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Michael Rapoport

A positivity property of the Satake isomorphism

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Abstract. Let G be an unramified reductive group over a local field. We consider the matrix describing the Satake isomorphism in terms of the natural bases of the source and the target. We prove that all coefficients of this matrix which are not obviously zero are in fact positive numbers. The result is then applied to an existence problem of F-crystals which is a partial converse to Mazur's theorem relating the Hodge polygon and the Newton polygon.

1. Introduction

Let F be a local field and G a connected reductive group over F which is quasisplit and which splits over an unramified extension. Let A be a maximal split torus in G. The centralizer T of A is a maximal torus contained in a Borel subgroup $B = T.U$. Let K be a hyperspecial maximal compact subgroup of $G(F)$ which fixes a vertex in the apartment corresponding to A in the Bruhat–Tits building of G_{ad}. We consider the constant term mapping between Hecke algebras with coefficients in $\mathbf{C}$,

$$\begin{aligned} b : \mathcal{H}(G(F)/\!\!/K) &\longrightarrow \mathcal{H}(T(F)/\!\!/T(F) \cap K) \\ f &\longmapsto b(f)(t) = \delta_B^{1/2}(t) \cdot \int_{U(F)} f(tu)du. \end{aligned} \tag{1.1}$$

Here δ_B denotes the modulus function and the Haar measures are normalized so that K, $T(F) \cap K$ and $U(F) \cap K$ all get volume 1. The Hecke algebra on the right in (1.1) can be identified with the group algebra $\mathbf{C}[X_*(A)]$. The theorem on the Satake isomorphism ([9], comp. also [3]) asserts that the image of b lies in the invariant ring under the relative Weyl group $W_0 = N(A)(F)/T(F)$ and that b induces an isomorphism of algebras

$$\mathcal{H}(G(F)/\!\!/K) \xrightarrow{\sim} \mathbf{C}[X_*(A)]^{W_0}. \tag{1.2}$$

That (1.2) is an isomorphism is proved as follows. A $\mathbf{C}$-basis for the space on the right of (1.2) is given by

$$m_\mu = |W_{0,\mu}|^{-1} \cdot \sum_{w \in W_0} w(\mu), \quad \mu \in X_*(A) \cap \overline{C}. \tag{1.3}$$

M. Rapoport: Mathematisches Institut der Universität zu Köln, Weyertal 86–90, 50931 Köln, Germany. e-mail: rapoport@mi.uni-koeln.de

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Here $\overline{C}$ denotes the closed Weyl chamber in $X_*(A)_{\mathbf{R}}$ corresponding to B and $W_{0,\mu}$ is the stabilizer of μ in W_0. On the other hand, by the Cartan decomposition of $G(F)$ with respect to K, a $\mathbf{C}$-basis for the space on the left of (1.2) is given by

$$f_\lambda = \text{char } K\varpi^\lambda K, \quad \lambda \in X_*(A) \cap \overline{C}. \tag{1.4}$$

Here $\varpi^\lambda = \lambda(\varpi)$, where ϖ is a fixed uniformizer of F. Let

$$C = M(b(f_\bullet), m_\bullet) \tag{1.5}$$

be the matrix describing the map (1.2) in terms of these bases, i.e. $C = (C_{\lambda\mu})$ with

$$b(f_\lambda) = \sum_\mu C_{\lambda\mu} \cdot m_\mu. \tag{1.6}$$

From the definition of b it follows that $C_{\lambda\mu} \geq 0$. The fact that (1.2) is an isomorphism now follows from the statement that the matrix C is upper triangular with strictly positive entries on the diagonal, i.e.

$$C_{\lambda\lambda} > 0 \quad \text{and if } C_{\lambda\mu} \neq 0, \quad \text{then } \lambda \overset{!}{\geq} \mu. \tag{1.7}$$

Here $\overset{!}{\geq}$ denotes the usual partial order on $X_*(A) \cap \overline{C}$, i.e.,

$$\lambda \overset{!}{\geq} \mu \Longleftrightarrow \lambda - \mu = \sum n_\alpha \cdot \alpha^\vee, \quad n_\alpha \in \mathbf{Z}_{\geq 0}. \tag{1.8}$$

The sum on the right is over the simple relative coroots.[1] The exclamation mark is meant to stress that the coefficients n_α are non-negative *integers* rather than non-negative real numbers. The purpose of the present note is to show that the converse to the statement (1.7) holds.

Theorem 1.1. *If* $\lambda \overset{!}{\geq} \mu$, *then* $C_{\lambda\mu} > 0$.

This is the positivity statement referred to in the title. The proof of the theorem is contained in Sect. 2. It proceeds by induction on the semi-simple rank of G, as suggested to me by Waldspurger who also communicated to me the induction step. In Sect. 3 the case of GL_n is considered. It is shown how the positivity assertion above can be deduced from the theory of symmetric functions [7]. In the final Sect. 4 we give an application of Theorem 1.1 to an existence problem of F-crystals.

Positivity statements on matrices describing a change of basis between natural bases have become a common phenomenon, starting with the seminal paper of Kazhdan and Lusztig, comp. [6]. They are also well-known for base change matrices between natural bases for the space of symmetric functions, comp. [7]. But the particular statement we prove here does not seem to have been pointed out before. A geometric interpretation of it would of course be very desirable.

In a companion paper [2] to the present one, T. Haines gives a different proof of Theorem 1.1 in the case of a split group by reducing it to a result of Dabrowski [1]. He also relates the matrix C to Kazhdan–Lusztig polynomials for the extended affine Weyl group of G.

[1] Throughout the paper, roots, coroots etc. are meant to be *relative* roots, *relative* coroots etc.

2. Proof of the theorem

We introduce the map

$$\mathrm{inv} : G(F)/K \times G(F)/K \longrightarrow X_*(A) \cap \overline{C} \tag{2.1}$$

which comes about by identifying the set of $G(F)$-orbits,

$$G(F) \setminus (G(F)/K \times G(F)/K) = K \setminus G(F)/K = X_*(A) \cap \overline{C}. \tag{2.2}$$

Our aim is to prove that $\lambda \overset{!}{\geq} \mu$ implies either of the following equivalent statements

$$\begin{aligned} K\varpi^\lambda K \cap \varpi^\mu U(F) \neq \emptyset &\Leftrightarrow K\varpi^\lambda K \cap U(F)\varpi^\mu \neq \emptyset \\ &\Leftrightarrow \exists u \in U(F) : \mathrm{inv}(u\varpi^\mu \cdot x_0, x_0) = \lambda. \end{aligned} \tag{2.3}$$

Here $x_0 \in G(F)/K$ denotes the base point.

The proof of this implication will proceed by reduction to the case of semi-simple rank equal to one. We first make a remark on the center. Let $\pi : G \to G_{\mathrm{ad}}$ be the natural homomorphism into the adjoint group. Assume Theorem 1.1 proved for G_{ad} and let us deduce it for G. From $\lambda \overset{!}{\geq} \mu$ we deduce for their images in $X_*(A_{\mathrm{ad}})$ that $\lambda_{\mathrm{ad}} \overset{!}{\geq} \mu_{\mathrm{ad}}$ and hence there exists $u \in U_{\mathrm{ad}}(F)$ with $\mathrm{inv}_{\mathrm{ad}}(u\varpi^{\mu_{\mathrm{ad}}} \cdot x_0, x_0) = \lambda_{\mathrm{ad}}$. The homomorphism π induces an isomorphism $U(F) \simeq U_{\mathrm{ad}}(F)$ and hence u may be considered as an element of $U(F)$. Let us compare the two elements of $X_*(A) \cap \overline{C}$,

$$\mathrm{inv}(u\varpi^\mu \cdot x_0, x_0) \quad \text{and} \quad \lambda. \tag{2.4}$$

By hypothesis both have the same image in $X_*(A_{\mathrm{ad}})$. Hence they differ by an element of $X_*(Z_0)$, where $Z_0 = Z \cap A$ denotes the split center of G. On the other hand, we have a natural inclusion into the cocharacter group of the maximal abelian factor group,

$$X_*(Z_0) \hookrightarrow X_*(G_{ab}). \tag{2.5}$$

But the images of the elements in (2.4) in $X_*(G_{ab})$ are μ_{ab} and λ_{ab} respectively which are equal since $\lambda - \mu \in X_*(A_{\mathrm{der}})$. Hence the elements in (2.4) are identical, as had to be shown.

A. The case of rank one

By the above remark we may assume that G is adjoint of relative rank one. It follows that G is of the form $G = R_{E/F}G'$, where E is an unramified extension and where

$$G' = PU_3, \quad \text{resp. } G' = PGL_2, \tag{2.6}$$

(quasisplit forms). Here PU_3 is the adjoint unitary group for a quasisplit hermitian form on a 3-dimensional vector space over an *unramified* quadratic extension of E. Since E/F is unramified, a uniformizer of F is also one of E. Hence we may replace F by E and G by G' in proving the assertion.

A1. Case $G = PU_3$

We identify $G(F)/K$ with the set of hyperspecial vertices of the Bruhat–Tits building of G, which is a tree. We choose an identification $X_*(A) = \mathbf{Z}$ such that $X_*(A) \cap \overline{C} = \mathbf{Z}_{\geq 0}$. Then the coroot $\alpha^\vee$ for the unique simple root α corresponds to $2 \in \mathbf{Z}$. Our hypothesis $\lambda \overset{!}{\geq} \mu$ is therefore equivalent to

$$\lambda - \mu \in 2 \cdot \mathbf{Z}_{\geq 0}. \tag{2.7}$$

Let $\mathcal{A}$ be the apartment corresponding to A and let $y = \varpi^\mu \cdot x_0 \in \mathcal{A}$. Let z be the point in $\mathcal{A}$ at distance $\frac{1}{2}(\lambda - \mu)$ from x_0 in the direction opposite to y (Fig. 1).

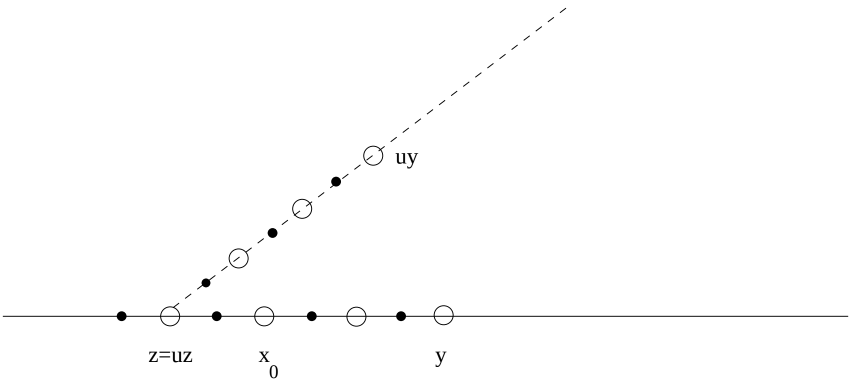

Fig. 1.

Note that z is a hyperspecial vertex. We indicated by larger dots the hyperspecial vertices and by smaller dots the non-hyperspecial vertices.

Now choose $u \in U(F)$ which fixes z and the unique edge in $\mathcal{A}$ containing z and pointing away from x_0, but which acts non-trivially on the other edges containing z. Then the unique geodesic from x_0 to uy passes through z and hence for the distance we get

$$\begin{aligned} d(uy, x_0) &= d(uy, z) + d(z, x_0) \\ &= d(y, z) + d(z, x_0) \\ &= d(y, x_0) + 2 \cdot d(z, x_0) \\ &= \mu + (\lambda - \mu) \\ &= \lambda. \end{aligned} \tag{2.8}$$

Here we normalized the distance so that hyperspecial vertices have distance one, which allows us to identify the distance function with the inv function. The claim in this case follows therefore from (2.8).

A2. Case $G = PGL_2$

In this case $X_*(A) = \mathbf{Z}$ and $\alpha^\vee = 2 \in \mathbf{Z}$, and again our hypothesis is $\lambda - \mu \in 2\cdot\mathbf{Z}_{\geq 0}$. The argument is identical, except that in the drawing above the larger dots have to be interpreted as vertices and the smaller dots as midpoints of edges.

B. The general case

To present Waldspurger's argument in the general case, we need a few lemmas.

Lemma 2.1. *Let $\lambda, \mu \in X_*(A)$ and $w \in W_0$. Then*

$$K\varpi^\lambda K \cap U(F)\varpi^\mu \neq \emptyset \Longleftrightarrow K\varpi^\lambda K \cap U(F)\varpi^{w(\mu)} \neq \emptyset.$$

Proof. In the notation of the Introduction this says

$$b(f_\lambda)(\varpi^\mu) \neq 0 \Longleftrightarrow b(f_\lambda)(\varpi^{w(\mu)}) \neq 0. \tag{2.9}$$

This is obvious since $b(f_\lambda) \in \mathbf{C}[X_*(A)]^{W_0}$. □

We denote by $R^\vee$ resp. $R^{\vee+}$ resp. $\Delta^\vee$ the set of coroots resp. of positive coroots resp. of simple coroots. They are in bijection with the sets R of roots, R^+ of positive roots and Δ of simple roots.

Lemma 2.2. *Let $\beta^\vee \in R^\vee$. Then there exists $w \in W_0$, $\alpha^\vee \in \Delta^\vee$ and $n \in \mathbf{Z}$, $n \geq 1$, such that $w(\beta^\vee) = n\alpha^\vee$.*

Proof. This is just Prop. 15 in Bourbaki: Groupes et Algèbres de Lie, ch. VI, 1.5.

The next lemma is due to Stembridge [10], Cor. 2.7. The following proof due to Waldspurger is even simpler than the alternative one by Steinberg reproduced in loc.cit.

Lemma 2.3 (Stembridge**).** *Let $\lambda, \mu \in X_*(A) \cap \overline{C}$ with $\lambda \overset{!}{\geq} \mu$ and $\lambda \neq \mu$. Then there exists $\beta \in R^{\vee+}$ such that $\mu + \beta^\vee \in X_*(A) \cap \overline{C}$ and $\lambda \overset{!}{\geq} \mu + \beta^\vee$.*

*Proof (*Waldspurger*).* Let $\lambda = \mu + \Sigma_{\alpha\in\Delta} n_\alpha \cdot \alpha^\vee$. Consider the set $\mathcal{X}$ of $\beta^\vee \in R^{\vee+}$ where $\beta^\vee = \Sigma_{\alpha\in\Delta} m_\alpha \alpha^\vee$ with $m_\alpha \leq n_\alpha$ for all α. Since $\lambda \neq \mu$ there exists α with $n_\alpha \neq 0$ and hence $\alpha^\vee \in \mathcal{X}$, hence $\mathcal{X} \neq \emptyset$. Let us partially order $\mathcal{X}$ by $\overset{!}{\geq}$. Let $\beta^\vee = \Sigma m_\alpha \alpha^\vee$ be a maximal element of $\mathcal{X}$. Then

$$\lambda = \mu + \beta^\vee + \Sigma_{\alpha\in\Delta}(n_\alpha - m_\alpha)\alpha^\vee, \tag{2.10}$$

with $n_\alpha - m_\alpha \geq 0$ for all α. Hence it suffices to show that $\mu + \beta^\vee \in \overline{C}$. Let $\alpha \in \Delta$. We want to show that $\langle \alpha, \mu + \beta^\vee \rangle \geq 0$. Since $\beta^\vee$ is maximal in $\mathcal{X}$, we have $\beta^\vee + \alpha^\vee \notin \mathcal{X}$. Hence there are two possibilities.

(i) $\beta^\vee + \alpha^\vee \notin R^\vee$.
(ii) $m_\alpha = n_\alpha$.

However, for any root system we have

$$\gamma_1^\vee, \gamma_2^\vee \in R^{\vee +} \quad \text{with } \langle \gamma_1, \gamma_2^\vee \rangle < 0 \Longrightarrow \gamma_1^\vee + \gamma_2^\vee \in R^\vee. \tag{2.11}$$

Under hypothesis (i) we have therefore $\langle \alpha, \beta^\vee \rangle \geq 0$. Hence

$$\langle \alpha, \mu + \beta^\vee \rangle = \langle \alpha, \mu \rangle + \langle \alpha, \beta^\vee \rangle \geq 0. \tag{2.12}$$

Under hypothesis (ii) we have

$$\langle \alpha, \mu + \beta^\vee \rangle = \langle \alpha, \lambda - \Sigma_{\gamma \in \Delta \setminus \{\alpha\}} (n_\gamma - m_\gamma) \cdot \gamma^\vee \rangle. \tag{2.13}$$

But $\langle \alpha, \lambda \rangle \geq 0$ and for $\gamma \in \Delta \setminus \{\alpha\}$ we have $\langle \alpha, \gamma^\vee \rangle \leq 0$. Hence again $\langle \alpha, \mu + \beta^\vee \rangle \geq 0$. $\square$

We can now complete the proof of the theorem. Fix $\lambda \in X_*(A) \cap \overline{C}$. Let $\mathcal{Y}$ be the set of $\mu \in X_*(A) \cap \overline{C}$ such that $\lambda \overset{!}{\geq} \mu$ and $K\varpi^\lambda K \cap U(F)\varpi^\mu = \emptyset$. We want to prove that $\mathcal{Y} = \emptyset$. Let us assume $\mathcal{Y} \neq \emptyset$ and deduce a contradiction. Let $\mu \in \mathcal{Y}$ be a maximal element with respect to $\overset{!}{\geq}$. Since $\varpi^\lambda \in K\varpi^\lambda K \cap U(F)\varpi^\lambda$, we have $\mu \neq \lambda$. By Lemma 2.3 we find $\beta^\vee \in R^{\vee +}$ with $\mu + \beta^\vee \in X_*(A) \cap \overline{C}$ and $\lambda \overset{!}{\geq} \mu + \beta^\vee$. Since $\mu + \beta^\vee \overset{!}{\geq} \mu$, by the maximality of μ we have $\mu + \beta^\vee \notin \mathcal{Y}$, hence

$$K\varpi^\lambda K \cap U(F)\varpi^{\mu + \beta^\vee} \neq \emptyset. \tag{2.14}$$

By Lemma 2.2 we find $w \in W_0$ and $\alpha^\vee \in \Delta^\vee$ such that $w(\beta^\vee)$ is a positive multiple of $\alpha^\vee$. By Lemma 2.1 we conclude from (2.14) that

$$K\varpi^\lambda K \cap U(F)\varpi^{w(\mu) + w(\beta^\vee)} \neq \emptyset. \tag{2.15}$$

Let $u \in U(F)$ such that $u \cdot \varpi^{w(\mu) + w(\beta^\vee)} \in K\varpi^\lambda K$. Let P be the standard parabolic subgroup corresponding to α with Levi subgroup M containing A. Then M has semisimple rank 1. Let U_P be the unipotent radical of P and $U^M = U \cap M$. Then

$$U(F) = U_P(F) U^M(F). \tag{2.16}$$

Let us write $u = u_P \cdot u^M$ according to this decomposition. Let $\overline{C}^M$ be the analogue of $\overline{C}$ for M, i.e.

$$\overline{C}^M = \{\nu \in X_*(A)_{\mathbf{R}} \ ; \ \langle \alpha, \nu \rangle \geq 0\}. \tag{2.17}$$

Now $w(\beta) = c\alpha$ with $c > 0$. Hence

$$\langle \alpha, w(\mu) \rangle = c^{-1} \langle w(\beta), w(\mu) \rangle = c^{-1} \langle \beta, \mu \rangle \geq 0, \tag{2.18}$$

since $\beta \in R^+$ and $\mu \in \overline{C}$. Hence $w(\mu) \in \overline{C}^M$. Since $w(\beta^\vee)$ is a positive multiple of $\alpha^\vee$ we conclude that

$$w(\mu) + w(\beta^\vee) \in \overline{C}^M, \quad w(\mu) + w(\beta^\vee) \overset{!}{\geq}{}^M \, w(\mu), \tag{2.19}$$

where $\stackrel{!}{\geq}^M$ is the analogue of $\stackrel{!}{\geq}$ with G replaced by M. By the Cartan decomposition for M we find $\nu \in X_*(A) \cap \overline{C}^M$ with

$$u^M \cdot \varpi^{w(\mu)+w(\beta^\vee)} \in K^M \varpi^\nu K^M, \quad \text{where } K^M = K \cap M(F). \tag{2.20}$$

By the converse of Theorem 1.1, the relation (2.20) implies $\nu \stackrel{!}{\geq}^M w(\mu) + w(\beta^\vee)$. Hence a fortiori $\nu \stackrel{!}{\geq}^M w(\mu)$. Applying now the statement of the Theorem for M which is of semisimple rank 1, we conclude

$$U^M(F)\varpi^{w(\mu)} \cap K^M \varpi^\nu K^M \neq \emptyset. \tag{2.21}$$

From (2.20) and (2.21) we deduce that there exists $u_0^M \in U^M(F)$ and $k_1, k_2 \in K^M$ such that

$$u_0^M \cdot \varpi^{w(\mu)} = k_1 \cdot u^M \cdot \varpi^{w(\mu)+w(\beta^\vee)} \cdot k_2. \tag{2.22}$$

Let $u_0 = k_1 u_P k_1^{-1} \cdot u_0^M$. Since $k_1 \in M(F)$ we have $k_1 u_P k_1^{-1} \in U_P(F)$, hence $u_0 \in U(F)$. Then

$$\begin{aligned} u_0 \varpi^{w(\mu)} &= k_1 u_P k_1^{-1} \cdot k_1 u^M \cdot \varpi^{w(\mu)+w(\beta^\vee)} k_2 \\ &= k_1 u \varpi^{w(\mu)+w(\beta^\vee)} k_2 \in K \varpi^\lambda K. \end{aligned} \tag{2.23}$$

Hence $U(F)\varpi^{w(\mu)} \cap K\varpi^\lambda K \neq \emptyset$. By Lemma 2.1 this contradicts the hypothesis that $\mu \in \mathcal{Y}$, and this contradiction proves the theorem. □

The following variant of Theorem 1.1 was pointed out to me by Kottwitz. Denote by $v : B(F) \to X_*(A)$ the composition of the following homomorphisms

$$B(F) \longrightarrow T(F) \longrightarrow X_*(A). \tag{2.24}$$

Here the first map comes from identifying $T(F)$ with $B(F)/U(F)$. The second map is the restriction to $T(F)$ of the canonical valuation map from $T(E)$ to $X_*(T)$, where E is an unramified extension of F that splits T; this restriction obviously takes values in the subgroup of Galois invariants in $X_*(T)$ which may be identified with $X_*(A)$. Let $\lambda, \mu \in X_*(A)$ be dominant. Then, obviously,

$$C_{\lambda\mu} \neq 0 \iff v^{-1}(\mu) \cap K\varpi^\lambda K \neq \emptyset. \tag{2.25}$$

Theorem 2.4. *Let $\lambda \stackrel{!}{\geq} \mu$. Then for any $b \in v^{-1}(\lambda)$ there exists $b' \in B(F) \cap KbK$ with $v(b') = \mu$.*

Proof. We first note this statement implies Theorem 1.1. Indeed, applying the assertion to $b = \varpi^\lambda$ we find $b' = k_1\varpi^\lambda k_2$ of the form $b' = k_3 \cdot \varpi^\mu \cdot u$ with $k_1, k_2 \in K$ and $k_3 \in T(F) \cap K$, $u \in U(F)$. Hence

$$k_3^{-1} \cdot k_1 \varpi^\lambda k_2 = \varpi^\mu \cdot u, \tag{2.26}$$

as desired, cf. (2.3).

Now let us deduce Theorem 2.4 from Theorem 1.1. Let $b \in v^{-1}(\lambda)$. Let $\lambda' \in X_*(A) \cap \overline{C}$ with

$$b \in v^{-1}(\lambda) \cap K\varpi^{\lambda'}K. \tag{2.27}$$

By the converse to Theorem 1.1 we have $\lambda' \overset{!}{\geq} \lambda$. Hence a fortiori we have $\lambda' \overset{!}{\geq} \mu$. By Theorem 1.1 there exists $b' \in B(F) \cap K\varpi^{\lambda'}K$ with $v(b') = \mu$, cf. (2.3). But then $b' \in B(F) \cap KbK$ has $v(b') = \mu$. $\square$

3. The case of GL_n

In this section we show that the positivity statement in Theorem 1.1 follows in the case of GL_n from facts on symmetric functions contained in Macdonald's book [7], which will be quoted "loc. cit." in this section. In the case of GL_n, with the usual choices of $A = T = \mathbf{G}_m^n$ and B (diagonal resp. upper triangular matrices) we have

$$X_*(A) \cap \overline{C} = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n;\ \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}. \tag{3.1}$$

We identify $\mathcal{H}(T(F)/\!/T(F) \cap K) = \mathbf{C}[X_*(A)]$ with the ring of Laurent polynomials $\mathbf{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Lemma 3.1. *Let q be the number of elements in the residue field of F. Then*

$$b(f_\lambda) = q^{\langle \varrho, \lambda \rangle} \cdot P_\lambda(x_1, \dots, x_n;\ q^{-1}).$$

Here $\langle\ ,\ \rangle$ is the standard scalar product on $\mathbf{R}^n$, and $\varrho = \frac{1}{2}(n-1, n-3, \dots, 1-n)$. By $P_\lambda(x_1, \dots, x_n;\ t)$ we denote the Hall–Littlewood polynomial corresponding to λ (loc. cit., p. 104).

Proof. It is proved in loc. cit., p. 161, that there is an algebra homomorphism $b' : \mathcal{H}(G(F)/\!/K) \to \mathbf{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{S_n}$ with

$$b'(f_\lambda) = q^{-n(\lambda)} \cdot P_\lambda(x_1, \dots, x_n;\ q^{-1}). \tag{3.2}$$

Here

$$n(\lambda) = \sum_i (i-1)\lambda_i = \langle \sigma, \lambda \rangle \quad \text{with } \sigma = (0, 1, \dots, n-1). \tag{3.3}$$

Let us first check that the right hand side in Lemma 3.1 indeed defines an algebra homomorphism b_S. First note that

$$\langle \varrho, \lambda \rangle = -n(\lambda) + \frac{1}{2}(n-1) \cdot |\lambda|. \tag{3.4}$$

We therefore have for $\lambda, \lambda' \in X_*(A) \cap \overline{C}$,

$$\begin{aligned} b_S(f_\lambda) \cdot b_S(f_{\lambda'}) &= q^{\langle \varrho, \lambda \rangle} \cdot P_\lambda \cdot q^{\langle \varrho, \lambda' \rangle} \cdot P_{\lambda'} \\ &= q^{\frac{1}{2}(n-1)(|\lambda| + |\lambda'|)} \cdot (q^{-n(\lambda)} \cdot P_\lambda) \cdot (q^{-n(\lambda')} \cdot P_{\lambda'}) \\ &= q^{\frac{1}{2}(n-1)(|\lambda| + |\lambda'|)} \cdot b'(f_\lambda) \cdot b'(f_{\lambda'}) \\ &= q^{\frac{1}{2}(n-1)(|\lambda| + |\lambda'|)} \cdot b'(f_\lambda * f_{\lambda'}), \end{aligned} \tag{3.5}$$

where we used that b' is an algebra homomorphism. On the other hand,

$$f_\lambda * f_{\lambda'} = \sum g^{\mu}_{\lambda,\lambda'} f_\mu \tag{3.6}$$

(loc. cit., p. 160), where the sum runs only over μ with $|\mu| = |\lambda| + |\lambda'|$, cf. loc. cit., p. 88. Hence the RHS in (3.5) is equal to $b_S(f_\lambda * f_{\lambda'})$.

To prove the Lemma it now suffices to check that b and b_S coincide on the algebra generators given by the fundamental coweights $\lambda_r = (1^r, 0^{n-r})$ $(r = 1, \dots, n)$. But

$$\langle \varrho, \lambda_r \rangle = \frac{1}{2} r(n-r) \ \text{ and } \ b_S(f_{\lambda_r}) = q^{\frac{1}{2} r(n-r)} \cdot e_r = b(f_{\lambda_r}). \tag{3.7}$$

Here $e_r = e_{\lambda_r}$ is the r-th elementary symmetric function. □

From the expression in Lemma 3.1 we see that the transition matrix $C = M(b(f_\bullet), m_\bullet)$ is of the form

$$C = D \cdot M(P_\bullet(q^{-1}), m_\bullet), \tag{3.8}$$

where D is the diagonal matrix with entry $q^{\langle \varrho, \lambda \rangle}$ in the place (λ, λ) and where $M(P_\bullet(q^{-1}), m_\bullet)$ is the transition matrix between the Hall–Littlewood symmetric functions $P_\lambda(q^{-1})$ and the monomial symmetric functions m_μ. Therefore the next lemma gives the desired positivity statement.

Lemma 3.2. *The matrix $M(P_\bullet(q^{-1}), m_\bullet)$ is unitriangular with all coefficients above the diagonal positive.*

Proof. We have by loc. cit., p. 128

$$M(P_\bullet(q^{-1}), m_\bullet) = K(q^{-1})^{-1} \cdot K = K(q^{-1})^{-1} \cdot K(1), \tag{3.9}$$

where

$$K(t) = M(s_\bullet, P_\bullet(t)) \tag{3.10}$$

is the transition matrix between the Schur symmetric functions s_λ and the Hall–Littlewood polynomials $P_\mu(t)$. Obviously $K(0) = \mathrm{Id}$ and $K(1) = K$ is the Kostka matrix expressing the Schur symmetric funtions s_λ in terms of the monomial symmetric functions m_μ. However, for $\lambda \overset{!}{\geq} \mu$,

$$(K(t)^{-1} \cdot K)_{\lambda\mu} = \sum_T \psi_T(t) \tag{3.11}$$

where the sum runs over all tableaux T of shape λ and weight μ, cf. loc. cit., p. 126. Note that this sum is non-empty. The polynomial $\psi_T(t)$ is a product of polynomials of the form

$$1 - t^m, \quad m \geq 0, \tag{3.12}$$

cf. loc. cit., p. 120. Inserting q^{-1} for t we see $\psi_T(q^{-1}) > 0$ and the claim follows. □

Remark 3.3. A deep fact (due to Lascoux and Schutzenberger) mentioned on p. 129 of loc. cit. is that $K_{\lambda\mu}(t)$ is of the form

$$K_{\lambda\mu}(t) = \sum_T t^{c(T)}; \tag{3.13}$$

in particular all coefficients above the diagonal are *monic* polynomials of positive degree $n(\lambda) - n(\mu)$ with non-negative integer coefficients. However, there seems to be no connection between this statement and that of Lemma 3.2.

4. An application

In this section we return to the situation considered in the Introduction. Let L be the completion of the maximal unramified extension of F and let σ denote the relative Frobenius automorphism. Recall Kottwitz's set [4]

$$B(G) = G(L)/\sigma - \text{conjugacy}. \tag{4.1}$$

Kottwitz has classified this set. We shall only use a weak form of his result. We recall the *Newton map* [5]

$$\overline{\nu} : B(G) \longrightarrow \overline{C}. \tag{4.2}$$

We also recall the algebraic fundamental group of G, which can be identified with the character group of the center of the Langlands dual group,

$$\pi_1(G) = X^*(Z(\hat{G})). \tag{4.3}$$

Let $\pi_1(G)_\Gamma$ be the coinvariants for the action of the absolute Galois group $\Gamma = \mathrm{Gal}(\overline{F}/F)$. The second ingredient of the classification is the *Kottwitz map* [4]

$$\kappa : B(G) \longrightarrow \pi_1(G)_\Gamma. \tag{4.4}$$

The maps (4.2) and (4.4) combine into an *injective map*

$$(\overline{\nu}, \kappa) : B(G) \longrightarrow \overline{C} \times \pi_1(G)_\Gamma. \tag{4.5}$$

Remark 4.1. In the case of GL_n the first component of (4.5) determines the second. Indeed, if $\tilde{b} \in GL_n(L)$ is a representative of $b \in B(G)$, then $\overline{\nu}(b)$ is the slope vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ of the isocrystal $(L^n, \tilde{b}\sigma)$. On the other hand, $\kappa(b) = \sum \lambda_i \in \mathbf{Z} = \pi_1(GL_n)_\Gamma$. At the other extreme, if $G = T$ is a torus, the second component of (4.5) determines the first. In fact, $\overline{\nu}(b)$ is the image of $\kappa(b)$ under the composition of maps

$$\pi_1(T)_\Gamma \to \pi_1(T)_\Gamma \otimes \mathbf{Q} = X_*(T)^\Gamma_{\mathbf{Q}} = X_*(A)_{\mathbf{Q}} \to X_*(A)_{\mathbf{R}} = \overline{C}. \tag{4.6}$$

By the Bruhat–Tits theory the hyperspecial maximal compact subgroup K of $G(F)$ determines a unique hyperspecial compact subgroup $\tilde{K}$ of $G(L)$ with $\tilde{K} \cap G(F) = K$. In analogy with (2.1) we have the map

$$\mathrm{inv} : G(L)/\tilde{K} \times G(L)/\tilde{K} \longrightarrow X_*(T) \cap \overline{C}(T). \tag{4.7}$$

Here $\overline{C}(T)$ denotes the closed Weyl chamber in $X_*(T)_{\mathbf{R}}$ corresponding to B. This map is equivariant for the action of σ on the source and the target. We note that $\overline{C} = \overline{C}(T)^{\Gamma} = \overline{C}(T)^{\langle\sigma\rangle}$.

Let now $\mu \in X_*(T) \cap \overline{C}(T)$. By [8] we have associated to μ its image in $\pi_1(G)_{\Gamma}$,

$$\mu^{\natural} \in \pi_1(G)_{\Gamma} \tag{4.8}$$

and its mean value over the Galois group

$$\overline{\mu} = [\Gamma : \Gamma_{\mu}]^{-1} \cdot \sum_{\Gamma/\Gamma_{\mu}} \gamma\mu \in \overline{C}. \tag{4.9}$$

The group-theoretic version of Mazur's theorem proved in [8] may now be stated as follows:

Let $b \in B(G)$ and let $\tilde{b} \in G(L)$ be a representative of b. Let $x \in G(L)/\tilde{K}$ and put $\mu = \mathrm{inv}(x, \tilde{b}\sigma(x))$. Then

(i) $\mu^{\natural} = \kappa(b)$ (4.10)

(ii) $\overline{\nu}(b) \leq \overline{\mu}$.

Here we use the partial order $\leq$ on $\overline{C}$ coarser than $\overset{!}{\leq}$, where in (1.8) the integrality condition is dropped and arbitrary coefficients $n_{\alpha} \in \mathbf{R}_{\geq 0}$ are allowed. If the derived group is simply connected, then both partial orders coincide on $X_*(A)$.

We now give the application of Theorem 1.1 which is a converse to (4.10) in special circumstances. Note that in the following statement the roles of λ and μ have changed as compared to the previous sections.

Proposition 4.2. *Let $\lambda, \mu \in X_*(A) \cap \overline{C}$ with $\lambda \overset{!}{\leq} \mu$. Let $\tilde{b} = \varpi^{\lambda}$. Then there exists $x \in G(L)/\tilde{K}$ with*

$$\mathrm{inv}(x, \tilde{b}\sigma(x)) = \mu.$$

Proof. By Theorem 1.1 there exists $u \in U(F)$ such that

$$\mathrm{inv}(x_0, u\tilde{b}x_0) = \mu.$$

Here as before $x_0 \in G(F)/K$ denotes the base point. However, the two elements $\tilde{b}$ and $u\tilde{b}$ of $B(L)$ are σ-conjugate, [5], 3.6,

$$u\tilde{b} = g^{-1}\tilde{b}\sigma(g), \quad g \in B(L). \tag{4.11}$$

Hence $x = gx_0$ satisfies the desired identity. $\square$

Remark 4.3. Under the hypotheses of the corollary we have

$$\kappa(b) = \lambda^\natural = \mu^\natural \tag{4.12}$$

and

$$\overline{\nu}(b) = \lambda \leq \mu = \overline{\mu}. \tag{4.13}$$

Hence the Proposition is indeed a converse to (4.10). However, we have shown this converse only under special circumstances: namely, we essentially impose on μ to be Γ-*invariant* and on $b \in B(G)$ to have *integral* image under the Newton map. This is implied by the following lemma which brings into focus the content of Proposition 4.2.

Lemma 4.4. *Let G_{der} be simply connected. Let $b \in B(G)$ with $\overline{\nu}(b) \in X_*(A) \cap \overline{C}$. Let $\mu \in X_*(A) \cap \overline{C}$ and assume the conditions (i) and (ii) of (4.10) satisfied. Then b is the σ-conjugacy class of ϖ^λ where $\lambda = \overline{\nu}(b)$.*

Proof. Since G_{der} is simply connected, the condition (ii) of (4.10) implies that $\lambda \overset{!}{\leq} \mu$. But then λ and μ have the same image in G_{ab} and hence $\lambda^\natural = \mu^\natural$. Put $\tilde{b} = \varpi^\lambda$. Then $\overline{\nu}(\tilde{b}) = \lambda = \overline{\nu}(b)$ and $\kappa(\tilde{b}) = \lambda^\natural = \mu^\natural = \kappa(b)$. We conclude by the injectivity of (4.5). □

In the end of this section we state (in the simplified setting of this section) the general problem to which Proposition 4.2 gives a solution in a special case. Let $K \subset G(F)$ be a parahoric subgroup which fixes a facet of the Bruhat–Tits building of G_{ad} over F inside the apartment corresponding to A, and let $\tilde{K}$ be the corresponding parahoric subgroup of $G(L)$. Let $T(L)_1$ be the subgroup of units of $T(L)$ and let

$$\tilde{W} = N(L)/T(L)_1 \tag{4.14}$$

be the Iwahori Weyl group. Let

$$\tilde{W}^K = N(L) \cap \tilde{K}/T(L)_1. \tag{4.15}$$

The analogue of (4.7) in this more general situation is a map

$$\mathrm{inv} : G(L)/\tilde{K} \times G(L)/\tilde{K} \longrightarrow \tilde{W}^K \setminus \tilde{W}/\tilde{W}^K. \tag{4.16}$$

If $K = I$ is an Iwahori subgroup, then the target set of (4.16) ist $\tilde{W}$.

For $\tilde{b} \in G(L)$ we introduce *the generalized affine Deligne–Lusztig variety associated to $w \in \tilde{W}^K \setminus \tilde{W}/\tilde{W}^K$*

$$X_w(\tilde{b}\sigma) = \{x \in G(L)/\tilde{K};\ \mathrm{inv}(x, \tilde{b}\sigma(x)) = w\}. \tag{4.17}$$

An element $\tilde{b}' \in G(L)$ which is σ-conjugate to $\tilde{b}$ yields a set which is in bijection with the one corresponding to $\tilde{b}$. In the case of an Iwahori subgroup we call this set *the affine Deligne–Lusztig variety associated to $w \in \tilde{W}$*. The reason for this terminology suggested by Kottwitz is that $\tilde{W}$ is the affine analogue of the Weyl group of a reductive group over a finite field and $\tilde{b}\sigma$ is the analogue of the Frobenius automorphism in that case. However, in contrast to that case $X_w(\tilde{b}\sigma)$ may well be empty.

Problem 4.5. Determine the pairs $(\tilde{b}, w)$ for which $X_w(\tilde{b}\sigma) \neq \emptyset$.

It is likely that the set $X_w(\tilde{b}\sigma)$ may be given the structure of an algebraic variety locally of finite type over the residue field of L. Going further than Problem 4.5, one may ask for the dimension of $X_w(\tilde{b}\sigma)$. I do not even have a conjecture.

We conclude this section by establishing a link of Problem 4.5 with a result of Dabrowski [1]. For this assume that G is split over F. *For a suitable* Iwahori subgroup I of $G(F)$ (comp. the companion paper of Haines [2]) we consider the two disjoint sum decompositions,

$$G(F) = \coprod_{w \in \tilde{W}} U(F)wI, \quad G(F) = \coprod_{w \in \tilde{W}} IwI. \tag{4.18}$$

Assume now that $\tilde{b} \in G(F)$ and let $x = x(\tilde{b}) \in \tilde{W}$ be such that $\tilde{b} \in U(F)xI$. The proof of Proposition 4.2 shows

$$U(F)xI \cap IwI \neq \emptyset \Longrightarrow X_w(\tilde{b}\sigma) \neq \emptyset. \tag{4.19}$$

On the other hand, by [1], Prop. 3.2, the condition on the LHS of (4.19) is equivalent to

$$x \in G(w) \tag{4.20}$$

(the set of terminal elements of good subexpressions of w, comp. [2]). However, this is not the kind of answer one would like to Problem 4.5. Just as for Proposition 4.2, one would like an answer which involves $\overline{\nu}(b)$ and $\kappa(b)$, where $b \in B(G)$ denotes the σ-conjugacy class of $\tilde{b}$.

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References

[1] Dabrowski, R.: Comparison of the Bruhat and the Iwahori decompositions of a p-adic Chevalley group. J. Algebra **167**, 704–723 (1994)

[2] Haines, T.: On matrix coefficients of the Satake isomorphism: complements to the paper of Rapoport. manuscripta math. **101**, 167–174 (2000) (same issue, next paper)

[3] Kottwitz, R.: Shimura varieties and twisted orbital integrals. Math. Ann. **169**, 287–300 (1984)

[4] Kottwitz, R.: Isocrystals with additional structure. Comp. Math. **56**, 210–220 (1985)

[5] Kottwitz, R.: Isocrystals with additional structure II. Comp. Math. **109**, 255–339 (1997)

[6] Lusztig, G.: Singularities, character formulas and a q-analogue of weight multiplicities. Astérisque **101–102**, 208–229 (1983)

[7] Macdonald, I.: Symmetric functions and Hall polynomials. New York: Oxford U. Press, 1979

[8] Rapoport, M., Richartz, M.: On the classification and specialization of F-isocrystals with additional structure. Comp. Math. **103**, 153–181 (1996)

[9] Satake, I.: Theory of spherical functions on reductive algebraic groups over p-adic fields. Pub. Math. IHES **18**, 5–69 (1963)
[10] Stembridge, J.: The partial order of dominant weights. Adv. Math. **136**, 340–364 (1998)