Let $\mathbb{Q}$ denote Cohen forcing. Suppose that (x, y) is $\mathbb{Q}^2$-generic over $L(\mathbb{R})$.

There is an elementary embedding $L(\mathbb{R}) \hookrightarrow L(\mathbb{R})^{V[x,y]}$ which fixes the ordinals by [001NZ, Theorem 1] . Therefore in $L(\mathbb{R})[x, y]$, ϕ defines a linear order on the equivalence classes of E_0 from α. Suppose that $(p, q) \Vdash^V_{\mathbb{Q}^2} \phi^{L(\mathbb{R})}(x, y, z, \alpha)$. Suppose that $(\bar{x}, x) \in E_0$, $(\bar{y}, y) \in E_0$, $p \subseteq \bar{y}$, and $q \subseteq \bar{x}$. Then $(p, q) \Vdash^V_{\mathbb{Q}^2} \phi^{L(\mathbb{R})}(\bar{y}, \bar{x}, z, \alpha)$. Since the definition of the linear order from α is invariant under E_0, this implies $(p, q) \Vdash^V_{\mathbb{Q}^2} \phi^{L(\mathbb{R})}(y, x, z, \alpha)$, contradicting the assumption. □

Let $\langle {}^{\omega}\omega, <\rangle$ denote ${}^{\omega}\omega$ partially ordered pointwise.

Theorem 1.25 (Paul Larson)**.** *Suppose that there is a measurable cardinal above ω Woodin cardinals and that U is $\mathbb{P}$-generic over $L(\mathbb{R})$. Then in $L(\mathbb{R})[U]$, the linear order $\langle {}^{\omega}\omega/U, < /U\rangle$ does not embed into $\langle {}^{\kappa}2, <_{lex}\rangle$ for any cardinal κ.*

Proof. Forcing with $\mathbb{P}$ preserves measurable cardinals by the Levy-Solovay theorem [010Cu, Theorem 9.6] and Woodin cardinals by [000HW]. Therefore $M^{\#}_{\omega}$ is absolute between V and $V[G]$, where G is generic over V for a forcing in V_δ, where δ is the least Woodin cardinal. Then the supremum of the Woodin cardinals of M_ω is countable. Therefore M_ω satisfies the assumption A_κ in [001NZ, Theorem 1], where κ is below the least Woodin cardinal. Hence forcing with $\mathbb{P}$ does not add new sequences of ordinals, and in particular $\langle {}^{\kappa}2, <_{lex}\rangle = \langle {}^{\kappa}2, <_{lex}\rangle^{V[G]}$ for any $\mathbb{P}$-generic filter G over V.

The theories of $L(\mathbb{R})$ and $L(\mathbb{R})^{V[H]}$ are both determined by M_ω by [010St, Theorem 7.19] and hence equal, where H is $Col(\omega, < \kappa)$-generic over V and κ is the least inaccessible cardinal. Therefore we can apply [003DT, Corollary 7.4] to any colouring in $L(\mathbb{R})$.

Suppose that $p \in \mathbb{P}$ forces that $\dot{f}$ is such an embedding. Let $\mathbb{P}/p = \{q \in \mathbb{P} \mid q \leqslant p\}$. Let $g\colon [\omega]^\omega \times (\mathbb{P}/p) \to 2$, $g(x, q) = 0$ if q decides $\dot{f}(x)$, and $g(x, q) = 1$ otherwise.

There is an infinite set $A \subseteq \omega$ and a sequence $(c_i)_{i\in\omega}$ of subsets of ω of size 2 such that g is constant on $[A]^\omega \times \prod_i c_i$ by [003DT]. It follows from the definition of g that the value is 0. Therefore in $L(\mathbb{R})$, there is a linear order on the equivalence classes of E_0, contradicting Lemma 1.24. □

Note that if there is a supercompact cardinal, then every Ramsey ultrafilter is $\mathbb{P}$-generic over $L(\mathbb{R})$ by a result of Todorcevic [998Fa, Theorem 4.9].

2. Partition relations for $\langle {}^\kappa\kappa, <_{lex}\rangle$

We consider the linear orders $\langle {}^\kappa\kappa, <_{lex}\rangle$ and $\langle {}^\kappa 2, <_{lex}\rangle$ for cardinals κ with $\kappa^{<\kappa} = \kappa > \omega$. These two linear orders are bi-embeddable and hence satisfy the same partition relations.

Definition 2.1. *(1) A* perfect subtree of ${}^{<\kappa}\kappa$ *is a $<\kappa$-closed subtree of ${}^{<\kappa}\kappa$ whose branching nodes are cofinal.*

(2) A perfect subset of ${}^\kappa\kappa$ *is a set of the form $[T]$, where T is a perfect subtree of ${}^{<\kappa}\kappa$.*

We identify $[{}^\kappa 2]^n$ with the set of injective n-tuples $(x_0, \dots, x_{n-1})$ in ${}^\kappa 2$ with $x_0 <_{lex} \dots <_{lex} x_{n-1}$.

2.1. Partition relations for $\langle {}^\omega 2, <_{lex}\rangle$.

We first consider the linear order $\langle {}^\omega 2, <_{lex}\rangle$. The following is a variant of a theorem of Mycielski and Taylor.

Lemma 2.2. *If $f\colon [{}^\omega 2]^m \to {}^\omega 2$ is Baire measurable, then there is a perfect set $C \subseteq {}^\omega 2$ such that $f \restriction [C]^m$ is continuous.*

Proof. Suppose that $(U_n)_{n\in\omega}$ is a sequence of open dense subsets of $({}^\omega 2)^n$ such that f is continuous on their intersection. We construct a family $(t_s)_{s\in 2^n,\ n\in\omega}$ by induction on n such that

(1) $t_s \subseteq t_u$ if $s \subseteq u$ and
(2) $N_{t_0} \times \dots \times N_{t_{m-1}} \subseteq U_n$ if $t_0, \dots, t_{m-1} \in \omega^n$ and $t_i \neq t_j$ for all $i < j < m$.

This is achieved by enumerating the tuples $(t_0, \dots, t_{m-1})$ with $t_0, \dots, t_{m-1} \in \omega^n$ and $t_i \neq t_j$ for all $i < j < m$ in step n and shrinking the sets U_t for $t \in \omega^n$ successively for each tuple. Let T denote the downwards closure of the set of t_s for $s \in 2^{<\omega}$. Let $C = [T]$. Then f is continuous on the set of m-tuples of distinct elements of C, and thus on $[C]^m$, by the construction. $\Box$

Theorem 2.3. *Suppose that all sets of reals have the property of Baire. Then $\langle {}^\omega 2, <_{lex}\rangle \longrightarrow (\langle {}^\omega 2, <_{lex}\rangle)^2_n$ for all n.*

Proof. Note that $\langle {}^\omega 2, <_{lex}\rangle \longrightarrow (\langle {}^\omega 2, <_{lex}\rangle)^2_2$ implies $\langle {}^\omega 2, <_{lex}\rangle \longrightarrow (\langle {}^\omega 2, <_{lex}\rangle)^2_n$ for all $n \in \omega$. Suppose that $f\colon ({}^\omega 2)^2 \to 2$ is Baire measurable. There is a perfect set C such that $f \restriction [C]^m$ is continuous by Lemma 2.2. Since C is order isomorphic with $\langle {}^\omega 2, <_{lex}\rangle$, we can assume that $C = {}^\omega 2$. We can assume that no interval is homogeneous for f in colour 0.

Using this assumption, we construct a family $(t_s)_{s\in 2^n,\ n\in\omega}$ by induction on n such that

(1) $t_s \subseteq t_u$ if $s \subseteq u$ and
(2) $f[N_{t_{s^\frown 0}} \times N_{t_{s^\frown 1}}] = \{1\}$ for all $s \in 2^n$.

This is possible since f is continuous. Let T denote that downwards closure of the set of t_s for $s \in 2^{<\omega}$. Then $f \restriction [T]^2$ is constant with value 1. $\Box$

Note that the assumption in Theorem 2.3 is consistent relative to ZF by [984Sh]. The consistency also follows as a special case of the result for cardinals κ with $\kappa^{<\kappa} = \kappa$ below.

The next two results are consequences of Theorem 1.16 and Lemma 2.2. The following result is used together with the negative partition relations in Section 3 to determine the consistent partition relations for $\langle {}^\omega 2, <_{lex}\rangle$ with exponent 3.

Theorem 2.4. *Suppose that all sets of reals have the property of Baire. Then*

(1) $\langle {}^\omega 2, <_{lex}\rangle \longrightarrow (\langle {}^\omega 2, <_{lex}\rangle, 1 + \omega^* \vee \omega + 1)^3$.
(2) $\langle {}^\omega 2, <_{lex}\rangle \longrightarrow (\langle {}^\omega 2, <_{lex}\rangle, n)^3$ *for all natural numbers n.*

Proof. Suppose that $f : [\langle {}^\omega 2, <_{lex}\rangle]^3 \to 2$ is a colouring. We can assume that f is continuous by Lemma 2.2. Moreover we can assume that the colour $f(\vec{x})$ of a triple $\vec{x}$ depends only on the splitting type of $\vec{x}$ by Theorem 1.16. Let X_i for $i = 0, 1$ denote the set of $x \in {}^\omega 2$ such that $x(n) = i$ for at most one n. Then X_0 and X_1 have order types $\omega + 1$ and $1 + \omega^*$, respectively and are homogenous. If the colour of the splitting types for triples in X_0 and in X_1 is 0, then there is a homogeneous set of order type $\langle {}^\omega 2, <_{lex}\rangle$ in colour 0.

Otherwise one of the splitting types has colour 1. In this case, there is a homogeneous set in colour 1 of order type $1 + \omega^*$ or $\omega + 1$. This shows (1).

(2) follows directly from (1). $\Box$

The following results are used in Section 4 to determine the consistent partition relations for $\langle {}^{\omega}2, <_{lex}\rangle$ with exponent 4.

Theorem 2.5. *Suppose that all sets of reals have the property of Baire. Then* $\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (\omega+1)^m_n$ *for all natural numbers* m *and* n.

Proof. Suppose that there is a colouring of $[{}^{\omega}2]^m$ in n colours. By lemma 2.2 and theorem 1.16 we may assume that the colour is given by the splitting-type. Consider the set

$$S := \{x \in {}^{\omega}2 \mid |\{n < \omega \mid x(n) = 0\}| < 2\}.$$

S has order-type $\omega + 1$ and $[S]^m$ only contains one splitting-type, the one where $\Delta(t_i, t_{i+1}) \sqsubseteq \Delta(t_{i+1}, t_{i+2})$ whenever $i + 2 < m$ where $\langle t_i \mid i < m\rangle$ is an order-preserving enumeration of an element of $[S]^m$. So S provides what was demanded. □

Note that the above is also a theorem in ZFC, cf.[970Ga, 986MP]. As stated before, a further problem is to determine the relations which allow finitely many order types linked by a disjunction, instead of a single order type. For example, assuming a fragment of choice, the occurence of $\omega^* \vee \omega$ in a partition relation for a linear order states that there is an infinite homogeneous set with arbitrary order type. The occurence of $\omega^* + \omega \vee \omega + \omega^*$ in a partition relation for a linear order states that there is an infinite homogeneous set such that L and L^* are not well-ordered.

Theorem 2.6. *Suppose that all sets of reals have the property of Baire. Then* $\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (1 + \omega^* + \omega + 1 \vee m + \omega^* \vee \omega + n, 6)^4$ *for all natural numbers* m *and* n.

Proof. Let $m, n < \omega$ be given and assume towards a contradiction that f is a colouring of $[{}^{\omega}2]^4$ in two colours not admitting any homogeneous set of a relevant order-type. By Lemma 2.2 and Theorem 1.16 we may assume without loss of generality that the f depends only on the splitting type. Consider the six possible splitting types of a quadruple $\{t_0, \ldots, t_3\}_{<_{lex}}$. We write $\mathrm{lt}(s)$ for the length of s.

(1) $\Delta(t_0, t_1) \sqsubseteq \Delta(t_1, t_2) \sqsubseteq \Delta(t_2, t_3)$,
(2) $\Delta(t_2, t_3) \sqsubseteq \Delta(t_1, t_2) \sqsubseteq \Delta(t_0, t_1)$,
(3) $\Delta(t_0, t_1) \sqsubseteq \Delta(t_2, q_3) \sqsubseteq \Delta(t_1, t_2)$,
(4) $\Delta(t_2, t_3) \sqsubseteq \Delta(t_0, t_1) \sqsubseteq \Delta(t_1, t_2)$,
(5) $\Delta(t_1, t_2) \sqsubseteq \Delta(t_0, t_1), \Delta(t_2, t_3)$ and $\mathrm{lt}(\Delta(t_2, t_3)) \leqslant \mathrm{lt}(\Delta(t_0, t_1)$,
(6) $\Delta(t_1, t_2) \sqsubseteq \Delta(t_0, t_1), \Delta(t_2, t_3)$ and $\mathrm{lt}(\Delta(t_0, t_1)) \leqslant \mathrm{lt}(\Delta(t_2, t_3)$.

Since it is easy to define arbitrarily large finite sets which either only contain quadruples of type (1) or (2) we may assume without loss of generality that $f(T) = 0$ for any such quadruple T. Consider the following sets:

$$\begin{aligned}
Z &:= \{x \in {}^{\omega}2 \,||\{k < \omega \mid x(k) \neq x(0)\}| < 2\},\\
M &:= \{x \in {}^{\omega}2 \,||\{k \in \omega \setminus m \mid x(k) = 1\}| < 2 \wedge \forall k < m(x(k) = 0 \to \forall l \in \omega \setminus k : x(l) = 0)\},\\
N &:= \{x \in {}^{\omega}2 \,||\{k \in \omega \setminus n \mid x(k) = 0\}| < 2 \wedge \forall k < n(x(k) = 1 \to \forall l \in \omega \setminus k : x(l) = 1\},\\
H_0 &:= \{x \in {}^{\omega}2 \,|\forall i < 6 \forall k < \omega(x(6k + i) = x(i)) \wedge \exists i < 2(x(3i) = x(3i + 1) = x(3i + 2) = 0)\\
&\qquad \wedge x(0) = 0 \to x(1) = x(2) = 0 \wedge \exists i < 3(x(i) = 0)\},\\
H_1 &:= \{x \in {}^{\omega}2 \,|\forall i < 6 \forall k < \omega(x(6k + i) = x(i)) \wedge \exists i < 2(x(3i) = x(3i + 1) = x(3i + 2) = 1)\\
&\qquad \wedge x(0) = 1 \to x(1) = x(2) = 1 \wedge \exists i < 3(x(i) = 1)\}.
\end{aligned}$$

Note that Z has order-type $1 + \omega^* + \omega + 1$, both H_0 and H_1 are sextuples and M has order-type $m + 1 + \omega^*$, even more than neccessary as does N which has type $\omega + n + 1$. Note that in $[Z]^4$ only the splitting-types (1),(2),(5) and (6) appear.

Consider Z. Since the splitting-types (1) and (2) got colour 0 and there is a $T \in [Z]^4$ with $f(T) = 1$ either splitting-type (5) or (6) get colour 1.

Now consider M, only the splitting-types (1),(2) and (3) appear in $[M]^4$. Since both the splitting-types (1) and (2) get colour 1 and by assumption there is a $T \in [M]^4$ we may assume that quadruples of the splitting-type (3) get colour 1.

Simlarly consider N, only the splitting-types (1),(2) and (4) appear in $[N]^4$. Since both the splitting-types (1) and (2) get colour 1 and by assumption there is a $T \in [N]^4$ we may assume that quadruples of the splitting-type (4) get colour 1.

But now notice that $[H_0]^4$ only contains the splitting types (3),(4) and (5) while $[H_1]^4$ only contains the splitting types (3),(4) and (6). So either H_0 or H_1 is a sextuple which is homogeneous in colour 1. $\square$

Note that in contrast to Theorem 2.5 the above theorem fails under ZFC by Theorem 4.22.

Theorem 2.7. *Suppose that all sets of reals have the property of Baire. Then* $\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (\omega+1+\omega^* \vee 1+\omega^*+\omega+1, 5)^4$ *holds.*

Proof. Suppose that there is a colouring f with no homogeneous sets of order types $\omega+1+\omega^*$ or $1+\omega^*+\omega+1$ in colour 0 and no homogeneous sets of size 5 in colour 1. We can assume that the colour of a tuple depends only on the splitting type by Lemma 2.2 and Theorem 1.16.

If tuples (q_0, q_1, q_2, q_3) with $q_0 <_{lex} q_1 <_{lex} q_2 <_{lex} q_3$ and type $\Delta_{q_0,q_1} < \Delta_{q_1,q_2} < \Delta_{q_2,q_3}$ have colour 1, then any set $\{q_i \mid i < 5\}$ with $q_0 <_{lex} q_1 <_{lex} q_2 <_{lex} q_3 <_{lex} q_4$ and $\Delta_{q_0,q_1} < \Delta_{q_1,q_2} < \Delta_{q_2,q_3} < \Delta_{q_3,q_4}$ is homogeneous in colour 1, contrary to our assumption. So the tuples of this type have colour 0. For the same reason, the tuples (q_0, q_1, q_2, q_3) with $q_0 <_{lex} q_1 <_{lex} q_2 <_{lex} q_3$ and type $\Delta_{q_3,q_2} < \Delta_{q_2,q_1} < \Delta_{q_1,q_0}$ have colour 0.

Let $z(n) = 0$ if n is even and $z(n) = 1$ if n is odd. We consider the sets

$$X := \{x \in {}^{\omega}2 \mid \forall i < j\ [(x(i) < z(i) \Rightarrow x(j) = 0) \wedge (x(i) > z(i) \Rightarrow x(j) = 0)]\}$$
$$Y := \{y \in {}^{\omega}2 \mid |\{n \mid y(n) \neq y(0)\}| \leqslant 1\}$$

s with order types $\omega+1+\omega^*$ and $1+\omega^*+\omega+1$, respectively. Consider the following types of triples (q_0, q_1, q_2) with $q_0 <_{lex} q_1 <_{lex} q_2$.

(1) (a) $\Delta_{q_1,q_2} < \Delta_{q_2,q_3} < \Delta_{q_0,q_1}$
 (b) $\Delta_{q_1,q_2} < \Delta_{q_0,q_1} < \Delta_{q_2,q_3}$

(2) (a) $\Delta_{q_0,q_1} < \Delta_{q_2,q_3} < \Delta_{q_1,q_2}$
 (b) $\Delta_{q_2,q_3} < \Delta_{q_0,q_1} < \Delta_{q_1,q_2}$

There is no triple in $[X]^3$ with a type in (1) and no triple in $[Y]^3$ with a type in (2). This implies that some type in (1) and some type in (2) has colour 1. If the types (1)(a) and (2)(a) have colour 1, then any set $\{q_i \mid i < 5\}$ with the type $\Delta_{q_1,q_2} < \Delta_{q_3,q_4} < \Delta_{q_2,q_3} < \Delta_{q_0,q_1}$ is homogeneous in colour 1. All other cases are symmetric, and in each case we obtain a homogeneous set of size 5 in colour 1. $\square$

The Theorem above is not provable in ZFC by Theorem 4.22 or Theorem 1.9. The following is analogous to Lemma 2.2 for Lebesgue measurable colourings.

Lemma 2.8. *Suppose that the Axiom of Dependent Choices* DC *holds. Suppose that* $f\colon [{}^{\omega}2]^m \to {}^{\omega}2$ *is a colouring such that* $f \restriction A$ *is Lebesgue measurable for all closed sets* $A \subseteq [{}^{\omega}2]^n$*. Then there is a perfect set* $C \subseteq {}^{\omega}2$ *such that* $f \restriction [C]^m$ *is continuous.*

Proof. We construct a family $(T_s)_{s\in 2^n,\ n\in\omega}$ of perfect subtrees of ${}^{<\omega}2$ by induction on n such that

(1) $T_s \subseteq T_u$ if $s \subseteq u$,
(2) T_s, T_u have the same i^{th} splitting levels for all $i \leqslant n$ if $s \subseteq u$ and $s \in 2^n$, and
(3) If $s_0, \ldots, s_{m-1} \in \omega^n$ and $s_i \neq s_j$ for all $i < j < m$, then there is some $u \in 2^n$ such that $f[[T_{s_0}] \times \cdots \times [T_{s_{m-1}}]] \subseteq N_u$.

If $A \subseteq ({}^{\omega}2)^n$ has positive measure, then there are perfect sets $C_0, \ldots, C_{n-1}$ with $\prod_i C_i \subseteq A$ by [967My, Theorem 1]. This is used in the successor step as follows. We enumerate the tuples $(s_0, \ldots, s_{m-1})$ with $s_0, \ldots, s_{m-1} \in \omega^n$ such that each s_i is on the n^{th} splitting level of some tree T_s for $s \in 2^n$ and $s_i \neq s_j$ for all $i < j < m$ in step n. We shrink the trees T_s above s_i for each i to perfect subtrees, successively for each tuple, and thus preserve the k^{th} splitting levels for all $k \leqslant n$. Let T denote the tree of nodes s such that $s \in T_u$ for some n and some $u \in 2^n$, and s is below the n^{th} splitting level of T_u. Let $C = [T]$. Then f is continuous on the set of m-tuples of distinct elements of C, and thus on $[C]^m$, by the construction. $\square$

Theorem 2.9. *Suppose that the Axiom of Dependent Choices* DC *holds and that all sets of reals are Lebesgue measurable. Then the conclusions of Theorem 2.3, Theorem 2.4, and Theorem 2.7 hold.*

Proof. The proofs are identical to those for Baire measurable colourings, using Lemma 2.8 instead of Lemma 2.2. $\square$

2.2. **Partition relations for $\langle {}^\kappa 2, <_{lex}\rangle$.** We now consider the analogous questions for $\langle {}^\kappa 2, <_{lex}\rangle$.

Lemma 2.10. *Suppose that κ is regular and V is a model of* ZFC.

(1) *Suppose that G is $Add(\kappa,1)$-generic over V. Then in $V[G]$, for every function $f\colon [{}^\kappa 2]^n \to {}^\kappa 2$ definable from ordinals, there is a perfect set C such that $f \restriction [C]^n$ is continuous.*
(2) *Suppose that H is $Add(\kappa,\lambda)$-generic over V and $\lambda \geqslant \kappa^+$. Then in $V[H]$, for every function $f\colon [{}^\kappa 2]^n \to {}^\kappa 2$ definable from ordinals and subsets of κ, there is a perfect set C such that $f \restriction [C]^n$ is continuous.*

Proof. For the first claim, note that there is a perfect set C of $Add(\kappa,1)$-generics in $V[G]$ such that the quotient forcing in $V[G]$ of each n-tuple $\vec{x} = (x_0,\ldots,x_{n-1})$ of distinct elements of C is equivalent to $Add(\kappa,1)$ by [014Sc]. Suppose that $\phi(\vec{x},\alpha,t)$ holds in $V[G]$ if and only if $f(\vec{x}) \restriction \alpha = t$, where ϕ is a formula with an ordinal parameter, which we omit. Then $V[G] \vDash \phi(\vec{x},\alpha,t) \Leftrightarrow 1 \Vdash^{V[\vec{x}]}_{Add(\kappa,1)} \phi(\vec{x},\alpha,t)$ for all $\vec{x} \in [C]^n$. Therefore $f(\vec{x}) \in V[\vec{x}]$ for all $\vec{x} \in [C]^n$.

Let $\psi(\vec{x},\alpha,t)$ denote the formula $1 \Vdash^{V[\vec{x}]}_{Add(\kappa,1)} \phi(\vec{x},\alpha,t)$. Let σ denote an $Add(\kappa,1)^n$-name for the n-tuple of $Add(\kappa,1)$-generic reals, so that $\sigma^{\vec{x}} = \vec{x}$ for all $\vec{x} \in [C]^n$.

Claim 2.11. *$f \restriction [C]^n$ is continuous.*

Proof. If $\vec{x} \in [C]^n$ and $\alpha < \kappa$, then there is a condition $p \in Add(\kappa,1)^n$ with $p \subseteq \vec{x}$ and $p \Vdash^V_{Add(\kappa,1)^n} \psi(\sigma,\alpha,f(\vec{x}) \restriction \alpha)$. So $f(\vec{x}) \restriction \alpha = f(\vec{y}) \restriction \alpha$ for all $\vec{y} \in C$ with $p \subseteq \vec{y}$. This proves that $f \restriction [C]^n$ is continuous. □

The proof of the second claim is analogous. We force with $Add(\kappa,1)^n$ over an intermediate model which contains the parameters and whose quotient forcing is equivalent to $Add(\kappa,\lambda)$. □

Theorem 2.12. *Suppose that κ is regular and V is a model of* ZFC.

(1) *Suppose that G is $Add(\kappa,1)$-generic over V. Then in $V[G]$*
$$\langle {}^\kappa 2, <_{lex}\rangle \longrightarrow (\langle {}^\kappa 2, <_{lex}\rangle)^2_n$$
holds for all n and for all colourings $f\colon [{}^\kappa 2]^2 \to 2$ definable from ordinals.
(2) *Suppose that H is $Add(\kappa,\lambda)$-generic over V and $\lambda \geqslant \kappa^+$. Then in $V[G]$*
$$\langle {}^\kappa 2, <_{lex}\rangle \longrightarrow (\langle {}^\kappa 2, <_{lex}\rangle)^2_n$$
holds in $HOD_{P(\kappa)}$ and therefore in $L(P(\kappa))$ for all n.

Proof. It is sufficient to prove $\langle {}^\kappa 2, <_{lex}\rangle \longrightarrow (\langle {}^\kappa 2, <_{lex}\rangle)^2_2$. Suppose that $f\colon [{}^\kappa 2]^2 \to 2$ is a colouring definable from ordinals in $V[G]$. There is a perfect set C such that $f \restriction [C]^2$ is continuous by Lemma 2.10. Since $\langle C, <_{lex}\rangle$ is order isomorphic to $\langle {}^\kappa 2, <_{lex}\rangle$, we can assume that f is continuous.

We can assume that no interval in $\langle {}^\kappa 2, <_{lex}\rangle$ is homogeneous for f in colour 0. Using this assumption, we construct a family $(t_s)_{s\in 2^\alpha,\ \alpha<\kappa}$ by induction on α such that

(1) $t_s \subseteq t_u$ if $s \subseteq u$ and
(2) $f[N_{t_{s^\frown 0}} \times N_{t_{s^\frown 1}}] = \{1\}$ for all $s \in 2^\alpha$.

The successor step is straightforward, since f is continuous. If $u \in 2^\beta$ and $\beta < \kappa$ is a limit, let $t_u = \bigcup_{s\subsetneq u} t_s$. Let T denote the downwards closure of the set of t_s for $s \in 2^{<\kappa}$. Then $f \restriction [T]^2$ is constant with value 1.

The proof of the second claim is analogous from the second claim in Lemma 2.10. □

The size of 2^κ is measured by the ordinal θ_κ in contexts without choice.

Definition 2.13. *Let θ_κ denote the supremum of the ordinals α such that there is a surjection $f\colon P(\kappa) \twoheadrightarrow \alpha$.*

The following result shows that the partition relation $\langle {}^\kappa 2, <_{lex}\rangle \longrightarrow (\langle {}^\kappa 2, <_{lex}\rangle)^2_n$ is not linked to the size of θ_κ.

Corollary 2.14. *Suppose that κ is regular and V is a model of* ZFC.

(1) *There is a $<\kappa$-closed forcing $\mathbb{P}$ such that for any $\mathbb{P}$-generic filter G over V, $HOD^{V[G]}_{P(\kappa)}$ and $L(P(\kappa))^{V[G]}$ satisfy*
   (a) $\kappa = \kappa^{<\kappa}$,

(b) $\theta_\kappa = \kappa^+$, *and*
(c) $\langle {}^\kappa 2, <_{lex}\rangle \longrightarrow (\langle {}^\kappa 2, <_{lex}\rangle)^2_n$.

(2) For any cardinal λ, there is a $< \kappa$-closed forcing $\mathbb{Q}$ such that for any $\mathbb{Q}$-generic filter H over V, $HOD^{V[H]}_{P(\kappa)}$ and $L(P(\kappa))^{V[G]}$ satisfy
(a) $\kappa = \kappa^{<\kappa}$,
(b) $\theta_\kappa \geqslant \lambda$, *and*
(c) $\langle {}^\kappa 2, <_{lex}\rangle \longrightarrow (\langle {}^\kappa 2, <_{lex}\rangle)^2_n$.

Moreover $HOD^{V[G]}_{P(\kappa)}$ and $L(P(\kappa))^{V[G]}$ satisfy dependent choice DC_κ for sequences of length κ.

Proof. For the first claim, we force GCH at κ with $Add(\kappa^+, 1)$ and then apply Theorem 2.12 for $\lambda = \kappa^+$.

For the second claim, we force $\theta_\kappa \geqslant \lambda$ with the forcing $\mathbb{P}$ given by [012Lü, Theorem 1.5] and again apply Theorem 2.12 for $\lambda = \kappa^+$. Forcing with $\mathbb{P}$ followed by $< \kappa$-closed forcing does not decrease θ_κ.

The model $HOD^{V[G]}_{P(\kappa)}$ in Theorem 2.12 is closed under κ-sequences in $V[G]$ and therefore satisfies DC_κ. Every element of $L(P(\kappa))^{V[G]}$ is definable in $L(P(\kappa))^{V[G]}$ from an ordinal and a subset of κ. To prove DC_κ in $L(P(\kappa))^{V[G]}$ for a given relation, we construct a witnessing sequence in $V[G]$ with the ordinals in the definitions chosen as minimal. This sequence is an element of $L(P(\kappa))^{V[G]}$. □

3. Negative partition relations for triples

Theorem 2.3 cannot be improved to exponent 3 for asymmetric partition relations.

Theorem 3.1. *$\langle {}^\alpha 2, <_{lex}\rangle \not\longrightarrow (\omega^*, \omega)^3$ for all ordinals α.*

Proof. Suppose that $x, y, z \in {}^\alpha 2$ with $x <_{lex} y <_{lex} z$. Let $f(x, y, z) = 0$ if $\Delta_{x,y} < \Delta_{y,z}$ and let $f(x, y, z) = 1$ otherwise. Suppose that H is homogeneous in colour 0 with order type ω^* and that $(x_i)_{i\in\omega}$ is the decreasing enumeration of H. Let $\alpha_i = \Delta_{x_i, x_{i+1}}$. Then $(\alpha_i)_{i\in\omega}$ is decreasing. The argument for colour 1 is symmetric. □

Theorem 2.4 shows that Theorem 3.1 is optimal. The relation $\langle {}^\omega 2, <_{lex}\rangle \longrightarrow (\langle {}^\omega 2, <_{lex}\rangle)^2_n$ holds for all n if all sets of reals have the property of Baire by Theorem 2.3. This cannot be improved to exponent 3 in symmetric partition relations (see Theorem 3.3 below).

Lemma 3.2. *If a regular initial ordinal κ embeds into $\langle {}^\alpha 2, <_{lex}\rangle$, then $\kappa \leqslant \alpha$.*

Proof. Suppose that this were false. Let κ be the smallest counterexample. Then κ is regular and there is an $\alpha < \kappa$ such that κ embeds into $\langle {}^\alpha 2, <_{lex}\rangle$ via an embedding which we call ι. Recall that in $\langle {}^\alpha 2, <_{lex}\rangle$, every strictly ascending sequence $\langle r_\nu \mid \nu < \gamma\rangle$ has a supremum. This is because the supremum can simply be defined as the function mapping an ordinal $\beta < \alpha$ to $\lim_{\nu<\gamma} r_\nu(\beta)$. These limits exist, since if there were a $\beta < \alpha$ on which this were not the case, then there would be the least one having this property, thus contradicting the assumption that $\langle r_\nu \mid \nu < \gamma\rangle$ is strictly ascending.

Consider the supremum s of the image of κ under ι in $\langle {}^\alpha 2, <_{lex}\rangle$. Since κ is regular, it is in particular a limit ordinal, so s is not attained, and κ even embeds into $X := \langle \{t \in {}^\alpha 2 \mid t <_{lex} s\}, <_{lex}\rangle$ via ι. Note that the cofinality of $\langle {}^\alpha 2 \setminus \{s\}, <_{lex}\rangle$ is equal to $\gamma := \mathrm{cof}(\alpha) < \kappa$ by the following argument. Let $\langle \xi_\nu \mid \nu < \gamma\rangle$ be cofinal in α. For $\nu < \gamma$, we define $f_\nu : \alpha \longrightarrow 2$, where

$$f_\nu(\beta) = \begin{cases} s(\beta) & \text{if } \beta < \xi_\nu, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle f_\nu \mid \nu < \gamma\rangle$ is cofinal in X and has order type γ. Let $\rho_\nu := \mathrm{otyp}(\iota``\kappa \cap f_\nu)$ for all $\nu < \gamma$. Then the sequence $\langle \rho_\nu \mid \nu < \gamma\rangle$is cofinal in κ, contradicting the assumption that κ is regular. □

Theorem 3.3. *Let κ be an infinite initial ordinal and $\alpha < \kappa^+$. Then $\langle {}^\alpha 2, <_{lex}\rangle \not\longrightarrow (2 + \kappa^* \vee \omega, \omega^* \vee \kappa + 2)^m$ for all $m \geqslant 3$.*

Proof. Suppose that $m = 3$. Let $h \colon \alpha \leftrightarrow \kappa$ be a bijection and $\beta(x, y) := h(\mathrm{lt}(\Delta(x, y)))$ for $x, y \in {}^\alpha 2$. We consider the following colouring $f : [{}^\alpha 2]^3 \to 2$. If $x, y, z \in {}^\alpha 2$ and $x <_{lex} y <_{lex} z$, let $f(\{x, y, z\}) = 0$ if $\beta(y, z) < \beta(x, y)$.

In the first case, suppose that $X = \{x_\nu \mid \nu < \kappa + 2\} \in [{}^\alpha 2]^{2+\kappa^*}$ and that $x_\gamma < x_\beta$ whenever $\beta < \gamma < \kappa + 2$. In the first subcase, suppose that S stabilises at $s \in {}^{<\alpha}2$ from $\gamma < \kappa$ onwards. Then Lemma 3.2 implies that $|\{\mathrm{lt}(\Delta(x_{\nu+1}, x_\nu)) \mid \nu \in \kappa \setminus \gamma\}| = \kappa$. Since h is one-to-one, $|\{\beta(x_{\nu+1}, x_\nu) \mid \nu \in \kappa \setminus \gamma\}| = \kappa$, so we may choose a $\xi \in \kappa \setminus \gamma$ with $\beta(x_{\xi+1}, x_\xi) > h(\mathrm{lt}(s))$. Then $f(\{x_{\kappa+1}, x_{\xi+1}, x_\xi\}) = 1$. Now suppose that S does not stabilise. The sequence $\langle \Delta(x_\nu, x_0) \mid \nu < \kappa\rangle$ stabilises at some s. Since S does not stabilise, Lemma 3.2 implies that $|\{\mathrm{lt}(\Delta(x_{\kappa+1}, x_\nu)) \mid \nu < \kappa\}| = \kappa$. Since h is one-to-one we have $|\{\beta(x_{\kappa+1}, x_\nu) \mid \nu < \kappa\}| = \kappa$, so we may choose a $\xi < \kappa$ with $\beta(x_{\kappa+1}, x_\xi) > h(\mathrm{lt}(s))$. Then $f(\{x_{\kappa+1}, x_\xi, x_0\}) = 1$.

In the second case, consider a set $Y = \{x_i \mid i < \omega\} \in [{}^\alpha 2]^\omega$ with $x_m < x_n$ for $m < n < \omega$. Assume towards a contradiction that Y were homogeneous in colour 0. Then for any $i < \omega$, we have $\beta(x_{i+1}, x_{i+2}) < \beta(x_i, x_{i+1})$, by considering the triple $\{x_i, x_{i+1}, x_{i+2}\}$. Then $\langle \beta(x_i, x_{i+1}) \mid i < \omega\rangle$ is an infinite decreasing sequence of ordinals, a contradiction.

The remaining cases in the proof for $m = 3$ are analogous.

The proof for $m \geqslant 4$ works similarly by considering the following colouring $f : [{}^\alpha 2]^m \to 2$. If $\vec{x} \in [{}^\alpha 2]^m$ and $x_0 <_{lex} \ldots < x_{m-1}$, let $f(\vec{x}) = 0$ if $\beta(x_0, x_1) < \beta(x_{m-2}, x_{m-1})$. $\square$

Unlike in other cases, assuming the Axiom of Choice, there is a linear ordering, even a well-ordering, satisfying the partition relation in Theorem 3.3. In fact, by the Erdős-Rado-Theorem [956ER, Theorem 39] the cardinal $(2^{\mathfrak{c}})^+$ is such a well-ordering and even $(2^{2^\kappa})^+ \longrightarrow (\kappa^+)^3_\kappa$. We do not know whether in some model of ZFC, the partition property considered in Theorem 3.3 holds for a linear order L such that neither $\omega_2 \leqslant L$ nor $\omega_2^* \leqslant L$.

The following result shows that the previous theorems solve the case of triple-colourings in the Cantor space completely, given that all sets of reals have the property of Baire.

We will only consider partition relations such that in no disjunction there are linear orders K, L with $K \leq L$, since in this case L can be omitted without changing the truth value of the partition relation.

Theorem 3.4. *Suppose that the principle of dependent choices* DC *holds true and all sets of reals have the property of Baire. Suppose that K_μ and L_ν are suborders of $\langle {}^\omega 2, <_{lex}\rangle$ for all $\mu < \kappa$ and $\nu < \lambda$. Then the partition relation*

$$\langle {}^\omega 2, <_{lex}\rangle \longrightarrow (\bigvee_{\nu<\kappa} K_\nu, \bigvee_{\nu<\lambda} M_\nu)^3$$

holds true if and only if one of the following cases applies.

(i) *$K_\xi \leq \omega + 1$ and $K_\rho \leqslant 1 + \omega^*$ for some $\xi, \rho < \kappa$,*
(ii) *$M_\xi \leq 1 + \omega^*$ and $M_\rho \leq \omega + 1$ for some $\xi, \rho < \lambda$,*
(iii) *$K_\xi, M_\rho \leq \omega + 1$ for some $\xi < \kappa$, $\rho < \lambda$,*
(iv) *$K_\xi, M_\rho \leq 1 + \omega^*$ for some $\xi < \kappa$, $\rho < \lambda$.*

Moreover, if none of these cases applies, then the relation is inconsistent with ZF*.*

Proof. Note that $K_\xi = K_\rho$ is finite if $\xi = \rho$ in (i), and similarly in (ii).

We first consider cases in which the partition relation fails. First assume that $K_\mu \nleqslant \omega + 1$ for all $\mu < \kappa$ and $M_\nu \nleqslant 1 + \omega^*$ for all $\nu < \lambda$. We claim that the partition relation in question fails. Note that by DC, for any linear order K, $K \leqslant \omega + 1$ is equivalent to $\omega^* \nleqslant K \wedge \omega + 2 \nleqslant K$, and symmetrically, $K \leqslant 1 + \omega^*$ is equivalent to $\omega \nleqslant K \wedge 2 + \omega^* \nleqslant K$. Hence the partition relation in question implies $\langle {}^\omega 2, <_{lex}\rangle \longrightarrow (\omega^* \vee \omega + 2, 2 + \omega^* \vee \omega)^3$, contradicting Theorem 3.3 for $\kappa = \omega$. Second, assume that $K_\mu \nleqslant 1 + \omega^*$ for all $\mu < \kappa$ and $M_\nu \nleqslant \omega + 1$ for all $\nu < \lambda$. This can be dealt with symmetrically.

The remaining cases are as follows, and in each case the partition relation holds. If there are $\xi, \rho < \kappa$ such that $K_\xi \leqslant \omega + 1$ and $K_\rho \leqslant 1 + \omega^*$, then the relation holds by Theorem 2.4. The argument is analogous if there are $\xi, \rho < \lambda$ such that $M_\xi \leqslant \omega + 1$ and $M_\rho \leqslant 1 + \omega^*$ If there are $\xi < \kappa$ and $\rho < \lambda$ with $K_\xi \leqslant \omega + 1$ and $M_\rho \leqslant \omega + 1$, then the relation holds by Theorem 2.5. An analogous argument works if there are $\xi < \kappa$ and $\rho < \lambda$ with $K_\xi \leqslant 1 + \omega^*$ and $M_\rho \leqslant 1 + \omega^*$. $\square$

4. Negative partition relations for quadruples

In this section, we prove several negative partition theorems for partitions of $[{}^\alpha 2]^4$ by providing colourings avoiding sets of certain order types in one colour and avoiding quintuples, sextuples, septuples, octuples or nonuples in the other. We first give an overview over the negative partition relations.

Theorem 4.1. *If α is an ordinal, then the following statements hold.*

$$\begin{aligned}
&\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\omega^* + \omega, 5)^4,\\
&\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\omega + \omega^*, 5)^4,\\
&\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\omega^* + \omega \vee \omega + \omega^*, 7)^4.
\end{aligned}$$

Theorem 4.2. *If κ is an infinite initial ordinal and $\alpha < \kappa^+$, then the following statements hold.*

$$\begin{aligned}
&\langle {}^{\alpha}2, <_{lex}\rangle \not\to (2 + \kappa^* \vee \kappa + 2 \vee \eta, 5)^4,\\
&\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\omega^* + \omega \vee \kappa + 2 + \kappa^* \vee (\kappa 2)^* \vee \kappa 2, 5)^4,\\
&\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\omega^* + \omega \vee \kappa + \omega \vee \omega^* + \kappa^*, 6)^4,\\
&\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\omega + \omega^* \vee 2 + \kappa^* \vee \kappa + 2, 6)^4,\\
&\begin{cases}\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\kappa^* + \kappa \vee 2 + \kappa^* \vee \kappa 2 \vee \omega\omega^*, 6)^4,\\
\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\kappa^* + \kappa \vee (\kappa 2)^* \vee \kappa + 2 \vee \omega^*\omega, 6)^4,\end{cases}\\
&\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\kappa^* + \kappa \vee \kappa + 2 \vee 2 + \kappa^* \vee \eta, 7)^4,\\
&\begin{cases}\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\omega^* + \omega \vee 2 + \kappa^* \vee \kappa + \omega, 7)^4,\\
\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\omega^* + \omega \vee \omega^* + \kappa^* \vee \kappa + 2, 7)^4,\end{cases}\\
&\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\omega^* + \omega \vee \omega + \omega^* \vee (\kappa 2)^* \vee \kappa 2, 8)^4,\\
&\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\kappa^* + \omega \vee \omega^* + \kappa \vee 2 + \kappa^* \vee \kappa + 2 \vee \omega\omega^* \vee \omega^*\omega, 8)^4,\\
&\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\omega^* + \omega \vee \omega + \omega^* \vee \kappa + 2 \vee 2 + \kappa^*, 9)^4.
\end{aligned}$$

4.1. **Various lemmata.** We first define some sets with respect to Δ as defined above.

Definition 4.3. *We consider the following sets.*

$$\begin{aligned}
E_0 =& \{\vec{x} \mid \Delta(x_0, x_1) \sqsubseteq \Delta(x_1, x_2) \sqsubseteq \Delta(x_2, x_3)\},\\
E_1 =& \{\vec{x} \mid \Delta(x_0, x_1) \sqsubseteq \Delta(x_2, x_3) \sqsubseteq \Delta(x_1, x_2)\},\\
E_2 =& \{\vec{x} \mid \Delta(x_2, x_3) \sqsubseteq \Delta(x_0, x_1) \sqsubseteq \Delta(x_1, x_2)\},\\
E_3 =& \{\vec{x} \mid \Delta(x_2, x_3) \sqsubseteq \Delta(x_1, x_2) \sqsubseteq \Delta(x_0, x_1)\},\\
E_4 =& \{\vec{x} \mid \Delta(x_1, x_2) \sqsubseteq \Delta(x_0, x_1), \Delta(x_2, x_3)\},
\end{aligned}$$

Note that there is a symmetry between every pair of E_i and E_{3-i}.

Lemma 4.4. *Let α be an ordinal number. Then for every set $Z \in [{}^{\alpha}2]^{\omega^*+\omega}$, there is some $\{z_0, z_1, z_2, z_3\}_{<_{lex}} \in [Z]^4 \cap E_4$.*

Proof. Let $\langle x_n \mid n < \omega\rangle$ be the order-reversing enumeration of the lower half of Z and $\langle y_n \mid n < \omega\rangle$ the order-preserving enumeration of its upper half such that $\Delta(x_0, y_0)$ is minimised. Then $\{x_1, x_0, y_0, y_1\}$ provides what was demanded. □

Lemma 4.5. *Let α be an ordinal number and $h : \alpha \hookrightarrow |\alpha|$ be an injection. Then for every $Z \in [{}^{\alpha}2]^{\omega^*+\omega}$ and both $i < 2$, there is some $\vec{x} = \{x_0, x_1, x_2, x_3\} \in [Z]^4 \cap E_4$ such that $\beta_h(x_1, x_2) < \beta_h(x_{2i}, x_{x2i+1})$, or there is a $\vec{x} = \{x_0, x_1, x_2, x_3\} \in [Z]^4 \cap E_{3-3i}$ such that $\beta_h(x_1, x_2) < \beta_h(x_{2i}, x_{2i+1}) < \beta_h(x_{2-2i}, x_{3-2i})$.*

Proof. Let $Z \in [{}^{\alpha}2]^{\omega^*+\omega}$ and $s \in {}^{<\alpha}2$ be the lowest splitting node of elements of Z. So let $\langle x_n \mid n < \omega\rangle$ be the enumeration of $\{x \in Z \mid x \sqsupset s^\frown\langle i\rangle\}$ which is order-respecting(order-reversing for $i = 0$ and order-preserving for $i = 1$). Let $y, z \in Z$ be such that $y, z \sqsupset s^\frown\langle 1 - i\rangle$. If there is an $n < \omega$ for which $\beta_h(x_{n+1}, x_n) > \beta_h(x_n, y)$ then $\{x_{n+1}, x_n, y, z\} \in E_4$ provides what was demanded. If not then by finitude of decreasing sequences of ordinals there has to be an $n < \omega$ such that $\beta_h(x_{n+2}, x_{n+1}) > \beta_h(x_{n+1}, x_n)$. Then $\{x_{n+2}, x_{n+1}, x_n, y\} \in E_{3-3i}$ provides what was demanded. □

Lemma 4.6. *Let α be an ordinal number and $h : \alpha \hookrightarrow |\alpha|$ an injection. Then for every $Q \in [{}^{\alpha}2]^{\eta}$ there is both an $i < 2$ and a $\vec{q} = \{q_0, q_1, q_2, q_3\}_{<_{lex}} \in [Q]^4 \cap E_{i+1}$ such that $\beta_h(q_{2i}, q_{2i+1}) < \beta_h(q_{2-2i}, q_{3-2i}) < \beta_h(q_1, q_2)$.*

Proof. Consider a $Q \in [{}^{\alpha}2]^{\eta}$. Let $s \in {}^{<\alpha}2$ be such that there are $p_0, r_0 \in Q$ with $\Delta(p_0, r_0) = s$ and $h(\mathrm{lt}(s))$ is minimised. Now inductively in step $n < \omega$ by density of Q there has to be a $t \in]p_n, r_n[\cap Q$. If $\Delta(p_n, t) = s$ then $p_{n+1} := p_n$ and $r_{n+1} := t$, otherwise $\Delta(t, r_n) = s$ and we define $p_{n+1} := t$ and $r_{n+1} := r_n$. At most one of the sequences $\vec{p} = \langle p_n \mid n < \omega\rangle$ and $\vec{r} = \langle r_n \mid n < \omega\rangle$ can stabilise. Suppose without loss of generality that $\vec{p}$ does not stabilise. Then there is an $n < \omega$ such that $\beta_h(p_n, p_{n+1}) < \beta_h(p_{n+1}, p_{n+2})$. Then $\{p_n, p_{n+1}, p_{n+2}, r_0\} \in E_2$ and provides what was demanded. □

Lemma 4.7. *Let α be an ordinal number and $h : \alpha \hookrightarrow |\alpha|$ an injection. then for every $Q \in [{}^{\alpha}2]^{\eta}$ there is a $\vec{q} = \{q_0, q_1, q_2, q_3\}_{<_{lex}} \in [Q]^4 \cap E_4$ such that $\beta_h(q_1, q_2) < \min(\beta_h(q_0, q_1), \beta_h(q_2, q_3))$.*

Proof. Let $s \in {}^{<\alpha}2$ be a splitting node of elements of Q such that $h(\mathrm{lt}(s))$ is minimised. Obviously for both $i < 2$ one has $Q_i := \mathrm{otyp}(\{q \in Q \mid q \sqsupset s^\frown\langle i\rangle\}) = \eta$. For both $i < 2$ take $\{a_i, b_i\}_{<_{lex}} \in [Q_i]^2$. Then $\{a_0, b_0, a_1, b_1\}$ provides what was demanded. □

Lemma 4.8. *Let α be an ordinal number and $h : \alpha \hookrightarrow |\alpha|$. Then for every $A \in [{}^{\alpha}2]^{2+\kappa^*}$ there is a $\{a_0, a_1, a_2, a_3\}_{<_{lex}} \in [A]^4 \cap (E_1 \cup E_3)$ such that $\beta_h(q_0, q_1) < \beta_h(q_2, q_3) < \beta_h(q_1, q_2)$. Then also, symmetrically, there is a $\{a_0, a_1, a_2, a_3\}_{<_{lex}} \in [A]^4 \cap (E_0 \cup E_2)$ such that $\beta_h(q_2, q_3) < \beta_h(q_0, q_1) < \beta_h(q_1, q_2)$ for every $A \in [{}^{\alpha}2]^{\kappa+2}$.*

Proof. Since the first half of the lemma is a symmetric statement, only the second half is going to be proved.

First let $\kappa := |\alpha|$ and consider a $B \in [{}^{\alpha}2]^{\kappa+2}$. Let $\langle b_\nu \mid \nu < \kappa + 2\rangle$ be the order-preserving enumeration of B. We distinguish two cases. First assume that the sequence $\vec{s} := \langle \Delta(b_\nu, b_{\kappa+1}) \mid \nu < \kappa\rangle$ stabilises, say at $s \in {}^{<\alpha}2$ from $\zeta < \kappa$ onwards. Since the domain and the range of h share their respective cardinality and by lemma 3.2 there has to be a $\rho \in \kappa \setminus \zeta$ such that $\beta_h(b_\rho, b_{\rho+1}) > h(\mathrm{lt}(s))$ and $|\{\nu < \kappa \mid b_\nu \sqsupset \Delta(b_\rho, b_{\rho+1})\}| = \kappa$. Then choose a $\xi \in \kappa \setminus \rho$ such that $\beta_h(b_\xi, b_{\xi+1}) > \beta_h(b_\rho, b_{\rho+1})$. Now $\{b_\rho, b_\xi, b_{\xi+1}, b_{\kappa+1}\} \in E_2$ provides what was demanded.

So assume that $\vec{s}$ does not stabilise. Then, using lemma 3.2, pick a $\zeta < \kappa$ such that $\beta_h(b_\zeta, b_{\zeta+1}) > \beta_h(b_\kappa, b_{\kappa+1})$ and $b_\kappa \sqsupset \Delta(b_\zeta, b_{\zeta+1})$. After that again pick a $\rho \in \kappa \setminus \zeta$ with $\beta_h(b_\rho, b_{\rho+1}) > \beta_h(b_\zeta, b_{\zeta+1})$ and $b_\kappa \sqsupset \Delta(b_\rho, b_{\rho+1})$. Then $\{b_\zeta, b_\rho, b_\kappa, b_{\kappa+1}\} \in E_0$ provide what was demanded. □

Lemma 4.9. *Let α be an ordinal number and $h : \alpha \hookrightarrow |\alpha|$ an injection. Then for every $Z \in [{}^{\alpha}2]^{\kappa^*+\omega}$ there is a $\{z_0, z_1, z_2, z_3\}_{<_{lex}} \in [Z]^4 \cap E_4$ with $\beta_h(z_1, z_2) < \beta_h(z_0, z_1)$. Then also, symmetrically, there is a $\{z_0, z_1, z_2, z_3\}_{<_{lex}} \in [Z]^4 \cap E_4$ with $\beta_h(z_1, z_2) < \beta_h(z_2, z_3)$ for every $Z \in [{}^{\alpha}2]^{\omega^*+\kappa}$.*

Proof. Since the two halves of the lemma are symmetric to one another, we are only going to prove the second one. So let Z be as in the lemma and let $s \in {}^{<\alpha}2$ be the minimal splitting node of elements of Z. Since Z has no least element there are $z_0, z_1 \sqsupset s^\frown\langle 0\rangle$. Let $\langle z_\nu \mid \nu < \kappa\rangle$ be the order-preserving enumeration of $\{z \in Z \mid z \sqsupset s^\frown\langle 1\rangle\}$. Let $\zeta < \kappa$ be such that $\beta_h(z_\zeta, z_{\zeta+1}) > h(\mathrm{lt}(s))$. Then $\{z_0, z_1, z_\zeta, z_{\zeta+1}\} \in E_4$ provides what was demanded. □

Lemma 4.10. *Let α be an ordinal number and $h : \alpha \hookrightarrow |\alpha|$ an injection. Then for every $Z \in [{}^{\alpha}2]^{\kappa^*+\kappa}$ there is a $\{z_0, z_1, z_2, z_3\}_{<_{lex}} \in [Z]^4 \cap E_4$ such that $\beta_h(z_1, z_2) < \min(\beta_h(z_0, z_1), \beta_h(z_2, z_3))$.*

Proof. Let Z be as in the lemma and let $s \in {}^{<\alpha}2$ be the minimal splitting node of elements of Z. Let $\langle x_\nu \mid \nu < \kappa\rangle$ be an order-reversing enumeration of elements of Z extending $s^\frown\langle 0\rangle$ and let $\langle y_\nu \mid \nu < \kappa\rangle$ be an order-presering enumeration of elements of Z extending $s^\frown\langle 1\rangle$. Then let $\zeta, \rho < \kappa$ be such that $\beta_h(x_{\zeta+1}, x_\zeta) > h(\mathrm{lt}(s))$ and $\beta_h(y_\rho, y_{\rho+1}) > h(\mathrm{lt}(s))$. Now $\{x_{\zeta+1}, x_\zeta, y_\rho, y_{\rho+1}\} \in E_4$ provides what was demanded. □

Lemma 4.11. *Let α be an ordinal number and $h : \alpha \hookrightarrow |\alpha|$ be an injection. Then for every $X \in [{}^{\alpha}2]^{\omega\omega^*}$ there is an $\vec{x} = \{x_0, x_1, x_2, x_3\}_{<_{lex}} \in [X]^4 \cap E_4$ such that $\beta_h(x_1, x_2) < \beta_h(x_0, x_1)$. Then also, symmetrically, there is an $\vec{x} = \{x_0, x_1, x_2, x_3\}_{<_{lex}} \in [X]^4 \cap E_4$ such that $\beta_h(x_1, x_2) < \beta_h(x_2, x_3)$ for every $X \in [{}^{\alpha}2]^{\omega^*\omega}$.*

Proof. Since the two halves of the lemma are symmetric to each other we only need to prove the first one. So let $X \in [{}^{\alpha}2]^{\omega\omega^*}$. Let s_0 be the first splitting node of elements of X and for every $k < \omega$ let s_{k+1} be the first splitting node of elements of X extending $s_k^\frown\langle 0\rangle$. Note that for infinitely many $k < \omega$ we have $\operatorname{otyp}\{x \in X \mid x \sqsupset s_k^\frown\langle 1\rangle\} \geqslant \omega$, otherwise we would have $1 + \omega^* \leqslant X$ which is false. So let $\langle k_i \mid i < \omega\rangle$ be an enumeration of these k. Since there is no decreasing sequence of ordinals there has to be an $i < \omega$ such that $h(\operatorname{lt}(s_{k_{i+1}})) > h(\operatorname{lt}(s_{k_i}))$. So pick an $a \in X$ with $a \sqsupset s_{k_{i+1}}^\frown\langle 0\rangle$, some $b \in X$ such that $b \sqsupset s_{k_{i+1}}^\frown\langle 1\rangle$ and $\{c, d\}_{<_{lex}} \in [X]^2$ satisfying $c, d \sqsupset s_{k_i}^\frown\langle 1\rangle$. Now clearly $\{a, b, c, d\}$ provides what was demanded. □

Lemma 4.12. *Let α be an ordinal number and $h : \alpha \hookrightarrow |\alpha|$ be an injection. Then for every $X \in [{}^{\alpha}2]^{\omega\omega^*}$ there is some $\vec{x} = \{x_0, x_1, x_2, x_3\}_{<_{lex}} \in [X]^4 \cap E_4$ such that $\beta_h(x_1, x_2) < \min(\beta_h(x_0, x_1), \beta_h(x_2, x_3))$ or a $\vec{x} = \{x_0, x_1, x_2, x_4\}_{<_{lex}} \in [X]^4 \cap E_0$ such that $\beta_h(x_1, x_2) < \beta_h(x_2, x_3) < \beta_h(x_0, x_1)$. Then also, symmetrically, for every $X \in [{}^{\alpha}2]^{\omega^*\omega}$ there is an $\vec{x} = \{x_0, x_1, x_2, x_3\}_{<_{lex}} \in [X]^4 \cap E_4$ such that $\beta_h(x_1, x_2) < \min(\beta_h(x_0, x_1), \beta_h(x_2, x_3))$ or a $\vec{x} = \{x_0, x_1, x_2, x_4\}_{<_{lex}} \in [X]^4 \cap E_3$ such that $\beta_h(x_1, x_2) < \beta_h(x_0, x_1) < \beta_h(x_2, x_3)$.*

Proof. Since the two halves of the lemma are symmetric, it suffices only to prove the first one. Suppose that $X \in [{}^{\alpha}2]^{\omega\omega^*}$. Let s_0 be the first splitting node of elements of X and for every $k < \omega$ let s_{k+1} be the first splitting node of elements of X extending $s_k^\frown\langle 0\rangle$. Note that as in the proof of Lemma 4.11 for infinitely many $k < \omega$ we have $\operatorname{otyp}\{x \in X \mid x \sqsupset s_k^\frown\langle 1\rangle\} \geqslant \omega$. So let $\langle k_i \mid i < \omega\rangle$ be an enumeration of these k. Since there is no decreasing sequence of ordinals there has to be an $i < \omega$ such that $h(\operatorname{lt}(s_{k_{i+1}})) > h(\operatorname{lt}(s_{k_i}))$. If there are $c, d \sqsupset s_{k_i}^\frown\langle 1\rangle$ with $\beta_h(c, d) > h(\operatorname{lt}(s_{k_i}))$ then for $a \sqsupset s_{k_{i+1}}^\frown\langle 0\rangle$ and $b \sqsupset s_{k_{i+1}}^\frown\langle 1\rangle$ the quadruple $\{a, b, c, d\} \in E_4$ provides what was demanded. So suppose now that for all $c, d \sqsupset s_{k_i}$ we have $c = d$ or $\beta_h(c, d) < h(\operatorname{lt}(s_{k_i}))$. Let $\langle c_i \mid i < \omega\rangle$ be an ascending enumeration of elements of $\{x \in X \mid x \sqsupset s_{k_i}^\frown\langle 1\rangle\}$. The finitude of decreasing sequences of ordinals implies that there has to be an $n < \omega$ such that $\beta_h(c_n, c_{n+1}) < \beta_h(c_{n+1}, c_{n+2})$. But then for any $b \sqsupset s_{k_{i+1}}$ the quadruple $\{b, c_n, c_{n+1}, c_{n+2}\} \in E_0$ provides what was demanded. □

Lemma 4.13. *Let α be an ordinal number and $h : \alpha \hookrightarrow |\alpha|$ an injection. Then for every $A \in [{}^{\alpha}2]^{\omega^*+\kappa^*}$ there is a $\{a_0, a_1, a_2, a_3\}_{<_{lex}} \in [A]^4 \cap E_3$ such that $\beta_h(q_1, q_2) < \beta_h(q_0, q_1) < \beta_h(q_2, q_3)$, or both a $\{a_0, a_1, a_2, a_3\}_{<_{lex}} \in [A]^4 \cap E_1$ such that $\beta_h(q_0, q_1) < \beta_h(q_2, q_3) < \beta_h(q_1, q_2)$ and a $\{a_0, a_1, a_2, a_3\}_{<_{lex}} \in [A]^4 \cap E_4$ such that $\beta_h(q_1, q_2) < \beta_h(q_0, q_1)$. Then also, symmetrically, there is a $\{b_0, b_1, b_2, b_3\}_{<_{lex}} \in [B]^4 \cap E_0$ such that $\beta_h(q_1, q_2) < \beta_h(q_2, q_3) < \beta_h(q_0, q_1)$ or both a $\{b_0, b_1, b_2, b_3\}_{<_{lex}} \in [B]^4 \cap E_2$ such that $\beta_h(q_2, q_3) < \beta_h(q_0, q_1) < \beta_h(q_1, q_2)$ and a $\{b_0, b_1, b_2, b_3\}_{<_{lex}} \in [B]^4 \cap E_4$ such that $\beta_h(q_1, q_2) < \beta_h(q_2, q_3)$.*

Proof. Since both halves of the the Lemma are symmetric to each other we are only going to prove the second one. First suppose that there is an $s \in {}^{<\alpha}2$ such that $\operatorname{otyp}(B_0) \geqslant \kappa$ and $\operatorname{otyp}(B_1) \geqslant \omega$ where $B_i := \{b \in B \mid b \sqsupset s^\frown\langle i\rangle\}$ for $i < 2$. Let $\langle x_\nu \mid \nu < \kappa\rangle$ be an ascending enumeration of elements of B_0 and $\langle y_n \mid n < \omega\rangle$ an ascending enumeration of elements of B_1. Then, using Lemma 3.2 one can pick a $\zeta < \kappa$ such that $\beta_h(x_\zeta, x_{\zeta+1}) > h(\operatorname{lt}(s))$ and $\{b \in B_0 \mid b \sqsupset \Delta(x_\zeta, x_{\zeta+1})\}$ has size κ. After that one can choose a $\rho \in \kappa \setminus \zeta$ such that $\beta_h(x_\rho, x_{\rho+1}) > \beta_h(x_\zeta, x_{\zeta+1})$. Then for any $y, z \in B_1$ the sets $\{x_\nu, x_{\nu+1}, y, z\} \in E_4$ and $\{x_\zeta, x_\rho, x_{\rho+1}, y\} \in E_2$ provide what was demanded.

Now assume that there is no such s. The nonexistence of infinite decreasing sequences of ordinals yields $m, n < \omega$ such that $\Delta(y_m, y_{m+1})^\frown\langle 1\rangle \sqsubseteq \Delta(y_n, y_{n+1})$ and $\beta_h(y_m, y_{m+1}) < \beta_h(y_n, y_{n+1})$. Now using Lemma 3.2 one can find a $\zeta < \kappa$ such that $\Delta(x_\zeta, x_{\zeta+1})^\frown\langle 1\rangle \sqsubseteq \Delta(y_n, y_{n+1})$ and $\beta_h(x_\zeta, x_{\zeta+1}) > \beta_h(y_n, y_{n+1})$. Now $\{x_\zeta, y_m, y_n, y_{n+1}\} \in E_0$ provides what was demanded. □

Lemma 4.14. *Let α be an ordinal number and $h : \alpha \hookrightarrow |\alpha|$ be an injection. Then for any set $A \in [{}^{\alpha}2]^{(\kappa 2)^*}$, there is some $\vec{x} = \{x_0, x_1, x_2, x_3\}_{<_{lex}} \in [A]^4 \cap E_4$ such that $\beta_h(x_1, x_2) < \min(\beta_h(x_0, x_1), \beta_h(x_2, x_3))$ or for all of the following β_h-relations there is a $\{x_0, x_1, x_2, x_3\}_{<_{lex}} \in [A]^4 \cap E_3$ satisfying them:*

$$\beta_h(x_1, x_2) < \beta_h(x_0, x_1) < \beta_h(x_2, x_3),$$
$$\beta_h(x_1, x_2) < \beta_h(x_2, x_3) < \beta_h(x_0, x_1),$$
$$\beta_h(x_2, x_3) < \beta_h(x_0, x_1) < \beta_h(x_1, x_2),$$
$$\beta_h(x_0, x_1) < \beta_h(x_2, x_3) < \beta_h(x_1, x_2).$$

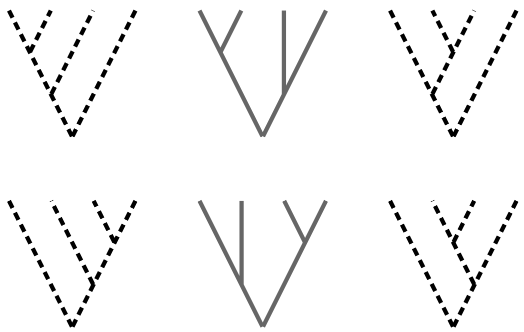

FIGURE 1. Colouring of the splitting types for the proof of theorem 4.16(1)

Symmetrically for any $B \in [^{\alpha}2]^{\kappa 2}$ *there is an* $\vec{x} = \{x_0, x_1, x_2, x_3\} \in [B]^4 \cap E_4$ *such that* $\beta_h(x_1, x_2) < \min(\beta_h(x_0, x_1) < \beta_h(x_2, x_3))$ *or for all of the* β_h*-relations above there is a* $\{x_0, x_1, x_2, x_3\}_{<_{lex}} \in [B]^4 \cap E_0$ *satisfying them.*

Proof. As the two halves of the lemma are symmetric to each other, it suffices to prove the second one. So let $B \in [^{\alpha}2]^{\kappa 2}$ and suppose that for all $\{t_0, t_1, t_2, t_3\}_{<_{lex}} \in [B]^4 \cap E_4$ there is an $i < 2$ with $\beta_h(t_{2i}, t_{2i+1}) < \beta_h(t_1, t_2)$. Via Lemma 3.2 this implies that there is a $\{b_\nu \mid \nu < \kappa 2\}_{<_{lex}} \in [B]^{\kappa 2}$ such that $\Delta(b_\zeta, b_{\zeta+1})^\frown\langle 1\rangle \sqsubseteq \Delta(b_\rho, b_{\rho+1})$ for every $\{\zeta, \rho\}_< \in [\kappa 2]^2$. Now for every β_h-relation mentioned above it is easy to choose ζ, ν, ξ, ρ such that $\{b_\zeta, b_\nu, b_\xi, b_\rho\}$ provides what was demanded. □

Lemma 4.15. *Let* α *be an ordinal number and* $h : \alpha \hookrightarrow |\alpha|$ *an injection. Then for every* $X \in [^{\alpha}2]^{\omega+\omega^*}$ *there is an* $i \in 3 \setminus 1$ *and an* $\vec{x} = \{x_0, x_1, x_2, x_3\} \in E_i$ *such that* $\beta_h(x_{4-2i}, x_{5-2i}) < \beta_h(x_1, x_2)$.

Proof. Let $X \in [^{\alpha}2]^{\omega+\omega^*}$ and let $s \in {}^{<\alpha}2$ be the least splitting node of elements of X. Then with $X_j := \{x \in X \mid x \sqsupset s^\frown\langle j\rangle\}$ we have $\mathrm{otyp}(X_0) \geqslant \omega$ or $\mathrm{otyp}(X_1) \geqslant \omega^*$. Suppose the former holds and let $\langle x_n \mid n < \omega\rangle$ be an ascending enumeration of elements in X_0. There is an $I \in [\omega]^\omega$ such that $\Delta(x_\ell, x_{\ell+1}) \sqsupset \Delta(x_k, x_{k+1})^\frown\langle 1\rangle$ for all $\{k, \ell\}_< \in [I]^2$. The finitude of decreasing sequences of ordinals implies that there is a pair $\{m, n\}_< \in [I]^2$ with $\beta_h(x_m, x_{m+1}) < \beta_h(x_n, x_{n+1})$. Now for any $y \in X_1$ the quadruple $\{x_m, x_n, x_{n+1}, y\}$ provides what was demanded for $i = 2$. □

4.2. **Quintuples.** In this section, we prove several negative partition relations with 5 on one side of the relation. These results are used in the classification in Section 4.6.

Theorem 4.16. (1) $\langle ^{\alpha}2, <_{lex}\rangle \not\to (\omega^* + \omega, n+1)^n$,
(2) $\langle ^{\alpha}2, <_{lex}\rangle \not\to (\omega + \omega^*, n+1)^n$.

Proof. Suppose that $\vec{x} = (x_0, ..., x_{n-1})$ is a tuple in $[^{\kappa}2]^n$ with $x_0 <_{lex} x_1 \ldots <_{lex} x_{n-1}$. For the first claim, let $f(\vec{x}) = 1$ if $\Delta_{x_1,x_2} \sqsubseteq \Delta_{x_0,x_1}$ and $\Delta_{x_1,x_2} \sqsubseteq \Delta_{x_2,x_3}$, and $f(x_0, x_1, x_2, x_3) = 0$ otherwise. We claim that there is no homogeneous set for f.

Suppose that there is a homogeneous set H isomorphic to $\omega^* + \omega$ in colour 0. Suppose that s is the largest common initial segment of all elements of H and that s has length β. Let $H_i = \{x \in H \mid x(\beta) = i\}$ for $i < 2$. Then H_0 has order type ω^* and H_1 has order type ω. Suppose that $x_0, x_1 \in H_0$ with $x_0 <_{lex} x_1$ and $x_2, \ldots, x_{n-1} \in H_1$ with $x_2 <_{lex} \ldots <_{lex} x_{n-1}$. Then $f(\vec{x}) = 1$, contradicting the choice of H.

Suppose that there is a set H of size $n+1$ homogeneous in colour 1. Suppose that $H = \{q_i \mid i < n+1\}$ with $q_i <_{lex} q_j$ for $i < j < n+1$. If $\Delta_{q_2,q_3} \sqsubseteq \Delta_{q_1,q_2}$, then $\{q_i \in H \mid i \neq n\}$ has colour 0, contradicting the assumption. If $\Delta_{q_1,q_2} \sqsubseteq \Delta_{q_2,q_3}$, then $\{q_i \in H \mid i \neq 0\}$ has colour 0, contradicting the assumption.

For the second claim, let $g(\vec{x}) = 1$ if $\mathrm{lt}(\Delta_{x_{n-1},x_n}) < \mathrm{lt}(\Delta_{x_0,x_1}) < \cdots < \mathrm{lt}(\Delta_{x_{n-2},x_{n-1}})$ or $\mathrm{lt}(\Delta_{x_0,x_1}) < \ldots \mathrm{lt}(\Delta_{x_{n-1},x_{n-2}}) < \cdots < \mathrm{lt}(\Delta_{x_1,x_0})$, and $g(\vec{x}) = 0$ otherwise. We claim that there is no homogeneous set for g.

Suppose that there is a homogeneous set H isomorphic to $\omega + \omega^*$ in colour 0. Suppose that s is the largest common initial segment of all elements of H and that s has length β. Let $H_i = \{x \in H \mid x(\beta) = i\}$ for $i < 2$. Then H_0 has a subset isomorphic to ω or H_1 has a

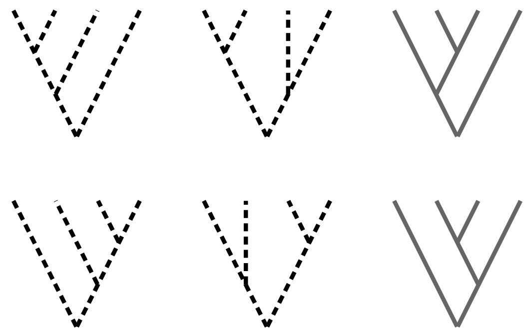

FIGURE 2. Colouring of the splitting types for the proof of theorem 4.16(2)

subset isomorphic to ω^*. We can assume that H_0 has a subset isomorphic to ω. Then there are $x_0, \dots x_{n-2} \in H_0$ with $x_0 <_{lex} \dots <_{lex} x_{n-2}$ and $\Delta_{x_i, x_{i+1}} \sqsubseteq \Delta_{x_{i+1}, x_{i+2}}$ for all $i < n-2$. Suppose that $x_{n-1} \in H_1$. Then $g(\vec{x}) = 1$, contradicting the choice of H.

Suppose that there is a set H of size $n+1$ homogeneous in colour 1. Suppose that $H = \{q_i \mid i < n+1\}$ with $q_i <_{lex} q_j$ for $i < j < n+1$. Since $\{q_i \mid i \neq n\}$ has colour 1, $\mathrm{lt}(\Delta_{q_2,q_3}) < \mathrm{lt}(\Delta_{q_1,q_2})$. Then $\{q_i \mid i \neq 0\}$ has colour 0, contradicting the assumption. $\square$

Theorem 4.16 implies that Theorem 2.4(2) does not lift to higher exponents. In the following, we weaken the requirement of an infinite homogeneous set in colour 0 to the requirement that the set has one of two, three, four, five and, in the case of Theorem 4.24, six given order types.

Theorem 4.17. *If κ be an infinite initial ordinal and $\alpha < \kappa^+$, then*

$$\langle {}^{\alpha}2, <_{lex} \rangle \not\to (\omega^* + \omega \vee \kappa + 2 + \kappa^* \vee (\kappa 2)^* \vee \kappa 2, 5)^4.$$

Proof. Suppose that there is an infinite initial ordinal κ and an $\alpha < \kappa^+$ such that the first partition property holds. Suppose that $h \colon \alpha \leftrightarrow \kappa$ is bijective. Let $\beta_{x,y} = \beta(x,y) = h(\mathrm{lt}(\Delta_{x,y}))$ for $x, y \in {}^{\alpha}2$. If $\alpha = \kappa$, we can choose $\beta_{x,y} = \Delta_{x,y}$ and obtain a simplified version of the following proof. We write $\vec{x}$ for (x_0, x_1, x_2, x_3) with $x_0 <_{lex} x_1 <_{lex} x_2 <_{lex} x_3$.

$$\begin{aligned} F_0 := F_2 :=& \{\vec{x} \mid \beta_h(x_1, x_2) < \beta_h(x_2, x_3) < \beta_h(x_0, x_1)\}, \\ F_1 := F_3 :=& \{\vec{x} \mid \beta_h(x_1, x_2) < \beta_h(x_0, x_1) < \beta_h(x_2, x_3)\}, \\ F_4 :=& \{\vec{x} \mid \beta_h(x_1, x_2) < \min(\beta_h(x_0, x_1), \beta_h(x_2, x_3))\}. \end{aligned}$$

Let $f(\vec{x}) = 1$ if $\vec{x}$ is in $E_i \cap F_i$ for an $i < 5$ and $f(\vec{x}) = 0$ otherwise. We will prove that there is no homogeneous set of the required type for f.

To see that there are no sets which are homogeneous for f in colour 0 of order-type $\omega^* + \omega$, see Lemma 4.5. In order to see that there are no such sets of order-type $(\kappa 2)^*$ or $\kappa 2$ use Lemma 4.14.

Now consider a $C \in [{}^{\alpha}2]^{\kappa + 2 + \kappa^*}$. We distinguish three cases. First assume that there is an $s \in {}^{<\alpha}2$ such that $\kappa \leqslant \mathrm{otyp}(\{t \in C \mid t \sqsupset s^\frown\langle 0 \rangle\})$ and $\kappa^* \leqslant \mathrm{otyp}(\{t \in C \mid t \sqsupset s^\frown\langle 1 \rangle\})$. Then one proceed essentially as in the proof of before and find an element Q in $[C]^4 \cap E_4 \cap F_4$. Then, again, $f(Q) = 1$.

For the second case, assume that there is no such s. Let $\langle c_\nu \mid \nu < \kappa + 1 \rangle$ be an ascending enumeration of the left half of C and let $\langle d_\nu \mid \nu < \kappa + 1 \rangle$ be a descending enumeration of its right half. Then, using Lemma 3.2 it is easy to choose $\{\nu, \zeta\} \in [\kappa]^2$ such that

$$\{c_\nu, c_\kappa, d_\kappa, d_\zeta\} \in \bigcup_{i \in 3 \setminus 1} (E_i \cap F_i).$$

Finally consider a $\vec{p} = \{p_0, \dots, p_4\}_{<_{lex}} \in [{}^{\alpha}2]^5$. Assume towards a contradiction that $f``[\vec{p}]^4 = \{1\}$. There are fourteen cases to check half of which are mirror images of the other half, to parameterise this let $i < 2$.

We assume in the first case that $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{i+1}, p_{i+2}) \sqsubseteq \Delta(p_{2-i}, p_{3-i}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i})$. Then $\{p_j \mid j < 4\}, \{p_j \mid j > 0\} \in E_0$ and by assumption $E_0 \cap [\vec{p}]^4 \subseteq F_0$ so $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the second case, we assume that $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{i+1}, p+i+2) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i}) \sqsubseteq \Delta(p_{2-i}, p_{3-i})$. Then $\{p_j \mid j < 4\} \in E_{2i}$ so $\{p_j \mid j < 4\} \in F_{2i}$. Also $\{p_j \mid j > 0\} \in E_{2i+1}$ so $\{p_j \mid j > 0\} \in F_{2i+1}$. But then $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the third case, assume that $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i}) \sqsubseteq \Delta(p_{2-i}, p_{3-i}) \sqsubseteq \Delta(p_{i+1}, p_{i+2})$. Then $\{p_j \mid j < 4\} \in E_{1-i}$ and $\{p_j \mid j > 0\} \in E_{3-i}$ so $\{p_j \mid j < 4\} \in F_{1-i}$ and $\{p_j \mid j > 0\} \in F_{3-i}$. This implies $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the fourth case, assume that $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i}) \sqsubseteq \Delta(p_{i+1}, p_{i+2}) \sqsubseteq \Delta(p_{2-i}, p_{3-i})$. Then $\{p_j \mid j < 4\} \in E_i$ and $\{p_j \mid j > 0\} \in E_{i+2}$ so $\{p_j \mid i < 4\} \in F_i$ and $\{p_j \mid j > 0\} \in F_{i+2}$ and hence $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the fifth case, assume that $\Delta(p_{i+1}, p_{i+2}) \sqsubseteq \Delta(p_{3i}, p_{3i+1}), \Delta(p_{2-i}, p_{3-i})$ and $\Delta(p_{2-i}, p_{3-i}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i})$. Then $\{p_j \mid j < 4\} \in E_{4-i}$ and $\{p_j \mid j > 0\} \in E_{4i}$ so $\{p_j \mid j < 4\} \in F_{4-i}$ and $\{p_j \mid j > 0 \in 5\} \in F_{4i}$. But this means that $\beta_h(p_{i+1}, p_{i+2}) < \min(\beta_h(p_{3i}, p_{3i+1}), \beta_h(p_{2-i}, p_{3-i})) \leqslant \beta_h(p_{2-i}, p_{3-i}) < \beta_h(p_{i+1}, p_{i+2})$, a contradiction.

In the sixth case, assume that $\Delta(p_{i+1}, p_{i+2}) \sqsubseteq \Delta(p_0, p_1), \Delta(p_3, p_4)$ and $\Delta(p_{3-3i}, p_{4-3i}) \sqsubseteq \Delta(p_{2-i}, p_{3-i})$. Then $\{p_j \mid j < 4\} \in E_{4-2i}$ and $\{p_j \mid j > 0\} \in E_{3i+1}$ and hence $\{p_j \mid j < 4\} \in F_{4-2i}$ and $\{p_j \mid j > 0\} \in F_{3i+1}$. So $\beta_h(p_{i+1}, p_{i+2}) < \min(\beta_h(p_{3i}, p_{3i+1}), \beta_h(p_{2-i}, p_{3-i})) \leqslant \beta_h(p_{2-i}, p_{3-i}) < \beta_h(p_{i+1}, p_{i+2})$, a contradiction.

In the final case, assume that $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{2-i}, p_{3-i})$ and $\Delta(p_{2-i}, p_{3-i}) \sqsubseteq \Delta(p_{i+1}, p_{i+2}), \Delta(p_{3-3i}, p_{4-3i})$. This means that $\{p_j \mid j < 4\} \in E_{3i+1}$ and $\{p_j \mid j > 0\} \in E_{4-2i}$. So $\beta_h(p_{2-i}, p_{3-i}) < \min(\beta_h(p_{i+1}, p_{i+2}), \beta_h(p_{3-i}, p_{4-i})) \leqslant \beta_h(p_{i+1}, p_{i+2}) < \beta_h(p_{2-i}, p_{3-i})$, a contradiction. $\square$

For the following Theorem (and Theorem 4.27) recall that η denotes the order-type of the rational numbers, the countable dense linear order without endpoints.

Theorem 4.18. *If κ is an infinite initial ordinal and $\alpha < \kappa^+$, then $\langle {}^{\alpha}2, <_{lex}\rangle \not\rightarrow (2+\kappa^* \vee \kappa+2 \vee \eta, n+1)^n$ for all $n \geqslant 4$.*

Proof. We write $\vec{x}$ for $(x_0, \dots, x_{n-1})$ with $x_0 <_{lex} \dots <_{lex} x_{n-1}$. Let

$$F_0 := F_2 := \{\vec{x} \mid \beta_h(x_{n-2}, x_{n-1}) < \beta_h(x_0, x_1) < \cdots < \beta_h(x_{n-3}, x_{n-2})\},$$
$$F_1 := F_3 := \{\vec{x} \mid \beta_h(x_0, x_1) < \beta_h(x_{n-2}, x_{n-1}) < \cdots < \beta_h(x_1, x_2)\}.$$

Let

$$\bar{E}_0 = \{\vec{x} \mid \Delta(x_0, x_1) \sqsubseteq \dots \sqsubseteq \Delta(x_{n-2}, x_{n-1})\},$$
$$\bar{E}_1 = \{\vec{x} \mid \Delta(x_0, x_1) \sqsubseteq \Delta(x_{n-2}, x_{n-1}) \sqsubseteq \dots \sqsubseteq \Delta(x_1, x_2)\},$$
$$\bar{E}_2 = \{\vec{x} \mid \Delta(x_{n-2}, x_{n-1}) \sqsubseteq \Delta(x_0, x_1) \sqsubseteq \dots \sqsubseteq \Delta(x_{n-3}, x_{n-2})\},$$
$$\bar{E}_3 = \{\vec{x} \mid \Delta(x_{n-2}, x_{n-1}) \sqsubseteq \dots \sqsubseteq \Delta(x_0, x_1)\},$$

Then $\bar{E}_i = E_i$ for $n = 4$. Let $f(\vec{x}) = 1$ if $\vec{x} \in \bigcup_{i<4}(\bar{E}_i \cap F_i)$ and $f(\vec{x}) = 0$ otherwise.

Claim 4.19. *There is no set homogeneous for f in colour 0 with order type $2+\kappa^*$ or κ^*+2.*

Proof. It is sufficient to consider the case $\kappa+2$ by symmetry. Let $\langle b_\nu \mid \nu < \kappa+1\rangle$ be the order-preserving enumeration of a subset of ${}^{\alpha}2$ of order type $\kappa+2$. We distinguish two cases.

First assume that the sequence $\vec{s} := \langle \Delta(b_\nu, b_{\kappa+1}) \mid \nu < \kappa\rangle$ stabilises, say at $s \in {}^{<\alpha}2$ from $\zeta < \kappa$ onwards. By Lemma 3.2, there is some $\rho_0 \in \kappa \setminus \zeta$ with $\beta_h(b_{\rho_0}, b_{\rho_0+1}) > h(\mathrm{lt}(s))$ and $|\{\nu < \kappa \mid b_\nu \sqsupseteq \Delta(b_{\rho_0}, b_{\rho_0+1})\}| = \kappa$. Similarly, we can find $\rho_1, ..., \rho_{n-2}$ with $\Delta(b_{\rho_0}, b_{\rho_1}) \sqsubseteq \dots \sqsubseteq \Delta(b_{\rho_{n-3}}, b_{\rho_{n-2}})$ and $\beta_h(b_{\rho_0}, b_{\rho_1}) \sqsubseteq \dots \sqsubseteq \beta_h(b_{\rho_{n-3}}, b_{\rho_{n-2}})$. Now $\vec{b} := \{b_{\rho_0}, \dots b_{\rho_{n-2}}, b_{\kappa+1}\} \in \bar{E}_2 \cap F_2$ and hence $f(\vec{b}) = 1$, a contradiction.

Second, assume that $\vec{s}$ does not stabilise. By Lemma 3.2, there is some $\zeta_0 < \kappa$ such that $\beta_h(b_{\zeta_0}, b_{\zeta_0+1}) > \beta_h(b_\kappa, b_{\kappa+1})$ and $b_\kappa \sqsupseteq \Delta(b_{\zeta_0}, b_{\zeta_0+1})$. Similarly, we can find $\zeta_1, ..., \zeta_{n-3}$ with $\Delta(b_{\zeta_0}, b_{\zeta_1}) \sqsubseteq \dots \sqsubseteq \Delta(b_{\zeta_{n-4}}, b_{\zeta_{n-3}})$ and $\beta_h(b_{\zeta_0}, b_{\zeta_1}) \sqsubseteq \dots \sqsubseteq \beta_h(b_{\zeta_{n-4}}, b_{\zeta_{n-3}})$. Now we have $\vec{b} := \{b_{\zeta_0}, \dots b_{\zeta_{n-3}}, b_\kappa, b_{\kappa+1}\} \in \bar{E}_0 \cap F_0$ and hence $f(\vec{b}) = 1$, a contradiction. $\square$

Claim 4.20. *There is no set of order-type η homogeneous for f in colour 0.*

Proof. This is analogous to the proof of Lemma 4.6. □

Claim 4.21. *There is no set* $\{q_0, \ldots, q_{n-1}\}$ *in* $[{}^{\alpha}2]^{n+1}$ *with* $q_0 <_{lex} \ldots <_{lex} q_n$ *homogeneous in colour* 1.

Proof. We first consider the case $n = 4$. Suppose that $\{p_0, \ldots, p_{n-1}\}_{<_{lex}} \in [{}^{\alpha}2]^{n+1}$ and assume towards a contradiction that it were homogeneous for f in colour 1. Regarding the splitting structure, one has to check eight cases, half of which are mirror images of the other half. We consider them using a parameter $i < 2$.

In the first case, assume that $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{i+1}, p_{i+2}) \sqsubseteq \Delta(p_{2-i}, p_{3-i}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i})$. Then $\{p_j \mid j < 4\}, \{p_j \mid j > 0\} \in E_{3i}$, so $\{p_j \mid j < 4\}, \{p_j \mid j > 0\} \in F_{3i}$ and hence $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the second case, assume that $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{i+1}, p_{i+2}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i}) \sqsubseteq \Delta(p_{2-i}, p_{3-i})$. So $\{p_i \mid j < 4\} \in E_{2i}$ but $\{p_j \mid j > 0\} \in E_{2i+1}$ and hence $\{p_j \mid j < 4\} \in F_{2i}$ and $\{p_j \mid j > 0\} \in F_{2i+1}$. But then $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the third case, assume that $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i}) \sqsubseteq \Delta(p_{2-i}, p_{3-i}) \sqsubseteq \Delta(p_{i+1}, p_{i+2})$. Then $\{p_j \mid j < 4\} \in E_{1-i}$ but $\{p_j \mid j > 0\} \in E_{3-i}$. So $\{p_j \mid j < 4\} \in F_{1-i}$ and $\{p_j \mid j > 0\} \in F_{3-i}$ and hence $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the final case, assume that $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i}) \sqsubseteq \Delta(p_{i+1}, p_{i+2}) \sqsubseteq \Delta(p_{2-i}, p_{3-i})$. So $\{p_j \mid j < 4\} \in E_i$ and $\{p_j \mid j > 0\} \in E_{2+i}$ which implies that $\{p_j \mid j < 4\} \in F_i$ and $\langle p_j \mid j > 0\rangle \in F_{2+i}$. So actually $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

Now suppose that $n > 4$. Again, we consider a parameter $i < 2$.

In the first case, assume that

$$\Delta(p_{(n-1)i}, p_{(n-1)i+1}) \sqsubseteq \Delta(p_{(n-3)i+1}, p_{(n-3)i+2}) \sqsubseteq \ldots \sqsubseteq \Delta(p_{-(n-1)i+(n-1)}, p_{-(n-1)i+(n-1)}).$$

Then $\{p_j \mid j < n+1\}, \{p_j \mid j > 0\} \in \bar{E}_{3i}$, so $\{p_j \mid j < n+1\}, \{p_j \mid j > 0\} \in F_{3i}$. If $i = 0$, then $\beta_h(p_{n-3}, p_{n-2}) < \beta_h(p_{n-2}, p_{n-1}) < \beta_h(p_{n-3}, p_{n-2})$, a contradiction. If $i = 1$, then $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

In the second case, assume that

$$\Delta(p_{-(n-1)i+(n-1)}, p_{-(n-1)i+(n-1)}) \sqsubseteq \Delta(p_{(n-1)i}, p_{(n-1)i+1}) \ldots \Delta(p_{-(n-3)i+(n-2)}, p_{-(n-3)i+(n-2)}).$$

Then $\{p_j \mid j < n+1\} \in \bar{E}_{2-i}$ and $\{p_j \mid j > 0\} \in \bar{E}_{3i}$, so $\{p_j \mid j < n+1\} \in F_{2-i}$ and $\{p_j \mid j > 0\} \in F_{3i}$. If $i = 0$, then $\beta_h(p_{n-3}, p_{n-3}) < \beta_h(p_{n-3}, p_{n-2}) < \beta_h(p_{n-4}, p_{n-3})$, a contradiction. If $i = 1$, then $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

These are the only possible cases, since $\{q_i \mid i < n+1\}$ and $\{q_i \mid i > 0\}$ are elements of $\bigcup_{i<4} \bar{E}_i$. □

This completes the proof of Theorem 4.18. □

4.3. **Choice, after all.** The following result shows that Theorem 2.7 fails in ZFC.

Theorem 4.22. *Suppose that the Axiom of Choice holds. Then* $\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (\omega^* + \omega \vee 2 + \omega^* \vee \omega + 2, 5)^4$ *for all* $\alpha < \omega_1$.

Proof. Let $\alpha < \omega_1$, let $g : {}^{\alpha}2 \hookrightarrow \gamma$ be one-to-one for a suitable ordinal number γ and let $h : \alpha \hookrightarrow \omega$ be one-to-one too, defining the function $\beta \colon {}^{\alpha}2 \times {}^{\alpha}2 \longrightarrow \omega$ by $\langle r, s\rangle \mapsto h(\mathrm{lt}(\Delta(r, s)))$. For any $\vec{q} = \{q_0, q_1, q_2, q_3\}_{<_{lex}} \in [{}^{\alpha}2]^4$ let $f(\vec{q}) = 1$ if and only if

(1) $\quad \forall i < 2(\Delta(q_1, q_2) \sqsubseteq \Delta(q_{2i}, q_{2i+1}) \wedge \beta(q_1, q_2) < \beta(q_{2i}, q_{2i+1}) \wedge g(q_{i+1}) < g(q_{3i}))$ or

(2) $\quad \exists i < 2(\Delta(q_{2i}, q_{2i+1}) \sqsubseteq \Delta(q_{2-2i}, q_{3-2i}) \sqsubseteq \Delta(q_1, q_2) \wedge$

$$\beta(q_{2i}, q_{2i+1}) < \beta(q_{2-2i}, q_{3-2i}) < \beta(q_1, q_2) \wedge g(q_{3-3i}) < g(q_{2-i}) < g(q_{i+1}))$$ or

(3) $\quad \exists i < 2(\Delta(q_{2i}, q_{2i+1}) \sqsubseteq \Delta(q_1, q_2) \sqsubseteq \Delta(q_{2-2i}, q_{3-2i}) \wedge$

$$\beta(q_{2-2i}, q_{3-2i}) < \beta(q_{2i}, q_{2i+1}) < \beta(q_1, q_2) \wedge g(q_{2i}) < g(q_{2i+1})).$$

Note that the colouring f is defined in a symmetric way. In the first case the index i is just a shorthand to define a situation where each of the two following cases each defines a pair of two situations which are symmetric to each other.

So let $Z \in [{}^{\alpha}2]^{\omega^*+\omega}$. We have to find a $\vec{q} = \{q_0, q_1, q_2, q_3\}_{<_{lex}} \in [Z]^4$ for which $f(\vec{q}) = 1$. To this end, let $\langle z_n^{(0)} \mid n < \omega\rangle$ the order-reversing enumeration of the lower half of Z and let

$\langle z^{(1)}_n \mid n < \omega\rangle$ be the order-preserving enumeration of its upper half. Suppose without loss of generality that for both $i < 2$ the sequence $\langle g(z^{(i)}_n) \mid n < \omega\rangle$ is ascending. Note that there has to be an $k < \omega$ such that for all $m \in \omega \setminus k$ and both $i < 2$ one has $\Delta(z^{(0)}_m, z^{(1)}_m) \sqsubseteq \Delta(z^{(i)}_m, z^{(i)}_{m+1})$. Furthermore, observe that there is an $m \in \omega \setminus k$ such that for all $n \in \omega \setminus m$ and both $i < 2$ one has $\beta(q_1, q_2) < \beta(q_{2i}, q_{2i+1})$. Let $\vec{q} := \{z^{(0)}_{m+1}, z^{(0)}_m, z^{(1)}_m, z^{(1)}_{m+1}\}$. Then, by (1), $f(\vec{q}) = 1$.

Now let $A \in [{}^{\alpha}2]^{2+\omega^*}$ and let $\langle a_\gamma \mid \gamma < \omega + 2\rangle$ be its order-reversing enumeration. We distinguisgh two cases. First assume that the sequence $\vec{s} := \langle \Delta(a_n, a_{\omega+1}) \mid n < \omega\rangle$ is stabilising, say at $s \in {}^{<\alpha}2$ from $k < \omega$ onwards. Because there is no infinite decreasing sequence of ordinals there is an $A_0 \in [A \setminus \{a_{\omega+1}\}]^{\omega^*}$ such that $g(c) < g(b)$ for any $\{b, c\}_< \in [A_0]^2$. Then there is an $A_1 \in [A_0]^{\omega^*}$ such that $\Delta(b, c) \sqsupset \Delta(c, d)^\frown\langle 0\rangle$ for any $\{b, c, d\}_< \in [A_1]^3$. One can find an $A_2 \in [A_1]^{\omega^*}$ such that $\beta(b, c) > h(\mathrm{lt}(s))$ for all $\{b, c\}_< \in [A_2]^2$. Finally there is a $\{b, c, d\}_< \in [A_2]^3$ such that $\beta(c, d) < \beta(b, c)$. Then for $\vec{q} := \{a_{\omega+1}, b, c, d\}$ we have $f(\vec{q}) = 1$ by (2) for $i = 0$. Now assume that $\vec{s}$ does not stabilise. Then there is is an $A_0 \in [A \setminus \{a_{\omega+1}\}]^{\omega^*}$ such that $\Delta(b, c) \sqsupset \Delta(c, d)^\frown\langle 0\rangle$ for all $\{b, c, d\}_< \in [A_0]^3$. There is an $A_1 \in [A_0]^{\omega^*}$ such that $g(c) < g(b)$ for every $\{b, c\}_< \in [A_1]^2$. Since $\beta(a_{\omega+1}, a_\omega)$ is finite there is an $\{b, c\}_< \in [A_1]^2$ such that $\beta(b, c) > \beta(a_{\omega+1}, a_\omega)$. Then for $\vec{q} := \{a_{\omega+1}, a_\omega, b, c\}$ we have $f(\vec{q}) = 1$ by (3) for $i = 1$.

Because f is defined in a symmetric way we do not have to deal with sets of order-type $\omega + 2$ separately.

Let $\{p_0, p_1, p_2, p_3, p_4\}_{<_{lex}} \in [{}^{\alpha}2]^5$. We distinguish seven pairs of cases, let $j < 2$.

First assume that $\Delta(p_{3j}, p_{3j+1}) \sqsubseteq \Delta(p_{j+1}, p_{j+2}) \sqsubseteq \Delta(p_{2-j}, p_{3-j}) \sqsubseteq \Delta(p_{3-3j}, p_{4-3j})$. Applying (3) to both $\{p_k \mid k < 4\}$ and $\{p_k \mid k \in 5\setminus 1\}$ one gets $\beta(p_{2-j}, p_{3-j}) < \beta(p_{j+1}, p_{j+2}) < \beta(p_{2-j}, p_{3-j})$, a contradiction.

Second suppose that $\Delta(p_{3j}, p_{3j+1}) \sqsubseteq \Delta(p_{j+1}, p_{j+2}) \sqsubseteq \Delta(p_{3-3j}, p_{4-3j}) \sqsubseteq \Delta(p_{2-j}, p_{3-j})$. Applying (2) with $i := j$ to $\{p_k \mid k \in 5 \setminus \{4i\}\}$ and (3) with $i := j$ to $\{p_k \mid k \in 5 \setminus \{2\}\}$ one gets $\beta(p_{3-3i}, p_{4-3i}) < \beta(p_1, p_3) = \beta(p_{i+1}, p_{i+2}) < \beta(p_{3-3i}, p_{4-3i})$, a contradiction.

Third assume that $\Delta(p_{3j}, p_{3j+1}) \sqsubseteq \Delta(p_{3-3j}, p_{4-3j}) \sqsubseteq \Delta(p_{j+1}, p_{j+2}) \sqsubseteq \Delta(p_{2-j}, p_{3-j})$. Applying (3) with $i := j$ to $\{p_k \mid k \neq 4 - 4j\}$ and (2) with $i := j$ to $\{p_k \mid k \neq 2j + 1\}$ one gets $\beta(p_{2-j}, p_{3-j}) < \beta(p_{3j}, p_{3j+1}) = \beta(p_{2j}, p_{2j+2}) < \beta(p_{2-j}, p_{3-j})$, a contradiction.

Fourth suppose $\Delta(p_{3j}, p_{3j+1}) \sqsubseteq \Delta(p_{3-3j}, p_{4-3j}) \sqsubseteq \Delta(p_{2-j}, p_{3-j}) \sqsubseteq \Delta(p_{j+1}, p_{j+2})$. Applying (3) with $i := 1 - j$ to $\{p_k \mid k \neq 4j\}$ and (2) with $i := j$ to $\{p_k \mid k \neq 3 - 2j\}$ yields $\beta(p_{j+1}, p_{j+2}) < \beta(p_{3-3j}, p_{4-3j}) = \beta(p_{2-2j}, p_{4-2j}) < \beta(p_{j+1}, p_{j+2})$, a contradiction.

Fifth assume that $\Delta(p_{3j}, p_{3j+1}) \sqsubseteq \Delta(p_{2-j}, p_{3-j}) \sqsubseteq \Delta(p_{1-j}, p_{2-j}), \Delta(p_{3-j}, p_{4-j})$. Applying (1) to $\{p_k \mid k \neq 4j\}$ and (3) with $i := j$ to $\{p_k \mid k \neq 2\}$ yields $\beta(p_{3-3i}, p_{4-3i}) < \beta(p_1, p_3) = \beta(p_{2-j}, p_{3-j}) < \beta(p_{3-3i}, p_{4-3i})$, a contradiction.

Sixth assume that $\Delta(p_{j+1}, p_{j+2}) \sqsubseteq \Delta(p_j, p_{j+1}), \Delta(p_{j+2}, p_{j+3})$ and further $\Delta(p_{2-j}, p_{3-j}) \sqsubseteq \Delta(p_{3-3j}, p_{4-3j})$. Applying (3) with $i := j$ to $\{p_k \mid k \neq 4j\}$ and (1) to $\{p_k \mid k \neq 2\}$ yields $\beta(p_{3-3j}, p_{4-3j}) < \beta(p_{j+1}, p_{j+2}) = \beta(p_1, p_3) < \beta(p_{3-3j}, p_{4-3j})$, a contradiction.

Last assume that $\Delta(p_{j+1}, p_{j+2}) \sqsubseteq \Delta(p_j, p_{j+1}), \Delta(p_{3-3j}, p_{4-3j})$ and that $\Delta(p_{3-3j}, p_{4-3j}) \sqsubseteq \Delta(p_{2-j}, p_{3-j})$. Applying (2) with $i := j$ to $\{p_k \mid k \neq 4j\}$ and (1) to $\{p_k \mid k \neq 2\}$ yields $g(p_{3-3i}) < g(p_{4-3i}) < g(p_{3-3i})$, a contradiction. □

Note that since λ and $\langle {}^{\omega}2, <_{lex}\rangle$ are mutually embeddable, Theorem 4.22 is a strengthening of [956ER, Theorem 28] which states that $\lambda \not\to (\omega + 2, 5)^4$.

In ZFC the statement of Theorem 2.5 is also provable, but Theorems 2.6 and 2.7 are falsified there by Theorem 4.22. For Theorem 2.7 this can also be shown using Theorem 1.9.

4.4. **Sextuples.** In this section, we prove several negative partition relations with 6 on one side of the relation. Most of these results are used in the classification in Section 4.6.

Theorem 4.23. *If κ is an infinite initial ordinal and $\alpha < \kappa^+$, then*

$$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\kappa^* + \kappa \vee 2 + \kappa^* \vee \kappa 2 \vee \omega\omega^*, 6)^4,$$
$$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\kappa^* + \kappa \vee (\kappa 2)^* \vee \kappa + 2 \vee \omega^*\omega, 6)^4.$$

Proof. Suppose that κ is as in the theorem and there is an ordinal α such that the first partition property holds. Suppose that $h\colon \alpha \leftrightarrow \kappa$ is bijective. Let $\beta_{x,y} = \beta(x, y) = h(\mathrm{lt}(\Delta_{x,y}))$ for $x, y \in {}^{\alpha}2$. If $\alpha = \kappa$, we can choose $\beta_{x,y} = \Delta_{x,y}$ and obtain a simplified version of the following proof. We write $\vec{x}$ for (x_0, x_1, x_2, x_3) with $x_0 <_{lex} x_1 <_{lex} x_2 <_{lex} x_3$. Let

$$\begin{aligned} F_0 :=& \{\vec{x} \mid \beta_h(x_1,x_2) < \beta_h(x_2,x_3) < \beta_h(x_0,x_1)\},\\ F_1 := F_3 :=& \{\vec{x} \mid \beta_h(x_0,x_1) < \beta_h(x_2,x_3) < \beta_h(x_1,x_2)\},\\ F_4 :=& \{\vec{x} \mid \beta_h(x_1,x_2) < \min(\beta_h(x_0,x_1), \beta_h(x_2,x_3))\}. \end{aligned}$$

Let $f(\vec{x}) = 1$ if $\vec{x}$ is in $E_i \cap F_i$ for an $i \in 5 \setminus \{2\}$ and $f(\vec{x}) = 0$ otherwise. We will prove that there is no homogeneous set of the required type for f.

To see that there is no set of order-type $\kappa^* + \kappa$ which is homogeneous for f in colour 0 consult Lemma 4.10. To show the nonexistence of such sets of order-type $2 + \kappa^*$ consider the first half of Lemma 4.8 and for the proof that f does not admit homogeneous sets in colour 0 of order-type $\kappa 2$ use the second half of Lemma 4.14 Finally, to see that there is no $X \in [{}^{\alpha}2]^{\omega\omega^*}$ homogeneous for f in colour 0 consider Lemma 4.12.

We consider sets homogeneous for f in colour 1. Assume towards a contradiction that $\vec{h} \in [{}^{\alpha}2]^6$ is homogeneous for f in colour 1. Since $[\vec{h}]^4 \subseteq [{}^{\alpha}2]^4 \setminus E_2$, there is a quintuple $\{p_0 \dots, p_4\}_{<_{lex}} \in [\vec{h}]^5$ for which one of the following six cases applies.

First assume that $\Delta(p_0,p_1) \sqsubseteq \Delta(p_1,p_2) \sqsubseteq \Delta(p_2,p_3) \sqsubseteq \Delta(p_3,p_4)$. Then $\{p_j \mid j < 4\}, \{p_j \mid j > 0\} \in E_0$ So $\beta_h(p_1,p_2) < \beta_h(p_2,p_3) < \beta_h(p_1,p_2)$, a contradiction.

Second assume that $\Delta(p_3,p_4) \sqsubseteq \Delta(p_2,p_3) \sqsubseteq \Delta(p_1,p_2) \sqsubseteq \Delta(p_0,p_1)$. Then $\{p_j \mid j < 4\}, \{p_j \mid j > 0\} \in E_3$ are elements of E_3 hence $\beta_h(p_1,p_2) < \beta_h(p_2,p_3) < \beta_h(p_1,p_2)$, a contradiction.

Third assume that $\Delta(p_0,p_1) \sqsubseteq \Delta(p_1,p_2) \sqsubseteq \Delta(p_3,p_4) \sqsubseteq \Delta(p_2,p_3)$. Then $\{p_j \mid j < 4\} \in E_0$ and $\{p_j \mid i \neq 1\} \in E_1$ from which we get $\beta_h(p_2,p_3) < \beta_h(p_0,p_1) = \beta_h(p_0,p_2) < \beta_h(p_2,p_3)$, a contradiction.

Fourth assume that $\Delta(p_0,p_1) \sqsubseteq \Delta(p_3,p_4) \sqsubseteq \Delta(p_2,p_3) \sqsubseteq \Delta(p_1,p_2)$, then $\{p_j \mid j > 0\} \in E_3$ and $\{p_j \mid j < 4\} \in E_1$. It follows that $\beta_h(p_1,p_2) < \beta_h(p_2,p_3) < \beta_h(p_1,p_2)$, a contradiction.

Fifth assume that $\Delta(p_1,p_2) \sqsubseteq \Delta(p_0,p_1), \Delta(p_2,p_3)$ and $\Delta(p_2,p_3) \sqsubseteq \Delta(p_3,p_4)$. Then $\{p_j \mid j > 0\} \in E_0$ and $\{p_j \mid j < 4\} \in E_4$ hence $\beta_h(p_2,p_3) < \beta_h(p_1,p_2) < \min(\beta_h(p_0,p_1), \beta_h(p_2,p_3)) \leqslant \beta_h(p_2,p_3)$, a contradiction.

Finally assume that $\Delta(p_2,p_3) \sqsubseteq \Delta(p_1,p_2), \Delta(p_3,p_4)$ and $\Delta(p_1,p_2) \sqsubseteq \Delta(p_0,p_1)$. This means that $\{p_j \mid j < 4\} \in E_3$ and $\{p_j \mid j \neq 2\} \in E_4$ yielding $\beta_h(p_0,p_1) < \beta_h(p_2,p_3) = \beta_h(p_1,p_3) < \min(\beta(p_0,p_1), \beta_h(p_3,p_4)) \leqslant \beta_h(p_0,p_1)$, a contradiction.

Once more the second part of the theorem follows immediately by consideration of symmetry. □

Theorem 4.24. *If κ is an infinite initial ordinal and $\alpha < \kappa^+$, then*

$$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\omega^* + \omega \vee \omega^* + \kappa^* \vee \kappa + \omega, 6)^4.$$

Proof. Suppose that κ is as in the theorem and there is an ordinal $\alpha < \kappa^+$ such that the theorem holds. Suppose that $h \colon \alpha \leftrightarrow \kappa$ is bijective. Let $\beta_{x,y} = \beta(x,y) = h(\Delta_{x,y})$ for $x, y \in {}^{\alpha}2$. If $\alpha = \kappa$, we can choose $\beta_{x,y} = \Delta_{x,y}$ and obtain a simplified version of the following proof. We write $\vec{x}$ for (x_0, x_1, x_2, x_3) with $x_0 <_{lex} x_1 <_{lex} x_2 <_{lex} x_3$. Let

$$\begin{aligned} F_0 =& \{\vec{x} \mid \beta_h(x_1,x_2) < \beta_h(x_2,x_3) < \beta_h(x_0,x_1)\},\\ F_3 =& \{\vec{x} \mid \beta_h(x_1,x_2) < \beta_h(x_0,x_1) < \beta_h(x_2,x_3)\},\\ F_4 =& \{\vec{x} \mid \beta_h(x_1,x_2) < \max(\beta_h(x_0,x_1), \beta_h(x_2,x_3))\}. \end{aligned}$$

Let $f(\vec{x}) = 1$ if $\vec{x} \in \bigcup_{i \in \{0,3,4\}} (E_i \cap F_i)$ and $f(\vec{x}) = 0$ otherwise. We will prove that there is no homogeneous set of the required type for f.

To see that there are no homogeneous sets of order-type $\omega^* + \omega$ in colour 0, consider Lemma 4.5. For sets of order-type $\omega^* + \kappa^*$ or $\kappa + \omega$, apply Lemma 4.13.

Finally consider a sextuple $\vec{h} = \{h_0, \dots, h_5\}_{<_{lex}} \in [{}^{\alpha}2]^6$. Since we have $[\vec{h}]^4 \subseteq [{}^{\alpha}2]^4 \setminus (E_1 \cup E_2)$ by the definition of f and f is symmetric it suffices to consider the following cases for $i < 2$:

First suppose that $\Delta(h_2,h_3) \sqsubseteq \Delta(h_1,h_2), \Delta(h_3,h_4)$ and $\Delta(h_{2j+1}, h_{2j+2}) \sqsubseteq \Delta(h_{4j}, h_{4j+1})$ for $j < 2$. Then $\{h_j \mid j \notin \{4j, 4j+1\}\} \in E_{3j}$ for $j < 2$ and $\{h_0, h_1, h_4, h_5\} \in E_4$. Then $\beta_h(h_{4j}, h_{4j+1}) < \beta_h(h_2,h_3)$ for $j < 2$ hence $\max(\beta_h(h_0,h_1), \beta_h(h_4,h_5)) < \beta_h(h_2,h_3) = \beta_h(h_1,h_4)$. But $\{h_0, h_1, h_4, h_5\} \in E_4$, a contradiction.

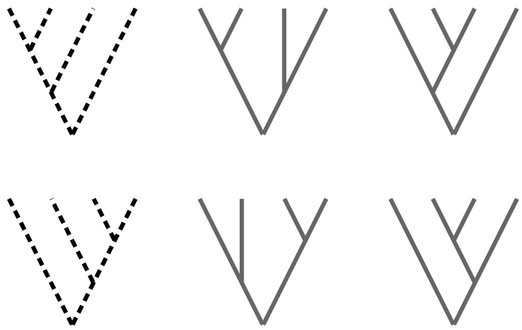

FIGURE 3. Colouring of the splitting types for the proof of theorem 4.26.

Now suppose that there is a quintuple $\{p_0, \ldots, p_4\}_{<_{lex}} \in \vec{s}$ and suppose that $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{i+1}, p_{i+2}) \sqsubseteq \Delta(p_{2-i}, p_{3-i}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i})$. Then $\{p_j \mid j < 4\}, \{p_j \mid j > 0\} \in E_{3i}$ so $\beta_h(p_2, p_3) < \beta_h(p_1, p_2) < \beta_h(p_2, p_3)$, a contradiction. $\square$

Theorem 4.25. *If κ is an infinite initial ordinal and $\alpha < \kappa^+$, then*

$$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\omega + \omega^* \vee 2 + \kappa^* \vee \kappa + 2, 6)^4.$$

Proof. Suppose that κ is as in the theorem and there is an ordinal $\alpha < \kappa^+$ such that the theorem holds. Suppose that $h \colon \alpha \leftrightarrow \kappa$ is bijective. Let $\beta_{x,y} = \beta(x, y) = h(\Delta_{x,y})$ for $x, y \in {}^{\alpha}2$. If $\alpha = \kappa$, we can choose $\beta_{x,y} = \Delta_{x,y}$ and obtain a simplified version of the following proof.

We write $\vec{x}$ for (x_0, x_1, x_2, x_3) with $x_0 <_{lex} x_1 <_{lex} x_2 <_{lex} x_3$. Let

$$\begin{aligned}
F_0 =& \{\vec{x} \mid \beta_h(x_2, x_3) < \beta_h(x_0, x_1) < \beta_h(x_1, x_2)\},\\
F_1 =& \{\vec{x} \mid \beta_h(x_2, x_3) < \beta_h(x_1, x_2)\},\\
F_2 =& \{\vec{x} \mid \beta_h(x_0, x_1) < \beta_h(x_1, x_2)\},\\
F_3 =& \{\vec{x} \mid \beta_h(x_0, x_1) < \beta_h(x_2, x_3) < \beta_h(x_1, x_2))\}.
\end{aligned}$$

Let $f(\vec{x}) = 1$ if $\vec{x} \in \bigcup_{i<4}(E_i \cap F_i)$ and $f(\vec{x}) = 0$ otherwise.

To see that there are no sets of order-type $\omega + \omega^*$ which are homogeneous for f in colour 0 consider Lemma 4.15. In order to show that there are no such sets of order-type $2 + \kappa^*$ or $\kappa + 2$, see Lemma 4.8.

So consider a sextuple $\vec{h} \in [{}^{\alpha}2]^6$ and suppose towards a contradiction that it were homogeneous for f in colour 1. Then clearly $[\vec{h}]^4 \subseteq [{}^{\alpha}2]^4 \setminus E_4$. Let $i < 2$. Then for some quintuple $\{p_0, \ldots, p_4\}_{<_{lex}} \in [\vec{h}]^5$ one of the following three cases holds.

First assume that $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{i+1}, p_{i+2}) \sqsubseteq \Delta(p_{2-i}, p_{3-i}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i})$. Then $\{p_j \mid j < 4\}, \{p_j \mid j > 0\} \in E_{3i}$ and hence $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

Second assume that $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i}) \sqsubseteq \Delta(p_{2-i}, p_{3-i}) \sqsubseteq \Delta(p_{i+1}, p_{i+2})$. Then $\{p_j \mid j < 4\} \in E_{1-i}$ while $\{p_j \mid j > 0\} \in E_{3-i}$. This implies $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) = \beta_h(p_1, p_2)$, a contradiction.

Last assume that $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i}) \sqsubseteq \Delta(p_{i+1}, p_{i+2}) \sqsubseteq \Delta(p_{2-i}, p_{3-i})$. Then $\{p_j \mid j < 4\} \in E_i$ while $\{p_j \mid j > 0\} \in E_{2+i}$. This implies $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) = \beta_h(p_1, p_2)$, a contradiction. $\square$

4.5. **Septuples.** In this section, we prove several negative partition relations with 7 on one side of the relation. Most of these results are used in the classification in Section 4.6.

Theorem 4.26. *If α is an ordinal, then $\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\omega^* + \omega \vee \omega + \omega^*, 7)^4$.*

Proof. Suppose that (x_0, x_1, x_2, x_3) is a tuple in $[{}^{\alpha}2]^4$ with $x_0 <_{lex} x_1 <_{lex} x_3 <_{lex} x_4$. We define $h(x_0, x_1, x_2, x_3) = 1$ if $f(x_0, x_1, x_2, x_3) = 1$ or $g(x_0, x_1, x_2, x_3) = 1$, where f and g are the colourings in the proof of Theorem 4.16, and $h(x_0, x_1, x_2, x_3) = 0$ otherwise. It was shown in the proof of Theorem 4.16 that there are no sets with order type $\omega + \omega^*$ or $\omega^* + \omega$ homogeneous for h in colour 0.

Suppose that $H \in [{}^{\alpha}2]^7$ is homogeneous for h in colour 1. Suppose that $H = \{x_i \mid i < 7\}$ and $x_i <_{lex} x_j$ for $i < j < 7$. Choose $i \leqslant 5$ such that $\Delta_{x_i,x_{i+1}}$ is least in $\{\Delta_{x_j,x_{j+1}} \mid j \leqslant 5\}$. We can assume that $n \leqslant 2$. If $\Delta_{x_6,x_5} < \Delta_{x_5,x_4} < \Delta_{x_4,x_3}$, then $h(\{x_j \mid 3 \leqslant j \leqslant 6\}) = 0$, contradicting the choice of H. Otherwise, there is some j with $3 \leqslant j \leqslant 5$ and $\Delta_{x_j,x_{j+1}} < \Delta_{x_{j+1}}, \Delta_{x_{j+2}}$. Then $h(\{x_i, x_j, x_{j+1}, x_{j+2}\}) = 0$, contradicting the choice of H. □

A variation of this theorem is the following.

Theorem 4.27. *If κ is an initial ordinal number and $\alpha < \kappa^+$, then*

$$\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\kappa^* + \kappa \vee \kappa + 2 \vee 2 + \kappa^* \vee \eta, 7)^4.$$

Proof. Suppose that there is an ordinal α such that the first partition property holds. Suppose that $h\colon \alpha \leftrightarrow \kappa$ is bijective. Let $\beta_{x,y} = \beta(x, y) = h(\Delta_{x,y})$ for $x, y \in {}^{\alpha}2$. If $\alpha = \kappa$, we can choose $\beta_{x,y} = \Delta_{x,y}$ and obtain a simplified version of the following proof.

We write $\vec{x}$ for (x_0, x_1, x_2, x_3) with $x_0 <_{lex} x_1 <_{lex} x_2 <_{lex} x_3$. Let

$$\begin{aligned} F_0 := F_2 :=& \{\vec{x} \mid \beta_h(x_2, x_3) < \beta_h(x_0, x_1) < \beta_h(x_1, x_2)\}, \\ F_1 := F_3 :=& \{\vec{x} \mid \beta_h(x_0, x_1) < \beta_h(x_2, x_3) < \beta_h(x_1, x_2)\}, \\ F_4 =& \{\vec{x} \mid \beta_h(x_1, x_2) < \min(\beta_h(x_0, x_1), \beta_h(x_2, x_3))\}. \end{aligned}$$

Let $f(\vec{x}) = 1$ if $\vec{x} \in \bigcup_{i<4}(E_i \cap F_i)$ and $f(\vec{x}) = 0$ otherwise. We will prove that there is no homogeneous set of the required type for f.

We can use Lemma 4.10 to show that there are no sets of order-type $\kappa^* + \kappa$ which are homogeneous for f in colour 0, Lemma 4.8 to see that there are no such sets of order-type $2 + \kappa^*$ or $\kappa + 2$ and Lemma 4.6 or Lemma 4.7 to see that there are no such sets of order-type η.

Finally consider some $S \in [{}^{\alpha}2]^7$. Let $\langle s_i \mid i < 7\rangle$ be the order-preserving enumeration of S. Note that $[S]^4 \cap (E_0 \cup E_1) \supsetneq \emptyset$. Without loss of generality suppose that $[S]^4 \cap E_0 \supsetneq \emptyset$. Assume towards a contradiction that $f``[S]^4 = 1$. Now notice that this implies that for no $[S]^5$ with order-preserving enumeration $\langle q_i \mid i < 6\rangle$ we have $\Delta(q_i, q_{i+1}) \sqsubseteq \Delta(q_{i+1}, q_{i+2})$ for all $i < 3$ or $\Delta(q_{i+1}, q_{i+2}) \sqsubseteq \Delta(q_i, q_{i+1})$ for all $i < 3$ since this would imply $\beta(q_1, q_2) < \beta(q_2, q_3) < \beta(q_0, q_1) < \beta(q_1, q_2)$ in the first case and $\beta(q_1, q_2) < \beta(q_0, q_1) < \beta(q_2, q_3) < \beta(q_1, q_2)$ in the second. But then there is a $Q \in [S]^5$ with order-preserving enumeration $\langle q_i \mid i < 5\rangle$ such that one of the two cases applies.

In the first case, $\Delta(q_0, q_1) \sqsubseteq \Delta(q_2, q_3) \sqsubseteq \Delta(q_1, q_2), \Delta(q_3, q_4)$, in this case consider the fact that $T_0 := \{q_i \mid i \in 5 \setminus \{1\}\} \in E_0$ and $T_1 := \{q_i \mid i \in 5 \setminus 1\} \in E_4$. So $T_0 \in F_0$ and $T_1 \in F_4$) so $\beta(q_3, q_4) < \beta(q_0, q_2) < \beta(q_2, q_3) < \min(\beta(q_1, q_2), \beta(q_3, q_4)) \leqslant \beta(q_3, q_4)$, a contradiction.

In the second case, $\Delta(q_1, q_2) \sqsubseteq \Delta(q_0, q_1), \Delta(q_2, q_3)$ and $\Delta(q_2, q_3) \sqsubseteq \Delta(q_3, q_4)$. Let $T_0 := \{q_i \mid i \in 5 \setminus 1\} \in E_0$ and $T_1 := \{q_i \mid 5 \setminus \{1\}\} \in E_4$. Then $T_0 \in F_0$ and $T_1 \in F_4$, hence $\beta(q_3, q_4) < \beta(q_1, q_2) < \beta(q_2, q_3) < \min(\beta(q_0, q_2), \beta(q_3, q_4)) \leqslant \beta(q_3, q_4)$. □

Theorem 4.28. *If κ is an initial ordinal and $\alpha < \kappa^+$, then*

$$\begin{aligned} &\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\omega^* + \omega \vee 2 + \kappa^* \vee \kappa + \omega, 7)^4, \\ &\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\omega^* + \omega \vee \omega^* + \kappa^* \vee \kappa + 2, 7)^4. \end{aligned}$$

Proof. Suppose that there is an ordinal α such that the first partition property holds. Suppose that $h\colon \alpha \leftrightarrow \kappa$ is bijective. Let $\beta_{x,y} = \beta(x, y) = h(\Delta_{x,y})$ for $x, y \in {}^{\alpha}2$. If $\alpha = \kappa$, we can choose $\beta_{x,y} = \Delta_{x,y}$ and obtain a simplified version of the following proof.

We write $\vec{x}$ for (x_0, x_1, x_2, x_3) with $x_0 <_{lex} x_1 <_{lex} x_2 <_{lex} x_3$. Let

$$\begin{aligned} F_0 :=& \{\vec{x} \mid \beta_h(x_1, x_2) < \beta_h(x_2, x_3) < \beta_h(x_0, x_1)\}, \\ F_1 := F_3 :=& \{\vec{x} \mid \beta_h(x_0, x_1) < \beta_h(x_2, x_3) < \beta_h(x_1, x_2)\}, \\ F_4 =& \{\vec{x} \mid \beta_h(x_1, x_2) < \max(\beta_h(x_0, x_1), \beta_h(x_2, x_3))\}. \end{aligned}$$

Let $f(\vec{x}) = 1$ if $\vec{x} \in \bigcup_{i \in 5 \setminus \{2\}}(E_i \cap F_i)$ and $f(\vec{x}) = 0$ otherwise. We will prove that there is no homogeneous set of the required type for f.

One can use Lemma 4.5 with $i = 1$ to show that there are no sets of order-type $\omega^* + \omega$ which are homogeneous for f in colour 0, the first half of Lemma 4.8 to see that there are no such sets of order-type $2+\kappa^*$ and the second half of Lemma 4.13 to see that there are no such sets of order-type $\kappa+\omega$.

Finally consider some $\vec{s} \in [{}^{\alpha}2]^7$ and assume towards a contradiction that it were homogeneous for f in colour 1. Note that $\vec{s} \in [{}^{\alpha}2]^4 \setminus E_2$. We distinguish some cases:

First suppose that there is a $\{h_0, \ldots, h_5\}_{<_{lex}} \in [\vec{s}]^6$ such that $\Delta(h_2,h_3) \sqsubseteq \Delta(h_1,h_2), \Delta(h_3,h_4)$ and $\Delta(h_{2i+1},h_{2i+2}) \sqsubseteq \Delta(h_{4i},h_{4i+1})$ for $i<2$. Then $\{h_0,h_1,h_2,h_3\} \in E_3$, $\{h_2,h_3,h_4,h_5\} \in E_0$ and $\{h_0,h_1,h_4,h_5\} \in E_4$. Together this implies $\beta_h(h_1,h_4) < \max(\beta_h(h_0,h_1),\beta_h(h_4,h_5)) \leqslant \beta_h(h_2,h_3) = \beta_h(h_1,h_4)$, a contradiction.

Now consider some $\{p_0,\ldots,p_4\}_{<_{lex}} \in [\vec{s}]^5$.

Second suppose that $\Delta(p_0,p_1) \sqsubseteq \Delta(p_1,p_2) \sqsubseteq \Delta(p_3,p_4) \sqsubseteq \Delta(p_2,p_3)$. Then $\{p_0,p_1,p_2,p_3\} \in E_0$ while $\{p_0,p_2,p_3,p_4\} \in E_1$ so $\beta_h(p_0,p_1) < \beta_h(p_2,p_3) < \beta_h(p_0,p_1)$, a contradiction.

Third suppose that $\Delta(p_0,p_1) \sqsubseteq \Delta(p_3,p_4) \sqsubseteq \Delta(p_2,p_3) \sqsubseteq \Delta(p_1,p_2)$. Then $\{p_0,p_1,p_2,p_3\} \in E_1$ while $\{p_1,p_2,p_3,p_4\} \in E_3$ so $\beta_h(p_1,p_2) < \beta_h(p_2,p_3) < \beta_h(p_1,p_2)$, a contradiction.

The two remaining cases are very similar to each other so we can shorten the discussion by using a parameter $i<2$.

So suppose that $\Delta(p_{3i},p_{3i+1}) \sqsubseteq \Delta(p_{i+1},p_{i+2}) \sqsubseteq \Delta(p_{2-i},p_{3-i}) \sqsubseteq \Delta(p_{3-3i},p_{4-3i})$. Then $\{p_0,p_1,p_2,p_3\}, \{p_1,p_2,p_3,p_4\} \in E_{3i}$ so $\beta_h(p_1,p_2) < \beta_h(p_2,p_3) < \beta_h(p_1,p_2)$, a contradiction.

The second half of the theorem can be proved in an analogous way. □

4.6. **The classification.** We will determine which partition relations of the forms

$$\langle {}^{\omega}2, <_{lex} \rangle \longrightarrow (K,L)^n$$
$$\langle {}^{\omega}2, <_{lex} \rangle \longrightarrow (\bigvee_{\nu<\lambda} K_\nu, \bigvee_{\nu<\mu} L_\nu)^4$$

for linear orders K, L, K_ν, L_ν are consistent with $\mathsf{ZF}+\mathsf{DC}$. We have the following positive relations by Theorems 2.5, 2.6 and 2.7.

$$\langle {}^{\omega}2, <_{lex} \rangle \longrightarrow (\omega+1)^4_2,$$
$$\langle {}^{\omega}2, <_{lex} \rangle \longrightarrow (1+\omega^*+\omega+1 \vee \omega+1+\omega^*, 5)^4,$$
$$\langle {}^{\omega}2, <_{lex} \rangle \longrightarrow (1+\omega^*+\omega+1 \vee m+\omega^* \vee \omega+n, 6)^4.$$

We have the following negative relations by Theorem 4.2.

$$\langle {}^{\omega}2, <_{lex} \rangle \not\longrightarrow (\omega^*+\omega \vee \kappa+2+\kappa^* \vee (\kappa 2)^* \vee \kappa 2, 5)^4,$$
$$\langle {}^{\omega}2, <_{lex} \rangle \not\longrightarrow (2+\kappa^* \vee \kappa+2 \vee \eta, 5)^4,$$
$$\begin{cases} \langle {}^{\omega}2, <_{lex} \rangle \not\longrightarrow (\kappa^*+\kappa \vee 2+\kappa^* \vee \kappa 2 \vee \omega\omega^*, 6)^4, \\ \langle {}^{\omega}2, <_{lex} \rangle \not\longrightarrow (\kappa^*+\kappa \vee (\kappa 2)^* \vee \kappa+2 \vee \omega^*\omega, 6)^4, \end{cases}$$
$$\langle {}^{\omega}2, <_{lex} \rangle \not\longrightarrow (\kappa^*+\kappa \vee \kappa+2 \vee 2+\kappa^* \vee \eta, 7)^4.$$

Theorem 4.29. *Suppose that the principle of dependent choices* DC *holds true and all sets of reals have the property of Baire. Suppose that K and L are suborders of $\langle {}^{\omega}2, <_{lex} \rangle$ and $n \geqslant 4$. Then the partition relation*

$$\langle {}^{\omega}2, <_{lex} \rangle \longrightarrow (K,M)^n$$

holds true if and only if $K, M \leq \omega+1$ or $K, M \leq 1+\omega^*$. Otherwise the relation is inconsistent with ZF.

Proof. Suppose that $K \not\leq \omega+1$ and $L \not\leq 1+\omega^*$. Then $\omega+2 \leq K$ or $\omega^* \leq K$ and $1+\omega^* \leq M$ or $\omega \leq M$, using DC. Then the partition relation fails by Theorem 3.3. If $K \not\leq 1+\omega^*$ and $L \not\leq \omega+1$, again the partition relation fails by Theorem 3.3.

If $K \leq \omega+1$ and $L \leq \omega+1$, then the relation holds by Theorem 2.5. Similarly, if $K \leq 1+\omega^*$ and $L \leq 1+\omega^*$, then the relation holds by Theorem 2.5.

In the other cases $K \leq \omega+1$ and $K \leq 1+\omega^*$, so that K is finite, or in the remaining symmetric case that M is finite, which we omit. Suppose that $|K| = n+1$. We can assume that none of the previous cases applies, so $\omega+2 \leq M$, $2+\omega^* \leq M$, or $\omega^*+\omega \leq M$. If $\omega^*+\omega \leq M$, then the relation fails by Theorem 4.16. If $\omega+2 \leq M$ or $2+\omega^* \leq M$, then the relation fails by Theorem 4.18. □

The following result shows that the previous theorems solve the case of quadruple-colourings in the Cantor space completely, given that all sets of reals have the property of Baire. We will only consider partition relations such that in no disjunction there are linear orders K, L with $K \leq L$, since in this case L can be omitted without changing the truth value of the partition relation.

Theorem 4.30. *Suppose that the principle of dependent choices* DC *holds true and all sets of reals have the property of Baire. Suppose that K_μ and L_ν are suborders of $\langle {}^{\omega}2, <_{lex}\rangle$ for all $\mu<\kappa$ and $\nu<\lambda$. Then the partition relation*

$$\langle {}^{\omega}2, <_{lex}\rangle \longrightarrow (\bigvee_{\mu<\kappa} K_\mu, \bigvee_{\nu<\lambda} M_\nu)^4$$

holds true if and only if one of the following cases applies.

(i) $K_\xi, M_\rho \leq \omega+1$ *for some* $\xi<\kappa$, $\rho<\lambda$,

(ii) $K_\xi, M_\rho \leq 1+\omega^*$ *for some* $\xi<\kappa$, $\rho<\lambda$.

(iii) $\kappa=1$, $K_0 \leq 6$, $\lambda=3$, *and for some* $i,j,k<3$ *and some* m,n
$$M_i \leq 1+\omega^*+\omega+1,\ M_j \leq \omega+m,\ M_k \leq n+\omega^*.$$

(iv) $\lambda=1$, $M_0 \leq 6$, $\kappa=3$, *and for some* $i,j,k<3$ *and some* m,n
$$K_i \leq 1+\omega^*+\omega+1,\ K_j \leq \omega+m,\ K_k \leq n+\omega^*.$$

(v) $\kappa=1$, $K_0 \leq 5$, $\lambda=2$, *and for some* $i,j<2$
$$M_i \leq 1+\omega^*+\omega+1,\ M_j \leq \omega+1+\omega^*.$$

(vi) $\lambda=1$, $M_0 \leq 5$, $\kappa=2$, *and for some* $i,j<2$
$$K_i \leq 1+\omega^*+\omega+1,\ K_j \leq \omega+1+\omega^*.$$

Moreover, if none of these cases applies, then the relation is inconsistent with ZF.

Proof. Suppose that $K_\mu \not\leq \omega+1$ and $M_\nu \not\leq 1+\omega^*$ for all $\mu<\kappa$ and $\nu<\lambda$. Then $\omega+2 \leq K_\mu$ or $\omega^* \leq K_\mu$ for all $\mu<\kappa$, and $2+\omega^* \leq M_\nu$ or $\omega \leq M_\nu$ for all $\nu<\lambda$, using DC. Then the partition relation fails by Theorem 3.3.

If $K_\mu \not\leq 1+\omega^*$ and $M_\nu \not\leq \omega+1$ for all $\mu<\kappa$ and $\nu<\lambda$, again the partition relation fails by Theorem 3.3.

If $K_\mu \leq \omega+1$ for some $\mu<\kappa$ and $M_\nu \leq \omega+1$ for some $\nu<\lambda$, then the relation holds by Theorem 2.5. Similarly, if $K_\mu \leq 1+\omega^*$ for some $\mu<\kappa$ and $M_\nu \leq 1+\omega^*$ for some $\nu<\lambda$, then the relation holds by Theorem 2.5. These are the first two cases in the classification.

It follows that $K_\mu \leq \omega+1$ and $K_\nu \leq 1+\omega^*$ for some $\mu,\nu<\kappa$, or the symmetric case for M_μ, M_ν and $\mu,\nu<\lambda$, which we omit. We can assume that none of the previous cases applies.

We first suppose that $\mu \neq \nu$, or that $\mu=\nu$ and $K_\mu \geqslant 7$. Let us consider the linear orders on the right side of the relation. Since none of the previous cases applies, the linear orders are neither embeddable into $1+\omega^*$ nor into $\omega+1$. Hence for each $\nu<\lambda$, $\omega+2 \leq M_\nu$, $2+\omega^* \leq M_\nu$, or $\omega^*+\omega \leq M_\nu$. Then the relation fails by Theorem 4.2.

Second, we suppose that $\kappa=1$ and $K_0=6$. Again, we consider the linear orders on the right. If every linear order contains $\omega+2$ or $2+\omega^*$, then the relation fails by Theorem 4.2. If every linear order contains $\omega^*+\omega$ or $\omega+2$, then the relation fails by Theorem 4.2. If every linear order contains $\omega^*+\omega$ or $2+\omega^*$, then the relation fails by Theorem 4.2. Any linear order which contains $\omega^*+\omega$, but neither $2+\omega^*$ nor $\omega+2$, is contained in $1+\omega^*+\omega+1$. Hence the linear orders on the right side of the relation are contained in $1+\omega^*+\omega+1$, $\omega+m$, and $n+\omega^*$ for some m,n. Then the partition relation holds by Theorem 2.6. This is the third case in the classification. The fourth case is symmetric and occurs when we exchange the left and right sides of the relation.

Finally, we consider the case $\kappa=0$ and $K_0=5$. If every linear order contains $\omega+2$ or $2+\omega^*$, then the relation fails by Theorem 4.2. If every linear order contains $\omega^*+\omega$, $\omega+2+\omega^*$, $\omega2$, or $(\omega2)^*$, then the relation fails by Theorem 4.2.

Otherwise, there are $\mu,\nu<\lambda$ such that M_μ contains neither $\omega^*+\omega$, $\omega+2+\omega^*$, $\omega2$, nor $(\omega2)^*$, and M_ν contains neither $2+\omega^*$ nor $\omega+2$. Then M_μ is embeddable into $\omega+1+\omega^*$.

First, suppose that M_ν contains $\omega^*+\omega$. Then M_ν is contained in $1+\omega^*+\omega+1$. If $\mu \neq \nu$, the relation holds by Theorem 2.7. This is the fifth case in the classification. The sixth case occurs symmetrically when the left and right sides in the relation are exchanged. If $\mu = \nu$, then M_μ embeds into $\omega+1$ or $1+\omega^*$, so the relation holds by Theorem 2.4. This is one of the first two cases of the classification.

Second, suppose that M_ν does not contain $\omega^*+\omega$. Then M_ν is contained in $\omega+n+\omega^*$ for some n, and therefore M_ν embeds into $\omega+1$ or $1+\omega^*$, so the relation holds by Theorem 2.4. This is one of the first two cases of the classification. □

4.7. **Octuples.** In the remaining sections, we prove three negative partition relations for octuples and nonuples. These relations follow from Theorem 4.27 for $\kappa=\omega$, but are new for $\kappa>\omega$.

Theorem 4.31. *if κ is an initial ordinal and $\alpha<\kappa^+$, then*

$$\langle{}^{\alpha}2, <_{lex}\rangle \not\to (\omega^*+\omega \vee \omega+\omega^* \vee (\kappa 2)^* \vee \kappa 2, 8)^4.$$

Proof. Suppose that there is an ordinal α such that the first partition property holds. Suppose that $h\colon \alpha \leftrightarrow \kappa$ is bijective. Let $\beta_{x,y}=\beta(x,y)=h(\Delta_{x,y})$ for $x,y\in {}^{\alpha}2$. If $\alpha=\kappa$, we can choose $\beta_{x,y}=\Delta_{x,y}$ and obtain a simplified version of the following proof.

We write $\vec{x}$ for (x_0,x_1,x_2,x_3) with $x_0 <_{lex} x_1 <_{lex} x_2 <_{lex} x_3$. Let

$$\begin{aligned}
F_0 :=&\{\vec{x} \mid \beta_h(x_1,x_2)<\beta_h(x_2,x_3)<\beta_h(x_0,x_1)\},\\
F_1 :=&\{\vec{x} \mid \beta_h(x_2,x_3)<\beta_h(x_1,x_2)\},\\
F_2 :=&\{\vec{x} \mid \beta_h(x_0,x_1)<\beta_h(x_1,x_2)\},\\
F_3 :=&\{\vec{x} \mid \beta_h(x_1,x_2)<\beta_h(x_0,x_1)<\beta_h(x_2,x_3)\},\\
F_4 :=&\{\vec{x} \mid \beta_h(x_1,x_2)<\min(\beta_h(x_0,x_1),\beta_h(x_2,x_3))\}.
\end{aligned}$$

Let $f(\vec{x})=1$ if $\vec{x}\in\bigcup_{i\in 5\setminus\{2\}}(E_i\cap F_i)$ and $f(\vec{x})=0$ otherwise.

We will prove that there is no homogeneous set of the required type for f. One can use Lemma 4.5 to show that there are no sets of order-type $\omega^*+\omega$ which are homogeneous for f in colour 0, Lemma 4.14 to see that there are no such sets of order-type $(\kappa 2)^*$ or $\kappa 2$ and Lemma 4.15 to see that there are no such sets of order-type $\omega+\omega^*$.

Finally consider some $\vec{o}\in[{}^{\alpha}2]^8$ and assume towards a contradiction that it were homogeneous for f in colour 1. There is an $i<2$ and a quintuple $\{p_0,\dots,p_4\}_{<_{lex}}\in[\vec{o}]^5$ for which one of the following three cases obtains:

First suppose that $\Delta(p_{3i},p_{3i+1})\sqsubseteq\Delta(p_{i+1},p_{i+2})\sqsubseteq\Delta(p_{2-i},p_{3-i})\sqsubseteq\Delta(p_{3-3i},p_{4-3i})$. Then $\{p_j \mid j<4\},\{p_j \mid j>0\}\in E_{3i}$ so $\beta_h(p_1,p_2)<\beta_h(p_2,p_3)<\beta_h(p_1,p_2)$, a contradiction.

Second suppose that $\Delta(p_{3i},p_{3i+1})\sqsubseteq\Delta(p_{3-3i},p_{4-3i})\sqsubseteq\Delta(p_{2-i},p_{3-i})\sqsubseteq\Delta(p_{i+1},p_{i+2})$. Then $\{p_j \mid j\neq i+1\}\in E_{i+1}$ while $\{p_j \mid j\neq 4i\}\in E_{3-3i}$ so $\beta_h(p_{2-i},p_{3-i})<\beta_h(p_{3-3i},p_{4-3i})<\beta_h(p_{2-i},p_{3-i})$, a contradiction.

Last suppose that $\Delta(p_{i+1},p_{i+2})\sqsubseteq\Delta(p_{3i},p_{3i+1}),\Delta(p_{2-i},p_{3-i})$ and further $\Delta(p_{2-i},p_{3-i})\sqsubseteq\Delta(p_{3-3i},p_{4-3i})$. Then $\{p_j \mid j<4\}\in E_{4-i}$ and $\{p_j \mid j>0\}\in E_{4i}$ so $\beta_h(p_{i+1},p_{i+2})<\min(\beta_h(p_{3i},p_{3i+1}),\beta_h(p_{2-i},p_{3-i}))\leqslant\beta_h(p_{2-i},p_{3-i})<\beta_h(p_{i+1},p_{i+2})$, a contradiction. □

Theorem 4.32. *If κ is an infinite initial ordinal and $\alpha<\kappa^+$, then*

$$\langle{}^{\kappa}2, <_{lex}\rangle \not\to (\kappa^*+\omega \vee \omega^*+\kappa \vee 2+\kappa^* \vee \kappa+2 \vee \omega\omega^* \vee \omega^*\omega, 8)^4.$$

Proof. Let α be any ordinal.

$$\begin{aligned}
F_0 := F_2 :=&\{\vec{x} \mid \beta_h(x_2,x_3)<\beta_h(x_0,x_1)<\beta_h(x_1,x_2)\},\\
F_1 := F_3 :=&\{\vec{x} \mid \beta_h(x_0,x_1)<\beta_h(x_2,x_3)<\beta_h(x_1,x_2)\},\\
F_4 :=&\{\vec{x} \mid \beta_h(x_1,x_2)<\max(\beta_h(x_0,x_1),\beta_h(x_2,x_3))\}.
\end{aligned}$$

For $\vec{x}\in[{}^{\alpha}2]^4$ let $f(\vec{x})=1$ if $\vec{x}$ is in $E_i\cap F_i$ for an $i<5$ and $f(\vec{x})=0$ otherwise. We will prove that there is no homogeneous set of the required type for f.

In order to see that there is no homogeneous set of the required type in colour 1, consider the Lemmata 4.9,4.8 and 4.11.

Now consider an octuple $O \in [{}^{\alpha}2]^8$ with order-preserving enumeration $\langle o_i \mid i < 8\rangle$. Note that O has has to contain one of the following splitting types of quintuples or sextuples. So let $\{p_0, \dots, p_4\}_{<_{lex}} \in [O]^5$ and $\{h_0, \dots, h_5\}_{<_{lex}} \in [O]^6$.

First suppose that $\Delta(h_2, h_3) \sqsubseteq \Delta(h_1, h_2), \Delta(h_3, h_4)$ and $\Delta(h_{2i+1}, h_{2i+2}) \sqsubseteq \Delta(h_{4i}, h_{4i+1})$ for $i < 2$. Then $\{h_i \mid i < 4\} \in E_3$ and $\{s_i \mid i \in 6 \setminus 2\} \in E_0$. This implies $\beta_h(h_0, h_1) < \beta_h(h_2, h_3)$ and $\beta_h(s_4, h_5) < \beta_h(h_2, h_3)$ so $\min(\beta_h(h_0, h_1), \beta_h(h_4, h_5)) < \beta_h(h_2, h_3) = \beta_h(h_1, h_4)$. But $\{h_i \mid i \in 6 \setminus \{1, 4\}\} \in E_4$, a contradiction.

The remaining cases come in two flavours, one for each $i < 2$.

Second suppose that $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{i+1}, p_{i+2}) \sqsubseteq \Delta(p_{2-i}, p_{3-i}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i})$. Then $\{p_i \mid i < 4\}, \{p_i \mid i > 0\} \in E_{3i}$ so $\beta_h(p_{2-i}, p_{3-i}) < \beta_h(p_{i+1}, p_{i+2}) < \beta_h(p_{2-i}, p_{3-i})$, a contradiction.

Third assume $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{i+1}, p_{i+2}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i}) \sqsubseteq \Delta(p_{2-i}, p_{3-i})$. Then $\{p_i \mid i < 4\} \in E_{3i}$ and $\{p_i \mid i > 0\} \in E_{i+1}$ so $\beta_h(p_{2-i}, p_{3-i}) < \beta_h(p_{i+1}, p_{i+2}) < \beta_h(p_{2-i}, p_{3-i})$, a contradiction.

Fourth suppose that $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{2-i}, p_{3-i}) \sqsubseteq \Delta(p_{i+1}, p_{i+2})$. Then $\{p_i \mid i < 4\} \in E_{i+1}$ and $\{p_i \mid i > 0\} \in E_{3-3i}$ so $\beta_h(p_{i+1}, p_{i+2}) < \beta_h(p_{2-i}, p_{3-i}) < \beta_h(p_{i+1}, p_{i+2})$, a contradiction.

Finally assume $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i}) \sqsubseteq \Delta(p_{i+1}, p_{i+2}) \sqsubseteq \Delta(p_{2-i}, p_{3-i})$. Then $\{p_i \mid i < 4\} \in E_{3i}$ and $\{p_i \mid i > 0\} \in E_{2-i}$ so $\beta_h(p_{2-i}, p_{3-i}) < \beta_h(p_{i+1}, p_{i+2}) < \beta_h(p_{2-i}, p_{3-i})$, a contradiction. $\square$

4.8. **Nonuples.** In the final section of this chapter, we prove negative partition relations for nonuples. These relations follow from Theorem 4.27 for $\kappa = \omega$, but are new for $\kappa > \omega$.

Theorem 4.33. *If κ is an infinite initial ordinal and $\alpha < \kappa^+$, then*

$$\langle {}^{\alpha}2, <_{lex}\rangle \not\longrightarrow (\omega^* + \omega \vee \omega + \omega^* \vee \kappa + 2 \vee 2 + \kappa^*, 9)^4.$$

Proof. Suppose that κ is as in the theorem and there is an ordinal $\alpha < \kappa^+$ such that the Theorem holds. Suppose that $h\colon \alpha \leftrightarrow \kappa$ is bijective. Let $\beta_{x,y} = \beta(x, y) = h(\Delta_{x,y})$ for $x, y \in {}^{\alpha}2$. If $\alpha = \kappa$, we can choose $\beta_{x,y} = \Delta_{x,y}$ and obtain a simplified version of the following proof.

We write $\vec{x}$ for (x_0, x_1, x_2, x_3) with $x_0 <_{lex} x_1 <_{lex} x_2 <_{lex} x_3$. Let

$$\begin{aligned} F_0 := F_2 :=& \{\vec{x} \mid \beta_h(x_2, x_3) < \beta_h(x_0, x_1) < \beta_h(x_1, x_2)\}, \\ F_1 := F_3 :=& \{\vec{x} \mid \beta_h(x_0, x_1) < \beta_h(x_2, x_3) < \beta_h(x_1, x_2)\}. \end{aligned}$$

Let $f(\vec{x}) = 1$ if $\vec{x} \in E_4 \cup \bigcup_{i<4}(E_i \cap F_i)$ and $f(\vec{x}) = 0$ otherwise. We will prove that there is no homogeneous set of the required type for f.

By Lemmata 4.4 and 4.8, there are no homogeneous sets of the order types $\omega^* + \omega$, $2 + \kappa^*$, and $\kappa + 2$ in colour 0.

Now suppose that there is a $Y \in [{}^{\alpha}2]^{\omega+\omega^*}$ and assume towards a contradiction that it were homogeneous for f in colour 0. We distinguish three cases. First assume that there is an $s \in {}^{<\alpha}2$ such that $\min(|\{y \in Y \mid y \sqsupset s^\frown\langle 0\rangle\}|, |\{y \in Y \mid y \sqsupset s^\frown\langle 1\rangle\}|) > 1$. Choose $y_{2i}, y_{2i+1} \sqsupset s^\frown\langle i\rangle$ for $i < 2$. Then $\{y_0, y_1, y_2, y_3\} \in E_4$. Now assume that there is no such s. This implies that all splitting nodes lie on a single branch. Let $\langle y_n^{(0)} \mid n < \omega\rangle$ be the ascending enumeration of the lower half of Y and $\langle y_n^{(1)} \mid n < \omega\rangle$ the descending one of the upper half. Consider the parameters $\gamma_i := \sup_{n<\omega} \delta(y_n^{(i)}, y_{n+1}^{(i)})$ and $\zeta_i := \limsup_{n<\omega} \beta(y_n^{(i)}, y_{n+1}^{(i)})$ for $i < 2$. Now the second case applies if $\gamma_0 \leqslant \gamma_1 \leftrightarrow \zeta_1 \leqslant \zeta_0$, otherwise the third case applies. Let $i < 2$ be such that $\gamma_i \leqslant \gamma_{1-i}$. Now choose $m < \omega$ such that $\Delta(y_m^{(1-i)}, y_{m+1}^{(1-i)}) > \Delta(y_0^{(i)}, y_1^{(i)})$ and $\beta(y_m^{(1-i)}, y_{m+1}^{(1-i)}) \in \zeta_{1-i} \setminus \beta(y_0^{(i)}, y_1^{(i)})$. Now choose an $n \in \omega \setminus m$ such that $\beta(y_n^{(1-i)}, y_{n+1}^{(1-i)}) > \beta(y_m^{(1-i)}, y_{m+1}^{(1-i)})$. Then $\{y_0^{(i)}, y_m^{(1-i)}, y_n^{(1-i)}, y_{n+1}^{(1-i)}\} \in E_{3-i} \cap F_{3-i}$. In the third and final case let $k < \omega$ be such that $\delta(y_k^{(1-i)}, y_{k+1}^{(1-i)}) > \gamma_i$. Then choose $m < \omega$ such that $\beta(y_m^{(i)}, y_{m+1}^{(i)}) \in \zeta_i \setminus \beta(y_k^{(i)}, y_{k+1}^{(i)})$ and finally $n \in \omega \setminus m$ such that $\beta(y_n^{(i)}, y_{n+1}^{(i)}) > \beta(y_m^{(i)}, y_{m+1}^{(i)})$. Then $\{y_m^{(i)}, y_n^{(i)}, y_k^{(1-i)}, y_{k+1}^{(1-i)}\} \in E_i \cap F_i$.

Finally assume towards a contradiction hat there were a nonuple $N \in [{}^{\alpha}2]^9$ homogeneous for f in colour 1. We consider the following four cases, specified by a parameter $i < 2$:

First suppose that there is some quintuple $\vec{p} = \{p_0, \ldots, p_4\}_{<_{lex}} \in [N]^5$ such that $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{i+1}, p_{i+2}) \sqsubseteq \Delta(p_{2-i}, p_{3-i}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i})$. Now $\{p_j \mid j < 4\}, \{p_j \mid j > 0\} \in E_{3i}$ so $\beta_h(p_1, q_2) < \beta_h(q_2, q_3) < \beta_h(q_1, q_2)$, a contradiction.

Second suppose that there is such a $\vec{p}$ with $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{i+1}, p_{i+2}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i}) \sqsubseteq \Delta(p_{2-i}, p_{3-i})$. Then $\{p_j \mid j < 4\} \in E_{2i}$ while $\{p_j \mid j > 0\} \in E_{i+2}$ so $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

Third suppose that there is such a $\vec{p}$ with $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i}) \sqsubseteq \Delta(p_{i+1}, p_{i+2}) \sqsubseteq \Delta(p_{2-i}, p_{3-i})$. Then $\{p_j \mid j < 4\} \in E_i$ while $\{p_j \mid j > 0\} \in E_{i+2}$ hence $\beta_h(p_1, p_2) < \beta_h(p_2, p_3) < \beta_h(q_1, q_2)$, a contradiction.

Finally suppose that there is such a $\vec{p}$ with $\Delta(p_{3i}, p_{3i+1}) \sqsubseteq \Delta(p_{3-3i}, p_{4-3i}) \sqsubseteq \Delta(p_{2-i}, p_{3-i}) \sqsubseteq \Delta(p_{i+1}, p_{i+2})$. Then $\{p_j \mid j < 4\} \in E_{1-i}$ while $\{p_j \mid j > 0\} \in E_{3-i}$ so $\beta_h(p_1, p_2) < \beta(p_2, p_3) < \beta_h(p_1, p_2)$, a contradiction.

These four pairs of cases exhaust all possibilities, since at least one of them occurs in every nonuple. □

5. Questions

We conclude this paper with the main open questions. The strong partition property for ω_1 implies $\langle {}^{\omega_1}2, <_{lex} \rangle \longrightarrow (\langle {}^{\omega_1}2, <_{lex} \rangle)^1_2$. This motivates the following question.

Question 5.1. *Does the axiom of determinacy imply* $\langle {}^{\omega_1}2, <_{lex} \rangle \longrightarrow (\langle {}^{\omega_1}2, <_{lex} \rangle)^2_2$*?*

The following question asks about an uncountable analogue of Blass' theorem. This seems necessary to generalise the positive partition results from $\langle {}^{\omega}2, <_{lex} \rangle$ to $\langle {}^{\kappa}2, <_{lex} \rangle$.

Question 5.2. *Is it consistent that* $\kappa = \kappa^{<\kappa} > \omega$ *and* $\langle {}^{\kappa}2, <_{lex} \rangle \longrightarrow_t (\langle {}^{\kappa}2, <_{lex} \rangle)^m_n$ *for all* m, n*?*

We ask whether the classifications in Theorems 3.4 and 4.30 generalise to exponent 5.

Question 5.3. *Which partition relations of the form*

$$\langle {}^{\omega}2, <_{lex} \rangle \longrightarrow (\bigvee_{\nu<\lambda} K_\nu, \bigvee_{\nu<\mu} L_\nu)^5$$

hold if all subsets of $\langle {}^{\omega}2, <_{lex} \rangle$ *have the property of Baire?*

It seems harder to generalise the classification to uncountable κ.

Question 5.4. *Which partition relations of the form*

$$\langle {}^{\kappa}2, <_{lex} \rangle \longrightarrow (\bigvee_{\nu<\lambda} K_\nu, \bigvee_{\nu<\mu} L_\nu)^n$$

for $n \geqslant 3$ *are (jointly) consistent with* ZF *(*$+\,\mathsf{DC}_\kappa$*), and which of the relations for* $\kappa = \omega_1$ *are provable in the theories* $\mathsf{ZF} + \mathsf{AD} + [V = L(\mathbb{R})]$ *and* $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}}$*?*

Theorems 1.20, 2.3, 2.4 and 2.7 suggest that models of determinacy are good candidates for obtaining positive partition relations. In particular $L(\mathbb{R})$ is a canonical model of $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$, provided that there are infinitely many Woodin cardinals and a measurable cardinal above them all, cf.[988MS].

The partition relations in Question 5.4 for which all K_ν for $\nu < \lambda$ are well-ordered hold for large ordinals on the left side of the relation by the Erdős-Rado Theorem. On the other hand it is unclear whether the existence of linear orderings K such that $K \longrightarrow (2 + \omega^* + \omega \vee \omega + \omega^*, 5)^4$, $K \longrightarrow (\omega^* + \omega \vee \omega + 2 + \omega^*, 5)^4$ or $K \longrightarrow (\omega^* + \omega \vee \omega + \omega^*, 6)^4$ is consistent with ZF. The relations fail in ZFC by Theorem 1.9. Moreover, if one of the relations holds for a linear order K of the form $K = \langle {}^{\gamma}2, <_{lex} \rangle$, then $\gamma \geqslant \omega_1$ by Theorem 4.2.

Finally, we ask about partition relations in the context of strong failures of the Axiom of Choice. The assumption in the following question is consistent from a proper class of strongly compact cardinals by [980Gi].

Question 5.5. *Which partition relations for linear orders hold if all uncountable cardinals are singular?*

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