GK SEMINAR ON CONTROLLED TOPOLOGY

SPIROS ADAMS-FLOROU, HENRIK RÜPING, AND WOLFGANG STEIMLE

The version with speakers is available at
http://www.math.uni-bonn.de/people/rueping/ControlledTopology.pdf

Introduction

The idea of controlled topology is to measure the "size" of homotopy-theoretic or algebraic objects in an auxiliary metric space. For instance, if a space Y comes equipped with a map to a metric space B, then an ε-homotopy $H\colon X \times I \to Y$ is a continuous map such that for each $x \in X$, the path $H(x, -)$ has diameter less than ε when mapped to B. This immediately leads to the notion of "ε-equivalence" which sits in between the usual notions of "homotopy equivalence" and "homeomorphism".

Somewhat surprisingly, this simple concept is extremely useful in many problems in geometric topology. For example, any ε-equivalence between closed manifolds is homotopic to a homeomorphism, provided ε is small enough (this is the α-approximation Theorem). On the other hand, one can prove that a homeomorphism between finite CW complexes is a simple homotopy equivalence using the fact that it is an ε-homotopy equivalence for each $\varepsilon > 0$.

In this seminar we would like to talk about four different important theorems where controlled topology plays a major role.

It would be great if the speakers of each section could collaborate in preparing their talks.

Part I: Topological invariance of the Whitehead torsion

The Whitehead torsion assigns to a homotopy equivalence $f : X \to Y$ of finite CW complexes an element $\tau \in \mathrm{Wh}(\mathbb{Z}\pi_1(X))$. The definition uses the CW-structure. It was for a long time a big open problem, whether the Whitehead torsion is actually independent of this choice. It was solved affirmatively by Chapman in [4]. Thus a homotopy equivalence cannot be homotopic to a homeomorphism, if its Whitehead torsion does not vanish.

Talk 1, Introduction to Whitehead torsion and controlled topology, 10.4.14 (Henrik Rüping). Start with a brief introduction to the basic notions of controlled topology (controlled maps and homotopies), recall the definitions of polyhedra, triangulations, PL-maps ([10] or [11]) and mention the simplicial approximation theorem [8]. Give a brief definition of the Whitehead torsion [7, Chapter 11].

Date: ,February 2014.

Talk 2, Topological Invariance of the Whitehead torsion I, 10.4.14 (Malte Pieper). Explain the notion of regular neighborhood of a subcomplex [7, Chapter 9]. Show that two polyhedra are simple homotopy equivalent, if and only if their regular neighborhoods in $\mathbb{R}^n$ are PL-homeomorphic for n large. Conclude with the proof of the Topological invariance of the Whitehead torsion using the main technical theorem. [1] This is covered in [7, 17.1-17.12].

Talk 3, Topological invariance of the Whitehead torsion II, 24.4.14 (Nils-Edwin Enkelmann). Proof of the main technical theorem, using the main technical lemma (MTL) and the immersion and splitting lemmas [7, Chapter 17.13ff]. For the idea of the torus trick used to prove the MTL refer to [7, Chapter 13-14] or Rob Kirby's lecture series [9, Talk 4,6].

Part II: The α-approximation Theorem

In this part we study the α-approximation Theorem which is due to Chapman and Ferry:

Theorem 1. *Let M^n, $n \geq 5$, be a closed topological manifold with a fixed topological metric d. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that if $f : N \to M$ is a δ-equivalence over M then f is ϵ-homotopic to a homeomorphism.*

Talk 4, The α-approximation Theorem I, 24.4.14 (Wolfgang Steimle). State the α-approximation theorem. Assuming the splitting theorem and a little surgery theory, prove the handle lemma. This is covered in [7, Lemma 21.3] as well as [5, Chapter 3]. More details about the surgery step can be found in [12, Page 280].

Talk 5, The α-approximation Theorem II, 15.5.14 (Spiros Adams-Florou). Using the splitting theorem and the handle lemma, prove the handle theorem. Use a handle decomposition of $\mathbb{R}^n$ to prove the α-approximation theorem over coordinate patches. Then, assume the strong form of local contractibility of the homeomorphism group as stated in the schedule for Talk 6 to extend the result to the whole manifold M.

Talk 6, The α-approximation Theorem III, 15.5.14 (Spiros Adams-Florou). Prove the following strong form of local contractibility of the homeomorphism group due to Edwards and Kirby:

Theorem 2. *Let M^n be a topological manifold. If C is a compact subset of M and U is an open neighbourhood of C in M, then for every $\epsilon > 0$ there is a $\delta > 0$ such that if $h : U \to M$ is an open embedding with $d(h(x), x) < \delta$, then there is a homeomorphism $\overline{h} : M \to M$ so that $\overline{h}|_C = h|_C$, $\overline{h}|_{M-U} = id$ and $d(\overline{h}(x), x) < \epsilon$.*

This is covered in [6, Proposition 3.2, Lemma 4.1 and Theorem 5.1]. See also [9, Talks 4, 6] for moving pictures.

[1] No need to be afraid of the name.

Part III: Topological invariance of rational Pontryagin classes

In this part we study the following theorem, which is due to Novikov:

Theorem 3. *If $f\colon M \to N$ is a homeomorphism between smooth manifolds, then $f^*\colon H^*(N;\mathbb{Q}) \to H^*(M;\mathbb{Q})$ sends the rational Pontryagin classes of N to the rational Pontryagin classes of M.*

In this part of the seminar we show, following [7, §22], that Theorem 3 is implied by the Novikov Conjecture for the group $\mathbb{Z}^n$. The Novikov Conjecture follows from the Farrell-Jones Conjectures (which are known for this class of groups); we will have a closer look at these conjectures in part IV of this seminar.

Talk 7, Pontryagin classes and signatures, 5.6.14 (Markus Land). Recall the role of Pontryagin classes in Hirzebruch's signature theorem and in Wall's theorem about the oriented cobordism ring [13]. State Theorem 3 and deduce the existence of a non-smoothable manifold. Explain [7, Lemma 22.13] and use it to transfer Theorem 3 into a statement about signatures of framed submanifolds of M and N.

Talk 8, Bounded splitting and the Novikov conjecture, 5.6.14 (Markus Land). Introduce the Novikov Conjecture. Explain why the stable bounded splitting theorem and the Novikov conjecture for the groups $\mathbb{Z}^n$ imply Theorem 3. (This uses again [7, Lemma 22.13].) Finally, sketch the proof of the stable bounded splitting theorem, going back to [7, §9] when necessary.

Part IV: The Farrell-Jones Conjectures

Talk 9, Controlled algebra used for the Farrell-Jones conjecture, 17.7.14 (Henrik Rüping). Give an introduction to controlled algebra (control conditions, pullbacks, ...) as in [2] or [1, Section 1,3,4]. Explain Karoubi filtrations and their quotients [3]. Sketch the proof how the continuous control condition can be used to establish excision on the K-groups. Define the obstruction category and show that the assembly map is an isomorphism if and only if K-theory of the obstruction category vanishes. Explain the categories appearing in the main diagram [1, Diagram 4.4] obtained from long and thin covers.

Talk 10, The core of the proof of the Farrell-Jones conjecture, 17.7.14 (Henrik Rüping). Explain the morphisms in the main diagram. If time permits, say something about the transfer. [1, Section 7] is quite important. This is where controlled methods are used a lot. Furthermore it is also the point, where the universal property of the classifying space is used. If there is still some time left, mention how geometry is used to contruct long and thin covers.

References

[1] A. Bartels, W. Lück, and H. Reich. The K-theoretic Farrell–Jones conjecture for hyperbolic groups. *Inventiones mathematicae*, 172(1):29–70, 2008.

[2] A. Bartels and H. Reich. Coefficients for the Farrell–Jones conjecture. *Advances in Mathematics*, 209(1):337–362, 2007.

[3] M. Cárdenas and E. K. Pedersen. On the Karoubi filtration of a category. *K-theory*, 12(2):165–191, 1997.

[4] T. Chapman. Topological invariance of Whitehead torsion. *Amer. J. Math*, 96(3):488–497, 1974.

[5] T. A. Chapman and S. Ferry. Approximating homotopy equivalences by homeomorphisms. *Amer. J. Math.*, 101(3):583–607, 1979.
[6] R. D. Edwards and R. C. Kirby. Deformations of spaces of imbeddings. *Ann. Math. (2)*, 93:63–88, 1971.
[7] S. Ferry. Geometric topology notes. http://www.math.rutgers.edu/ sferry/ps/geotop.pdf.
[8] A. Hatcher. *Algebraic topology*. 2002.
[9] R. Kirby. Topological manifolds: their triangulations and smoothings. http://www.maths.ed.ac.uk/ aar/kirby.htm.
[10] J. Lurie. Lecture 2: PL-topology. http://www.math.harvard.edu/ lurie/937notes/937Lecture2.pdf.
[11] C. P. Rourke and B. J. Sanderson. *Introduction to piecewise-linear topology*. Springer, 1972.
[12] L. C. Siebenmann. Approximating cellular maps by homeomorphisms. *Topology*, 11:271–294, 1972.
[13] C. T. C. Wall. Determination of the cobordism ring. *Ann. of Math. (2)*, 72:292–311, 1960.

E-mail address: spirosaf@mpim-bonn.mpg.de

E-mail address: henrik.rueping@hcm.uni-bonn.de

E-mail address: steimle@math.uni-bonn.de