A rough outline of the proof of the K-theoretic Farrell-Jones conjecture for CAT(-1) groups

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July 17th, 2014

An axiomatic view of algebraic K-theory

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4. The natural inclusions induce isomorphisms $\bigoplus_i K_*(\mathcal{A}_i) \cong K_*(\bigoplus_i \mathcal{A}_i)$. This is also called continuity of algebraic K-theory.

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3. every map $g : U \to A$ to an object $A \in \mathcal{A}$ factors through one of the compositions;
4. For $U, V \in \mathcal{U}$ the posets $I_{U \oplus V}$ and $I_U \times I_V$ are equivalent (i.e. cofinal in each other as subposets of all decompositions).

Now we can define a quotient category $\mathcal{U}/\mathcal{A}$ whose objects are the same as the objects from $\mathcal{U}$, but whose morphism sets $\mathrm{Hom}_{\mathcal{U}/\mathcal{A}}(U, U')$ is the quotient of $\mathrm{Hom}_{\mathcal{U}}(U, U')$ by the subgroup of all morphisms that factor through some $A \in \mathcal{A}$,

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$$U \xrightarrow{f_i} E_i \oplus U_i \xrightarrow{pr} E_i \xrightarrow{f'} E'_j \hookrightarrow E'_j \oplus U'_j \xrightarrow{f_j} U'$$

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Exercise

Show that equivalent choices of decompositions give the same quotient category.

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If all $\mathcal{A}_i$ are equal, then we could also allow morphisms that mix the degree. This will be a very important example.

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- let for a subset S of $\mathbb{N}$ and an object $U = (B_i)_{i\in\mathbb{N}}$ the restriction $U|_S$ be the object with $(U|_S)_i = U_i$ for $i \in S$ and $(U|_S)_i = 0$ otherwise.

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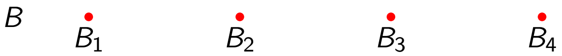
Remark

Thus $K_n\mathcal{B} = K_{n+1}(\mathcal{U}/\mathcal{B})$ and hence we could use these constructions to define negative K-theory in terms of K_0 of other categories.

Pictures of the swindle



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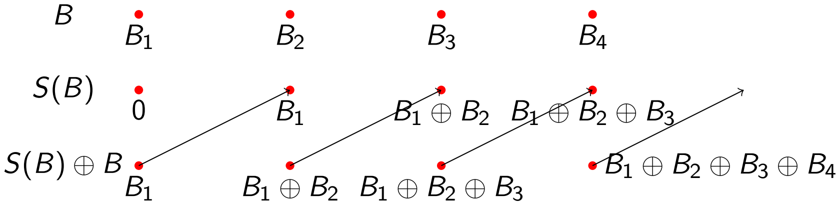

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The full subcategories of objects with compact support are the same. Thus the left arrow is an isomorphism. The middle arrow is an isomorphism since both categories are flasque. Thus the third arrow is also an isomorphism. This is not obvious! The underlying functor is not a equivalence of additive categories.

Controlled algebra

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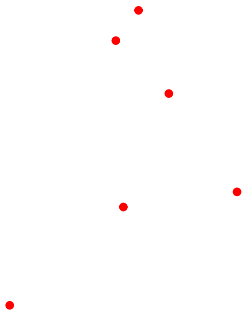
- A morphism $\varphi : (A_x)_{x \in X} \to (B_y)_{y \in Y}$ is a collection of morphisms $\varphi_{x,y} : A_x \to B_y$ such that its *support* $\{(x, y) \mid \varphi_{x,y} \neq 0\}$ is row and column finite.

Generic picture of a morphism



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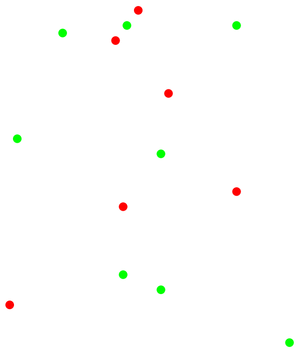

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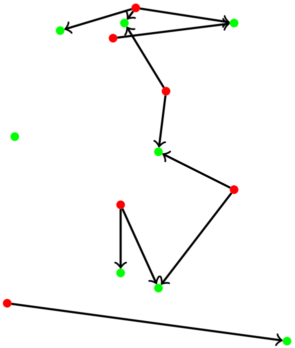

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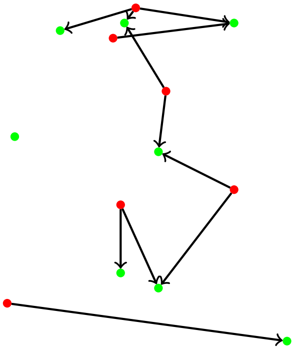

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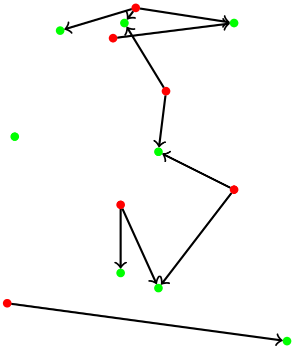

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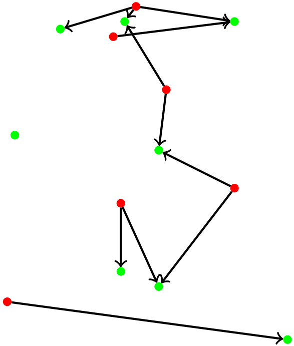

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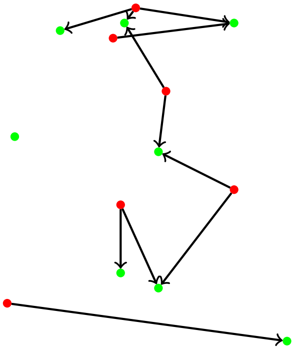

Figure: Sketch of a morphism

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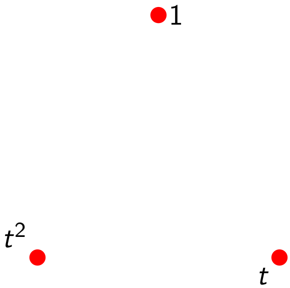
Let $\mathcal{C}(X, \mathcal{E}, \mathcal{F}; \mathcal{A})^G$ denote the fixed point category, i.e. the subcategory with $A_x = g^* A_{gx}$ for all $x \in X$ and $(g^* \varphi)_{x,y} := (g^* \varphi_{gx,gy})$.

Free RG-Modules



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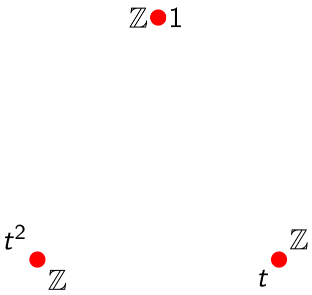

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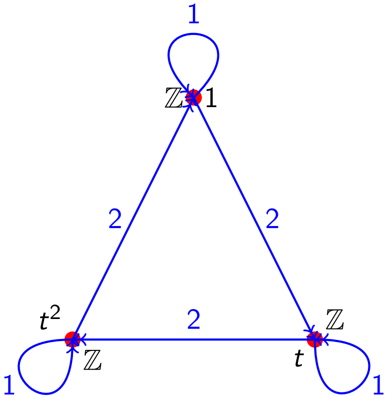

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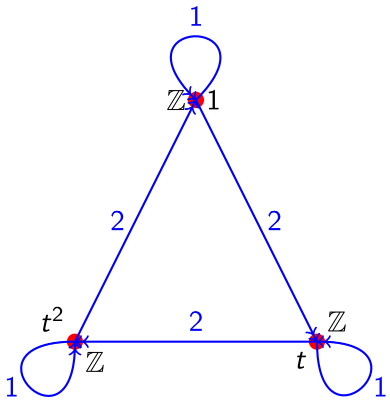

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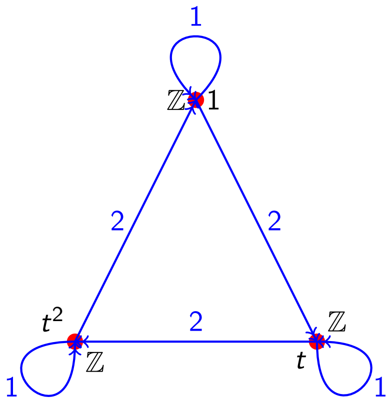

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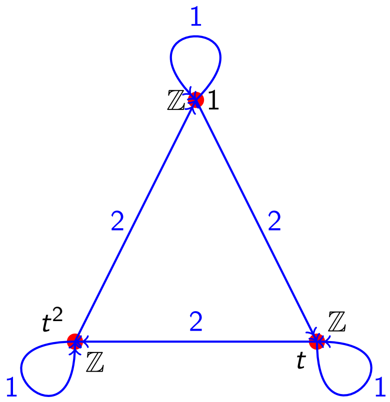

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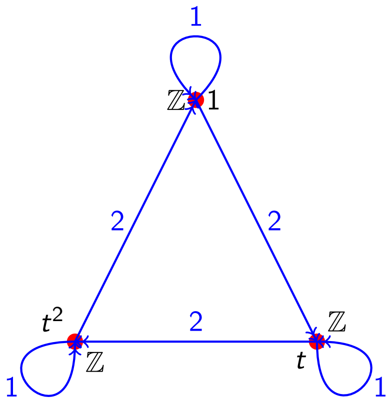

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$$(U^c \times V) \cap E = \emptyset \wedge (V^c \times U) \cap E = \emptyset.$$

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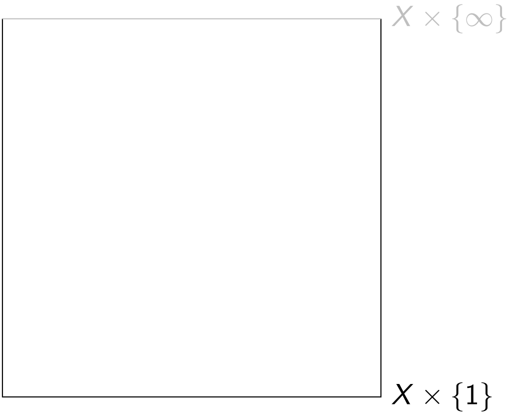

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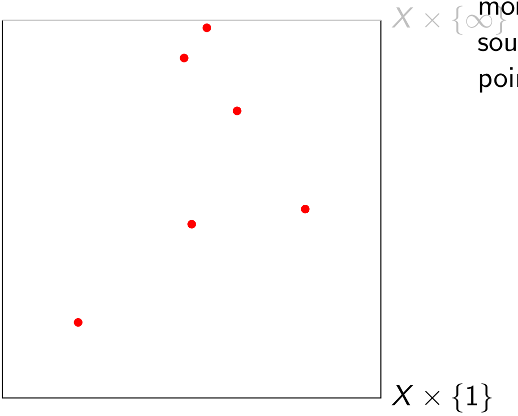

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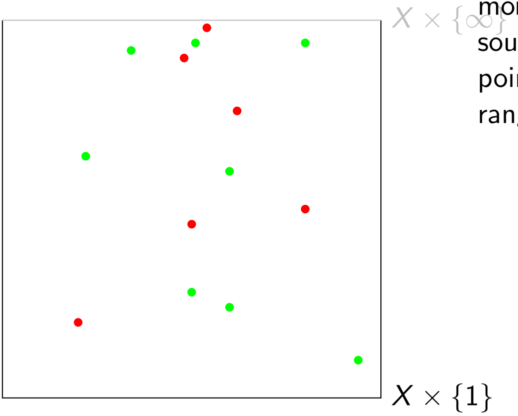

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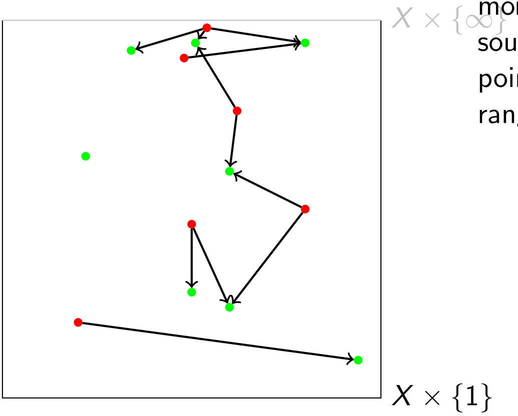

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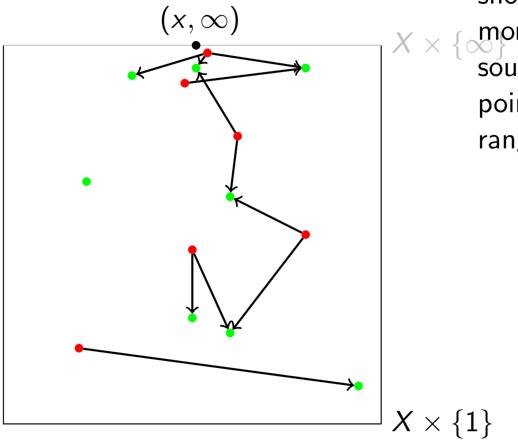

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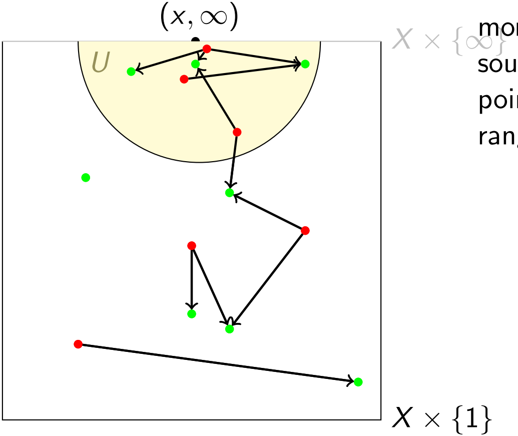

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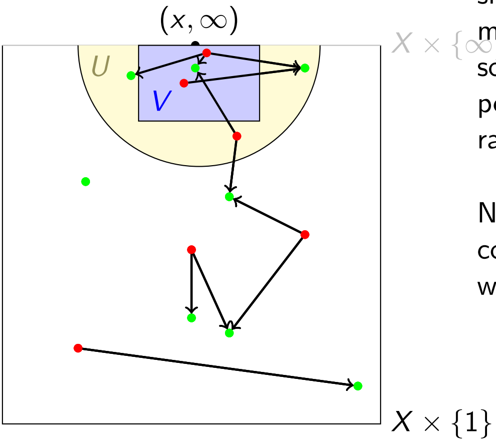

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Let us now use controlled algebra to construct a homology theory. Let Y be a G-CW complex and $\mathcal{A}$ be any additive category. Let $\mathcal{D}^G(Y;\mathcal{A})$ be the following controlled category:

- The underlying space is $G \times Y \times [1,\infty)$;
- every object should be G-cocompactly controlled over $G \times Y$, i.e. we pull the cocompact control condition on $G \times Y$ back along the projection;
- every morphism should be metrically controlled over G with respect to some word metric
- and it should satisfy the equivariant continuous control condition over Y.
- Furthermore we want to take germs at infinity, which means that we take the Karoubi-quotient by the full subcategory $\mathcal{T}^D(Y)$ of objects with cocompact support over the whole of $G \times Y \times [1;\infty)$. Or in one line:

$$\mathcal{C}^G(G\times Y\times[1,\infty), p^{-1}_{Y\times[1,\infty)}(\mathcal{E}^Y_{G-cc})\cap p^{-1}_G(\mathcal{E}^G_{d_G}), p^{-1}_{G\times Y}(\mathcal{F}^{G\times Y}_{G-c}); \mathcal{A})^\infty.$$

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- We can simplify the continuous control condition. For a given point (gH, ∞) we can choose U of the form $\{gH\} \times [C, \infty)$. The resulting V is WLOG also of the form $\{gH\} \times [C', \infty)$. Thus for any point gH we can find a bound C'.
- By G-equivariance we can even assume that there is a uniform bound C' such that all arrows $\varphi_{(g,gH,t),(g',g'H,t')}$ that start or end above C' have $gH = g'H$.

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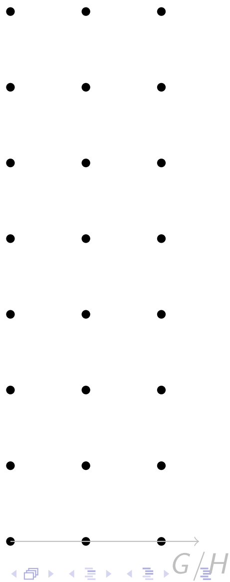

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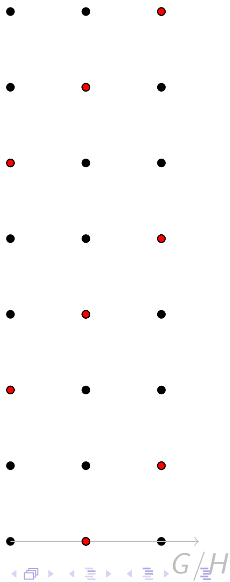

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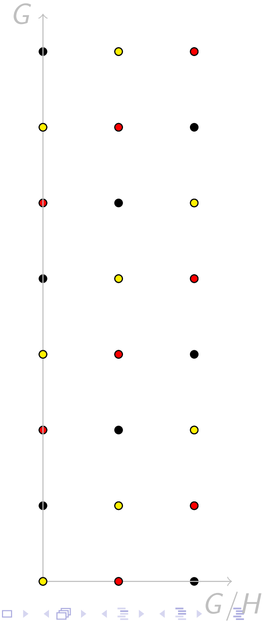

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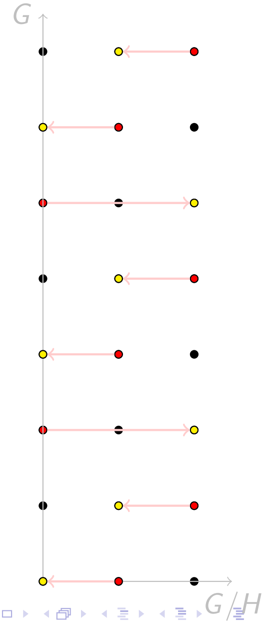

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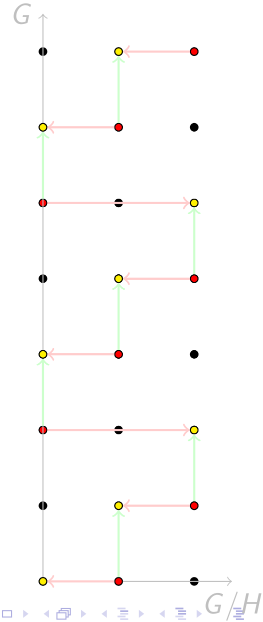

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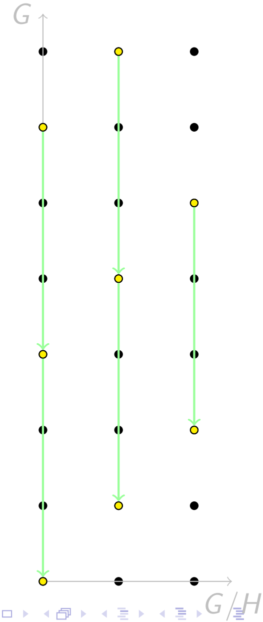
- Now we have simplified the situation as far as possible; if we assume that $\mathcal{A}$ was the category of f.g., free R-modules, then this category is equivalent to the category of f.g. free RH-modules, i.e.

$$K_{*+1}(\mathcal{D}^G(G/H;\mathcal{A})) = K_*^{alg}(RH).$$

After moving the support to one orbit the G-equivariant control condition ensures that nonzero morphisms $\varphi_{(g,gH,t),(g'g,g'gH,t')}$ can only appear for $g' \in H$ for t, t' large.



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Thus we have a long exact sequence of a pair.

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The crucial idea is that the continuous control ensures that if we fix a morphism in $\mathcal{D}^G(Y, B)$ and go far enough in the $[1, \infty)$-direction, any arrow starting in A will end in B.

Homotopy Invariance

- We first claim that it suffices to show that both inclusions $i_0 : X \to X \times \{0\} \subset X \times [0,1]$ and i_1 induce isomorphisms $K_*(\mathcal{D}^G(X)) \to K_*(\mathcal{D}^G(X \times [0,1]))$.

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- $f_* = (i_0 \circ F)_* = (i_0 \circ G)_* = g_*$ and thus homotopy invariance follows.

We have to show that $\mathcal{D}^G(X) \to \mathcal{D}^G(X \times [0,1])$ is an isomorphism. We can view the left category as a full subcategory of the right category and after fattening (replacing it by the the subcategory of all objects isomorphic to something on the left) we obtain a Karoubi-filtration.

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Homotopy invariance: Pictures

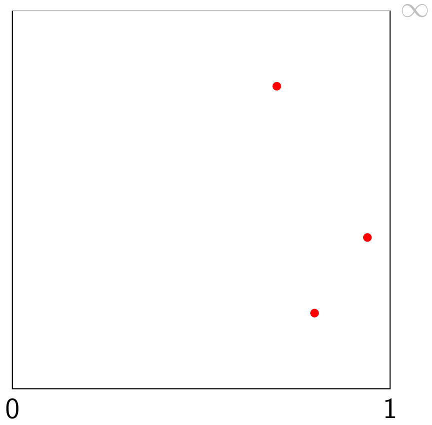

Figure: red: A , green: $S(A)$, both: $A \oplus S(A)$, black: $S(A) \oplus A \to S(A)$, blue: $S(A) \to S(A) \oplus A$

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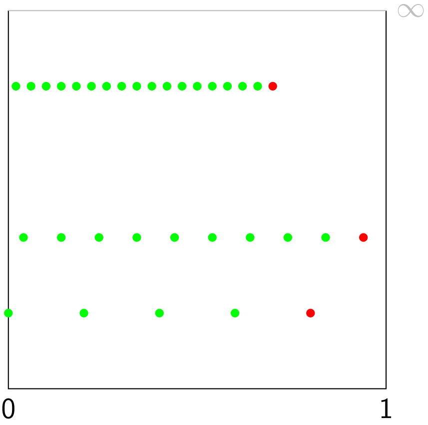

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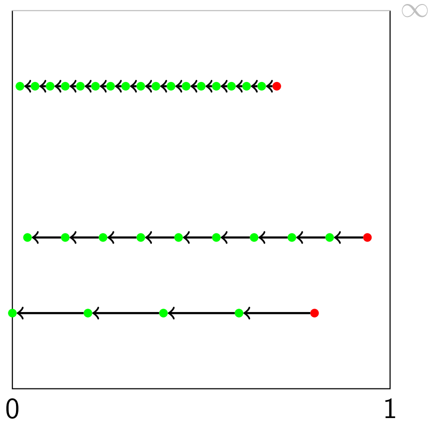

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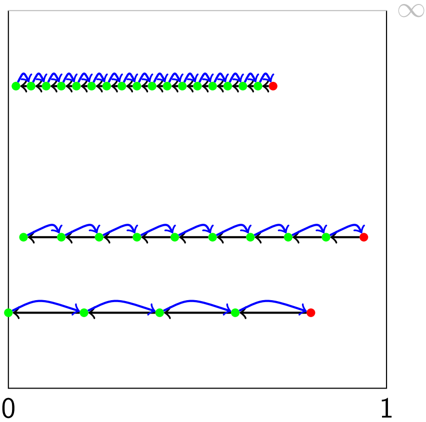

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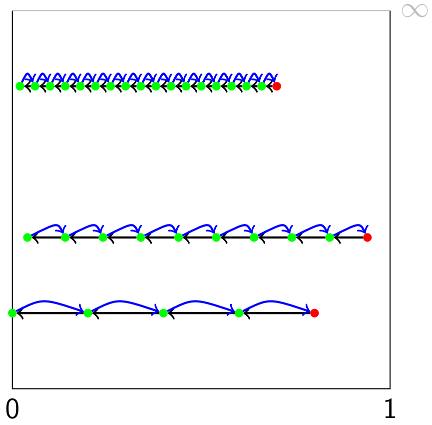

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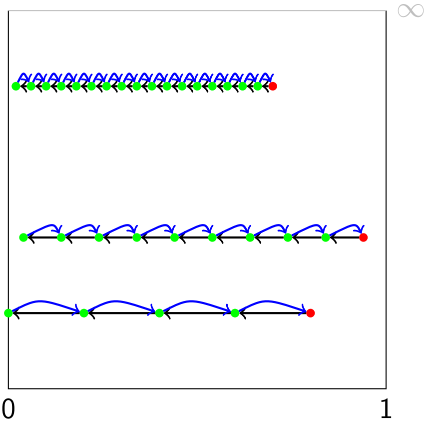

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Lemma

$\mathcal{O}^G(*; \mathcal{A})$ *is flasque. Hence its K-theory vanishes.*

Proof of the lemma

The continuous control condition is an empty condition for $X = pt$. For this reason the naive Eilenberg swindle given by

$$((g, pt, t), k) \mapsto (g, pt, t + k)$$

works.

Proof of the equivalence

Let $\mathcal{T}^G(X; \mathcal{A})$ be the full subcategory of $\mathcal{O}^G(X; \mathcal{A})$ whose objects have compact support over $[1, \infty)$.

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Thus the five lemma implies that the middle map is an isomorphism everywhere if and only if $K_*(\mathcal{O}^G(X);\mathcal{A})$ vanishes.

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For a sequence (Z_n, d_n) of metric spaces we can consider the product category

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$$\mathcal{O}^G(X, (Z_n, d_n)_{n \in \mathbb{N}}; \mathcal{A})$$

denote the lluf subcategory consisting of those morphisms that admit a uniform bound α.

Functoriality in Z

A sequence of maps $f_n : Z_n \to Z'_n$ induces a functor

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if for every α there is a $\beta(\alpha)$ such that
$d_n(z_n, z'_n) < \alpha \Rightarrow d'_n(f_n(z_n), f_n(z'_n)) < \beta(\alpha)$.

The diagram

$$
\begin{tikzcd}
 & \bigoplus_n \mathcal{O}^G(E, G \times |\mathcal{U}(n)|, d_1) \arrow[d, "(3)"] \\
\mathcal{O}^G(E, (G \times \overline{X}, d_{C(n)})_{n \in \mathbb{N}}) \arrow[r, "(2)"] \arrow[d, "inc"] & \mathcal{O}^G(E, (G \times |\mathcal{U}(n)|, d_n^1)_{n \in \mathbb{N}}) \arrow[d, "inc"] \\
\prod_{\in \mathbb{N}} \mathcal{O}^G(E, G \times \overline{X}, d_{C(n)}) \arrow[r, "\prod_n F_{\mathcal{U}(n)}"] \arrow[d, "pr_k"] & \prod_{\in \mathbb{N}} \mathcal{O}^G(E, G \times |\mathcal{U}(n)|, d_n^1) \arrow[d, "pr_k"] \\
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The diagram:Step 1

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The map (1) exists after applying K-theory and it is a section of $pr_k \circ inc$ for all k. Morally this means that (1) is a kind of a diagonal embedding.

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The inclusion (3) is an equivalence on K-theory for a sequence $|\mathcal{U}(n)|$ of simplicial complexes of dimension at most N (independent of n) and metrics on $G \times |\mathcal{U}(n)|$ with certain properties.

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The diagram:Step 2

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Now we need maps $F_{\mathcal{U}(n)}$ to simplicial complexes. Given a locally finite open cover $\mathcal{U}$ of a metric space Z, we obtain a continuous (with l^1-topology) map

$F_{\mathcal{U}} : Z \to |\mathcal{U}(n)| \qquad z \mapsto [[\sum_{U \in \mathcal{U}, z \in U} d(z, U^c) U]].$

The conditions above on the functoriality in the sequence of open covers give rise to certain conditions on those covers.

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Let (Y_n, d_n^∞) be the disjoint union of the N-simplices where the distance is the n-times the l^1-distance, if the two points lie in the same simplex and infinite otherwise.

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by Induction on N. The attaching of N-simplices gives a diagram

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After fattening both categories on the left (as before) both rows turn into Karoubi-filtrations. The induced map on the quotient will be an equivalence ($\blacksquare_1$).

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After fattening both categories on the left (as before) both rows turn into Karoubi-filtrations. The induced map on the quotient will be an equivalence ($\blacksquare_1$). Furthermore the K-theory of $\mathcal{L}((Y_n, d_n^\infty)_{n\in\mathbb{N}})^{>\oplus}$ vanishes ($\blacksquare_2$).

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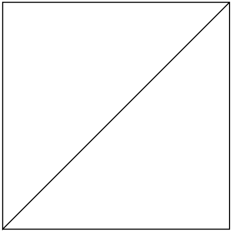

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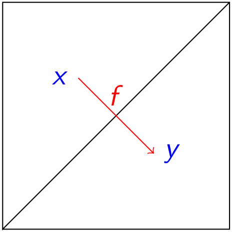

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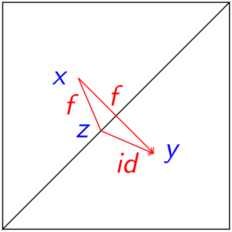

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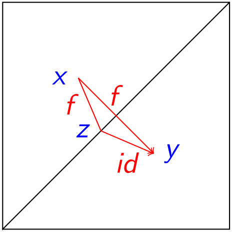

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The naive idea just to shift in the $[1, \infty)$ direction (without changing the other ones) fails - the equivariant control condition is violated. We are already in the situation where arrows cannot connect different simplices. Thus we have to make simplices smaller.

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$$H : [0,1] \times Y_n \times E \to E$$

such that

$$H_0(y, e) = e; \qquad H_1(y, e) = i(p(y))$$

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Now define for every point (g, y, e, t) in $G \times Y_n \times E \times [1, \infty)$ a sequence of points

$$(y, H_{\frac{k}{k+t}}(y, e), t + k)_{k \in \mathbb{N}}$$

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and define an Eilenberg swindle along those; i.e. if we consider an object A and some module M sits over a point (y, e, t) we define put in $S(A)$ that module over all those points in that sequence. The natural isomorphism $S \oplus Id \cong S$ is given by shifting along those lines.

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This completes the proof of $\blacksquare_2$ and thus the proof that (3) is an equivalence in K-theory.

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Excursion: Geometry

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The free group is CAT(-1) as it acts on a tree; Surface groups are CAT(-1) as the universal cover of a surface with the hyperbolic metric is $\mathbb{H}^2$.

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- The normalizers of the open sets in the cover lie in $\mathcal{F}$ (Here $\mathcal{F} = \mathcal{VC}\text{yc}$).

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- For CAT(0)-groups there can be fixed points in $\overline{X}$ and a lot of the diagram has to be reworked.

The flow space

Definition

For a CAT(-1) space X define

$$FS(X) := \{f : \mathbb{R} \to X \mid \exists -\infty \leq a \leq b \leq \infty :$$
$$f_{[a,b]} \text{ is a unit speed geodesic, } f_{(-\infty,a]}, f_{[b,\infty)} \text{ are constant}\}.$$

The function $d(f, f') := \int_{\mathbb{R}} \frac{d_X(f(t), f'(t))}{2e^{|t|}}$ defines a metric on it which generates the compact open topology. A (geometric) G-action on X induces a (geometric) G-action on $FS(X)$. Furthermore precomposition with translations gives a $\mathbb{R}$-action Φ on $FS(X)$ commuting with the G action.
The evaluation map $f \mapsto f(0)$ is proper with compact, contractible fibers; Heuristically the flow space wants to mimic the unit ball bundle in the tangent bundle of a Riemannian manifold.

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The function $d(f, f') := \int_{\mathbb{R}} \frac{d_X(f(t), f'(t))}{2e^{|t|}}$ defines a metric on it which generates the compact open topology. A (geometric) G-action on X induces a (geometric) G-action on $FS(X)$. Furthermore precomposition with translations gives a $\mathbb{R}$-action Φ on $FS(X)$ commuting with the G action.

The flow space

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The evaluation map $f \mapsto f(0)$ is proper with compact, contractible fibers; Heuristically the flow space wants to mimic the unit ball bundle in the tangent bundle of a Riemannian manifold.

Theorem (Bartels-Lück-Reich)

Let G act geometrically on a $CAT(-1)$-space X. There is a natural number N such that for any $R > 0$ there is an $\varepsilon > 0$ and a $\mathcal{VC}yc$ cover of $FS(X)$ of dimension at most N such that for every $f \in FS(X)$ there is an open set containing $B_\varepsilon(\Phi_{[-R,R]}f)$.

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The goal is now to define a map $G \times \overline{X} \to FS(X)$ such that every set of the form $\{(gh^{-1}, hx) \mid |h| \leq \beta\}$ gets mapped into one of the sausages $B_\varepsilon(\Phi_{[-R,R]}f)$.

The map to the flow space

Choose a basepoint $x_0 \in X$ and define a map

$$i_{x_0} : G \times \overline{X} \to FS(X), \qquad i_{x_0}(g,x)(t) := \begin{cases} gx_0 & t \leq 0 \\ gx & t \geq d(x,y) \\ c_{gx_0,gx}(t) & \text{else} \end{cases},$$

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The key idea is now to postcompose with flowing by a large enough number T towards x.

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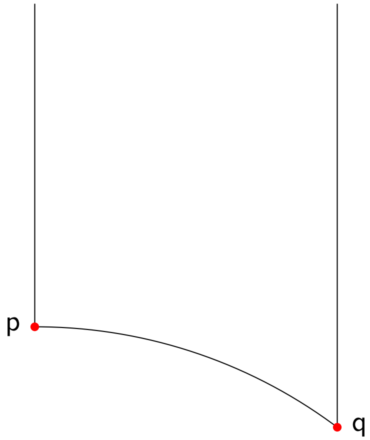
Picture of the flow



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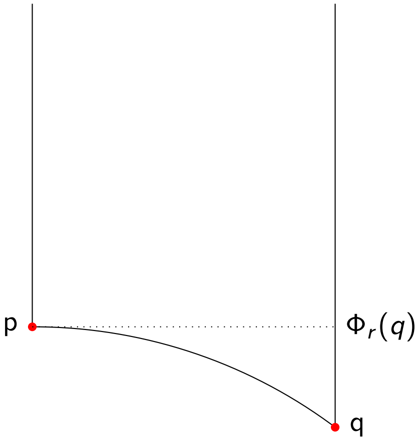

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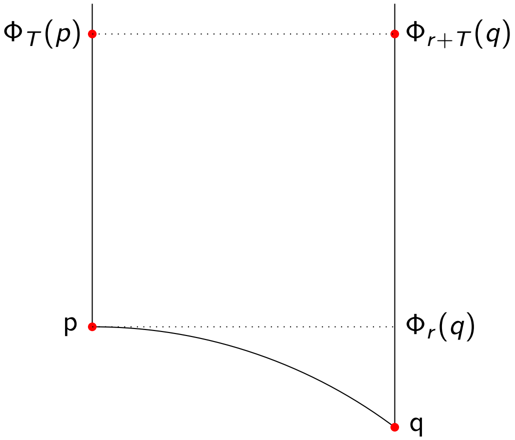

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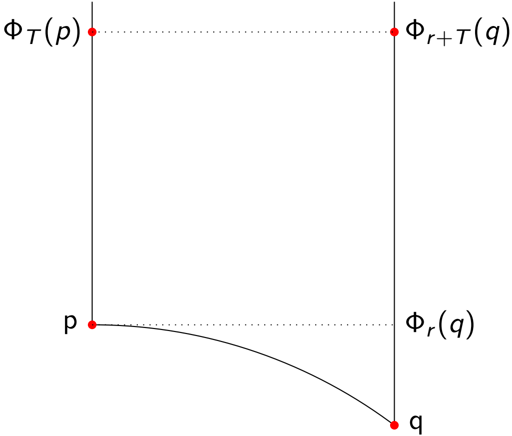
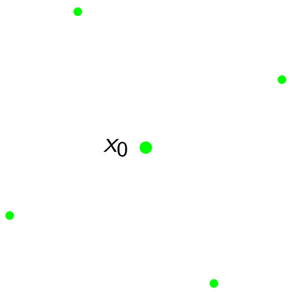


A picture in the Poincaré - half plane of a generic ideal triangle. Recall that in this model points of the same y coordinate (i.e. of with the value of the Busemann function of the point at infinity of the triangle) get arbitrarily close together if we move them upwards.

Those sets around $(1, x)$ with $x \in \partial X$



The green dots are the set $B = \{hx_0 \mid h \in B_\beta(1)\}$, then $i_{x_0}(\{(h^{-1}, hx) \mid h \in B_\beta(1)\})$ consists of those geodesics that start at some point in B at time 0 and go towards the point at infinity x. Two elements f, f' in the flow space are close, if $\sup_{t\in[-R,R] d_X(f(t),f'(t))}$ is small. Basically $d(f(0), f'(0))$ has the highest weight

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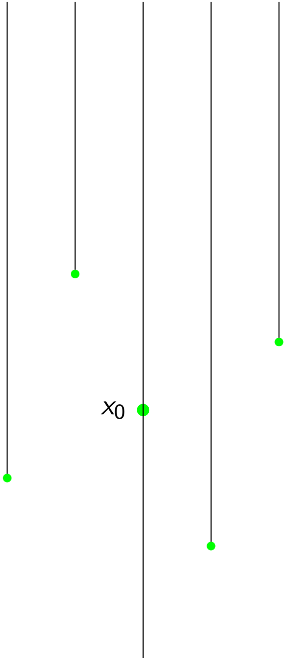

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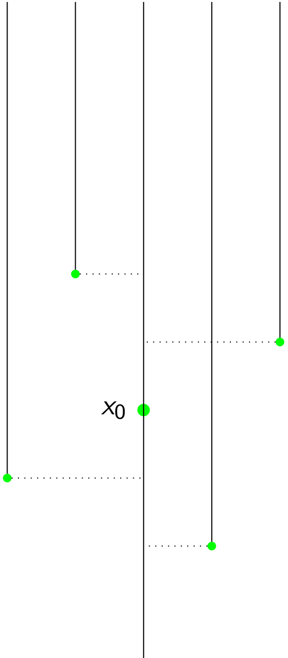

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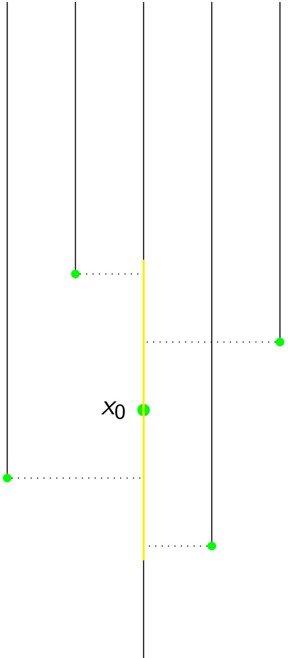

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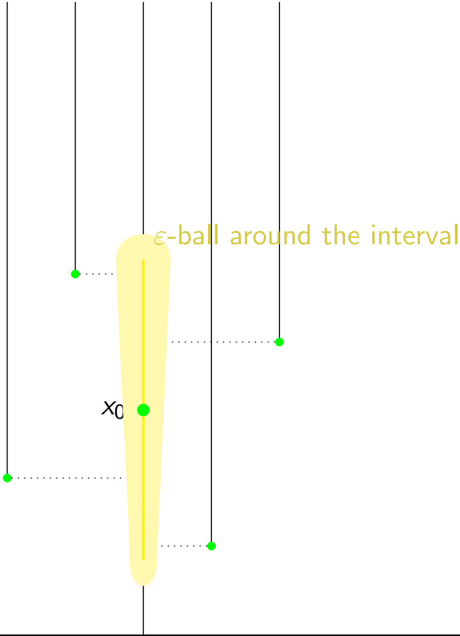

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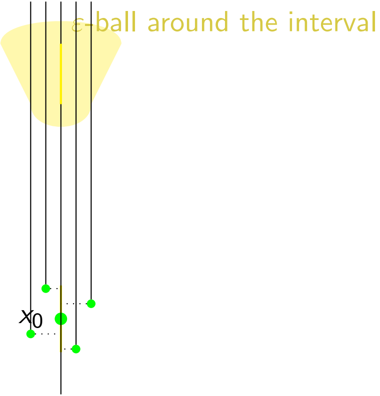

If you let the interval flow long enough before taking the ε-neighborhood, the resulting set will be much bigger in the horizontal direction. This comes from the Poincare-halfplane model.

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The whole talk is based on the following papers:

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