Minneapolis, March 01

<u>Talk: The arithmetic significance of some Eisenstein Series.</u>

- Thanks for invit. to Coll., Ordway: very productive visit
- Some of deepest results in arithm. alg. geom. come about by identif. object of alg.-geo./arithm. nature with object of analytic nature.

  One of most spectacular instances is identific. of $\zeta(E/\mathbb{Q}, s)$ with $L(\pi, s)$.

  Here at prime p of good reduction, have $\zeta(E/\mathbb{Q}, s) = \prod_p \zeta_p(E/\mathbb{Q}, s)$

$$\log \zeta_p(E, s) = \sum_{n \geq 1} \# E(\mathbb{F}_{p^n}) \cdot \frac{T^n}{n} \quad , \quad T = p^{-s}.$$

- In this talk we go the other way: start with analytic object, search for arithm. interpretation. Again we will be counting points, but this time with multiplicities.

<u>Simplest case</u> (Kudla, Rapo, Yang): Let d prime number, $d \equiv 3\ (4)$, $d>3$. For $\tau = u+iv \in \mathfrak{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s)>1$, let <u>Eisenstein series of wt 1 for</u> $\Gamma = SL_2(\mathbb{Z})$

$$E_\pm(\tau,s) = v^{s/2} \cdot \sum_{\Gamma_\infty \backslash \Gamma} (c\tau+d)^{-1} \cdot |c\tau+d|^{-s} \cdot \Phi_\pm(\gamma)$$

where for $\begin{pmatrix} a & b \\ c & * \end{pmatrix} \in \Gamma$

$$\Phi_\pm(\gamma) = \begin{cases} \chi_d(a) & c \equiv 0\ (d) \\ \pm i \cdot d^{-1/2} \cdot \chi_d(c) & (c,d)=1. \end{cases}$$

Can normalize this Eisenstein series by $E^*_\pm(\tau,s) = d^{\frac{s+1}{2}} \cdot \underbrace{\Lambda(s+1,\chi_d)}_{\Gamma\text{-factor}} \cdot E_\pm$

Then have analytic continuation + fctl. equation

$$E^*_\pm(\tau,-s) = \pm E^*_\pm(\tau,s)$$

$$\left[\text{Hecke}: \quad E^*_+(\tau,0) = 2 \cdot \sum_{a \in \mathcal{Cl}(k)} \theta(\tau,a) \right].$$

We are interested in

$$\phi(\tau) = -\frac{\partial}{\partial s} E^*_-(\tau,s)\Big|_{s=0}.$$

Then $\phi(\tau)$ is a non-holomorphic modular form of weight 1 for $SL_2(\mathbb{Z})$.

Consider Fourier expansion of $\phi(\tau)$,

$$\phi(\tau) = \sum_{t \in \mathbb{Z}} a_t(v) \cdot q^t \qquad q = e^{2\pi i \tau}.$$

It turns out that for $t > 0$, $a_t(v) = a_t$ constant.

<u>Let us give a arithm.-geom. interpretation of these coefficients.</u>

$k = \mathbb{Q}(\sqrt{-d})$, consider

$\mathcal{M}$ = moduli space of elliptic curves E with CM ι by $\mathcal{O}_k$.

$\rightarrow$ $\mathcal{M} = \mathrm{Spec}\, \mathcal{O}_H$, H/k Hilbert class field.

1-dimensional

Let

$Z(t)$ = moduli space of (E, ι) with $j \in \mathrm{End}(E)$ anti-commuting with ι, s.t. $j^2 = -t$ (special endomorph)

Then $Z(t)$ is 0-dimensional $\leadsto$ $\deg Z(t) = \log |\mathcal{O}_{Z(t)}|$.
$= \sum_{x \in \overline{Z(t)}} \mathrm{mult}_x(Z(t))$.

<u>Theorem</u>: <u>For</u> $t > 0$ <u>have</u>

$$a_t = \deg Z(t).$$

<u>More precisely, we have case by case:</u>

a) If t is a norm from $k/\mathbb{Q}$, then t/p is not a norm $\forall$ inert p

and $Z(t) \subset \mathcal{M} \otimes \mathcal{O}_k/(d)$.

b) If p is inert s.t. t/p is a norm, then t/p' is not a

norm $\forall\, p' \neq p$, p' inert and $Z(t) \subset \mathcal{M} \otimes \mathcal{O}_k/(p)$.

$Z(t) \cap (\mathcal{M} \otimes \mathcal{O}/\mathfrak{p}) \neq \emptyset$, if $\deg \mathfrak{p} = 1$. c) If neither t nor t/p is a norm $\forall p$ inert, then

$Z(t) = \emptyset$ (and $a_t = 0$).

In all cases $\deg Z(t) = a_t$.

For negative t, one can give a somewhat artificial interpretation

More striking is $a_0(v)$. Namely,

$$\boxed{a_0(v) = -h_k\,(\log v + 4\,h_{Fal}(E) + C)}\ , \quad C \text{ const. indep't of } d.$$

Remarkable, since heights are in general just introduced for estimates – but

here a precise formula. Reminiscent of Zagier's formula (geometric, not arithm

$$\phi_{Zagier}(\tau) = \sum_{t \in \mathbb{Z}} b_t(v) \cdot q^t\ , \text{ where for } t > 0\ \ b_t(v) \equiv b_t = \deg Y(t),$$

where $Y(t)$ 0-cycle on modular curve $\mathfrak{H}/\Gamma$. Here ϕ_{Zagier} = value

at $s = 1/2$ of Eisenstein series of wt. $3/2$.

Kudlas's dream: He has a systematic way of producing Siegel-Eisenstein series of genus vanishing at center of symmetry s=0. Want

series of genus n vanishing at center of symmetry $s = 0$. Want

to relate the derivative at $s=0$ to degrees of special cycles on arithmetic models of Shimura varieties assoc. to orthogonal groups of signature $(n-1, 2)$.

The case just considered corresponds to $n = 1$ (one can eliminate the special hypotheses on d)

$n = 2$: modular curves + Heegner points on them.

$n = 3$: HB-surfaces + HZ-cycles on them.

$n = 4$ Siegel 3-folds + Humbert cycles on them

After this, no more results. One diffic. is to construct arithm. models of these Sh.-varieties. Another is: what are special cycles?

From now on let $n = 2$. Let

B = indefinite quat. division algebra over $\mathbb{Q}$

$V = \{x \in B;\ tr(x) = 0\}$ w. Nm° is quadr. space sgn $(1,2)$

$H = B^\times = GSpin(V)$ acts on V.

Let $O_B \subset B$ maximal order

$\rightsquigarrow$ $X = X_K$ Shimura curve for $K = (O_B \otimes \hat{\mathbb{Z}})^\times$.

<u>Moduli space of</u> (A, ι), where

A = 2-dim'l abelian variety

$\iota : O_B \longrightarrow \mathrm{End}(A)$ "special"

Hence $X/\mathbb{Z}$ with $X(\mathbb{C}) = \Gamma_B \backslash \mathfrak{H}$, $\Gamma_B = \mathrm{Ker}(\mathrm{Nm} : O_B^\times \to \mathbb{Z}^\times)$

<u>Definition:</u> Let (A, ι). The <u>group of special endomorphisms of</u> (A,ι) is

$$V(A,\iota) = \{ x \in \mathrm{End}(A) ;\ \iota(b)x = x\iota(b) ;\ \mathrm{tr}^0(x) = 0 \}$$

Has quadratic form Q, via

$$x^2 = -Q(x) \cdot 1_A .$$

For $T \in \mathrm{Sym}_2(\mathbb{Z})_{>0}$, let

$Z(T)$ = moduli space of (A, ι), with $\underline{x} = [x_1, x_2] \in V(A, \iota)^2$ s.t.

$$Q(\underline{x}) = \tfrac{1}{2}((x_i, x_j))_{ij} = T.$$

Proposition: If $Z(T) \neq \emptyset$, then $\exists!\, p$ s.t. $Z(T) \subset X \otimes \mathbb{F}_p$

If $p \nmid D(B)$ (prime of good reduction), then $Z(T)$ has dimension 0 and is supported by the supersingular locus in $X \otimes \mathbb{F}_p$.

The prime p attached in this way can be characterized as follows in terms of the quadratic form Q resp. by V: $\forall p$ have $B^{(p)}$ and $V^{(p)}$. Then p is the unique prime number s.t. T represented by $V^{(p)}$.

Theorem (imprecise version) Kudla, Annals: Let $\varphi = \mathrm{char}(\hat{O}_B \cap V(\mathbb{A}_f))$. Then have Siegel-Eisenstein series of genus 2 and weight $\frac{3}{2}$, $E(\tau, s, \varphi)$ ($\tau \in \mathfrak{H}_2$) which vanishes at $s = 0$. And have

$$E'(\tau, 0, \varphi) = \sum_{T \in \mathrm{Sym}_2(\mathbb{Z})} a_T(v) \cdot q^T \qquad q^T = e^{2\pi i \mathrm{tr}(T\tau)}$$

where

if $T \in \mathrm{Sym}_2(\mathbb{Z})_{>0}$ have: $a_T(v) \equiv a_T$ constant and either T not represented by

$V^{(p)}$, for any p and then $a_T = 0$

or T represented by a unique $V^{(p)}$ and then

$$a_T = \deg Z(T),$$

provided $p \nmid 2 \cdot D(B)$.

Proof: calculate both sides explicitly: analytic side Kitaoka ⇝ Yang, alg-geom. side Gross / Keating

- If $p \mid D(B)$ have variant: Kudla / Rapo.
- For "T" $\in \mathrm{Sym}_2(\mathbb{Z})$ with $\det T \neq 0$: have interpretation à la Arakelov (Kudla).
- $\mathrm{rk}\, T = 1$: current work.
- $T = 0$: complete mystery.

For higher n have new phenomena...