A rough outline of the proof of the K-theoretic Farrell-Jones conjecture for CAT(-1) groups

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An axiomatic view of algebraic K-theory

We will view the algebraic K-groups as a functor from the category $\mathcal{A}dd\mathcal{C}at$ of small additive categories to the category of $\mathbb{Z}$-graded, Abelian groups. The n-th K-group $K_n(\mathcal{A})$ is defined as the n-th stable homotopy group of the non-connective algebraic K-theory spectrum of $\mathcal{A}$. However, during this talk we will only make use of the following properties:

1. Equivalences of additive categories induce isomorphisms;
2. There is a natural long exact sequence associated to a *Karoubi*-filtration of additive categories.
3. If an additive category $\mathcal{A}$ is *flasque*, i.e. there is an endofunctor $F : \mathcal{A} \to \mathcal{A}$ and a natural isomorphism $F \oplus \mathrm{Id} \cong F$, then $K_*(\mathcal{A}) = 0$.
4. The natural inclusions induce isomorphisms $\bigoplus_i K_*(\mathcal{A}_i) \cong K_*(\bigoplus_i \mathcal{A}_i)$. This is also called continuity of algebraic K-theory.

Karoubi-filtrations

Let $\mathcal{A} \subset \mathcal{U}$ be a full additive subcategory. An $\mathcal{A}$-filtration of $\mathcal{U}$ consists of the following data: For each $U \in \mathrm{Obj}(\mathcal{U})$ we have a family of decompositions $(f_i : U \cong E_i \oplus U_i)_{i \in I_U}$ with $E_i \in \mathcal{A}$. Note that if we define

$$i \leq j :\Leftrightarrow E_i \subset E_j \text{ and } U_i \supset U_j,$$

we obtain a partial order on each of the sets I_U. Furthermore we require that

1. any two elements in I_U have an common upper bound,
2. every map $g : A \to U$ from an object $A \in \mathcal{A}$ factors through one of the decompositions, i.e. there is an $i \in I_U$ and an $g' \in \mathrm{Hom}_{\mathcal{A}}(A, E_i)$ such that $g : A \xrightarrow{g'} E_i \hookrightarrow E_i \oplus U_i \xrightarrow{f_i} U$;
3. every map $g : U \to A$ to an object $A \in \mathcal{A}$ factors through one of the compositions;
4. For $U, V \in \mathcal{U}$ the posets $I_{U \oplus V}$ and $I_U \times I_V$ are equivalent (i.e. cofinal in each other as subposets of all decompositions).

Now we can define a quotient category $\mathcal{U}/\mathcal{A}$ whose objects are the same as the objects from $\mathcal{U}$, but whose morphism sets $\mathrm{Hom}_{\mathcal{U}/\mathcal{A}}(U, U')$ is the quotient of $\mathrm{Hom}_{\mathcal{U}}(U, U')$ by the subgroup of all morphisms that factor through some $A \in \mathcal{A}$, i.e. that have the form:

$$U \xrightarrow{f_i} E_i \oplus U_i \xrightarrow{pr} E_i \xrightarrow{f'} E'_j \hookrightarrow E'_j \oplus U'_j \xrightarrow{f_j} U'$$

for some $i \in I_U, j \in I_{U'}, f' \in \mathrm{Hom}_{\mathcal{A}}(E_i, E'_j)$.

Exercise

Show that equivalent choices of decompositions give the same quotient category.

A small example

Let $(\mathcal{A}_i)_{i \in \mathbb{N}}$ be a family of additive categories.

Exercise

Find decompositions that turn $\bigoplus_i \mathcal{A}_i \subset \prod_i \mathcal{A}_i$ into a Karoubi-filtration.

If all $\mathcal{A}_i$ are equal, then we could also allow morphisms that mix the degree. This will be a very important example.

An example

- Let $\mathcal{B}$ be any additive category;
- let $\mathcal{U}$ be the category whose objects are collections $(B_i)_{i \in \mathbb{N}}$ of objects from $\mathcal{B}$;
- its morphisms $\mathrm{Hom}_{\mathcal{U}}((B_i)_{i \in \mathbb{N}}, (C_i)_{i \in \mathbb{N}})$ are families of morphisms $(\varphi_{i,j} : B_i \to C_j)$ such that there is some natural number r with $|i - j| \geq r \Rightarrow \varphi_{i,j} = 0$. We can think of those morphisms as band matrices indexed over $\mathbb{N}$;
- composition is given by matrix multiplication;
- let $\mathcal{A}$ be the full subcategory consisting of all objects $(B_i)_{i \in \mathbb{N}}$ where almost all B_i are zero;
- let for a subset S of $\mathbb{N}$ and an object $U = (B_i)_{i \in \mathbb{N}}$ the restriction $U|_S$ be the object with $(U|_S)_i = U_i$ for $i \in S$ and $(U|_S)_i = 0$ otherwise.

An example

Exercise

We have:

1. The inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ is a equivalence;
2. the family of decompositions $U = U|_{1,\ldots,n} \oplus U|_{n+1,\ldots}$ indexed over $n \in \mathbb{N}$ gives an $\mathcal{A}$-filtration of $\mathcal{U}$;
3. the endofunctor (aka *Eilenberg-swindle*)

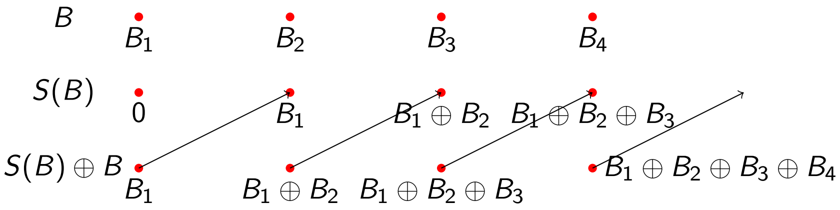
$$F : \mathcal{U} \to \mathcal{U} \quad (B_i)_{i\in\mathbb{N}} \mapsto (\bigoplus_{j<i} B_i)_{i\in\mathbb{N}}$$

   shows that $\mathcal{U}$ is flasque.

Remark

Thus $K_n\mathcal{B} = K_{n+1}(\mathcal{U}/\mathcal{B})$ and hence we could use these constructions to define negative K-theory in terms of K_0 of other categories.

Pictures of the swindle



A remarkable remark

We could drop the finite propagation condition on $\mathcal{U}$. Let us call the resulting additive category $\mathcal{U}'$. We would still get a Karoubi-filtration and a long exact ladder:

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & K_*(\mathcal{B}) & \longrightarrow & K_*(\mathcal{U}) & \longrightarrow & K_*(\mathcal{U}/\mathcal{B}) & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & K_*(\mathcal{B}) & \longrightarrow & K_*(\mathcal{U}') & \longrightarrow & K_*(\mathcal{U}'/\mathcal{B}) & \longrightarrow & \ldots \end{array}$$

The full subcategories of objects with compact support are the same. Thus the left arrow is an isomorphism. The middle arrow is an isomorphism since both categories are flasque. Thus the third arrow is also an isomorphism. This is not obvious! The underlying functor is not a equivalence of additive categories.

Controlled algebra

- The last example is really important. We can construct a lot of additive categories that way. We will define an additive category $\mathcal{C}(X, \mathcal{A})$ depending on a space X and an additive category $\mathcal{A}$ (thought of as coefficients). In the last example the space was $\mathbb{N}$.
- An object in $\mathcal{C}(X; \mathcal{A})$ is a collection of objects $(A_x)_{x \in X}$ of objects of $\mathcal{A}$ such that its *support* $\{x \mid A_x \neq 0\}$ is locally finite.
- A morphism $\varphi : (A_x)_{x \in X} \to (B_y)_{y \in Y}$ is a collection of morphisms $\varphi_{x,y} : A_x \to B_y$ such that its *support* $\{(x, y) \mid \varphi_{x,y} \neq 0\}$ is row and column finite.

Generic picture of a morphism

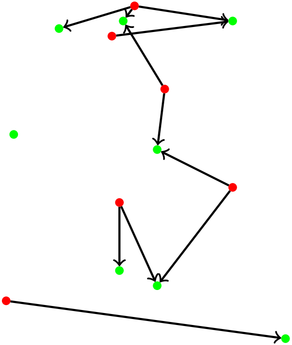


Figure: Sketch of a morphism

The figure on the left hand side shows a morphism. Its source is indicated by the red dots; at every red dot there should be a nonzero module attached. The range is indicated by the green points. We draw an arrow between x, y whenever $\varphi_{x,y}$ is nonzero.

Object control conditions

- We can also generalize the notion of having compact support (that appeared in the example). An *object control condition* on a space X is a collection of subsets of X such that for any two such subsets we can find a third that contains their union.
- Examples of such object control conditions are the *compact support condition* $\mathcal{F}_c$, or all subsets of a metric space which have bounded distance to a fixed set. If there is also a G-action on X we also have the cocompact support condition $\mathcal{F}_{G-c}$.
- Object control conditions can be pulled back. The intersection of two object control conditions $\mathcal{F}, \mathcal{F}'$ consists of all subsets of the form $F \cap F'$ for some $F \in \mathcal{F}, F' \in \mathcal{F}'$.
- Now we can look at the subcategory of $\mathcal{C}(X, \mathcal{A})$ that consists of all objects such that their support is contained in a subset of our object condition.

Morphism control conditions

- We can also impose conditions on the morphisms (analogous to the condition $|i - j| < r$ from the example). A *morphism control condition* $\mathcal{E}$ on a space X is a collection of subsets of $X \times X$ such that
  1. For all $E, E' \in \mathcal{E}$ there is a $E'' \in \mathcal{E}$ with $E \cup E' \subset E''$;
  2. For all $E, E' \in \mathcal{E}$ there is a $E'' \in \mathcal{E}$ with $E \circ E' = \{(x, z) \mid \exists y : (x, y) \in E \text{ and } (y, z) \in E'\} \subset E''$;
  3. There is some $E \in \mathcal{E}$ with $\Delta(X) \subset E$.
- An example of such a morphism control condition is the *metric control condition* $\mathcal{E}_d$. Morphism control conditions are also known as *coarse structures*. Pull backs and intersections of morphism control conditions work the same way.
- Now we can look at the subcategory $\mathcal{C}(X, \mathcal{E}, \mathcal{F}; \mathcal{A})$ of $\mathcal{C}(X; \mathcal{A})$ that consists of all objects such that their support is contained in a set in our object control condition $\mathcal{F}$ and of all morphisms whose support is contained in a subset of our morphism control condition $\mathcal{E}$.

The equivariant version

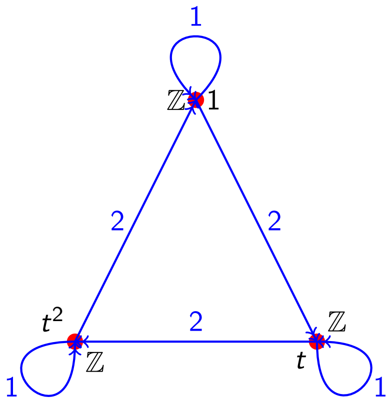
A G-action on X induces an action on all object (morphism) control conditions. Suppose now we have

1. a G-action on the space X;
2. a G-invariant object control condition $\mathcal{F}$;
3. a G-invariant morphism control condition $\mathcal{E}$;
4. an additive category $\mathcal{A}$ with a strict (right) G-action (could be trivial).

Then $\mathcal{C}(X, \mathcal{E}, \mathcal{F}; \mathcal{A})$ also has a strict (right) G-action via

$$(g^* C)_x := g^* C_{gx}; \qquad (g^* \varphi)_{x,y} := (g^* \varphi_{gx,gy}).$$

Let $\mathcal{C}(X, \mathcal{E}, \mathcal{F}; \mathcal{A})^G$ denote the fixed point category, i.e. the subcategory with $A_x = g^* A_{gx}$ for all $x \in X$ and $(g^* \varphi)_{x,y} := (g^* \varphi_{gx,gy})$.



Let us examine what $\mathcal{C}(G;\mathcal{A})^G$ is for $\mathcal{A}$ the additive category of (f.g.) free R-Modules. The action is transitive, thus by G-invariance we have to attach at every point the same module. The arrows going out of some point also look the same at every point. On the right you can see the endomorphism given by $1 + 2t$.

Free RG-modules

Then $\mathcal{C}(G;\mathcal{A})^G$ is equivalent to the category of free RG-modules. The functor F sends $A = (A_g)_g \in G$ to $\bigoplus_{g\in G} A_g$ equipped with the G-action that permutes the coordinates. To see that we really hit all morphisms write an element $f \in \mathrm{Hom}_{RG}(F(A), F(A'))$ in the form $\sum_{g\in G} f_g \cdot g$. The preimage is given by the morphism φ with $\varphi_{g,g'} := f_{g^{-1}g'}$.

What is this good for ?

Finally we want to show that the so-called *assembly map*

$$H_*^G(E_{\mathcal{VC}\text{yc}}G;\mathbf{K}) \to K_*(RG)$$

is an isomorphism. The goal is to find an additive category, such that the left hand side is the K-theory of this additive category. At first glance this might have made things worse; it will turn out that the assembly map appears in the long exact sequence associated to a Karoubi-filtration. Thus we have to focus on the third term.

Excision via Continuous control

We do not have excision on $K_*(\mathcal{C}(-;\mathcal{A})^G)$. The equivariant *continuous control condition* can be used to fix this. For a G-space X let $\mathcal{E}_{G-cc}$ denote the collection of those subsets E of $(X \times [1,\infty))^2$ such that

1. E is symmetric and invariant under the diagonal G-action;
2. $pr_{[1,\infty)^2}$ has bounded (Euclidean) distance to the diagonal;
3. For every $x \in X$ and any G-invariant neighborhood U of (x,∞) in $X \times [1,\infty)$ there is a (smaller) G-invariant neighborhood V such that

$$(U^c \times V) \cap E = \emptyset \wedge (V^c \times U) \cap E = \emptyset.$$

Continuous control

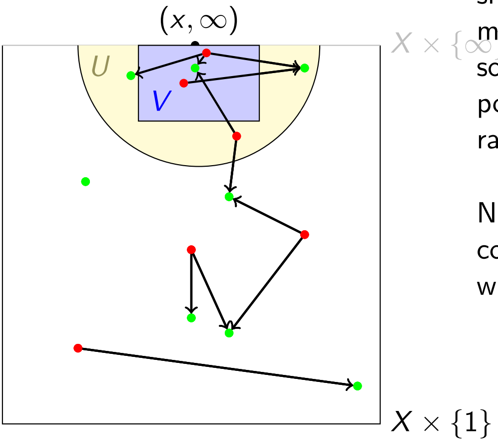


Figure: A morphism satisfying the continuous control condition

The figure on the left hand side shows a continuously controlled morphism. The support of the source is indicated by the red points and the support of the range by the green points.

Note that there is no arrow connecting a point inside V with a point outside U.

G-homology theories

A G-homology theory is a covariant functor H_* from the category of G-CW-pairs to the category of $\mathbb{Z}$-graded abelian groups satisfying:

- Excision,.i.e. for a G-CW-pair (X, A) and a G-subcomplex $B \subset A^\circ$ we have $H_*(X \setminus B, A \setminus B) \cong H_*(X, A)$ induced by the inclusion;
- G-homotopy invariance;
- The long exact sequence associated to a G-CW-pair;
- Continuity, i.e. $H_*(\amalg_i X_i) \cong \bigoplus_i H_*(X_i)$.

Construction of the homology theory

Let us now use controlled algebra to construct a homology theory. Let Y be a G-CW complex and $\mathcal{A}$ be any additive category. Let $\mathcal{D}^G(Y;\mathcal{A})$ be the following controlled category:

- The underlying space is $G \times Y \times [1,\infty)$;
- every object should be G-cocompactly controlled over $G \times Y$, i.e. we pull the cocompact control condition on $G \times Y$ back along the projection;
- every morphism should be metrically controlled over G with respect to some word metric
- and it should satisfy the equivariant continuous control condition over Y.
- Furthermore we want to take germs at infinity, which means that we take the Karoubi-quotient by the full subcategory $\mathcal{T}^D(Y)$ of objects with cocompact support over the whole of $G \times Y \times [1;\infty)$. Or in one line:

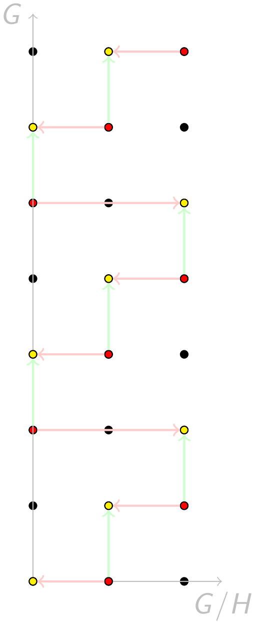
$$\mathcal{C}^G(G\times Y\times[1,\infty), p^{-1}_{Y\times[1,\infty)}(\mathcal{E}^Y_{G-cc})\cap p^{-1}_G(\mathcal{E}^G_{d_G}), p^{-1}_{G\times Y}(\mathcal{F}^{G\times Y}_{G-c}); \mathcal{A})^\infty.$$

Computing the coefficients

Let us first examine, what happens for a homogeneous space G/H.

- Since $G \times G/H$ is discrete, we can consider by cocompactness the (equivalent) subcategory consisting of all objects, whose support lies above the orbit of $(1, H) \in G \times G/H$. This subcategory could also be written as $\mathcal{C}^G(G; \dots)$ where we pull back the control conditions.
- We can simplify the continuous control condition. For a given point (gH, ∞) we can choose U of the form $\{gH\} \times [C, \infty)$. The resulting V is WLOG also of the form $\{gH\} \times [C', \infty)$. Thus for any point gH we can find a bound C'.
- By G-equivariance we can even assume that there is a uniform bound C' such that all arrows $\varphi_{(g,gH,t),(g',g'H,t')}$ that start or end above C' have $gH = g'H$.

Here we see how any object (whose support is red) is isomorphic to an object supported at the orbit (yellow) of $(1, 1H)$. We need to put the modules at different spots, such that the morphism control holds. Think of a third $[1, \infty)$-direction coming out of the board; very far away the red arrows would violate the continuous control. However the yellow arrows do the job.

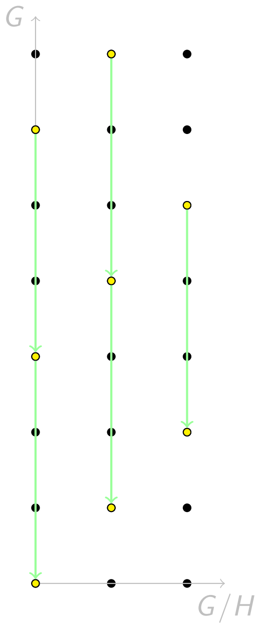


Computing the coefficients

- Since we took germs at infinity, our category is equivalent to the category, where no propagation in the G/H- direction is allowed.
- The K-theory of this category is (as in our example above) just the shifted K-theory of the following category: Its objects are given by collections $(A_g)_{g\in G}$ with $A_1 = A_g$ for all g. Thus its objects are just the objects from $\mathcal{A}$. Its morphisms are G-invariant collections $(\varphi_{g,g'})_{(g,g')\in G^2}$ with $g^{-1}g' \notin H \Rightarrow \varphi_{g,g'} = 0$.
- Now we have simplified the situation as far as possible; if we assume that $\mathcal{A}$ was the category of f.g., free R-modules, then this category is equivalent to the category of f.g. free RH-modules, i.e.

$$K_{*+1}(\mathcal{D}^G(G/H;\mathcal{A})) = K_*^{alg}(RH).$$

After moving the support to one orbit the G-equivariant control condition ensures that nonzero morphisms $\varphi_{(g,gH,t),(g'g,g'gH,t')}$ can only appear for $g' \in H$ for t, t' large.



A relative version

In order to verify the Eilenberg-Steenrod axioms, we should first define a relative version. Recall that the inclusion of $\mathcal{C}(\{0\}; \mathcal{A})$ into $\mathcal{C}(\mathbb{N}; \mathcal{A})$ is not a Karoubi-filtration; we have to take all objects isomorphic to an object from $\mathcal{C}(\{0\}; \mathcal{A})$. Let (Y, Z) be a G-CW-pair. The same happens for $\mathcal{D}^G(Z; \mathcal{A}) \to \mathcal{D}^G(Y; \mathcal{A})$. Let $\mathcal{D}^G(Y, Z; \mathcal{A})$ denote the quotient.

Thus we have a long exact sequence of a pair.

Excision

Let Y be a G -CW- complex and let $A \subset B \subset Y$ be two open subsets with $\overline{A} \subset B$. We want to show that the canonical map $\mathcal{D}^G(Y \setminus A, B \setminus A) \to \mathcal{D}^G(Y, B; \mathcal{A})$ is an equivalence. For an object M in $\mathcal{D}^G(Y, B; \mathcal{A})$ we can delete all modules sitting over A. For a morphism we can set those arrows to 0 which start or end at a point in A.

Exercise

Show that this functor is a equivalence.

The crucial idea is that the continuous control ensures that if we fix a morphism in $\mathcal{D}^G(Y, B)$ and go far enough in the $[1, \infty)$-direction, any arrow starting in A will end in B.

Homotopy Invariance

- We first claim that it suffices to show that both inclusions $i_0 : X \to X \times \{0\} \subset X \times [0,1]$ and i_1 induce isomorphisms $K_*(\mathcal{D}^G(X)) \to K_*(\mathcal{D}^G(X \times [0,1]))$.
- Given a homotopy $H : X \times [0,1] \to Y$ between f, g and let $F, G : X \times [0,1] \to Y$ denote the maps $(x,t) \mapsto f(x)$ resp. $g(x)$.

$$\begin{array}{ccc} X & \xrightarrow{i_0} & X \times [0,1] \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle H} \\ X \times [0,1] & \xrightarrow{F} & Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{i_1} & X \times [0,1] \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle H} \\ X \times [0,1] & \xrightarrow{G} & Y \end{array}$$

- After applying $K_*(\mathcal{D}^G(-))$ the maps i_0 and i_1 turn into isomorphisms and thus F, H, G induce the same maps.
- $f_* = (i_0 \circ F)_* = (i_0 \circ G)_* = g_*$ and thus homotopy invariance follows.

Homotopy invariance

We have to show that $\mathcal{D}^G(X) \to \mathcal{D}^G(X \times [0,1])$ is an isomorphism. We can view the left category as a full subcategory of the right category and after fattening (replacing it by the subcategory of all objects isomorphic to something on the left) we obtain a Karoubi-filtration.
Recall that objects are modules over $G \times X \times [0,1] \times [1,\infty)$. Let me sketch an Eilenberg-swindle on the quotient. Suppose a module sits over (g,x,t,h). In the Eilenberg-swindle S we glue the same module over the sequence of points $k \mapsto (g,x,t-\frac{k}{h},h)$. Of course we stop if $t - k/h < 0$.

Homotopy invariance: Pictures

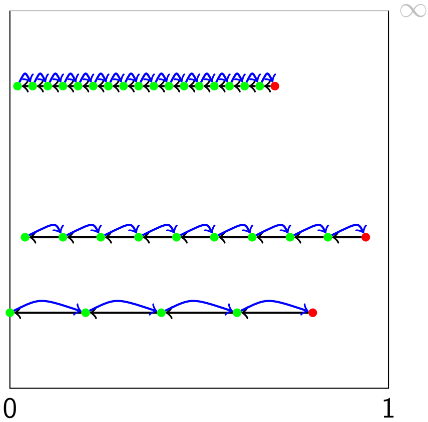


Figure: red: A , green: $S(A)$, both: $A \oplus S(A)$, black: $S(A) \oplus A \to S(A)$, blue: $S(A) \to S(A) \oplus A$

Making the steps arbitrarily small for $h \to \infty$ we obtain equivariant continuous control. One composition is the identity, the other one is identity everywhere except on the leftmost dots. The difference to the identity factors through an object over $X \times \{0\}$ and thus it is also the identity in the Karoubi-quotient. We obtain a well defined natural isomorphism $S \oplus ID \cong S$.

Continuity

It follows immediately from the definition that $\mathcal{D}^G(\amalg_i X_i) \cong \bigoplus_i \mathcal{D}^G(X_i)$ as additive categories. Continuity of the algebraic K-theory then gives the result.

The obstruction category $\mathcal{O}^G(X;\mathcal{A})$ differs from $\mathcal{D}^G(X;\mathcal{A})$ only a bit. Namely we will not take germs at infinity. We will show now, that the K-theory of $\mathcal{O}^G(X)$ vanishes if and only if the map $\mathcal{D}^G(X;\mathcal{A}) \to \mathcal{D}^G(*;\mathcal{A})$ is an isomorphism.

Lemma

$\mathcal{O}^G(*;\mathcal{A})$ *is flasque. Hence its K-theory vanishes.*

Proof of the lemma

The continuous control condition is an empty condition for $X = pt$. For this reason the naive Eilenberg swindle given by

$$((g, pt, t), k) \mapsto (g, pt, t + k)$$

works.

Proof of the equivalence

Let $\mathcal{T}^G(X;\mathcal{A})$ be the full subcategory of $\mathcal{O}^G(X;\mathcal{A})$ whose objects have compact support over $[1,\infty)$. Thus we have (by definition) a Karoubi-filtration $\mathcal{T}^G(X;\mathcal{A}) \subset \mathcal{O}^G(X;\mathcal{A})$ with quotient $\mathcal{D}^G(X;\mathcal{A})$.

Exercise

Show that $\mathcal{T}^G(X;\mathcal{A}) \to \mathcal{T}^G(*;\mathcal{A})$ is an equivalence of categories.

Thus we obtain the following natural ladder:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & K_n(\mathcal{O}^G(X)) & \longrightarrow & K_n(\mathcal{D}^G(X)) & \longrightarrow & K_{n-1}(\mathcal{T}^G(X)) & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \cong & & \downarrow \\
\dots & \longrightarrow & K_n(\mathcal{O}^G(*)) & \longrightarrow & K_n(\mathcal{D}^G(*)) & \longrightarrow & K_{n-1}(\mathcal{T}^G(*)) & \longrightarrow & \dots
\end{array}$$

Thus the five lemma implies that the middle map is an isomorphism everywhere if and only if $K_*(\mathcal{O}^G(X);\mathcal{A})$ vanishes.

Recapitulation

So we have now found an additive category whose K-theory vanishes if and only if the Farrell-Jones assembly map is an isomorphism. If you could show that the K-theory of $\mathcal{O}^G(E_{VCyc}G)$ vanishes in general (say just using Karoubi-filtrations and long exact ladders), then you would have shown the Farrell-Jones conjecture in general. Still at some point one has to use the universal property of a classifying space. Life is complicated.

For a metric space (Z, d) with free (left) G-action define

$$\mathcal{O}^G(Y, Z, d; \mathcal{A}) := \mathcal{C}^G(Z \times Y \times [1, \infty), \mathcal{E}^Y_{G-cc} \cap \mathcal{E}^Z_d, \mathcal{F}^{Z \times Y}_{G-c}; \mathcal{A}).$$

For a sequence (Z_n, d_n) of metric spaces we can consider the product category

$$\prod_n \mathcal{O}^G(X, Z_n, d_n; \mathcal{A})$$

For a morphism φ we can consider its n-th coordinate. There is a bound α_n such that if φ_n has a nonzero arrow going from (y, z, t) to (y', z', t') then $d(z, z') < \alpha_n$. Let

$$\mathcal{O}^G(X, (Z_n, d_n)_{n \in \mathbb{N}}; \mathcal{A})$$

denote the lluf subcategory consisting of those morphisms that admit a uniform bound α.

Functoriality in Z

A sequence of maps $f_n : Z_n \to Z'_n$ induces a functor

$$\mathcal{O}^G(X, (Z_n, d_n)_{n \in \mathbb{N}}; \mathcal{A}) \to \mathcal{O}^G(X, (Z'_n, d'_n)_{n \in \mathbb{N}}; \mathcal{A}),$$

if for every α there is a $\beta(\alpha)$ such that
$d_n(z_n, z'_n) < \alpha \Rightarrow d'_n(f_n(z_n), f_n(z'_n)) < \beta(\alpha)$.

The diagram

$$
\begin{array}{ccc}
 & & \bigoplus_n \mathcal{O}^G(E, G \times |\mathcal{U}(n)|, d_1) \\
 & & \Big\downarrow {\scriptstyle (3)} \\
\mathcal{O}^G(E, (G \times \overline{X}, d_{C(n)})_{n \in \mathbb{N}}) & \xrightarrow{(2)} & \mathcal{O}^G(E, (G \times |\mathcal{U}(n)|, d_n^1)_{n \in \mathbb{N}}) \\
\Big\downarrow {\scriptstyle inc} & & \Big\downarrow {\scriptstyle inc} \\
\prod_{\in \mathbb{N}} \mathcal{O}^G(E, G \times \overline{X}, d_{C(n)})) & \xrightarrow{\prod_n F_{\mathcal{U}(n)}} & \prod_{\in \mathbb{N}} \mathcal{O}^G(E, G \times |\mathcal{U}(n)|, d_n^1) \\
\Big\downarrow {\scriptstyle pr_k} & & \Big\downarrow {\scriptstyle pr_k} \\
\mathcal{O}^G(E) & \xrightarrow{id} & \mathcal{O}^G(E)
\end{array}
$$

(1): $\mathcal{O}^G(E) \to \mathcal{O}^G(E, (G \times \overline{X}, d_{C(n)})_{n \in \mathbb{N}})$

The diagram:Step 1

$$
\begin{array}{ccc}
 & & \bigoplus_n \mathcal{O}^G(E, G \times |\mathcal{U}(n)|, d_1) \\
 & & \big\downarrow (3) \\
\mathcal{O}^G(E, (G \times \overline{X}, d_{C(n))})_{n\in\mathbb{N}}) & \xrightarrow{(2)} & \mathcal{O}^G(E, (G \times |\mathcal{U}(n)|, d^1_n)_{n\in\mathbb{N}}) \\
\big\downarrow inc & & \big\downarrow inc \\
(1) \quad \prod_{\in\mathbb{N}} \mathcal{O}^G(E, G \times \overline{X}, d_{C(n))}) & \xrightarrow{\prod_n F_{\mathcal{U}(n)}} & \prod_{\in\mathbb{N}} \mathcal{O}^G(E, G \times |\mathcal{U}(n)|, d^1_n) \\
\big\downarrow pr_k & & \big\downarrow pr_k \\
\mathcal{O}^G(E) & \xrightarrow{id} & \mathcal{O}^G(E)
\end{array}
$$

The map (1) exists after applying K-theory and it is a section of $pr_k \circ inc$ for all k. Morally this means that (1) is a kind of a diagonal embedding.

$$\begin{array}{cccc}
 & & & \bigoplus_n \mathcal{O}^G(E, G \times |\mathcal{U}(n)|, d_1) \\
 & & & \big\downarrow (3) \\
 & \mathcal{O}^G(E, (G \times \overline{X}, d_{C(n))})_{n\in\mathbb{N}}) & \xrightarrow{(2)} & \mathcal{O}^G(E, (G \times |\mathcal{U}(n)|, d_n^1)_{n\in\mathbb{N}}) \\
 & \big\downarrow inc & & \big\downarrow inc \\
(1) & \prod_{\in\mathbb{N}} \mathcal{O}^G(E, G \times \overline{X}, d_{C(n))}) & \xrightarrow{\prod_n F_{\mathcal{U}(n)}} & \prod_{\in\mathbb{N}} \mathcal{O}^G(E, G \times |\mathcal{U}(n)|, d_n^1) \\
 & \big\downarrow pr_k & & \big\downarrow pr_k \\
 & \mathcal{O}^G(E) & \xrightarrow{id} & \mathcal{O}^G(E)
\end{array}$$

The inclusion (3) is an equivalence on K-theory for a sequence $|\mathcal{U}(n)|$ of simplicial complexes of dimension at most N (independent of n) and metrics on $G \times |\mathcal{U}(n)|$ with certain properties. This is the crucial step!

$$\begin{array}{ccc}
 & & \bigoplus_n \mathcal{O}^G(E, G \times |\mathcal{U}(n)|, d_1) \\
 & & \big\downarrow (3) \\
\mathcal{O}^G(E, (G \times \overline{X}, d_{C(n))})_{n\in\mathbb{N}}) & \xrightarrow{(2)} & \mathcal{O}^G(E, (G \times |\mathcal{U}(n)|, d_n^1)_{n\in\mathbb{N}}) \\
\big\downarrow inc & & \big\downarrow inc \\
\prod_{\in\mathbb{N}} \mathcal{O}^G(E, G \times \overline{X}, d_{C(n))}) & \xrightarrow{\prod_n F_{\mathcal{U}(n)}} & \prod_{\in\mathbb{N}} \mathcal{O}^G(E, G \times |\mathcal{U}(n)|, d_n^1) \\
\big\downarrow pr_k & & \big\downarrow pr_k \\
\mathcal{O}^G(E) & \xrightarrow{id} & \mathcal{O}^G(E)
\end{array}$$

$$(1): \mathcal{O}^G(E) \to \mathcal{O}^G(E, (G \times \overline{X}, d_{C(n))})_{n\in\mathbb{N}})$$

Now we need maps $F_{\mathcal{U}(n)}$ to simplicial complexes. Given a locally finite open cover $\mathcal{U}$ of a metric space Z, we obtain a continuous (with l^1-topology) map

$F_{\mathcal{U}} : Z \to |\mathcal{U}(n)| \qquad z \mapsto [[\sum_{U\in\mathcal{U}, z\in U} d(z, U^c)U]].$

The conditions above on the functoriality in the sequence of open covers give rise to certain conditions on those covers.

Details on (3)

First: again more notation. For a quasi-metric space (∞ is allowed) (X, d) define a metric $\tilde{d}$ on $G \times X$ via $\tilde{d}(g, x, g', x') = d_G(g, g') + d(x, x')$. Let $(X_n, d_n)_n$ be a sequence of N-dimensional metric simplicial complexes. Now abbreviate:

$$\mathcal{L}_{\oplus}((X_n, d_n)_{n \in \mathbb{N}}) := \bigoplus_{n \in \mathbb{N}} \mathcal{O}^G(E, G \times X_n, \tilde{d}_n);$$

$$\mathcal{L}((X_n, d_n)_{n \in \mathbb{N}}) := \mathcal{O}^G(E, (G \times X_n, \tilde{d}_n)_{n \in \mathbb{N}}).$$

Let $\mathcal{L}((X_n, d_n)_{n \in \mathbb{N}})^{>\oplus}$ denote their Karoubi-quotient (Exercise: The inclusion is a Karoubi-filtration).
Let (Y_n, d_n^{∞}) be the disjoint union of the N-simplices where the distance is the n-times the l^1-distance, if the two points lie in the same simplex and infinite otherwise.

Theorem ([BLR08, Theorem 7.2])

Given $N \in \mathbb{N}$ and a sequence $(X_n, d_n)_{n\in\mathbb{N}}$ of simplicial complexes of dimension at most N equipped with a simplicial, cell preserving, isometric G-action with isotropy in $\mathcal{F}$ such that $d_n(x,y) \geq n \cdot d^1(x,y)$ with equality whenever $x, y \in |X_n|$ lie in a common (closed) simplex. Then the inclusion

$$\mathcal{L}_{\oplus}((X_n, d_n)_{n\in\mathbb{N}}) \hookrightarrow \mathcal{L}((X_n, d_n)_{n\in\mathbb{N}})$$

induces an equivalence on the level of K-theory; equivalently, the K-theory of $\mathcal{L}((X_n, d_n)_{n\in\mathbb{N}})^{>\oplus}$ vanishes.

Proof

by Induction on N. The attaching of N-simplices gives a diagram

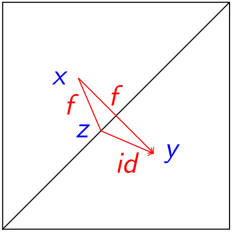
$$\begin{array}{ccc}
\mathcal{L}((\partial Y_n, d_n^\infty)_{n\in\mathbb{N}})^{>\oplus} & \longrightarrow & \mathcal{L}((Y_n, d_n^\infty)_{n\in\mathbb{N}})^{>\oplus} \\
\downarrow & & \downarrow \\
\mathcal{L}((X_n^{N-1}, d_n)_{n\in\mathbb{N}})^{>\oplus} & \longrightarrow & \mathcal{L}((X_n, d_n)_{n\in\mathbb{N}})^{>\oplus},
\end{array}$$

After fattening both categories on the left (as before) both rows turn into Karoubi-filtrations. The induced map on the quotient will be an equivalence ($\blacksquare_1$). Furthermore the K-theory of $\mathcal{L}((Y_n, d_n^\infty)_{n\in\mathbb{N}})^{>\oplus}$ vanishes ($\blacksquare_2$). By induction assumption the K-theory of both left entries vanish. Applying the 5-Lemma to the long exact ladder gives the result.

Given an object either of the two quotient categories, we can delete all modules sitting over ∂Y_n (resp. X_n^{N-1}) without changing the isomorphism type.
The functor is injective on morphism sets since the metric in the top row is larger and thus there are stronger conditions on the morphisms.
To see the surjectivity let us pick a morphism in $\mathcal{L}((X_n, d_n)_{n \in \mathbb{N}})^{>\oplus}$. Wlog we may assume that the objects have no modules sitting over ∂Y_n (resp. X_n^{N-1}). We have to show that it is the sum of a morphism from the top and a morphism from the left. The morphisms from the top are exactly those which only connect modules sitting over the same N-simplex. Subtracting those gives a morphism which connects only different simplices. We have to show that this one factors though an object from the left.

Exercise

For any two points x, y in different N-simplices of a simplicial complex there is a point z in an $n-1$-simplex with $d^1(x,z) \leq 2d^1(x,y)$ (and thus $d^1(y,z) \leq 3d^1(x,y)$).



The idea to factorize a morphism is shown in the picture above; glue the modules over y at $z = z(x,y)$. The estimations from the lemma give the control conditions.

Now we have to construct an Eilenberg swindle on $\mathcal{L}((Y_n, d_n^\infty)_{n\in\mathbb{N}})^{>\oplus}$. We will construct in each degree $\mathcal{O}^G(E, Y_n, \tilde{d}_n^{\infty})$ separately. It will not change the Y_n-coordinate at all. Thus those swindles combine to an Eilenberg-swindle on $\mathcal{L}((Y_n, d_n^\infty)_{n\in\mathbb{N}})$.

Recall that objects in this category are modules over $Y_n \times E \times [1, \infty)$.

The naive idea just to shift in the $[1, \infty)$ direction (without changing the other ones) fails - the equivariant control condition is violated. We are already in the situation where arrows cannot connect different simplices. Thus we have to make simplices smaller.

The swindle

Let R_n be the set of N-simplices of Y_n, i.e. $Y_n = \bigcup_{i \in R_n} \Delta^N$ and let $p : Y_n \to R_n$ be the projection. Fix a G-map $i : R_n \to E := E_{\mathcal{F}} G$. By the universal property of $E := E_{\mathcal{F}} G$ there is a homotopy

$$H : [0,1] \times Y_n \times E \to E$$

such that

$$H_0(y,e) = e; \qquad H_1(y,e) = i(p(y))$$

The swindle

Now define for every point (g, y, e, t) in $G \times Y_n \times E \times [1, \infty)$ a sequence of points

$$(y, H_{\frac{k}{k+t}}(y, e), t + k)_{k \in \mathbb{N}}$$

and define an Eilenberg swindle along those; i.e. if we consider an object A and some module M sits over a point (y, e, t) we define put in $S(A)$ that module over all those points in that sequence. The natural isomorphism $S \oplus Id \cong S$ is given by shifting along those lines.

The only problem is to show that the continuous control conditions are satisfied - to show that S is a functor and that the shift defines a natural transformation $S \oplus Id \to S$. Let us pick a sequence of arrows such that the $[1, \infty)$-coordinate of the source (which is of the form $t_i + k_i$) converges to ∞, and whose E-coordinate of the source converges, then we get:

- By cocompact control, we can pass to a subsequence where the Y_n coordinates lie in one orbit of a simplex $r_n \in R_n$;
- If t_i is bounded, then the E-coordinate of the endpoints converges to $i(r_n)$. Thus continuous control holds.
- Otherwise use that the morphism that we started with is continuously controlled and that H is continuous.

This completes the proof of $\blacksquare_2$ and thus the proof that (3) is an equivalence in K-theory.

Excursion:Geometry

Definition

A geodesic metric space is *CAT(-1)*, if triangles are thinner than in $\mathbb{H}^2$.

Definition

A group is called *CAT(-1)*, if it acts geometrically (proper, isometrically and cocompact) on a CAT(-1)-space.

Example

The free group is CAT(-1) as it acts on a tree; Surface groups are CAT(-1) as the universal cover of a surface with the hyperbolic metric is $\mathbb{H}^2$.

Definition

The boundary of a CAT(-1) space X (works also for CAT(0)) is the set of equivalence classes of geodesic rays under the equivalence relation given by having bounded distance. It may as well be thought of as $\lim_{\leftarrow} S(*, r)$ the inverse limit of spheres of radius r around a basepoint x.

Definition

The *compactification* of X is the inverse limit of the system of balls around some basepoint (x).

The compactification inherits an $Isom(X)$-action although the point x is not a fixed point. Furthermore since it is an inverse limit of metric spaces, it is metrizable. And of course it is compact.

G-covers

Let X be a G-space and $\mathcal{F}$ be a family of subgroups of G. A G-cover of X is a collection $\mathcal{U}$ of open sets such that:

- For $g \in G, u \in \mathcal{U}$ we have $gU \in \mathcal{U}$;
- If $gU \cap U \neq \emptyset$, then $gU = U$;
- For every U we have that the subgroup $\{g \in G \mid gU = U\}$ lies in the family $\mathcal{F}$;

Constructing Covers

Now let G be a group acting geometrically on a CAT(-1)-space X (for example $G = F_2$). Let $\overline{X}$ be its compactification. For (2) to hold, we have to construct the following:

- A natural number N such that we can find for every $\beta > 0$:
- An N-dimensional G- cover $\mathcal{U}$ (left action on G, trivial action on $\overline{X}$) of $G \times \overline{X}$ such that we can find for every (g, x) an open set $U \in \mathcal{U}$ with

$$\{(gh^{-1}, hx) \mid |h| \leq \beta\} \subset U.$$

- The normalizers of the open sets in the cover lie in $\mathcal{F}$ (Here $\mathcal{F} = \mathcal{VC}$yc).

Big isotropy causes problems

- Let $x \in \overline{X}$ be a point and let H be its isotropy subgroup.
- Pick β so large a generating system S of H lies in $B_\beta(1)$ and which is symmetric under inversion.
- Thus we find an open set $U \in \mathcal{U}$ such that $\{(h^{-1}, hx) \mid |h| \leq \beta\} \subset U$.
- Recall that $\mathcal{U}$ was a G-cover under the left action and thus $s(s^{-1}, x) = (1, x) \in sU \cap U$.
- A G-cover satisfies by definition $gU \cap U \neq \emptyset \Rightarrow gU = U$.
- Thus we see that the normalizer of U is the whole of H. Even if we drop that condition from above we get that $\{hU \mid h \in B_\beta(H)\}$ has cardinality at most N, i. e. one set U is fixed by a finite index subgroup of H.
- Luckily for a $CAT(-1)$ group all point stabilizers in $\overline{X}$ are virtually cyclic and thus there is some hope that this approach works.
- For CAT(0)-groups there can be fixed points in $\overline{X}$ and a lot of the diagram has to be reworked.

The flow space

Definition

For a CAT(-1) space X define

$$FS(X) := \{f : \mathbb{R} \to X \mid \exists -\infty \leq a \leq b \leq \infty :$$
$$f_{[a,b]} \text{ is a unit speed geodesic, } f_{(-\infty,a]}, f_{[b,\infty)} \text{ are constant}\}.$$

The function $d(f, f') := \int_{\mathbb{R}} \frac{d_X(f(t), f'(t))}{2e^{|t|}}$ defines a metric on it which generates the compact open topology. A (geometric) G-action on X induces a (geometric) G-action on $FS(X)$. Furthermore precomposition with translations gives a $\mathbb{R}$-action Φ on $FS(X)$ commuting with the G action.
The evaluation map $f \mapsto f(0)$ is proper with compact, contractible fibers; Heuristically the flow space wants to mimic the unit ball bundle in the tangent bundle of a Riemannian manifold.

$\blacksquare_3$: long covers of the flow space

Theorem (Bartels-Lück-Reich)

Let G act geometrically on a $CAT(-1)$-space X. There is a natural number N such that for any $R > 0$ there is an $\varepsilon > 0$ and a $\mathcal{VC}yc$ cover of $FS(X)$ of dimension at most N such that for every $f \in FS(X)$ there is an open set containing $B_\varepsilon(\Phi_{[-R,R]} f)$.

The goal is now to define a map $G \times \overline{X} \to FS(X)$ such that every set of the form $\{(gh^{-1}, hx) \mid |h| \leq \beta\}$ gets mapped into one of the sausages $B_\varepsilon(\Phi_{[-R,R]} f)$.

The map to the flow space

Choose a basepoint $x_0 \in X$ and define a map

$$i_{x_0} : G \times \overline{X} \to FS(X), \qquad i_{x_0}(g,x)(t) := \begin{cases} gx_0 & t \leq 0 \\ gx & t \geq d(x,y) \\ c_{gx_0,gx}(t) & \text{else} \end{cases},$$

where $c_{gx_0,gx} : [0, d(x,y)) \to X$ denotes the unique geodesic connecting gx_0 to gx. This map is G-equivariant, i. e. $i_{x_0}(g'g,x)(t) = g' i_0(g,x)$. If we insert a point of the form $(gh, h^{-1}x)$ we get the geodesic connecting ghx_0 to gx. Let

$$R := \max\{d(gx_0, ghx_0) \mid h \in B_\beta(1)\} = \max\{d(x_0, hx_0) \mid h \in B_\beta(1)\}.$$

The key idea is now to postcompose with flowing by a large enough number T towards x.

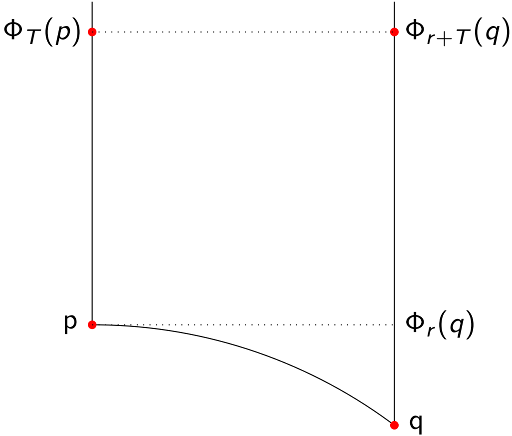
Flowing makes things better

Let us just consider the harder case where $x \in \partial X$. There is for every $\varepsilon > 0$ a $T \in \mathbb{R}$ such that for any two points p, q of distance at most R (here gx_0 and ghx_0) we have that the two generalized geodesics f, f' starting at time 0 at p and q and going towards x satisfy

$$d_{FS}(\Phi_T f, \Phi_{T+r}(f')) < \varepsilon,$$

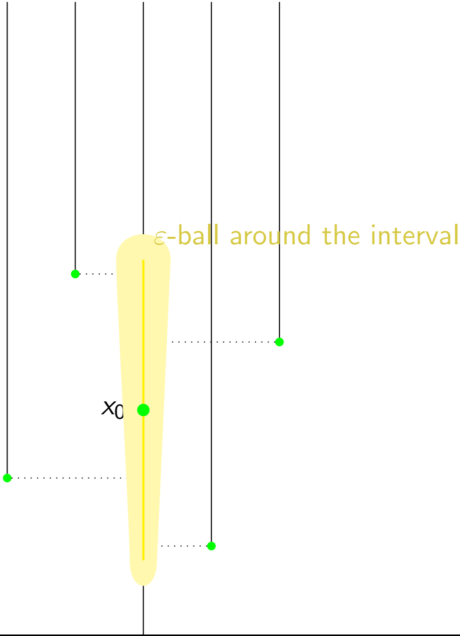
where $r := B_x(p) - B_x(q)$ is the difference of the Busemann function at x and y. We would first have to relate the metric on the flow space to the metric on X, then relate the metric on X to the metric on $\mathbb{H}^2$ using comparison triangles and then finally use geometry in $\mathbb{H}^2$ to get this result. Or we can just draw a picture...

Picture of the flow



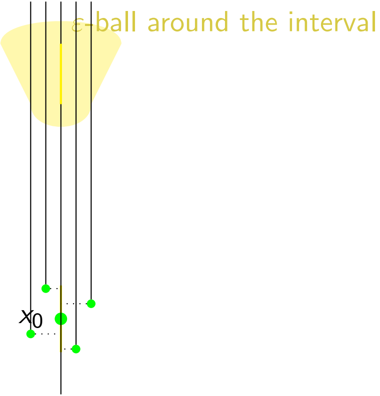
A picture in the Poincaré - half plane of a generic ideal triangle. Recall that in this model points of the same y coordinate (i.e. of with the value of the Busemann function of the point at infinity of the triangle) get arbitrarily close together if we move them upwards.

Those sets around $(1, x)$ with $x \in \partial X$



The green dots are the set $B = \{hx_0 \mid h \in B_\beta(1)\}$, then $i_{x_0}(\{(h^{-1}, hx) \mid h \in B_\beta(1)\})$ consists of those geodesics that start at some point in B at time 0 and go towards the point at infinity x. Two elements f, f' in the flow space are close, if $\sup_{t\in[-R,R] d_X(f(t),f'(t))}$ is small. Basically $d(f(0), f'(0))$ has the highest weight

Those sets around $(1, x)$ with $x \in \partial X$



If you let the interval flow long enough before taking the ε-neighborhood, the resulting set will be much bigger in the horizontal direction. This comes from the Poincare-halfplane model.

Thank you for your attention.

The whole talk is based on the following papers:

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