Local models in the ramified case
I. The EL-case

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1 Introduction

In the arithmetic theory of Shimura varieties it is of interest to have a model over $\operatorname{Spec} \mathcal{O}_E$, where E is the completion of the reflex field at some finite prime of residue characteristic p. If the Shimura variety is the moduli space of abelian varieties of PEL type and the level structure at p is of parahoric type, such a model was defined in [RZ] by posing the moduli problem over $\operatorname{Spec} \mathcal{O}_E$. In [RZ] it was conjectured that this model is flat over $\operatorname{Spec} \mathcal{O}_E$, but in [P] it was shown that this is not always true. It still seems reasonable to expect this flatness property when the group G defining the Shimura variety splits over an unramified extension of $\mathbf{Q}_p$, and this is supported by the theorem of Görtz [G] in the EL-case. On the other hand, in [P] it is shown that this conjecture fails in general if the group G defining the Shimura variety has localization over $\mathbf{Q}_p$ isomorphic to the group of unitary similitudes corresponding to a *ramified* quadratic extension of $\mathbf{Q}_p$. Furthermore, a modified moduli problem was proposed in loc. cit. which defines a closed subscheme of the original model and which stands a better chance of being flat over $\operatorname{Spec} \mathcal{O}_E$.

Our main purpose of the present paper is to come to grips with the phenomenon of non-flatness by investigating the simplest case in which it can occur. As usual, the problem can be reduced to the consideration of the associated *local model* which locally for the étale topology around each point of the special fiber coincides with the model of the Shimura variety. In the rest of the paper we only consider the local models which can be defined in terms of linear algebra as schemes over the spectrum of a complete discrete valuation ring with perfect residue field. However, in view of the fact that the models proposed in [RZ] are not "the right ones" in general we shall term them *naive local models* and reserve the name of *local models* for certain closed subschemes of the naive local models defined in the body of the paper. Both of them have as generic fibers a closed subscheme of a Grassmannian, and as special fiber a closed subvariety of the affine partial flag variety over the residue field corresponding to the fixed parahoric.

As a by-product of our investigations, as they concern the special fibers, we also obtain several results on the structure of Schubert varieties in affine Grassmannians and their relation to nilpotent orbit closures.

The simplest case of a naive local model occurs for the *standard models*. Let us define them. Let F_0 be a complete discretely valued field with ring of integers $\mathcal{O}_{F_0}$ and perfect residue field. Let F/F_0 be a totally ramified extension of degree e, with ring of integers $\mathcal{O}_F$. Let V be a F-vector space of dimension d, and let Λ be a $\mathcal{O}_F$-lattice in V. Choose for each embedding φ of F into a separable closure F_0^{sep} of F_0 an integer r_φ with

$$(1.1)\qquad 0 \leq r_\varphi \leq d \ \ , \ \ \forall \varphi \ \ .$$

Put $r = \sum_\varphi r_\varphi$. Then the *standard model for* GL_d *and* $\mathbf{r} = (r_\varphi)$, denoted $M(\Lambda, \mathbf{r})$, parametrizes the points in the Grassmannian of subspaces $\mathcal{F}$ of rank r of Λ which are $\mathcal{O}_F$-stable and on which the representation of $\mathcal{O}_F$ is prescribed in terms of $\mathbf{r}$ (comp. (2.4)). It is defined over $\operatorname{Spec}\mathcal{O}_E$ where $E = E(V, \mathbf{r})$ is the reflex field, the field of definition of the prescribed representation of $\mathcal{O}_F$. Over F_0^{sep} the scheme $M = M(\Lambda, \mathbf{r})$ is isomorphic to the product over all φ of Grassmannians of subspaces of dimension r_φ in a vector space of dimension d. Over the residue field k of $\mathcal{O}_E$ the scheme $\overline{M} = M \otimes_{\mathcal{O}_E} k$ is a closed subvariety of the affine Grassmannian of GL_d over k, and is in fact a union of Schubert strata which are enumerated by the following dominant coweights of GL_d

$$(1.2)\qquad \mathcal{S}^0(r, e, d) = \{ \ \mathbf{s} = s_1 \geq \ldots \geq s_d ; \ e \geq s_1, \ s_d \geq 0, \ \sum_i s_i = r \ \} \ .$$

An easy dimension count shows that the dimension of the generic fiber and the special fiber are unequal, unless all integers r_φ differ by at most 1. In this last case we conjecture, and often can prove that M is flat over $\operatorname{Spec}\mathcal{O}_E$. In particular, in the Hilbert-Blumenthal case ($d = 2$, and $r_\varphi = 1$, $\forall \varphi$) the standard model coincides with the flat model constructed in this case in [DP]. In all cases when two r_φ differ from each other by more than one, M is not flat over $\operatorname{Spec}\mathcal{O}_E$.

To analyze the standard model we construct a diagram

$$(1.3)\qquad M \xleftarrow{\pi} \widetilde{M} \xrightarrow{\phi} N \ \ .$$

Here $\widetilde{M}$ is the GL_r-torsor over M which fixes a basis of the variable subspace $\mathcal{F}$. The morphism ϕ is defined as follows. Let π be a uniformizer of $\mathcal{O}_F$ which satisfies an Eisenstein polynomial $Q(\pi) = 0$. Let

$$(1.4)\qquad N = \{A \in \mathrm{Mat}_{r\times r} ; \ \det(T \cdot I - A) \equiv \prod_\varphi (T - \varphi(\pi))^{r_\varphi} \ , Q(A) = 0\}.$$

By considering the action of π on the variable subspace $\mathcal{F}$ and expressing it as a matrix in terms of the fixed basis of $\mathcal{F}$, we obtain the morphism ϕ. We show the following result (Theorem 4.1).

Theorem A: *The morphism ϕ is smooth of relative dimension rd.*

Using Theorem A many structure problems on M can be reduced to corresponding questions on N. After a finite extension $\mathcal{O}_E \to \mathcal{O}_K$, the variety N can be seen as a rank variety in the sense of Eisenbud and Saltman ([ES]), whereas the special fiber $\overline{N} = N \otimes_{\mathcal{O}_E} k$ is a subscheme of the nilpotent variety,

$$(1.5) \qquad \overline{N} = \{A \in \mathrm{Mat}_{r\times r};\ \det(T\cdot I - A) \equiv T^r,\ A^e = 0\}\ .$$

Using now results of Mehta and van der Kallen ([M-vdK]) on the structure of the closures of nilpotent conjugacy classes and basing ourselves on the methods of Eisenbud and Saltman, we use this reduction procedure to prove the following result (Theorem 5.4).

Theorem B: *Let $M^{\rm loc}$ be the scheme theoretic closure of $M \otimes_{\mathcal{O}_E} E$ in M. Then*

(i) $M^{\rm loc}$ is normal and Cohen-Macaulay.

(ii) The special fiber $\overline{M}^{\rm loc}$ is reduced, normal with rational singularities and is the union of all those strata of $\overline{M}$ which correspond to $\mathbf{s}$ in (1.2) with $\mathbf{s} \leq \mathbf{r}^\vee$ (dual partition to $\mathbf{r}$).

(iii) If the scheme $\overline{N}$ is reduced, then $M^{\rm loc} = M$ provided that all r_φ differ by at most 1.

We conjecture that the hypothesis made in (iii) is automatically satisfied, i.e. that $\overline{N}$ is always reduced. This is true by a classical result of Kostant if $r \leq e$ (in which case the second condition in (1.5) is redundant). For $e < r$ it seems a difficult problem. In a companion paper to ours, J. Weyman proves our conjecture for $e = 2$, and for arbitrary e when char $k = 0$.

Recall that Lusztig [L] has interpreted certain Schubert varieties in the affine Grassmannian of GL_r as a compactification of the nilpotent variety of GL_r (namely the Schubert variety corresponding to the coweight $(r, 0, \ldots, 0)$), compatible with the orbit stratifications of both varieties. In particular, as used by Lusztig in his paper, all singularities of nilpotent orbit closures occur in certain Schubert varieties in the affine Grassmannians. However, the main thrust of Theorem A (when restricted to the special fibers) goes in the other direction. Namely, we prove:

Theorem C: *Any Schubert variety in the affine Grassmannian of GL_d is smoothly equivalent to a nilpotent orbit closure for GL_r, for suitable r. In particular, it is normal with rational singularities.*

We point out the recent preprint by Faltings [F] in which he proves basic results (like normality) on Schubert varieties in affine flag varieties for arbitrary reductive groups.

The disadvantage of $M^{\rm loc}$ compared to M is that $M^{\rm loc}$ is not defined by a moduli problem in general. However, assume that $e = 2$ and order the embeddings so that $r_{\varphi_1} \geq r_{\varphi_2}$. If $r_{\varphi_1} = r_{\varphi_2}$, then $E = F_0$ and it follows from (iii) above and Weyman's result that $M^{\rm loc} = M$. If $r_{\varphi_1} > r_{\varphi_2}$, we may use φ_1 to identify E with F. Then using work of Strickland (see Cor. 5.10) we can see that $M^{\rm loc}$ is defined inside M by the following condition on $\mathcal{F}$,

$$(1.6) \qquad \wedge^{r_{\varphi_2}+1}\left(\pi - \varphi_1(\pi)\cdot \mathrm{Id}|\mathcal{F}\right) = 0\ .$$

In general, there is a connection to a conjecture of De Concini and Procesi describing the ideal of the closure of a nilpotent conjugacy class. Assuming this conjecture to be true and under a technical hypothesis we can write down a number of conditions which would define $M^{\rm loc}$ inside M. Since these conditions are highly redundant, the interest of such a description may be somewhat limited, however.

Let $N^{\rm loc}$ be the scheme theoretic closure of $N\otimes_{\mathcal{O}_E}E$ in N. Then the special fiber $\overline{N}^{\rm loc}$ can be identified with the closure in $\overline{N}$ of the nilpotent conjugacy class corresponding to $\mathbf{r}^\vee$. Let K be the Galois closure of F/F_0 with ring of integers $\mathcal{O}_K$ and residue field k'. Then $N\otimes_{\mathcal{O}_E}\mathcal{O}_K$ has a canonical resolution of singularities,

$$\mu:\mathcal{N}\longrightarrow N\otimes_{\mathcal{O}_E}\mathcal{O}_K\ , \tag{1.7}$$

i.e. a morphism with source a smooth $\mathcal{O}_K$-scheme $\mathcal{N}$ which is an isomorphism on the generic fiber and whose special fiber can be identified with the Springer-Spaltenstein resolution of $\overline{N}^{\rm loc}\otimes_k k'$. Using the fact that $\mu\otimes_{\mathcal{O}_K}k'$ is semi-small, one can calculate the direct image of the constant perverse sheaf on $\overline{\mathcal{N}}$ (Borho, MacPherson [BM], Braverman, Gaitsgory [BG]). Transporting back the result to M via Theorem A, we obtain the following result (comp. Theorem 7.1).

Theorem D: *Let $R\psi'$ denote the complex of nearby cycles of the $\mathcal{O}_K$-scheme $M\otimes_{\mathcal{O}_E}\mathcal{O}_K$. Then there is the following identity of perverse sheaves pure of weight 0 on $\overline{M}\otimes_k k'$,*

$$R\psi'[\dim\overline{M}](\tfrac{1}{2}\dim\overline{M})=\bigoplus_{\mathbf{s}\leq\mathbf{r}^\vee}K_{\mathbf{r}^\vee,\mathbf{s}}\cdot IC_{M_{\mathbf{s}}\otimes_k k'}\quad .$$

Here $K_{\mathbf{r}^\vee,\mathbf{s}}$ is a Kostka number.

We refer to section 7 for an explanation of the notation and for the question of descending this result from $\mathcal{O}_K$ to $\mathcal{O}_E$.

Via Theorem A, the resolution of singularities (1.7) is closely related to an analogous resolution of singularities of $M\otimes_{\mathcal{O}_E}\mathcal{O}_K$ whose special fiber can be identified with the Demazure resolution of the affine Schubert variety corresponding to the coweight $\mathbf{r}^\vee$ of GL_d. As pointed out to us by Ngô, from this identification one obtains another formula for the complex of nearby cycles $R\psi'$.

This concludes the discussion of our results on the standard models for GL_d and $\mathbf{r}$. The general naive local models of EL-type are projective schemes $M^{\rm naive}$ over $\operatorname{Spec}\mathcal{O}_E$ which at least over $\operatorname{Spec}\mathcal{O}_{\breve{E}}$ (where $\breve{E}$ is the completion of the maximal unramified extension of E) are closed subschemes of products of standard models. We define closed subschemes $M^{\rm loc}$ of $M^{\rm naive}$ by demanding that the projection into any standard model lies in the flat closure considered above. The modification of the flatness conjecture of [RZ] in the present case is that $M^{\rm loc}$ is flat over $\operatorname{Spec}\mathcal{O}_E$. We have nothing to say about this conjecture, except to point out that the description of $M^{\rm loc}(k)$ implicit in [KR], in terms of μ-permissible elements in the Iwahori-Weyl group is correct, as follows from Theorem B, (ii).

In the original version of this paper Theorem A was proved by exhibiting affine charts around the worst singularities of M over which the morphisms π and ϕ can be made completely explicit. The present simple proof of Theorem A is based on ideas from [FGKV]. We thank D. Gaitsgory for pointing it out to us. We would also like to thank G. Laumon, T. Haines and B.C. Ngô for helpful discussions on the material of sections 6 and 7. We are grateful to J. Weyman for his interest in our conjecture and for making his results available in a companion paper to ours. We thank G. Pfister for computer calculations using the symbolic algebra package REDUCE, which gave us the courage to elevate an initially naive question to the rank of a conjecture. We also thank the Max-Planck-Institut Bonn for its hospitality and support. The first named author was also partially supported by NSF grant DMS99-70378 and by a Sloan Research Fellowship.

General notational convention: If X is a scheme over Spec R and R' is an R-algebra, we often write $X \otimes_R R'$ or $X_{R'}$ for $X \times_{\operatorname{Spec} R} \operatorname{Spec} R'$.

2 Standard models for GL_d

In this section and the sections 3 – 7 we will use the following notation. Let F_0 be a complete discretely valued field with ring of integers $\mathcal{O}_{F_0}$ and uniformizer π_0, and perfect residue field. Let F be a totally ramified separable extension of degree e of F_0, with ring of integers $\mathcal{O}_F$. Let π be a uniformizer of $\mathcal{O}_F$ which is a root of the Eisenstein polynomial

$$(2.1) \qquad Q(T) = T^e + \sum_{k=0}^{e-1} b_k T^k, \quad b_0 \in \pi_0 \cdot \mathcal{O}_{F_0}^\times,\ b_k \in (\pi_0).$$

Let V be an F-vector space of dimension d and Λ an $\mathcal{O}_F$-lattice in V. We fix a separable closure $F_0^{\rm sep}$ of F_0. Finally, we choose for each embedding $\varphi : F \to F_0^{\rm sep}$ an integer r_φ with

$$(2.2) \qquad 0 \le r_\varphi \le d \quad .$$

Associated to these data we have the *reflex field* E, a finite extension of F_0 contained in $F_0^{\rm sep}$ with

$$(2.3) \qquad \mathrm{Gal}(F_0^{\rm sep}/E) = \{\sigma \in \mathrm{Gal}(F_0^{\rm sep}/F_0);\ r_{\sigma\varphi} = r_\varphi,\ \forall \varphi\} \quad .$$

Let $\mathcal{O}_E$ be the ring of integers in E. We now formulate a moduli problem on $(\mathrm{Sch}/\operatorname{Spec}\mathcal{O}_E)$:

$$(2.4) \qquad \begin{aligned} M(S) \quad = \quad \{ & \mathcal{F} \subset \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S;\ \text{a } \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S\text{-submodule,} \\ & \text{which is locally on } S \text{ a direct summand as} \\ & \mathcal{O}_S\text{-module, with } \det(a \mid \mathcal{F}) = \prod_\varphi \varphi(a)^{r_\varphi} \}. \end{aligned}$$

The last identity is meant as an identity of polynomial functions on $\mathcal{O}_F$ (comp. [K], [RZ]).

It is obvious that this functor is representable by a projective scheme $M = M(\Lambda, \mathbf{r})$ over $\operatorname{Spec} \mathcal{O}_E$. This scheme is called the *standard model for GL_d corresponding to $\mathbf{r} = (r_\varphi)_\varphi$ (and to F_0, F and π).*

Let us analyze the geometric general fiber and the special fiber of M. We have a decomposition

$$F \otimes_{F_0} F_0^{\text{sep}} = \bigoplus_{\varphi: F \to F_0^{\text{sep}}} F_0^{\text{sep}} \quad . \tag{2.5}$$

Correspondingly we get a decomposition of $V \otimes_{F_0} F_0^{\text{sep}}$ into F_0^{sep}-vector spaces

$$V \otimes_{F_0} F_0^{\text{sep}} = \bigoplus_{\varphi} V_\varphi \quad . \tag{2.6}$$

Each summand is of dimension d. The determinant condition in (2.4) can now be interpreted as saying that $M \otimes_{\mathcal{O}_E} F_0^{\text{sep}}$ parametrizes subspaces $\mathcal{F}_\varphi$ of V_φ, one for each φ, of dimension r_φ. In other words,

$$M \otimes_{\mathcal{O}_E} F_0^{\text{sep}} = \prod_{\varphi} \operatorname{Grass}_{r_\varphi}(V_\varphi) \quad . \tag{2.7}$$

In particular,

$$\dim M \otimes_{\mathcal{O}_E} E = \sum_{\varphi} r_\varphi (d - r_\varphi) \ . \tag{2.8}$$

Denote by k the residue field of $\mathcal{O}_E$. Let us consider $\overline{M} = M \otimes_{\mathcal{O}_E} k$. Put

$$W = \Lambda \otimes_{\mathcal{O}_{F_0}} k \quad , \quad \Pi = \pi \otimes \mathrm{id}_k \quad . \tag{2.9}$$

Then W is a k-vector space of dimension de and Π is a nilpotent endomorphism with $\Pi^e = 0$. The conditions that the subspace $\mathcal{F} \subset W$ gives a point of $\overline{M}$ translate into the following: $\mathcal{F}$ is Π-stable, $\dim_k \mathcal{F} = r := \sum_\varphi r_\varphi$ and $\det(T - \Pi \mid \mathcal{F}) \equiv T^r$. In other words $M \otimes_{\mathcal{O}_E} k$ is the closed subscheme of Π-stable subspaces $\mathcal{F}$ in $\operatorname{Grass}_r(W)$ which satisfy the above condition on the characteristic polynomial. We point out that the k-scheme $\overline{M} = M \otimes_{\mathcal{O}_E} k$ only depends on r, not on the partition (r_φ) of r.

3 Relation to the affine Grassmannian

We denote by $\widetilde{\operatorname{Grass}}_k$ the affine Grassmannian over k associated to GL_d. Recall ([BL]) that this is the Ind-scheme over $\operatorname{Spec} k$ whose k-rational points parametrize the $k[[\Pi]]$-lattices in $k((\Pi))^d$. Here $k[[\Pi]]$ denotes the power series ring in the indeterminate Π over k. On $\widetilde{\operatorname{Grass}}_k$ we have an action of the group scheme $\tilde{\mathcal{G}}$ over k with k-rational points equal to $GL_d(k[[\Pi]])$. The orbits of this action are finite-dimensional irreducible locally closed subvarieties (with the reduced scheme structure) which are parametrized by the dominant coweights of GL_d, i.e. by d-tuples of integers $\mathbf{s} = (s_1, \dots, s_d)$ with

$$s_1 \geq \dots \geq s_d \quad . \tag{3.1}$$

Furthermore ([BL]), if $\mathcal{O}_{\mathbf{s}}$ denotes the orbit corresponding to $\mathbf{s}$, we have

$$\dim \mathcal{O}_{\mathbf{s}} = \langle \mathbf{s}, 2\varrho \rangle \;\text{ and }\; \mathcal{O}_{\mathbf{s}'} \subset \text{ closure } (\mathcal{O}_{\mathbf{s}}) \Leftrightarrow \mathbf{s}' \leq \mathbf{s} \quad . \tag{3.2}$$

Here $2\varrho = (d-1, d-3, \dots, 1-d)$ and $\langle\ ,\ \rangle$ denotes the standard scalar product on $\mathbf{R}^d$. Furthermore, $\mathbf{s}' \leq \mathbf{s}$ denotes the usual partial order on dominant coweights, i.e.

$$\mathbf{s}' \leq \mathbf{s} \Leftrightarrow s_1' \leq s_1,\ s_1' + s_2' \leq s_1 + s_2, \dots, s_1' + \dots + s_d' = s_1 + \dots + s_d \quad .$$

Let us fix an isomorphism of $k[[\Pi]]$-modules

$$\Lambda \otimes_{\mathcal{O}_{F_0}} k \simeq (k[[\Pi]]/\Pi^e)^d \quad . \tag{3.3}$$

Then we obtain a closed embedding (of Ind-schemes),

$$\iota : M \otimes_{\mathcal{O}_E} k \longrightarrow \widetilde{\text{Grass}}_k \quad . \tag{3.4}$$

On k-rational points, ι sends a point of $M(k)$, corresponding to a $k[\Pi]$-submodule $\mathcal{F}$ of $\Lambda \otimes_{\mathcal{O}_{F_0}} k = (k[[\Pi]]/\Pi^e)^d$, to its inverse image $\tilde{\mathcal{F}}$ in $k[[\Pi]]^d$, which is a $k[[\Pi]]$-lattice contained in $k[[\Pi]]^d$,

$$\begin{array}{ccc} \tilde{\mathcal{F}} & \subset & k[[\Pi]]^d \\ \downarrow & & \downarrow \\ \mathcal{F} & \subset & (k[[\Pi]]/\Pi^e)^d \end{array} \quad . \tag{3.5}$$

The embedding ι is equivariant for the action of $\tilde{\mathcal{G}}$ in the following sense. Consider the smooth group scheme $\mathcal{G}$ over $\operatorname{Spec} \mathcal{O}_{F_0}$,

$$\mathcal{G} = \underline{\text{Aut}}_{\mathcal{O}_F}(\Lambda) \quad . \tag{3.6}$$

In fact, we will only need the base change of $\mathcal{G}$ to $\operatorname{Spec} \mathcal{O}_E$ which we denote by the same symbol. The group scheme $\mathcal{G}$ acts on M by

$$(g, \mathcal{F}) \longmapsto g(\mathcal{F}) \quad . \tag{3.7}$$

Let

$$\begin{aligned} \overline{\mathcal{G}} = \mathcal{G} \otimes_{\mathcal{O}_E} k \;&=\; \underline{\text{Aut}}_{k[\Pi]/\Pi^e}(\Lambda \otimes_{\mathcal{O}_{F_0}} k) \\ &\simeq\; GL_d(k[[\Pi]]/\Pi^e) \quad . \end{aligned} \tag{3.8}$$

In this way $\overline{\mathcal{G}}$ becomes a factor group of $\tilde{\mathcal{G}}$ and the equivariance of ι means that the action of $\tilde{\mathcal{G}}$ stabilizes the image of ι, that the action on this image factors through $\overline{\mathcal{G}}$ and that ι is $\overline{\mathcal{G}}$-equivariant.

A point of M with values in a field extension k' of k, corresponding to a Π-stable subspace $\mathcal{F}$ of $\Lambda \otimes_{\mathcal{O}_{F_0}} k'$, has image in $\mathcal{O}_{\mathbf{s}_\mathcal{F}}$, where $\mathbf{s}_\mathcal{F}$ is the Jordan type of the nilpotent endomorphism $\Pi \mid \mathcal{F}$. It follows that the orbit decomposition of $\overline{M} = M \otimes_{\mathcal{O}_E} k$ under the action of $\overline{\mathcal{G}}$ has the form

$$\overline{M} = \bigcup_{\mathbf{s}} M_{\mathbf{s}} \quad . \tag{3.9}$$

Here $M_{\mathbf{s}} \simeq \mathcal{O}_{\mathbf{s}}$ via ι and $\mathbf{s}$ ranges over the subset of (3.1) given by

$$\mathcal{S}^0(r,e,d)=\{\mathbf{s}=s_1 \geq s_2 \geq \ldots \geq s_d;\ e \geq s_1,\ s_d \geq 0,\ \sum_i s_i=r\} \ , \tag{3.10}$$

i.e. the partitions of r into at most d parts bounded by e. In all of the above we have ignored nilpotent elements.

Proposition 3.1 *The special fibre $\overline{M} = M \otimes_{\mathcal{O}_E} k$ is irreducible of dimension $dr - ec^2 - (2c+1)f$.*

Here we have written $r = c \cdot e + f$ with $0 \leq f < e$.

Proof. Among the coweights in $\mathcal{S}^0(r,e,d)$ there is a unique maximal one,

$$\mathbf{s}_{\max} = \mathbf{s}_{\max}(r,e) = (e,\ldots,e,f,0\ldots 0) = (e^c, f) \quad . \tag{3.11}$$

Hence $M_{\mathbf{s}_{\max}}$ is open and dense in $\overline{M}$. Its dimension is equal to $\langle \mathbf{s}_{\max}, 2\varrho\rangle$, which gives the result. □

Sometimes for convenience we shall number the embeddings φ in such a way that the $r_i = r_{\varphi_i}$ form a decreasing sequence $\mathbf{r} = (r_1 \geq r_2 \geq \ldots \geq r_e)$. Then $\mathbf{r}$ is a partition of r into at most e parts bounded by d.

Let $\mathbf{r}_{\min} = \mathbf{r}_{\min}(r,e) = \mathbf{s}_{\max}^\vee$ be the dual partition to $\mathbf{s}_{\max}$, i.e.

$$\mathbf{r}_{\min} = (c+1,\ldots,c+1,c,\ldots,c) = ((c+1)^f, c^{e-f}) \quad . \tag{3.12}$$

Proposition 3.2 *We have*

$$\dim\ M(\Lambda,\mathbf{r}) \otimes_{\mathcal{O}_E} E \leq \dim\ M(\Lambda,\mathbf{r}) \otimes_{\mathcal{O}_E} k \quad ,$$

with equality if and only if $\mathbf{r} = \mathbf{r}_{\min}(r,e)$ (after renumbering $\mathbf{r}$), i.e. iff all r_φ differ by at most one.

Proof. If $\mathbf{r} = \mathbf{r}_{\min}(r,e)$, then

$$\begin{aligned}
\dim\ M(\Lambda,\mathbf{r}) \otimes_{\mathcal{O}_E} E &= dr - \sum_\varphi r_\varphi^2 = dr - f(c+1)^2 - (e-f)c^2 \\
&= dr - f((c+1)^2 - c^2) - ec^2 \\
&= dr - ec^2 - (2c+1)f \\
&= \dim\ M(\Lambda,\mathbf{r}) \otimes_{\mathcal{O}_E} k \quad .
\end{aligned}$$

Here we used (2.8) in the first line and the previous proposition in the last line. Now let $\mathbf{r}$ be arbitrary and let $\mathbf{t} = \mathbf{r}^\vee$ be the dual partition to $\mathbf{r}$, i.e.

$$t_1 = \#\{\varphi; r_\varphi \geq 1\},\ t_2 = \#\{\varphi; r_\varphi \geq 2\}\ ,\ \text{etc.} \tag{3.13}$$

By (2.2) the partition $\mathbf{t}$ lies in $\mathcal{S}^0(r,e,d)$. Hence $\mathbf{t} \leq \mathbf{s}_{\max}(r,e)$ with equality only if $\mathbf{r} = \mathbf{r}_{\min}(r,e)$. Hence, if $\mathbf{r} \neq \mathbf{r}_{\min}(r,e)$, we have

$$\dim M_{\mathbf{t}} < \dim M_{\mathbf{s}_{\max}} = \dim \overline{M}\ \ . \tag{3.14}$$

Now

$$\begin{aligned} \dim M_{\mathbf{t}} &= \langle \mathbf{t}, 2\varrho\rangle = t_1(d-1) + t_2(d-3) + \ldots + t_d(1-d) \\ &= d \cdot \sum_i^d t_i - \sum_{i=1}^d (2i-1)t_i\ \ . \end{aligned}$$

But $\sum_i t_i = \sum_\varphi r_\varphi = r$. The second sum on the right hand side can be written as a sum of contributions of each φ. Each fixed φ contributes $\sum_{j=1}^{r_\varphi}(2j-1) = r_\varphi^2$. Hence

$$\dim M_{\mathbf{t}} = dr - \sum_\varphi r_\varphi^2 = \dim M(\Lambda, \mathbf{r}) \otimes_{\mathcal{O}_E} E\ \ . \tag{3.15}$$

Taking into account (3.14), the result follows. □

Corollary 3.3 *If $\mathbf{r} \neq \mathbf{r}_{\min}(r,e)$, the corresponding standard model M is not flat over* Spec $\mathcal{O}_E$. □

There remains the question whether if $\mathbf{r} = \mathbf{r}_{\min}(r,e)$, the corresponding standard model is flat over $\mathcal{O}_E$; we will return to it in section 5.

We end this section with the following remark. Among all strata of $\overline{M}$ enumerated by $\mathcal{S}^0(r,e,d)$ there is a unique minimal one,

$$\mathbf{s}_{\min} = \mathbf{s}_{\min}(r,d) = ((u+1)^j, u^{d-j})\ \ . \tag{3.16}$$

Here we have written $r = u \cdot d + j,\ 0 \leq j < d$. The corresponding stratum $M_{\mathbf{s}_{\min}}$ is closed and lies in the closure of any other stratum.

4 Relation to the nilpotent variety

In this section we describe the connection of the standard models for GL_d with the nilpotent variety. Consider the GL_r-torsor $\pi : \widetilde{M} \to M$ where

$$\widetilde{M}(S) = \{(\mathcal{F}, \psi)\ ;\ \mathcal{F} \in M(S), \psi : \mathcal{F} \xrightarrow{\simeq} \mathcal{O}_S^r\} \tag{4.1}$$

and GL_r acts via $(\mathcal{F}, \psi) \mapsto (\mathcal{F}, \gamma \cdot \psi)$. Then $\mathcal{G}$ acts on $\widetilde{M}$ via

$$(g, (\mathcal{F}, \psi)) \longmapsto (g(\mathcal{F}), \psi \cdot g^{-1})\ \ ,$$

and the morphism π is equivariant.

Note that $\prod_\varphi (T - \varphi(\pi))^{r_\varphi}$ has coefficients in $\mathcal{O}_E$; let us define a scheme $N = N(\mathbf{r})$ over $\operatorname{Spec} \mathcal{O}_E$ via the functor which to the $\mathcal{O}_E$-algebra R associates the set

$$\{A \in M_{r\times r}(R);\ \det(T\cdot I - A) \equiv \Pi_\varphi (T-\varphi(\pi))^{r_\varphi},\ Q(A){=}0\}. \tag{4.2}$$

The group scheme GL_r acts on N via conjugation $A \mapsto \gamma\cdot A\cdot \gamma^{-1}$. There is a morphism

$$\phi : \widetilde{M} \to N, \quad \phi((\mathcal{F},\psi)) = \psi(\pi|\mathcal{F})\psi^{-1} \quad , \tag{4.3}$$

which is GL_r-equivariant. We have for $g \in \mathcal{G}$,

$$\phi(g\cdot(\mathcal{F},\psi)) = \psi g^{-1}\cdot(\pi|g(\mathcal{F}))\cdot(\psi g^{-1})^{-1} = \psi\cdot(\pi|\mathcal{F})\cdot\psi^{-1} = \phi((\mathcal{F},\psi))$$

since g commutes with π. Therefore, the morphism ϕ is $\mathcal{G}$-equivariant with trivial $\mathcal{G}$-action on the target N. We therefore obtain a diagram of morphisms

$$M \xleftarrow{\ \pi\ } \widetilde{M} \xrightarrow{\ \phi\ } N \quad , \tag{4.4}$$

which is equivariant for the action of $\mathcal{G}\times GL_r$, where the first factor acts trivially on the right hand target and the second factor acts trivially on the left hand target.

Since $Q(T) = \prod_\varphi (T-\varphi(\pi))$ with all elements $\varphi(\pi)$ pairwise distinct, we see that the generic fiber of N consists of all *semisimple* matrices with eigenvalues $\varphi(\pi)$ with multiplicity r_φ. It follows that GL_r acts transitively on $N\otimes_{\mathcal{O}_E} E$, which is smooth of dimension $r^2 - \Sigma_\varphi r_\varphi^2$ over $\operatorname{Spec} E$. The special fiber $\overline{N}$ of N is the subscheme of $\mathrm{Mat}_{r\times r}$ defined by the equations

$$\overline{N} = \{A \in \mathrm{Mat}_{r\times r};\ A^e = 0,\ \det(T\cdot I - A) \equiv T^r\} \quad . \tag{4.5}$$

This is a closed subscheme of the variety of nilpotent $r\times r$ matrices (which is defined by $\det(T\cdot I - A) \equiv T^r$).

The following result exhibits a close connection between the standard model and the nilpotent variety.

Theorem 4.1 *The morphism $\phi : \widetilde{M} \longrightarrow N$ is smooth of relative dimension rd.*

Proof. Let $\mathfrak{Mod} = \mathfrak{Mod}(\mathcal{O}_F, \mathbf{r})$ be the algebraic stack over $\operatorname{Spec} \mathcal{O}_E$ given by the fibered category of $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-modules $\mathcal{F}$ which are locally free $\mathcal{O}_S$-modules of rank r (comp. [LMoB] 3.4.4, 4.6.2.1) and for which

$$\det(T\cdot I - \pi|\mathcal{F}) \equiv \prod_\phi (T - \phi(\pi))^{r_\phi}$$

as polynomials. There is an isomorphism

$$\mathfrak{Mod} \;\simeq\; [N/GL_r]$$

(where the quotient stack is for the conjugation action) given by $\mathcal{G} \mapsto$ the conjugation GL_r-torsor of matrices A giving the action of π on $\mathcal{G}$. By the definition of the quotient stack the diagram (4.4) corresponds to a morphism

$$\bar{\phi} : M \longrightarrow [N/GL_r] \quad . \tag{4.6}$$

The morphism ϕ is smooth if and only if $\bar{\phi}$ is a smooth morphism of algebraic stacks. Under the identification above the morphism $\bar{\phi}$ becomes

$$\bar{\phi} : M \to \mathfrak{Mod} \quad ; \quad (\mathcal{F} \subset \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \mapsto \mathcal{F} \ .$$

Now consider the morphism $\phi^* : \widetilde{M} \to N$ obtained by composing ϕ with the automorphism of N given by $A \mapsto {}^tA$; ϕ is smooth if and only if ϕ^* is smooth. Set $\mathcal{F}^* := Hom_{\mathcal{O}_S}(\mathcal{F}, \mathcal{O}_S)$ which is also naturally an $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-module. By regarding $\widetilde{M}$ as the GL_r-torsor over M giving the $\mathcal{O}_S^r$-trivializations of the dual $\mathcal{F}^*$ we see that ϕ^* descends to

$$\bar{\phi}^* : M \to \mathfrak{Mod} \quad ; \quad (\mathcal{F} \subset \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \mapsto \mathcal{F}^* \ .$$

As before, it is enough to show that $\bar{\phi}^*$ is smooth.

The stack $\mathfrak{Mod}$ supports the universal $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_{\mathfrak{Mod}}$-module $\mathcal{F}$. Let us consider the $\mathcal{O}_{\mathfrak{Mod}}$-module

$$\mathcal{T} = Hom_{\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_{\mathfrak{Mod}}}(\Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_{\mathfrak{Mod}}, \mathcal{F}) \simeq \mathcal{F}^{\oplus d} \ .$$

This defines a vector bundle $\mathfrak{V}(\mathcal{T}^*)$ over $\mathfrak{Mod}$ (see [LMoB] 14.2.6). The structure morphism $\mathfrak{V}(\mathcal{T}^*) \to \mathfrak{Mod}$ is representable and smooth. Its effect on objects is given by

$$(\mathcal{H},\ \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \xrightarrow{f} \mathcal{H}) \mapsto \mathcal{H} \ .$$

Consider now the perfect $\mathcal{O}_{F_0}$-bilinear pairing

$$(\ ,\ ) : \mathcal{O}_F \times \mathcal{O}_F \to \mathcal{O}_{F_0} \quad ; \quad (x, y) = \mathrm{Tr}_{F/F_0}(\delta^{-1} xy)$$

where δ is an $\mathcal{O}_F$-generator of the different $\mathcal{D}_{F/F_0}$. This pairing gives an $\mathcal{O}_F$-module isomorphism $\mathcal{O}_F^* = Hom_{\mathcal{O}_{F_0}}(\mathcal{O}_F, \mathcal{O}_{F_0}) \simeq \mathcal{O}_F$. Now choose an $\mathcal{O}_F$-module isomorphism $\Lambda \simeq \mathcal{O}_F^d$. We obtain functorial $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-module isomorphisms $(\Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S)^* \simeq \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ and a morphism $i : M \to \mathfrak{V}(\mathcal{T}^*)$ given by

$$(\mathcal{F} \subset \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S) \ \mapsto \ (\mathcal{F}^*,\ \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \simeq (\Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S)^* \to \mathcal{F}^*).$$

We therefore see that i is representable and an open immersion (it gives an isomorphism between M and the open substack of $\mathfrak{V}(\mathcal{T}^*)$ given by the full subcategory of objects for which the morphism f is surjective). The morphism $\bar{\phi}^*$ is the composition

$$M \to \mathfrak{V}(\mathcal{T}^*) \to \mathfrak{Mod}$$

and therefore is smooth. The relative dimension of ϕ is equal to the relative dimension of $\bar{\phi}$; this, in turn, is equal to the relative dimension of the composition above. However, this is the same as the relative dimension of the vector bundle $\mathfrak{V}(\mathcal{T}^*) \to \mathfrak{Mod}$ which is equal to rd.

Remarks 4.2 (i) The proof of Theorem 4.1 given above follows ideas which appear in the paper [FGKV] (see especially loc.cit. §4.2) and were brought to the attention of the authors by D. Gaitsgory. A previous version of the paper contained a more complicated proof which used explicit matrix calculations to describe affine charts for the scheme M.

(ii) Let us consider the conjugation action of GL_r on the special fibre $\overline{N}$. The orbits of this action are parametrized by

$$\mathcal{S}(r,e) = \{\mathbf{s} = (s_1 \geq \ldots \geq s_r);\ e \geq s_1,\ s_r \geq 0,\ \Sigma_i s_i = r\} \quad . \tag{4.7}$$

We denote the corresponding orbit by $N_{\mathbf{s}}$. Again $N_{\mathbf{s}'}$ lies in the closure of $N_{\mathbf{s}}$ if and only if $\mathbf{s}' \leq \mathbf{s}$. Obviously $\mathcal{S}^0(r,e,d) \subset \mathcal{S}(r,e)$ and the $\mathcal{G} \times GL_r$-equivariance of the diagram (4.4) shows that for $\mathbf{s} \in \mathcal{S}^0(r,e,d)$,

$$\phi(\pi^{-1}(M_{\mathbf{s}})) = N_{\mathbf{s}} \quad . \tag{4.8}$$

In particular, the image of $\tilde{M} \otimes_{\mathcal{O}_E} k$ under ϕ is the union of orbits corresponding to $\mathbf{s} \in \mathcal{S}^0(r,e,d)$ and the diagram (4.4) induces an injection of the set of $\mathcal{G}$-orbits in $\overline{M}$ into the set of GL_r-orbits in $\overline{N}$. It is easy to see that the complement of $\mathcal{S}^0(r,e,d)$ in $\mathcal{S}(r,e)$ is closed under the partial order $\leq$ on $\mathcal{S}(r,e)$, i.e. corresponds to a closed subset of $\overline{N}$.

(iii) The dimension of $N_{\mathbf{s}}$ is given by the formula

$$\dim N_{\mathbf{s}} = r^2 - \sum_{i=1}^{e} r_i^2 \quad , \tag{4.9}$$

where $r_1 \geq \ldots \geq r_e \geq 0$ is the dual partition to $\mathbf{s}$.
For $\mathbf{s}$ in $\mathcal{S}^0(r,e,d)$, this formula is compatible with the one of (3.14) via Theorem 4.1: we have

$$\langle \mathbf{s}, 2\varrho \rangle + r^2 - rd = r^2 - \sum_{i=1}^{e} r_i^2 \quad ,$$

comp. (3.15) (which is "dual" to (4.9)).

(iv) The equivariance of (4.4) can be rephrased (via descent theory for $\mathcal{G}$-torsors) by saying that the morphism (4.6) factors through a morphism of algebraic stacks

$$[M/\mathcal{G}] \longrightarrow [N/GL_r] \quad . \tag{4.10}$$

Note that both these stacks have only finitely many points. (The set of points in the special fiber of $[M/\mathcal{G}]$ resp. $[N/GL_r]$ is $\mathcal{S}^0(r,e,d)$ resp. $\mathcal{S}(r,e)$.)

Corollary 4.3 *Let $r_\varphi \leq 1$, $\forall \varphi$. Then the standard model for GL_d corresponding to $\mathbf{r}$ is flat over* Spec $\mathcal{O}_E$*, with special fiber a normal complete intersection variety.*

PROOF. In this case the first condition $A^e = 0$ in the definition (4.5) of $\overline{N}$ is a consequence of the second condition. Hence $\overline{N}$ is the variety of nilpotent matrices, which is a reduced and irreducible, normal and complete intersection variety. On the other hand, obviously $\mathbf{r} = \mathbf{r}_{\min}$, hence by Proposition 3.2 we have dim $N \otimes_{\mathcal{O}_E} E =$ dim $N \otimes_{\mathcal{O}_E} k$. hence the generic point of $\overline{N}$ is the specialization of a point of $N \otimes_{\mathcal{O}_E} E$. Since $N \otimes_{\mathcal{O}_E} k$ is reduced, the flatness of N follows from EGA IV 3.4.6.1. By Theorem 4.1 this implies the corresponding assertions for M. $\square$

Remark 4.4 Let us fix an isomorphism $\Lambda \simeq \mathcal{O}_F^d$ and suppose that $r = d = e$. Write

$$\Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \simeq (\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S)^r = 1 \cdot \mathcal{O}_S^r \oplus \pi \cdot \mathcal{O}_S^r \oplus \cdots \oplus \pi^{r-1} \cdot \mathcal{O}_S^r \ .$$

Now let us consider $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-submodules $\mathcal{F} \subset \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ which are locally free $\mathcal{O}_S$-locally direct summands and are such that the composition

$$F \ : \ \mathcal{F} \subset \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \xrightarrow{pr} \pi^{r-1} \cdot \mathcal{O}_S^d$$

is a surjection (and therefore an isomorphism). The inverse of the isomorphism F can be written

$$F^{-1}(\pi^{r-1} \cdot v) = (1 \cdot f_{r-1}(v),\ \pi \cdot f_{r-2}(v),\ \ldots,\ \pi^{r-1} \cdot f_0(v))$$

where $f_i : \mathcal{O}_S^r \to \mathcal{O}_S^r$, $i = 0, \cdots, r-1$, are $\mathcal{O}_S$-linear homomorphisms and $f_0 = \mathrm{Id}$. Recall $Q(T) = T^e + \sum_{k=0}^{e-1} b_k T^k$. The condition that $\mathcal{F}$ is stable under multiplication by π translates to

$$f_{k+1} - b_{r-k-1} = f_k \cdot (f_1 - b_{r-1} \cdot f_0), \quad k = 1, \ldots, r-2 \ ,$$

$$-b_0 = f_{r-1} \cdot (f_1 - b_{r-1} \cdot f_0) = 0 \ .$$

Therefore, all the f_k are determined by f_1 and we have $Q(f_1 - b_{r-1} \cdot \mathrm{Id}) = 0$. We also have $F \cdot (\pi|\mathcal{F}) \cdot F^{-1} = f_1 - b_{r-1} \cdot \mathrm{Id}$. Using these facts, we see that after choosing a basis of $\mathcal{O}_S^r$, the modules $\mathcal{F}$ for which the composition F is an isomorphism are in 1–1 correspondence with $r \times r$-matrices A which satisfy $Q(A) = 0$. Let U denote the open subscheme of M whose S-points correspond to modules $\mathcal{F}$ for which the homomorphism F above is an isomorphism. In terms of the diagram (4.4), the above implies that, in this case, there is a section $s : N \to \widetilde{M}$ to the morphism $\phi : \widetilde{M} \to N$ such that the composition $\pi \circ s : N \to M$ is an open immersion identifying N with U. The smoothness of the morphism ϕ then amounts to the smoothness of the conjugation action morphism $N \times GL_r \to N$. Also, since M is projective, this shows that, in this case, the local model M may be considered as a relative compactification over Spec $\mathcal{O}_E$of the variety N. This last result for the special fibers is precisely the scheme-theoretic version of Lusztig's result [L], section 2. Hence the special fiber of any standard model for GL_d, for which $r = d$ and $e = d$ (they are all identical), may be considered as a compactification of the nilpotent variety. We return in section 6 to the consequences of Theorem 4.1 for the special fibers.

5 The canonical flat model

We have seen in Corollary 3.3 that a standard model is rarely flat over $\operatorname{Spec}\mathcal{O}_E$. By Theorem 4.1 the same can be said of the scheme N. In this section we will first show that the flat scheme theoretic closure of the generic fiber $N\otimes_{\mathcal{O}_E}E$ in N has good singularities. The idea is to use a variant of the Springer resolution of the nilpotent variety, as also in the work of Eisenbud and Saltman ([E-S]).

Recall our notations from the beginning of section 2. Let K be the Galois hull of F inside F_0^{sep}. Let us order the different embeddings $\phi : F \to K$. Then we can write

$$P(T) = \prod_{\phi}(T-\phi(\pi))^{r_\phi} = \prod_{i=1}^{e}(T-a_i)^{r_i}\ ,\ Q(T) = \prod_{i=1}^{e}(T-a_i)\ . \tag{5.1}$$

Here $\phi(\pi) = a_i \in \mathcal{O}_K$ are distinct roots.

Let us set $n_k = \sum_{i=1}^{k} r_i$, for $1 \le k \le e$. Let $\boldsymbol{\mathcal{F}}$ be the scheme which classifies flags:

$$(0) = \mathcal{F}_e \subset \mathcal{F}_{e-1} \subset \cdots \subset \mathcal{F}_0 = \mathcal{O}_S^r \tag{5.2}$$

where $\mathcal{F}_k$ is locally on S a direct summand of $\mathcal{O}_S^r$ of corank n_k. Following [E-S], we consider the subscheme $\mathcal{N}$ of $((\mathrm{Mat}_{r\times r})\times\boldsymbol{\mathcal{F}})_{\mathcal{O}_K}$ classifying pairs $(A,\{\mathcal{F}_\bullet\})$ such that

$$(A - a_k\cdot I)\cdot\mathcal{F}_{k-1} \subset \mathcal{F}_k, \quad 1 \le k \le e\ . \tag{5.3}$$

The scheme $\mathcal{N}$ supports an action GL_r by $g\cdot(A,\{\mathcal{F}_i\}) = (gAg^{-1},\{g(\mathcal{F}_i)\})$.

Obviously this is a variant of the Grothendieck-Springer construction. It differs from the original in two aspects: we consider partial flags instead of complete flags, and we fix the (generalized) eigenvalues a_i of A.

Lemma 5.1 *i) $\mathcal{N}$ is smooth over* $\operatorname{Spec}\mathcal{O}_K$.

ii) There is a projective GL_r-equivariant morphism $\mu : \mathcal{N} \to N\otimes_{\mathcal{O}_E}\mathcal{O}_K$, given by $(A,\{\mathcal{F}_i\}) \mapsto A$.

iii) The morphism $\mu_{\operatorname{Spec}K}$ is an isomorphism between the generic fibers $\mathcal{N}\otimes_{\mathcal{O}_K}K$ and $N\otimes_{\mathcal{O}_E}K$.

Proof. Part (i) follows from the fact that the projection to the second factor is smooth

$$\mathcal{N} \longrightarrow \boldsymbol{\mathcal{F}}\ , \tag{5.4}$$

comp. [E-S], p. 190 (the fiber over $\{\mathcal{F}_i\}$ can be identified with the cotangent space of $\boldsymbol{\mathcal{F}}$ at $\{\mathcal{F}_i\}$). Now suppose that A is in $M_{r\times r}(R)$ for some $\mathcal{O}_K$-algebra R and that locally on Spec R there is a filtration $\{\mathcal{F}_\bullet\}$ of R^r as described above. Then the characteristic polynomial of A is equal to $P(T)$ and we have

$$\prod_{k=1}^{N}(A - a_k\cdot I) = 0 \ \in\ M_{r\times r}(R). \tag{5.5}$$

This implies $Q(A) = 0 \in M_{r\times r}(R)$, cf. (5.1). This shows that the natural morphism $\mathcal{N} \to (\mathrm{Mat}_{r\times r})_{\mathcal{O}_K}$ factors through $N\otimes_{\mathcal{O}_E}\mathcal{O}_K$; the claim (ii) follows. Now $N\otimes_{\mathcal{O}_E}K$

consists of all semisimple matrices with eigenvalues a_i with multiplicity r_i (comp. remarks before (4.5)). Hence the fiber of $\mu_{\mathrm{Spec}\ K}$ over A is the filtration $\mathcal{F}_\bullet$ associated to the eigenspace decomposition corresponding to A, i.e., is uniquely determined by A. $\square$

Now set $N' = \mathrm{Spec}(\mu_*(\mathcal{O}_{\mathcal{N}}))$ and denote by $\mu(\mathcal{N})$ the scheme-theoretic image of $\mu : \mathcal{N} \to N \otimes_{\mathcal{O}_E} \mathcal{O}_K$; this is a reduced closed subscheme of $N \otimes_{\mathcal{O}_E} \mathcal{O}_K$ since $\mathcal{N}$ is smooth (therefore reduced) and μ is proper. The scheme N' supports an action of GL_r; there is a natural GL_r-equivariant morphism $q' : N' \to N \otimes_{\mathcal{O}_E} \mathcal{O}_K$. The morphism q' is finite and factors as follows

$$q' : N' \to \mu(\mathcal{N}) \to N \otimes_{\mathcal{O}_E} \mathcal{O}_K. \tag{5.6}$$

Lemma 5.1 (iii) implies that $q'_{\mathrm{Spec}\, K}$ is an isomorphism.

Let k' be the residue field of $\mathcal{O}_K$.

Proposition 5.2 *a) The special fiber $\overline{N}' = N' \otimes_{\mathcal{O}_K} k'$ is normal and has rational singularities.*

b) The scheme N' is normal, Cohen-Macaulay and flat over $\mathrm{Spec}\, \mathcal{O}_K$.

c) $N' = \mu(\mathcal{N})$.

d) N' is the scheme theoretic closure of $N \otimes_{\mathcal{O}_E} K$ in $N \otimes_{\mathcal{O}_E} \mathcal{O}_K$. Its special fiber is the reduced closure of the orbit $N_{\mathbf{t}}$, where $\mathbf{t} = \mathbf{r}^\vee$ is the dual partition to $\mathbf{r} = (r_i)$.

Proof. This follows closely the arguments of [E-S] (see p. 190-192, proof of Theorem 2.1) with new input the results of Mehta-van der Kallen ([M-vdK]). They show (using Frobenius splitting) that the closure of the orbit of a nilpotent matrix is normal and Cohen-Macaulay also in positive characteristic. For the duration of this proof, we will denote by ϖ a uniformizer of $\mathcal{O}_K$. We will first consider the situation over $\mathrm{Spec}\, k'$ and use a bar to denote base change from $\mathrm{Spec}\, \mathcal{O}_K$ to $\mathrm{Spec}\, k'$. Consider the morphism $\overline{\mu} : \overline{\mathcal{N}} \to \overline{N}$ and the scheme $\mathrm{Spec}\, (\overline{\mu}_*(\mathcal{O}_{\overline{\mathcal{N}}})) \to \overline{N}$. The morphism $\overline{\mu}$ factors as

$$\overline{\mu} : \overline{\mathcal{N}} \to \mathrm{Spec}\, (\overline{\mu}_*(\mathcal{O}_{\overline{\mathcal{N}}})) \to \overline{\mu}(\overline{\mathcal{N}}) \to \overline{N}\ , \tag{5.7}$$

where again $\overline{\mu}(\overline{\mathcal{N}})$ denotes the scheme theoretic image of $\overline{\mu}$. The morphism $\overline{\mathcal{N}} \to \overline{\mu}(\overline{\mathcal{N}})$ is one to one on the open subset of $\overline{\mathcal{N}}$ of those $(A, \{\mathcal{F}_\bullet\})$ such that $A\mathcal{F}_{k-1} = \mathcal{F}_k$, $k = 1, \dots, e$ and so it is birational; therefore the morphism $\mathrm{Spec}\, (\overline{\mu}_*(\mathcal{O}_{\overline{\mathcal{N}}})) \to \overline{\mu}(\overline{\mathcal{N}})$ is finite and birational. Since $\overline{\mathcal{N}}$ is reduced, $\overline{\mu}(\overline{\mathcal{N}})$ is also reduced. As in [E-S], we see that $\overline{\mu}(\overline{\mathcal{N}}) \subset (\mathrm{Mat}_{r\times r})_{k'}$ is the reduced closure of the conjugation orbit $N_{\mathbf{t}}$ of the Jordan form for the dual partition $\mathbf{t} = \mathbf{r}^\vee$. By [M-vdK], the closure of $N_{\mathbf{t}}$ is normal and has rational singularities. We conclude that $\mathrm{Spec}\, (\overline{\mu}_*(\mathcal{O}_{\overline{\mathcal{N}}})) \to \overline{\mu}(\overline{\mathcal{N}})$ is an isomorphism which we use to identify these two schemes. We will first show that $\mathrm{Spec}\, (\overline{\mu}_*(\mathcal{O}_{\overline{\mathcal{N}}})) = \overline{\mu}(\overline{\mathcal{N}})$ actually gives the special fiber of N'. The statement (a) then follows from the results of Mehta-van der Kallen. The cohomology exact sequence obtained by applying μ_* to

$$0 \to \mathcal{O}_{\mathcal{N}} \stackrel{\varpi}{\to} \mathcal{O}_{\mathcal{N}} \to \mathcal{O}_{\overline{\mathcal{N}}} \to 0$$

gives an injective homomorphism

$$\mathcal{O}_{N'}/\varpi\mathcal{O}_{N'} = \mu_*(\mathcal{O}_{\mathcal{N}})/\varpi\mu_*(\mathcal{O}_{\mathcal{N}}) \to \overline{\mu}_*(\mathcal{O}_{\overline{\mathcal{N}}})$$

and it is enough to show that this is an isomorphism. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mu(\mathcal{N})}/\varpi\mathcal{O}_{\mu(\mathcal{N})} & \to & \mathcal{O}_{\overline{\mu}(\overline{\mathcal{N}})} \\ \downarrow & & \downarrow \\ \mathcal{O}_{N'}/\varpi\mathcal{O}_{N'} & \to & \overline{\mu}_*(\mathcal{O}_{\overline{\mathcal{N}}}). \end{array}$$

From the definition of the scheme theoretic image, the upper horizontal homomorphism is an isomorphism. On the other hand, we have seen above that the right vertical homomorphism is an isomorphism, whence the claim. We will now show that $N' \to \mu(\mathcal{N})$ is an isomorphism; this will establish (c). From the above it follows that the homomorphism

$$\mathcal{O}_{\mu(\mathcal{N})}/\varpi\mathcal{O}_{\mu(\mathcal{N})} \to \mathcal{O}_{N'}/\varpi\mathcal{O}_{N'}$$

is surjective. Since μ is proper, $\mathcal{O}_{N'} = \mu_*(\mathcal{O}_{\mathcal{N}})$ is finite over $\mathcal{O}_{\mu(\mathcal{N})}$. Therefore, using Nakayama's lemma, we conclude that $\mathcal{O}_{\mu(\mathcal{N})} \to \mathcal{O}_{N'}$ is surjective locally over all points at the special fiber. Since $N' \otimes_{\mathcal{O}_K} K \to \mu(\mathcal{N}) \otimes_{\mathcal{O}_K} K$ is an isomorphism it follows that $\mathcal{O}_{\mu(\mathcal{N})} \to \mathcal{O}_{N'}$ is surjective; it now follows from the definition of the scheme-theoretic image that $\mathcal{O}_{\mu(\mathcal{N})} \to \mathcal{O}_{N'}$ is an isomorphism. This shows (c).

Now let us show part (b). By (a) the special fiber $\overline{N}'$ is normal and Cohen-Macaulay; in fact, it has dimension $r^2 - \sum_i r_i^2$. This is equal to the dimension of the generic fiber $N \otimes_{\mathcal{O}_E} K = N' \otimes_{\mathcal{O}_K} K$. As a result the special fiber is reduced and its unique generic point lifts to the generic fiber; this implies that N' is flat over $\operatorname{Spec}\mathcal{O}_K$ (EGA IV 3.4.6.1). Since $\overline{N}'$ is Cohen-Macaulay and $N' \to \operatorname{Spec}\mathcal{O}_K$ is flat, N' is Cohen-Macaulay. Now $N' \otimes_{\mathcal{O}_K} K = N \otimes_{\mathcal{O}_E} K$ is smooth and $\overline{N}'$ generically smooth; this shows that N' is regular in codimensions 0 and 1 and therefore, by Serre's criterion, normal.

Finally, since $N' = \mu(\mathcal{N}) \subset N \otimes_{\mathcal{O}_E} \mathcal{O}_K$ with identical generic fibers, and since N' is flat over $\operatorname{Spec}\mathcal{O}_K$, N' is the (flat) scheme theoretic closure of $N \otimes_{\mathcal{O}_E} K$ in $N \otimes_{\mathcal{O}_E} \mathcal{O}_K$. This shows (d). □

Proposition 5.3 *Let N^{loc} be the (flat) scheme theoretic closure of $N \otimes_{\mathcal{O}_E} E$ in N. Then the scheme N^{loc} is normal and Cohen-Macaulay. Its special fiber is the reduced closure of the orbit $N_{\mathbf{t}}$ with $\mathbf{t} = \mathbf{r}^\vee$ the dual partition to $\mathbf{r} = (r_\varphi)$. The special fiber is normal with rational singularities.*

PROOF. Denote by $N'' \subset N$ the scheme theoretic image of the finite composite morphism $N' \to N \otimes_{\mathcal{O}_E} \mathcal{O}_K \to N$. This is a GL_r-equivariant closed subscheme of N. We have $\mathcal{O}_{N''} \subset \mathcal{O}_{N'}$ and so since N' is flat over $\operatorname{Spec}\mathcal{O}_K$, we conclude that N'' is also flat over $\operatorname{Spec}\mathcal{O}_E$. We have $N'' \otimes_{\mathcal{O}_E} E = N \otimes_{\mathcal{O}_E} E$; hence N'' is the flat scheme theoretic closure of $N \otimes_{\mathcal{O}_E} E$ in N, that is $N'' = N^{\mathrm{loc}}$. The base change $N^{\mathrm{loc}} \otimes_{\mathcal{O}_E} \mathcal{O}_K$ is a closed subscheme of $N \otimes_{\mathcal{O}_E} \mathcal{O}_K$ which is flat over $\operatorname{Spec}\mathcal{O}_K$. Since we have $N^{\mathrm{loc}} \otimes_{\mathcal{O}_E} K = N \otimes_{\mathcal{O}_E} K$ by Proposition 5.2 (d) we have

$$N^{\mathrm{loc}} \otimes_{\mathcal{O}_E} \mathcal{O}_K = N' \subset N \otimes_{\mathcal{O}_E} \mathcal{O}_K \quad . \tag{5.8}$$

By Proposition 5.2 (b) N' is normal. Therefore, $N^{\mathrm{loc}} = N''/\mathrm{Gal}(K/E)$ is also normal. Denoting by $\overline{N}^{\mathrm{loc}}$ the special fiber of N^{loc}, we have

$$\overline{N}^{\mathrm{loc}} \otimes_k k' = \overline{N}' \ .$$

Hence the remaining assertions also follow from Proposition 5.2. $\square$

We note that we can define analogues of $\mathcal{N}$ for M and $\widetilde{M}$. Namely, we consider the $\mathcal{O}_K$-scheme $\mathcal{M}$, which for an $\mathcal{O}_K$-scheme S classifies the filtrations of $\mathcal{O}_S$-submodules

$$\{(0) = \mathcal{F}_e \subset \mathcal{F}_{e-1} \subset \ldots \subset \mathcal{F}_0 = \mathcal{F} \subset \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S\} \ , \tag{5.9}$$

where $\mathcal{F} \in M^{\mathrm{loc}}(S)$ and where $\mathcal{F}_k$ is locally on S a direct summand of $\mathcal{F}$ of corank $n_k = \sum_{i=1}^k r_i$ such that

$$(\pi - a_k \cdot \mathrm{Id}) \cdot \mathcal{F}_{k-1} \subset \mathcal{F}_k \ , \quad k = 1, \ldots, e \ . \tag{5.10}$$

Similarly we define $\widetilde{\mathcal{M}}$ by fixing in addition an isomorphism $\psi : \mathcal{F} \to \mathcal{O}_S^r$. We thus obtain a diagram with cartesian squares in which the vertical morphisms are projective with source a smooth $\mathcal{O}_K$-scheme,

$$\begin{array}{ccccc} \mathcal{M} & \longleftarrow & \widetilde{\mathcal{M}} & \longrightarrow & \mathcal{N} \\ \downarrow & & \downarrow & & \downarrow \\ M \otimes_{\mathcal{O}_E} \mathcal{O}_K & \longleftarrow & \widetilde{M} \otimes_{\mathcal{O}_E} \mathcal{O}_K & \longrightarrow & N \otimes_{\mathcal{O}_E} \mathcal{O}_K \end{array} \quad . \tag{5.11}$$

We now use Theorem 4.1 to transfer the previous results to the standard models.

Theorem 5.4 *Let M^{loc} be the (flat) scheme theoretic closure of $M \otimes_{\mathcal{O}_E} E$ in M. Then*
(i) M^{loc} is normal and Cohen-Macaulay.
(ii) The special fiber $\overline{M}^{\mathrm{loc}}$ is reduced, normal with rational singularities, and we have

$$\overline{M}^{\mathrm{loc}} = \bigsqcup_{\mathbf{s} \le \mathbf{t}} M_{\mathbf{s}}^{\mathrm{loc}}.$$

(iii) There is a diagram with cartesian squares of $\mathcal{G} \times GL_r$-equivariant morphisms in which the horizontal morphisms are smooth and the vertical morphisms are closed embeddings,

$$\begin{array}{ccccc} M^{\mathrm{loc}} & \xleftarrow{\pi^{\mathrm{loc}}} & \widetilde{M}^{\mathrm{loc}} & \xrightarrow{\phi^{\mathrm{loc}}} & N^{\mathrm{loc}} \\ \downarrow & & \downarrow & & \downarrow \\ M & \xleftarrow{\pi} & \widetilde{M} & \xrightarrow{\phi} & N \end{array} \quad . \quad \square$$

The $\mathcal{O}_E$-scheme M^{loc} is called the *canonical model* associated to the standard model for GL_d corresponding to $\mathbf{r} = (r_i)$.

Remark 5.5 The use of Theorem 4.1 in proving parts (i) and (ii) of the above theorem is to enable us to appeal to the results of Mehta-van der Kallen on nilpotent orbit closures. An alternative approach which would not appeal to Theorem 4.1 might be obtained by studying directly the composite morphism

$$\mathcal{M} \to M \otimes_{\mathcal{O}_E} \mathcal{O}_K \to M$$

and using the theory of generalized Schubert varieties in the affine Grassmannian for GL_d. One can show directly that $\mathcal{M}$ is smooth over $\operatorname{Spec}\mathcal{O}_K$ (comp. the proof of Lemma 5.1); the main point is that the special fiber $\overline{\mathcal{M}}$ can be written as a composite of smooth fibrations with fibers Grassmannian varieties (indeed, we may think of $\overline{\mathcal{M}}$ as a generalized Demazure-Bott-Samelson variety). Then, the arguments in the proofs of Propositions 5.2 and 5.3 can be repeated to obtain a direct proof of parts (i) and (ii) of Theorem 5.4, provided we know that the reduced closure of the stratum $M_{\mathbf{t}}$ in the special fiber $\overline{M}$ is normal and has rational singularities. By the discussion in §3, this reduced closure is isomorphic to the reduced closure $X_{\mathbf{t}} := \overline{\mathcal{O}}_{\mathbf{t}}$ of the Schubert cell $\mathcal{O}_{\mathbf{t}}$ in the affine Grassmannian (defined as in [BL], comp. §3). It remains to show that the generalized Schubert variety $X_{\mathbf{t}}$ is normal and has rational singularities. Results of this type have been shown (in positive characteristic) by Mathieu [Mat]. Unfortunately, it is not clear that in positive characteristic the Schubert varieties he considers have the same scheme structure as the $X_{\mathbf{t}}$ (see the remarks on p. 410 of [BL]). Since this point is not cleared up, we use Theorem 4.1 to deduce results on the affine Grassmannian from results on nilpotent orbit closures, comp. section 6 below.

Suppose now that $(|\Gamma|, \operatorname{char} k) = 1$. Under this hypothesis we will show that a conjecture of de Concini and Procesi implies a rather explicit description of the scheme M^{loc}.

We follow [deC-P] §1. Let $x_1, \ldots, x_r$ be a set of variables; for every pair of integers t, h with $h \geq 0$, $1 \leq t \leq r$, we can consider the total symmetric function of degree h in the first t variables; this is defined to be the sum of all monomials in $x_1, \ldots, x_t$ of degree h and will be denoted by $S^t_h(x_i)$. We also indicate by the symbols σ_h the elementary symmetric function of degree h in the variables $x_1, \ldots, x_r$ with the convention that $\sigma_h = 0$ if $h > r$. Write

$$S^t_h(x_i) = \sum a_{(h_1,\cdots,h_t)} x_1^{h_1} \cdots x_t^{h_t} \tag{5.12}$$

For $A \in \operatorname{Mat}_{r\times r}(R) = \operatorname{End}(R^r)$, we set

$$S^t_h(A)(e_{i_1} \wedge \cdots \wedge e_{i_t}) = \sum a_{(h_1,\cdots,h_t)} A^{h_1} e_{i_1} \wedge \cdots \wedge A^{h_t} e_{i_t} \tag{5.13}$$

Since $S^t_h(x_i)$ is symmetric this defines a R-linear operator

$$S^t_h(A) : \wedge^t(R^r) \to \wedge^t(R^r) \tag{5.14}$$

Now let us indicate by

$$T^r - \sigma_1(A)T^{r-1} + \sigma_2(A)T^{r-2} - \cdots + (-1)^r \sigma_r(A) \tag{5.15}$$

the characteristic polynomial $\det(T \cdot I - A)$ of A. For t, h as above, we now define the following element of $\operatorname{End}(\wedge^t(R^r))$:

$$F^t_h(A) := S^t_h(A) - \sigma_1(A) S^t_{h-1}(A) + \cdots + (-1)^h \sigma_h(A)$$

For each function $f : \{1, 2, \ldots, e\} \to \mathbf{N}$ with $0 \leq f(i) \leq r_i$, for all $1 \leq i \leq e$, consider

$$Q_f(t) = \prod_{i=1}^{e} (T - a_i)^{f(i)} \tag{5.16}$$

(a divisor of the polynomial $P(T)$).

Let us consider the subscheme N_0 of $\mathrm{Mat}_{r\times r}$ over $\operatorname{Spec}\mathcal{O}_E$ which is defined by the equations given by the conditions

$$\det(T\cdot I - A)\equiv P(T), \text{ and } \sum_{\sigma\in\Gamma} F_h^t(A)\cdot\wedge^t(\sigma Q_f(A)) = 0, \tag{5.17}$$

$$\text{for all } f, \quad \text{and for } \ t+h = r - \sum_{i,f(i)\neq 0} r_i + 1, \quad t\geq 1,\ h\geq 0.$$

(it is obvious that the generators of the ideal defining N_0 have all coefficients in $\mathcal{O}_E$). This is actually a closed subscheme of N; indeed, consider the second set of equations for $f\equiv 1$, $t=1$, $h=0$. Since $F_0^t(A)=I$, we obtain

$$|\Gamma|\cdot\prod_{i=1}^{e}(A - a_i\cdot I) = 0\ . \tag{5.18}$$

Since $(|\Gamma|, \mathrm{char}\, k)=1$, this equation implies $Q(A)=0$. It is straightforward to see that N_0 is a GL_r-invariant subscheme of N.

Proposition 5.6 *i) The schemes N and N_0 over* $\operatorname{Spec}\mathcal{O}_E$ *have the same generic fiber.*

ii) The reduced special fiber of N_0 is equal to the reduced closure of the conjugation orbit $N_{\mathbf{t}}$ of the Jordan form with partition $\mathbf{t}=\mathbf{r}^\vee$.

Proof. We first show (i). By descent, we can check this after base changing to K. Consider the diagonal $r\times r$ matrix A_0 in which the element a_i appears with multiplicity r_i. We have

$$\mathrm{rank}(Q_f(A_0)) = r - \sum_{i,f(i)\neq 0} r_i. \tag{5.19}$$

It now follows from [deC-P], Proposition on p. 206, that

$$F_h^t(A_0)\cdot\wedge^t(Q_f(A_0)) = 0 \tag{5.20}$$

if $t+h\geq r-\sum_{i,f(i)\neq 0} r_i + 1$ (comp. the proof of the Theorem on p. 207 loc. cit). This shows that A_0 satisfies the conditions defining $N_0\otimes_{\mathcal{O}_E} K$. Since $N\otimes_{\mathcal{O}_E} K$ is the reduced GL_r-orbit of such a matrix and N_0 is GL_r-equivariant, the result follows.

Now we prove (ii). Reducing the equations defining N_0 modulo the maximal ideal of $\mathcal{O}_E$ gives the following equations:

$$\det(T\cdot I - A)\equiv T^r, \text{ and } |\Gamma|\cdot F_h^t(A)\cdot\wedge^t(A^{\sum_i f(i)}) = 0 \tag{5.21}$$

$$\text{for all } f, \quad \text{and for } \ t+h = r - \sum_{i,f(i)\neq 0} r_i + 1, \quad t\geq 1,\ h\geq 0.$$

By the definition of $F_h^t(A)$, the difference $F_h^t(A) - S_h^t(A)$ is in the ideal given by $\det(T \cdot I - A) \equiv T^r$. Using this and taking $J = \mathrm{supp}(f) \subset I = \{1, \ldots, e\}$ we see that the above set of equations generates the same ideal as the following one:

$$\det(T \cdot I - A) \equiv T^r, \text{ and } S_h^t(A) \cdot \wedge^t(A^{|J|}) = 0 \tag{5.22}$$

$$\text{for all subsets } J \subset I, \quad t + h = r - \sum_{i \in J} r_i + 1, \quad t \geq 1, \ h \geq 0.$$

If the r_i are arranged in decreasing order, it is the same to consider

$$\det(T \cdot I - A) \equiv T^r, \text{ and } S_h^t(A) \cdot \wedge^t(A^k) = 0 \tag{5.23}$$

$$k = 0, \ldots, e, \quad t + h = r - n_k + 1, \quad t \geq 1, \ h \geq 0.$$

(recall our notation $n_k = \sum_{i=1}^k r_i$, $n_0 = 0$). In fact, we could also omit the first equation and write this as

$$S_h^t(A) \cdot \wedge^t(A^k) = 0 \tag{5.24}$$

$$\text{for } k = 0, \ldots, e, \quad t + h = r - n_k + 1, \quad t \geq 1, \ h \geq 0.$$

Indeed, by [deC-P], Lemma p. 206, the ideal generated by $S_h^t(A)$ for $t + h = r + 1$ is the same as the one generated by the coefficients of the characteristic polynomial $\sigma_1(A)$, ..., $\sigma_r(A)$.

Let us denote by $\mathcal{I}$ the ideal of $k[a_{ij}]_{1 \leq i,j \leq r}$ generated by the equations (5.24). Then it follows as in [deC-P], Theorem on p. 207, that its radical $\mathrm{rad}(\mathcal{I})$ defines the (reduced) closure of the orbit $N_{\mathbf{t}}$ of the Jordan form for the dual partition $\mathbf{t} = \mathbf{r}^\vee$. This shows (ii). ◻

De Concini and Procesi conjecture (loc. cit.) that when $\mathrm{char}\, k = 0$, we have $\mathrm{rad}(\mathcal{I}) = \mathcal{I}$. Assume that this is true even if $\mathrm{char}\, k > 0$. Then

$$N_0 \otimes_{\mathcal{O}_E} k = \overline{N}_{\mathbf{t}} \quad , \tag{5.25}$$

(i.e, the special fiber of N_0 is already reduced). Since by [M-vdK] the reduced closure of $N_{\mathbf{t}}$ is normal, Cohen-Macaulay and has dimension $r^2 - \sum_i r_i^2$, an argument as in the proof of Proposition 5.2 (ii) shows that N_0 is normal and flat over $\mathrm{Spec}\, \mathcal{O}_E$. It follows that N_0 is the scheme theoretic closure of $N \otimes_{\mathcal{O}_E} E$ in N, i.e. $N_0 = N^{\mathrm{loc}}$. Transferring this result to the standard model we obtain the following result.

Theorem 5.7 *Assume that $(|\Gamma|, \mathrm{char}\, k) = 1$ and that the conjecture of De Concini-Procesi holds over k. Then the canonical flat model M^{loc} can be described as the moduli scheme for the following moduli functor on $(\mathrm{Sch}/\mathcal{O}_E)$: The S-valued points are given by $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-submodules $\mathcal{F}$ of $\Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ which are locally direct summands as $\mathcal{O}_S$-modules, with*

$$\det(T \cdot \mathrm{Id} - \pi \mid \mathcal{F}) \equiv \prod_{\varphi} (T - \varphi(\pi))^{r_\varphi}$$

and
$$\sum_{\sigma\in\Gamma} F_h^t(\pi \mid \mathcal{F}) \cdot \wedge^t(\sigma Q_f(\pi \mid \mathcal{F})) = 0,$$
for all f as before, and for $t+h = r - \sum_{\varphi, f(\varphi)\neq 0} r_\varphi + 1$, $t \geq 1, h \geq 0$. □

Let us fix integers d, r and e and let us state the following conjecture.

Conjecture 5.8 *Consider the closed subscheme $\overline{N}$ of $\mathrm{Mat}_{r\times r}$ over k defined in (4.5),*
$$\overline{N} = \{A \in \mathrm{Mat}_{r\times r};\ A^e = 0,\ \det(T\cdot I - A) \equiv T^r\}\ .$$
Then this scheme is reduced.

Note that when $r \leq e$, the first condition describing $\overline{N}$ follows from the second (Cayley-Hamilton) and in this case the statement is a classical theorem of Kostant on the nilpotent variety. In a companion paper to ours, J. Weyman proves this conjecture when $e = 2$, and for arbitrary e when char $k = 0$. We note that the reduced subscheme of $\overline{N}$ is simply the orbit closure corresponding to the partition (e^c, f) of r, where we have written as usual $r = c\cdot e + f$, $0 \leq f < e$. Now De Concini and Procesi [deC-P] have determined the length of the intersection of any nilpotent orbit closure with the diagonal matrices (in arbitrary characteristic). Therefore, if Conjecture 5.8 holds we would obtain from their formula that

$$\dim_k k[X_1, \ldots, X_r]/(e_1, \ldots, e_r, X_1^e, \ldots, X_r^e) = \frac{r!}{((c+1)!)^f \cdot (c!)^{e-f}}\ . \tag{5.26}$$

Here $e_1, \ldots, e_r$ are the elementary symmetric functions in the indeterminates $X_1, \ldots, X_r$.

Corollary 5.9 *Let $\mathbf{r} = \mathbf{r}_{\min}(r, e)$. The corresponding standard model for GL_d is flat over* $\mathrm{Spec}\,\mathcal{O}_E$ *if Conjecture 5.8 holds true.*

Proof. If the conjecture holds true, we conclude from Theorem 4.1 that $M \otimes_{\mathcal{O}_E} k$ is reduced. On the other hand, by Proposition 3.2 the generic and the special fiber of M are irreducible of the same dimension. The flatness of M follows as in the proof of Proposition 5.2 from EGA IV.3.4.6.1. □

Corollary 5.10 *Assume that $e = 2$ and order the embeddings so that $r_1 \geq r_2$. Then the canonical flat model M^{loc} represents the following moduli problem on $(Sch/\mathcal{O}_E)$: The S-valued points are given by $\mathcal{O}_F \otimes_{F_0} \mathcal{O}_S$-submodules $\mathcal{F}$ of $\Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ which are locally direct summands as $\mathcal{O}_S$-modules with*
$$\det(T\cdot \mathrm{Id} - \pi|\mathcal{F}) \equiv (T - \varphi_1(\pi))^{r_1}(T - \varphi_2(\pi))^{r_2}$$
and
$$\wedge^{r_2+1}(\pi - \varphi_1(\pi)\cdot \mathrm{Id}|\mathcal{F}) = 0 \quad \textit{if } r_1 > r_2\ .$$
Here in the last case we used φ_1 to identify F with E.

Proof. In this case E is a Galois extension, namely $E = F_0$ if $r_1 = r_2$ and $E = F$ via φ_1 if $r_1 > r_2$. Let us discuss the case when $r_1 > r_2$. In this case Γ is trivial, and the second identity above is the one in Theorem 5.7 corresponding to f with $f(1) = 1$, $f(2) = 0$ and to $t = r_2 + 1$, $h = 0$. It follows that the conditions above define a closed subscheme M' of M and it is easy to see that the generic fibers coincide. The special fiber of M' is defined by the conditions

$$\det(T \cdot \mathrm{Id} - \pi|\mathcal{F}) \equiv T^r \quad , \quad \wedge^{r_2+1}(\pi|\mathcal{F}) = 0 \quad . \tag{5.27}$$

Consider the closed subscheme N' of N defined by the condition $\wedge^{r_2+1}A = 0$. Then the special fiber of N' is given as

$$\overline{N}' = \{A \in \mathrm{Mat}_{r\times r};\ A^2 = 0, \wedge^{r_2+1}A = 0, \det(T \cdot I - A) \equiv T^r\}. \tag{5.28}$$

We then obtain a diagram of morphisms

$$M' \xleftarrow{\pi'} \widetilde{M}' \xrightarrow{\phi'} N' \tag{5.29}$$

in which π' is a GL_r-torsor and ϕ' is smooth of relative dimension rd. According to Strickland ([St]), the special fiber of N' is irreducible and reduced and in fact equal to the reduced closure of the nilpotent orbit corresponding to the partition $\mathbf{s} = (2^{r_2}, 1^{r_1-r_2})$, hence of dimension $r^2 - (r_1^2 + r_2^2)$. It follows that the special fiber of M' is irreducible and reduced of dimension $dr - (r_1^2 + r_2^2) = \dim\, M' \otimes_{\mathcal{O}_E} E$. Hence M' is $\mathcal{O}_E$-flat and therefore coincides with M^{loc}.

Now assume $r_1 = r_2$. In this case we are asserting that the standard model is flat over $\mathcal{O}_{F_0}$. This follows via Corollary 5.9 from Weyman's result on Conjecture 5.8 concerning $e = 2$. □

Remark: The second identity above first appeared in [P] with the purpose of defining a flat local model for the unitary group corresponding to a ramified quadratic extension of $\mathbf{Q}_p$. It was the starting point of the present paper.

Remark 5.11 Corollary 5.10 illustrates the fact that the identities in Theorem 5.7 are extremely redundant. It is an open problem to find a shorter list of identities, with coefficients in $\mathcal{O}_E$, in suitable tensor powers of $\pi|\mathcal{F}$ which describe the canonical flat model.

6 Applications to the affine Grassmannians

In this section we spell out some consequences of the results of the previous sections, as they pertain to the special fibers. For the first application we take up again the notation of section 3; in particular, we denote for a dominant coweight $\mathbf{s}$ of GL_d, by $\mathcal{O}_\mathbf{s}$ the corresponding orbit of $\tilde{\mathcal{G}}$ on Grass_k.

Theorem 6.1 *The reduced closure $\overline{\mathcal{O}}_\mathbf{s}$ is normal with rational singularities. Its singularities are in fact smoothly equivalent to singularities occurring in nilpotent orbit closures for a general linear group.*

PROOF. Let $\mathbf{s} = (s_1, \ldots, s_d)$. After translation by a scalar matrix we may assume $s_d \geq 0$. Let e be any integer $\geq s_1$, and put $r = \sum s_i$. Then $\mathbf{s} \in \mathcal{S}^0(r, e, d)$. Hence $\mathcal{O}_{\mathbf{s}}$ may be identified with the corresponding stratum $M_{\mathbf{s}}$ of the special fiber $\overline{M}$ of any standard model $M(\Lambda, \mathbf{r})$, where $[F : F_0] = e$ and $\Lambda = \mathcal{O}_F^d$ and $\sum_{j=1}^e r_j = r$ (all of them have identical special fibers). And the closure $\overline{\mathcal{O}}_{\mathbf{s}}$ of $\mathcal{O}_{\mathbf{s}}$ can be identified with the closure $\overline{M}_{\mathbf{s}}$ of $M_{\mathbf{s}}$ in $\overline{M}$. From Theorem 4.1 it follows that $\overline{M}_{\mathbf{s}}$ is smoothly equivalent to the closure of the corresponding nilpotent orbit $N_{\mathbf{s}}$ of $\overline{N}$, which by Mehta - van der Kallen is normal with rational singularities. □

Remark 6.2 Results of this type (normality of Schubert varieties) have been shown in positive characteristic in the context of Kac-Moody algebras by Mathieu [Mat]. However, it is not clear whether the Schubert varieties he considers have the same scheme structure as the $\overline{\mathcal{O}}_{\mathbf{s}}$ considered here (the corresponding statement is known in characteristic zero, by the integrability result of Faltings, comp. [BL], app. to section 7). It follows from the methods of Görtz [G] that, once a statement of the kind of Theorem 6.1 is known, it follows that *any* Schubert variety in *any* parahoric flag variety for $GL_d(k((\Pi)))$ is normal (and much more, comp. [G1]). For a completely different approach see Faltings's paper [F].

In fact, we can be more precise than in Theorem 6.1. Let (V_e, Π_e) be the standard vector space of dimension e over k, with the standard regular nilpotent endomorphism. For $d \geq 1$ let

$$(W, \Pi) = (V_e, \Pi_e)^d \quad . \tag{6.1}$$

Let $0 \leq r \leq ed$ and consider the projective scheme $X = X(r, e, d)$ over k which represents the following functor on the category of k-algebras. It associates to a k-algebra R the set of R-submodules,

$$\begin{aligned} \{\mathcal{F} \subset W \otimes_k R; \qquad & \mathcal{F} \text{ is locally on Spec } R \text{ a direct summand,} \\ & \mathcal{F} \text{ is } \Pi\text{-stable and} \\ & \det(T - \Pi|\mathcal{F}) \equiv T^r\} \ . \end{aligned}$$

By the end of section 3, X is the special fiber of any standard model $M(\Lambda, \mathbf{r})$ where $[F : F_0] = e$, where $\Lambda = \mathcal{O}_F^d$ and $r = \sum r_\varphi$. We obtain as a special fiber of the diagram (4.4) the diagram

$$X \xleftarrow{\pi} \tilde{X} \xrightarrow{\phi} \overline{N} \quad , \tag{6.2}$$

in which π is a GL_r-torsor and ϕ is smooth of relative dimension rd. It is equivariant with respect to the action of the product group $\overline{\mathcal{G}} \times GL_r$, where $\overline{\mathcal{G}} = \mathcal{R}_{k[\Pi]/(\Pi^e)/k}(GL_d)$, cf. (3.8). We therefore obtain an extremely close relationship between affine Grassmannians and nilpotent varieties. Indeed, X is the Schubert variety in the affine Grassmannian for GL_d corresponding to the coweight $(e^c, f, 0, \ldots, 0)$ (where we have written as usual $r = c \cdot e + f$, $0 \leq f < e$), and the image of $\tilde{X}$ is the closure of the nilpotent orbit corresponding to the partition (e^c, f) of r.

Remark 6.3 For this result the fact that the nilpotent endomorphism Π in (6.1) is "homogeneous" is essential. Indeed, let $r = e \geq 2$ and change momentarily notations to consider the following inhomogeneous example. Let

$$(W, \Pi) = k[\Pi]/(\Pi^e) \oplus k[\Pi]/(\Pi) \quad , \tag{6.3}$$

and define X as above. Then it is easy to see that any $\mathcal{F} \in X$ satisfies

$$\mathrm{span}\{\Pi v_1, \ldots, \Pi^{e-1} v_1\} \underset{\neq}{\subset} \mathcal{F} \underset{\neq}{\subset} W \quad ,$$

where v_1 resp. v_2 denotes the generator as $k[\Pi]$-module of the first resp. second summand of (6.3). Hence $X \simeq \mathbf{P}^1$. Let $o \in X$ be the special point corresponding to

$$\mathcal{F}_o = \mathrm{span}\{v_2, \Pi v_1, \ldots, \Pi^{e-1} v_1\} \quad .$$

For $\mathcal{F}_o$, the Jordan type of $\Pi|\mathcal{F}_o$ is $(e-1, 1)$ whereas for $\mathcal{F} \neq \mathcal{F}_0$, the Jordan type of $\Pi|\mathcal{F}$ is (e). It follows that the fiber of ϕ through a point of $\pi^{-1}(\mathcal{F}_o)$ has dimension equal to $\dim GL_r - \dim N_{(e-1,1)} = e + 2$. On the other hand, the fiber of ϕ through a point of $\pi^{-1}(X \setminus \{o\})$ has dimension equal to $(\dim GL_r + 1) - \dim N_{(e)} = e + 1$. Hence ϕ is not smooth in this case.

Remark 6.4 Let us fix $\mathbf{r}$ with $r = \sum r_\varphi$, and let us consider a standard model $M(\Lambda, \mathbf{r})$ with special fiber $X = X(r, e, d)$. Let us order $\mathbf{r} = (r_1, \ldots, r_e)$. After extension of scalars from k to k' we obtain as special fiber of the diagram (5.11) the following diagram with cartesian squares,

$$\begin{array}{ccccc} \overline{\mathcal{M}} & \longleftarrow & \overline{\tilde{\mathcal{M}}} & \longrightarrow & \overline{\mathcal{N}} \\ \downarrow & & \downarrow & & \downarrow \\ X \otimes_k k' & \longleftarrow & \tilde{X} \otimes_k k' & \longleftarrow & \overline{N} \otimes_k k' \end{array} \quad . \tag{6.4}$$

Here $\overline{\mathcal{N}}$ is the Springer resolution of the nilpotent orbit closure corresponding to $\mathbf{t} = \mathbf{r}^\vee$, comp. beginning of section 5. On the other hand, as Ngô pointed out to us, the variety $\overline{\mathcal{M}}$ is an object which is well-known in the theory of the affine Grassmannians, comp. [N-P]. Namely, let us introduce the e minuscule coweights $\mu_i = (1^{r_i}, 0^{d-r_i})$ of GL_d. Corresponding to μ_i we have the Schubert variety $\overline{\mathcal{O}}_{\mu_i}$ in the affine Grassmannian for GL_d *over* k'. Then we may identify the variety $\overline{\mathcal{M}}$ with the *convolution* in the sense of Lusztig, Ginzburg, Mirkovic and Vilonen

$$\overline{\mathcal{O}}_{(\mu_1, \ldots, \mu_e)} := \overline{\mathcal{O}}_{\mu_1} \tilde{\times} \ldots \tilde{\times} \overline{\mathcal{O}}_{\mu_e} \tag{6.5}$$

and the morphism from $\overline{\mathcal{M}}$ to $X \otimes_k k'$ factors through a proper surjective morphism which may be identified with the natural morphism ([N-P], §9)

$$m_{(\mu_1, \ldots, \mu_e)} : \overline{\mathcal{O}}_{(\mu_1, \ldots, \mu_e)} \longrightarrow \overline{\mathcal{O}}_{\mu_1 + \ldots + \mu_e} \quad . \tag{6.6}$$

Note that $\overline{\mathcal{O}}_{\mu_1 + \ldots + \mu_e}$ is just the Schubert variety corresponding to the coweight $\mathbf{t} = \mathbf{r}^\vee$.

We note the following consequence of (6.2).

Proposition 6.5 *If $r \leq e$, the scheme $X(r,e,d)$ is reduced and locally a complete intersection. If $r > e$ and Conjecture 5.8 is true, then X is still reduced.*

PROOF. We argue with the special fiber $\overline{M}$ of a standard model as described before. If $r \leq e$, then we may take $\mathbf{r}$ such that $r_j \leq 1$, $\forall j$. The result then follows from Corollary 4.3 If $r > e$, then X is smoothly equivalent to (an open subscheme of) $\overline{N}$, and the assertion follows from a positive answer to Conjecture 5.8. □

It is well-known that the Grassmannian over k associated to GL_d is not reduced (this happens already for $d = 1$). Recall ([BL]) that

$$\widetilde{\mathrm{Grass}}_k = GL_d(k((\Pi))\,)/GL_d(k[[\Pi]]) \quad , \tag{6.7}$$

where $GL_d(k((\Pi))\,)$ resp. $GL_d(k[[\Pi]])$ is the ind-group scheme resp. group scheme which to a k-algebra R associates $GL_d(R((\Pi))\,)$ resp. $GL_d(R[[\Pi]])$, and where the quotient is taken in the category of k-spaces and turns out to be an ind-scheme, [BL], 2.2. On the other hand, the analogous quotient for SL_d instead of GL_d is an ind-scheme which is reduced and even integral ([BL], 6.4.),

$$\widetilde{\mathrm{Grass}}_k^{(0)} = SL_d(k((\Pi))\,)/SL_d(k[[\Pi]]) \quad . \tag{6.8}$$

(At this point the blanket assumption in loc. cit. that char $k = 0$ is not used.) This means that $\widetilde{\mathrm{Grass}}_k^{(0)}$ can be obtained as an increasing union of integral k-schemes. However, the rather indirect proof of this fact in loc.cit. does not give an explicit presentation of $\widetilde{\mathrm{Grass}}_k^{(0)}$ as such an increasing union. Based on Proposition 6.5 we are able to give such a presentation, provided Conjecture 5.8 holds true. For this recall ([BL], 2.3) that $\widetilde{\mathrm{Grass}}_k^{(0)}$ represents the functor on k-algebras which to a k-algebra R associates the set of special lattices in $R((\Pi))^d$. (A *lattice* is a $R[[\Pi]]$-submodule W of $R((\Pi))^d$ with $\Pi^f R[[\Pi]]^d \subset W \subset \Pi^{-f} R[[\Pi]]^d$ for some f and such that the R-module $\Pi^{-f}R[[\Pi]]^d/W$ is projective. A lattice W is *special* if the lattice $\wedge^d W$ in $\wedge^d R((\Pi))^d = R((\Pi))$ is trivial, i.e. equal to $R[[\Pi]]$). Let $\widetilde{\mathrm{Grass}}_k[0]$ be the space of lattices of total degree 0 (i.e. the rank of $\Pi^{-f}R[[\Pi]]^d/W$ is equal to fd. Then $\widetilde{\mathrm{Grass}}_k[0]$ is a connected component of $\widetilde{\mathrm{Grass}}_k$ and

$$\widetilde{\mathrm{Grass}}_k^{(0)} = (\widetilde{\mathrm{Grass}}_k[0])_{\mathrm{red}} \quad , \tag{6.9}$$

(cf. [BL], 2.2. and 6.4.).

Let $\tilde{X}_f$ be the subscheme of $\widetilde{\mathrm{Grass}}_k[0]$ which parametrizes the lattices W with $\Pi^f R[[\Pi]]^d \subset W \subset \Pi^{-f} R[[\Pi]]^d$ and let X_f be the closed subscheme of W in $\tilde{X}_f$ such that $\det(T - \Pi | (W/\Pi^f R[[\Pi]]^d)) \equiv T^{fd}$. We have an exact sequence

$$0 \to \Pi^f R[[\Pi]]^d/\Pi^{f+1}R[[\Pi]]^d \to W/\Pi^{f+1}R[[\Pi]]^d \to W/\Pi^f R[[\Pi]]^d \to 0 \ ,$$

which implies that

$$\det(T - \Pi|(W/\Pi^{f+1}R[[\Pi]]^d)) = \det(T - \Pi|(W/\Pi^f R[[\Pi]]^d)) \cdot T^d \quad .$$

We therefore obtain a chain of closed embeddings of k-schemes

$$X_0 \subset X_1 \subset \dots \tag{6.10}$$

Proposition 6.6 *For $d \leq 2$, the chain (6.10) presents $\widetilde{\mathrm{Grass}}_k^{(0)}$ as an increasing union of integral k-schemes. The same is true for arbitrary d, if Conjecture 5.8 holds true. In particular, this holds (by the theorem of Weyman) if char $k = 0$.*

PROOF. We note that $X_f = X(fd, 2f, d)$, hence is integral if $d \leq 2$ and for arbitrary d, if Conjecture 5.8 holds true. Hence $X_f = (\tilde{X}_f)_{\mathrm{red}}$. The claim follows from $\widetilde{\mathrm{Grass}}_k[0] = \varinjlim \tilde{X}_f$ and

$$\widetilde{\mathrm{Grass}}_k^{(0)} = \varinjlim (\tilde{X}_f)_{\mathrm{red}} = \varinjlim X_f \quad . \qquad \square$$

7 The complex of nearby cycles

In this section we suppose that the residue field k of $\mathcal{O}_E$ is finite and shall aim for an expression for the complex of nearby cycles of a standard model for GL_d corresponding to $\mathbf{r}$, by exploiting in more depth the resolution of singularities given by the scheme $\mathcal{N}$ in (5.3).

Recall that K denotes the Galois hull of F in F_0^{sep} and that k' is the residue field of $\mathcal{O}_K$. Also, Γ denotes the Galois group of K over E. We fix a prime number ℓ which is invertible in $\mathcal{O}_E$ and denote by $R\psi = R\psi\overline{\mathbf{Q}}_\ell$ the complex of nearby cycles of the $\mathcal{O}_E$-scheme M. Since M^{loc} is the scheme-theoretic closure of the generic fiber in M, this complex has support in $\overline{M}^{\mathrm{loc}} := M^{\mathrm{loc}} \otimes_{\mathcal{O}_E} k$. The complex is equipped with an action of $\mathrm{Gal}(F_0^{\mathrm{sep}}/E)$ which lifts the action on $\overline{M}^{\mathrm{loc}}$. We also fix a square root of the cardinality $|k|$ in $\overline{\mathbf{Q}}_\ell$.

Theorem 7.1 *There is an isomorphism between perverse sheaves pure of weight zero,*

$$R\psi[\dim \overline{M}^{\mathrm{loc}}]\left(\tfrac{1}{2}\dim \overline{M}^{\mathrm{loc}}\right) = \bigoplus_{\mathbf{s}\leq\mathbf{t}} \mathcal{M}_{\mathbf{s}} \otimes IC_{M_{\mathbf{s}}} \ ,$$

where $IC_{M_{\mathbf{s}}}$ denotes the intermediate extension of the constant sheaf $\overline{\mathbf{Q}}_\ell[\dim M_{\mathbf{s}}]\left(\frac{1}{2}\dim M_{\mathbf{s}}\right)$, equipped with the action of $\mathrm{Gal}(F_0^{\mathrm{sep}}/E)$ which factors through $\mathrm{Gal}(\overline{k}/k)$, and $\mathcal{M}_{\mathbf{s}}$ is a $\overline{\mathbf{Q}}_\ell$-vector space equipped with an action of $\mathrm{Gal}(F_0^{\mathrm{sep}}/E)$. The action of $\mathrm{Gal}(F_0^{\mathrm{sep}}/E)$ on $\mathcal{M}_{\mathbf{s}}$ factors through Γ, and the degree of this representation is given as a Kostka number,

$$m_{\mathbf{s}} = \dim \mathcal{M}_{\mathbf{s}} = K_{\mathbf{s}^\vee,\mathbf{r}}$$

(cf. [Mc], p. 115).

By making use of naturality properties of the complex of nearby cycles, the diagram in Theorem 5.4 (iii) with smooth horizontal morphisms reduces us to proving the corresponding statements for N^{loc} instead of M^{loc} (or M). To be more precise, let us change notations and denote now by $R\psi$ the complex of nearby cycles of the $\mathcal{O}_E$-scheme N^{loc}. We wish to prove the formula

$$R\psi[\dim \overline{N}^{\text{loc}}]\left(\tfrac{1}{2}\dim\overline{N}^{\text{loc}}\right) = \bigoplus_{\mathbf{s}\leq\mathbf{t}} \mathcal{M}_{\mathbf{s}} \otimes IC_{N_{\mathbf{s}}} \quad , \tag{7.1}$$

with $\mathcal{M}_{\mathbf{s}}$ as above.

The left hand side is a perverse sheaf of weight zero on $\overline{N}^{\text{loc}}$ ([I] Thm. 4.2 and Cor. 4.5), which is GL_r-equivariant. By [BBD], 5.3.8 its inverse image $R\overline{\psi}$ on $\overline{N}^{\text{loc}} \otimes_k \overline{k}$ is semisimple and its simple constituents are all of the form $IC_{N_{\mathbf{s}}\otimes_k\overline{k}}$, for some $\mathbf{s} \leq \mathbf{t}$, since $N_{\mathbf{s}} \otimes_k \overline{k}$ admits no non-trivial GL_r-equivariant irreducible local system. Therefore, the isotypical decomposition of $R\overline{\psi}$ has the form

$$R\overline{\psi} = \bigoplus_{\mathbf{s}\leq\mathbf{t}} K_{\mathbf{s}} \quad , \tag{7.2}$$

where $K_{\mathbf{s}}$ is a multiple of $IC_{N_{\mathbf{s}}\otimes_k\overline{k}}$. Obviously $K_{\mathbf{s}}$ is of the form

$$K_{\mathbf{s}} = \mathcal{M}_{\mathbf{s}} \otimes IC_{N_{\mathbf{s}}\otimes_k\overline{k}} \quad , \tag{7.3}$$

where $\mathcal{M}_{\mathbf{s}}$ is a $\overline{\mathbf{Q}}_\ell$-vector space with an action of the inertia group $I \subset \mathrm{Gal}(F_0^{\text{sep}}/E)$. The fact that both sides of (7.3) come by extension of scalars from k, implies now that $\mathrm{Gal}(F_0^{\text{sep}}/E)$ acts on $\mathcal{M}_{\mathbf{s}}$ and that we have a decomposition of the form (7.1).

It remains to show that the restriction of $\mathcal{M}_{\mathbf{s}}$ to $\mathrm{Gal}(F_0^{\text{sep}}/K)$ is trivial and to determine its degree. By Deligne [D], Prop.3.7., we have

$$R\psi \otimes_k k' = R\psi' \tag{7.4}$$

with $R\psi'$ the complex of nearby cycles of the $\mathcal{O}_K$-scheme $N^{\text{loc}}\otimes_{\mathcal{O}_E}\mathcal{O}_K$. Now $N^{\text{loc}}\otimes_{\mathcal{O}_E}K$ has the smooth model $\mathcal{N}$ which maps via μ to $N^{\text{loc}}\otimes_{\mathcal{O}_E}\mathcal{O}_K$. Since the complex of nearby cycles of a smooth scheme is the constant sheaf placed in degree 0, the functoriality with respect to push-forward under a proper morphism gives a natural identification of complexes on $\overline{N}^{\text{loc}} \otimes_k k'$,

$$R\psi' = R\overline{\mu}_*\overline{\mathbf{Q}}_\ell \quad . \tag{7.5}$$

In particular, the action of $\mathrm{Gal}(F_0^{\text{sep}}/K)$ on $R\psi'$ is through $\mathrm{Gal}(\overline{k}/k')$. Now the morphism $\overline{\mu} : \overline{\mathcal{N}} \otimes_k k' \to \overline{N}^{\text{loc}} \otimes_k k'$ comes by base change from the moment map $\overline{\overline{\mu}}$ for the variety $\overline{\boldsymbol{\mathcal{F}}}$ of partial flags over k, cf. (5.2). This map is semi-small with all strata $N_{\mathbf{s}}$

of $\overline{N}^{\rm loc}$ relevant, comp. [BM]. Base changing the decomposition (7.1) from k to k' and identifying the left hand side with $R\overline{\mu}_*\overline{\mathbf{Q}}_\ell[\dim\overline{N}^{\rm loc}](\frac{1}{2}\dim\,\overline{N}^{\rm loc})$, we see that

$$\mathcal{M}_{\mathbf{s}} = R^{2d_{\mathbf{s}}}\overline{\mu}_*\overline{\mathbf{Q}}_\ell(d_{\mathbf{s}})_{|N_{\mathbf{s}}\otimes_k\overline{k}} \quad , \tag{7.6}$$

as representations of $\mathrm{Gal}(F_0^{\rm sep}/K)$, i.e. of $\mathrm{Gal}(\overline{k}/k')$. Here $d_{\mathbf{s}} = \frac{1}{2}\mathrm{codim}\ N_{\mathbf{s}}$ is the relative dimension of $\overline{\mu}$ over $N_{\mathbf{s}}$. But by Spaltenstein [Sp] all irreducible components of $\overline{\overline{\mu}}^{-1}(N_{\mathbf{s}}\otimes_k\overline{k})$ are defined over k, hence $\mathrm{Gal}(\overline{k}/k')$ acts trivially on the right hand side of (7.6). This shows that $\mathrm{Gal}(F_0^{\rm sep}/K)$ acts trivially on $\mathcal{M}_{\mathbf{s}}$. (Spaltenstein works over an algebraically closed field, but his results are valid over an arbitrary field, comp. [HS], §2.) A different argument, pointed out by G.Laumon, is to transpose the result of Braverman and Gaitsgory [BG] from $\mathbf{C}$ to a finite field and appeal to [HS], Corollary 2.3.

For the degree $m_{\mathbf{s}}$ of $\mathcal{M}_{\mathbf{s}}$ Borho and MacPherson give the formula [BM], 3.5.,

$$m_{\mathbf{s}} = \dim\ \mathrm{Hom}_{S_r}(\chi^{\mathbf{s}}, \mathrm{Ind}_{S_{\mathbf{r}}}^{S_r}(\mathrm{sgn}))\ . \tag{7.7}$$

Here $S_{\mathbf{r}} = S_{r_1}\times\ldots\times S_{r_e}$ is a subgroup of the symmetric group S_r and sgn is the sign character on all factors. Furthermore, $\chi^{\mathbf{s}}$ denotes the unique irreducible representation of S_r which occurs both in $\mathrm{Ind}_{S_{\mathbf{s}}}^{S_r}(1)$ and in $\mathrm{Ind}_{S_{\mathbf{s}}}^{S_r}(\mathrm{sgn})$, cf. [Mc], p. 115. By loc.cit. (7.7) can be identified with the Kostka number occurring in Theorem 7.1. Note that this agrees with the formula of Braverman and Gaitsgory [BG], Cor. 1.5.,

$$m_{\mathbf{s}} = \dim\ V(\mathbf{s}^\vee)_{\mathbf{r}} \quad , \tag{7.8}$$

(where, however, in their formula $V(\mathbf{P})_{\mathbf{d}}$ should be replaced by $V(\mathbf{P}^\vee)_{\mathbf{d}}$).

Here $V(\mathbf{s}^\vee)$ denotes the rational representation of GL_e,

$$V(\mathbf{s}^\vee) = \mathrm{Hom}_{S_r}(\chi^{\mathbf{s}^\vee}, (\mathrm{nat}_{GL_e})^{\otimes r})$$

where nat is the natural representation of GL_e and $V(\mathbf{s}^\vee)_{\mathbf{r}}$ the weight space corresponding to $\mathbf{r}$. By [Mc], p. 163 the character of $V(\mathbf{s}^\vee)$ is given by the Schur function $s_{\mathbf{s}^\vee}$ and hence (7.8) is indeed equal to $K_{\mathbf{s}^\vee,\mathbf{r}}$ by [Mc], p. 101. (Both sources [BM] and [BG] work over $\mathbf{C}$, but can be transposed to the present context.) □

Remark 7.2 We have used here Theorem 4.1 in order to deduce Theorem 7.1 from the smoothness of $\mathcal{N}$ and the fact that the Springer resolution of the nilpotent orbit closure corresponding to $\mathbf{t} = \mathbf{r}^\vee$ is semi-small. Of course, Theorem 4.1 implies also that the scheme $\mathcal{M}$ is smooth and that the special fiber of $\mathcal{M}$ is a semi-small resolution of $\overline{M}_{\mathbf{t}}$. On the other hand, this last fact has a direct proof, cf. [N-P], Lemma 9.3. In fact, the Remark 6.4 establishes an equivalence between the semi-smallness properties.

Remark 7.3 (B. C. Ngô) Denote by $R\psi'$ the complex of nearby cycles of the $\mathcal{O}_K$-scheme $M^{\rm loc} \otimes_{\mathcal{O}_E} \mathcal{O}_K$. Let us fix an $\mathcal{O}_F$-basis of the module Λ. Recall that then by (6.5), the special fiber $\overline{\mathcal{M}}$ can be identified with the convolution

$$\overline{\mathcal{O}}_{\mu_1} \tilde{\times} \ldots \tilde{\times} \overline{\mathcal{O}}_{\mu_e}$$

and the morphism $\overline{\mathcal{M}} \to \overline{M}^{\rm loc} \times_k k'$ with the natural morphism

$$m_{(\mu_1,\ldots,\mu_e)} : \overline{\mathcal{O}}_{\mu_1} \tilde{\times} \ldots \tilde{\times} \overline{\mathcal{O}}_{\mu_e} \to \overline{\mathcal{O}}_{\mu_1+\cdots+\mu_e}$$

of (6.6); here $\overline{\mathcal{O}}_\mu$ denotes the reduced closure of the orbit which corresponds to the dominant coweight μ in the affine Grassmanian for GL_d over k'. The same arguments as in the proof of Theorem 7.1, applied to the morphism $\pi : \mathcal{M} \to M^{\rm loc} \otimes_{\mathcal{O}_E} \mathcal{O}_K$ in place of $\mathcal{N} \to N^{\rm loc} \otimes_{\mathcal{O}_E} \mathcal{O}_K$, show now that

$$(7.9) \qquad R\psi'[\dim \overline{M}^{\rm loc}]\Big(\tfrac{1}{2}\dim \overline{M}^{\rm loc}\Big) = R\overline{\pi}_*\overline{\mathbf{Q}}_l[\dim \overline{\mathcal{M}}]\big(\tfrac{1}{2}\dim \overline{\mathcal{M}}\big).$$

By the above discussion, the right hand side of (7.9) is the convolution $IC_{\mathcal{O}_{\mu_1}} * \cdots * IC_{\mathcal{O}_{\mu_e}}$. Hence, we obtain a relation between nearby cycles and convolution:

$$(7.10) \qquad R\psi'[\dim \overline{M}^{\rm loc}]\Big(\tfrac{1}{2}\dim \overline{M}^{\rm loc}\Big) = IC_{\mathcal{O}_{\mu_1}} * \cdots * IC_{\mathcal{O}_{\mu_e}}.$$

Recall that there is an equivalence of tensor categories between the category of $\tilde{\mathcal{G}}_{k'}$-equivariant pure perverse $\overline{\mathbf{Q}}_l$-sheaves of weight 0 on the affine Grassmanian $\widetilde{\rm Grass}_{k'}$ (with tensor structure given by the convolution product) and the category of finite dimensional $\overline{\mathbf{Q}}_l$-representations of the Langlands dual group $GL_d = \widehat{GL_d}$ (see [Gi], and especially [M-V] §7). Under this equivalence the perverse sheaf $IC_{\mathcal{O}_\mu}$ corresponds to the representation $V(\mu)$ of GL_d of highest weight μ, and the convolution in (7.10) to the tensor product

$$V(\mu_1) \otimes \cdots \otimes V(\mu_e).$$

This decomposes

$$V(\mu_1) \otimes \cdots \otimes V(\mu_e) \ = \bigoplus_{\lambda \leq \mu_1+\cdots+\mu_e} \mathcal{M}_\lambda \otimes V(\lambda)\ ,$$

where $\mathcal{M}_\lambda$ is a finite dimensional $\overline{\mathbf{Q}}_l$-vector space. Using again the above equivalence of categories and (7.10), we obtain

$$(7.11) \qquad R\psi'[\dim \overline{M}^{\rm loc}]\Big(\tfrac{1}{2}\dim \overline{M}^{\rm loc}\Big) = \bigoplus_{\lambda \leq \mu_1+\cdots+\mu_e} \mathcal{M}_\lambda \otimes IC_{\mathcal{O}_\lambda}\ .$$

The right hand side of (7.11) corresponds to the expression in Theorem 7.1. Indeed, $\{\lambda \mid \lambda \leq \mu_1 + \cdots + \mu_e\}$ corresponds to $\{\mathbf{s} \mid \mathbf{s} \leq \mathbf{r}^\vee\}$, and we can see directly that the Littlewood-Richardson number $\dim(\mathcal{M}_\lambda)$ is equal to the Kostka number $K_{\mathbf{s}^\vee,\mathbf{r}}$ of 7.1.

Remark 7.4 As before, let us choose an ordering of the set of embeddings $\phi : F \to F_0^{\text{sep}}$. The Galois group $\Gamma = \text{Gal}(K/E)$ can be identified with the subgroup of elements σ of the symmetric group S_e which satisfy $r_{\sigma(i)} = r_i$. The group Γ acts by permutation of the factors on the tensor product $V(\mu_1) \otimes \cdots \otimes V(\mu_e)$. Let us denote by ρ the corresponding representation

$$\rho \ : \ \Gamma \to GL(V(\mu_1) \otimes \cdots \otimes V(\mu_e)).$$

Since the permutation action commutes with the action of GL_d on the tensor product, the representation ρ decomposes as

$$\rho = \bigoplus_{\lambda \leq \mu_1 + \cdots + \mu_e} \rho_\lambda \otimes \text{id}_{V(\lambda)}$$

where ρ_λ is a representation of Γ on the vector space $\mathcal{M}_\lambda$. We conjecture that the representation of $\text{Gal}(K/E)$ on $\mathcal{M}_{\mathbf{s}}$ (see Theorem 7.1) is isomorphic to ρ_λ (with λ corresponding to $\mathbf{s}$).

After our choice of an $\mathcal{O}_F$-basis of Λ, the generic fiber of M^{loc} can be identified with the product of Grassmanians

$$M^{\text{loc}} \otimes_{\mathcal{O}_K} K = \prod_{i=1}^{e} \text{Grass}_{r_i}(K^d).$$

For $\sigma \in \Gamma$, permutation of the factors gives an isomorphism of K-schemes

$$\kappa(\sigma) : M^{\text{loc}} \otimes_{\mathcal{O}_E} K \to M^{\text{loc}} \otimes_{\mathcal{O}_E} K.$$

The isomorphism $\kappa(\sigma)$ induces an automorphism of the sheaf of vanishing cycles $R\psi'$. Therefore, by (7.10) we obtain a "commutativity" isomorphism

$$\kappa(\sigma) : IC_{\mathcal{O}_{\mu_1}} * \cdots * IC_{\mathcal{O}_{\mu_e}} \to IC_{\mathcal{O}_{\mu_1}} * \cdots * IC_{\mathcal{O}_{\mu_e}} = IC_{\mathcal{O}_{\mu_{\sigma(1)}}} * \cdots * IC_{\mathcal{O}_{\mu_{\sigma(e)}}}.$$

(notice here that, since $\sigma \in \Gamma$, $\mu_{\sigma(i)} = \mu_i$). As was pointed out by Ngô, the conjecture follows, if $\kappa(\sigma)$ coincides with the isomorphism given using the permutation σ and the "commutativity constraint" (for the tensor category of perverse sheaves of Remark 7.3) of [M-V].

Remark 7.5 Let x be a point of M^{loc} with values in a finite extension $\mathbf{F}_q$ of k. Let $\text{Tr}^{ss}(\text{Fr}_q, R\psi_x^M)$ be the semi-simple trace of the geometric Frobenius on the stalk at x of the complex of nearby cycles of M^{loc}. It should be possible to transfer the conjecture of Kottwitz [HN] to this case to obtain a group-theoretic expression for this (the group $R_{F/F_0}(GL_d)$ relevant for this conjecture here is not split so that the original formulation does not apply directly). Note that

$$\text{Tr}^{ss}(\text{Fr}_q, R\psi_x^M) = \text{Tr}^{ss}(\text{Fr}_q, R\psi_y^N) \tag{7.12}$$

where $y = \phi(\tilde{x})$ for an arbitrary point $\tilde{x} \in \widetilde{M}^{\text{loc}}(\mathbf{F}_q)$ mapping to x, and where $R\psi^N$ is the complex of nearby cycles of N^{loc}. The expression (7.12) only depends on the stratum $M_{\mathbf{s}}$ containing x resp. $N_{\mathbf{s}}$ containing y and may therefore be denoted by $\text{Tr}^{ss}(Fr_q, R\psi_{\mathbf{s}}^N)$.

We also consider the analogous semi-simple traces for points with values in a finite extension $\mathbf{F}_q$ of k',

$$\begin{aligned}
\text{(7.13)}\qquad & \mathrm{Tr}^{ss}(Fr_q, R\psi_x^{M\otimes K}),\ \mathrm{Tr}^{ss}(Fr_q, R\psi_x^{N\otimes K})\ ,\\
& \mathrm{Tr}^{ss}(Fr_q, R\psi_{\mathbf{s}}^{M\otimes K}),\ \mathrm{Tr}^{ss}(Fr_q, R\psi_{\mathbf{s}}^{N\otimes K})\ .
\end{aligned}$$

(these semi-simple traces may differ from the preceding ones when K/E is ramified).

¿From Theorem 7.1 we obtain

$$\mathrm{Tr}^{ss}(\mathrm{Fr}_q;\ R\psi_{\mathbf{s}}^{N\otimes K}) = q^{\frac{1}{2}\dim\ \overline{N}^{\mathrm{loc}}} \sum_{\mathbf{s}\leq\mathbf{s}'\leq\mathbf{t}} m_{\mathbf{s}'}\cdot \mathrm{Tr}^{ss}(\mathrm{Fr}_q, (IC_{N_{\mathbf{s}'}})_{\mathbf{s}})\ , \tag{7.14}$$

provided $\mathbf{F}_q \supset k'$. By Lusztig [L], the entity $\mathrm{Tr}^{ss}(\mathrm{Fr}_q, (IC_{N_{\mathbf{s}'}})_{\mathbf{s}})$ can be expressed in terms of Kostka polynomials $K_{\mathbf{s}',\mathbf{s}}(q^{-1})$, comp. also [Mc], p. 245.

Consider the spectral sequence of nearby cycles,

$$E_2^{pq} = H^p(M^{\mathrm{loc}}\otimes_k \overline{k}, R^q\psi^M) \Rightarrow H^{p+q}(M^{\mathrm{loc}}\otimes_{\mathcal{O}_E} F_0^{\mathrm{sep}}, \overline{\mathbf{Q}}_\ell)\ . \tag{7.15}$$

We obtain an identity of semi-simple traces (cf. [HN])

$$\mathrm{Tr}^{ss}(\mathrm{Fr}_q, H^*(M^{\mathrm{loc}}\otimes_{\mathcal{O}_E} F_0^{\mathrm{sep}}, \overline{\mathbf{Q}}_\ell)) = \sum_{x\in M^{\mathrm{loc}}(\mathbf{F}_q)} \mathrm{Tr}^{ss}(\mathrm{Fr}_q, R\psi_x^M). \tag{7.16}$$

Similarly we may consider the spectral sequence of nearby cycles for $M^{\mathrm{loc}}\otimes_{\mathcal{O}_E}\mathcal{O}_K$. We then obtain an identity of semi-simple traces (as $\mathrm{Gal}(F_0^{\mathrm{sep}}/K)$-modules), provided that $\mathbf{F}_q\supset k'$,

$$\mathrm{Tr}^{ss}(Fr_q; H^*((M\otimes_{\mathcal{O}_E}\mathcal{O}_K)\otimes_{\mathcal{O}_K} F_0^{\mathrm{sep}}, \overline{\mathbf{Q}}_\ell)) = \sum_{x\in M(\mathbf{F}_q)} \mathrm{Tr}^{ss}(\mathrm{Fr}_q, R\psi_x^{M\otimes K}). \tag{7.17}$$

However,

$$M^{\mathrm{loc}}\otimes_{\mathcal{O}_E} K = \prod_\varphi \mathrm{Grass}_{r_\varphi}(V_\varphi)\ \ ,$$

cf. (2.7), and hence the left hand side of (7.17) is known. Taking into account (7.12) we obtain therefore from (7.16) a combinatorial identity involving Kostka polynomials, Kostka numbers etc. It might be interesting to identify this combinatorial identity explicitly.

8 Local models of EL-type

In this section we use the following notation (following [RZ]).

F_0 a complete discretely valued field with perfect residue field
F a finite separable field extension of F_0,
B a simple algebra with center F,
V a finite B-module,
$G = GL_B(V)$, as algebraic group over F_0,
$\mu : \mathbf{G}_{mK} \to G_K$ a one parameter subgroup, defined over some sufficiently big extension K contained in a fixed separable closure F_0^{sep} of F_0, given up to conjugation. We assume that the eigenspace decomposition of $V \otimes_{\mathbf{Q}_p} K$ is given by

$$V \otimes_{F_0} K = V_0 \oplus V_1 \quad . \tag{8.1}$$

E the field of definition of the conjugacy class of μ.
$\mathcal{L}$ a periodic O_B-lattice chain in V.

In the sequel we denote by $\mathcal{O}_{F_0}, \mathcal{O}_F, \mathcal{O}_B, \mathcal{O}_E$ the respective rings of integers.

To these data we associate the following functor on $(Sch/\mathcal{O}_E)$: The S-valued points are given by

i) a functor $\Lambda \mapsto t_\Lambda$ to the category of $\mathcal{O}_B \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-modules on S

ii) a morphism of functors $\varphi_\Lambda : \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S \to t_\Lambda$.

The requirements on these data are:

a) t_Λ is locally on S a free $\mathcal{O}_S$-module of finite rank, and we have the following identity of polynomial functions on $\mathcal{O}_B$:

$$\det_{\mathcal{O}_S}(a|t_\Lambda) = \det_K(a|V_1) \quad . \tag{8.2}$$

b) φ_Λ is surjective, for all $\Lambda \in \mathcal{L}$.

We remark that the standard models considered in sections 2 – 7 are a special case. Indeed, let $B = F$ and let $\mathcal{L}$ consist of the $F^\times$-multiples of a fixed $\mathcal{O}_F$-lattice Λ_0 in the d-dimensional F-vector space V and assume that under the decomposition (2.5) of $F \otimes_{F_0} F_0^{\mathrm{sep}}$ we have

$$V \otimes_{F_0} F_0^{\mathrm{sep}} = \bigoplus_\varphi V_\varphi, \ V_0 \otimes_K F_0^{\mathrm{sep}} = \bigoplus_\varphi V_{0,\varphi}, \ V_1 \otimes_K F_0^{\mathrm{sep}} = \bigoplus_\varphi V_{1,\varphi}, \tag{8.3}$$

with $\dim_{F_0^{\mathrm{sep}}} V_{0,\varphi} = r_\varphi$. If $(t_\Lambda, \varphi_\Lambda)_\Lambda$ is an S-valued point of the moduli problem above, then $\mathcal{F} = \mathrm{Ker}\ \varphi_{\Lambda_0}$ is an S-valued point of the standard model for GL_d corresponding to $\mathbf{r} = (r_\varphi)$ and this establishes an isomorphism of moduli problems. Furthermore, E has indeed the description given in section 2 in terms of $\mathbf{r}$.

In general, the above moduli problem is representable by a projective scheme over $\mathrm{Spec}\,\mathcal{O}_E$ which is a closed subscheme of a form over $\mathcal{O}_E$ of a product of Grassmannians. Let us make this more precise.

Let us first consider the case where $B = F$ is a totally ramified extension of degree e of F_0. Let $v_1, \dots, v_d$ be a basis of V. For $i = 0, \dots, d-1$ consider the $\mathcal{O}_F$-lattice Λ_i in V spanned by

$$\Lambda_i = \text{span}\ \{\pi^{-1}v_1, \dots, \pi^{-1}v_i, v_{i+1}, \dots, v_d\}\ . \tag{8.4}$$

Here π denotes a uniformizer in F. This yields a complete periodic lattice chain

$$\dots \to \Lambda_0 \to \Lambda_1 \to \dots \to \Lambda_{d-1} \to \pi^{-1}\Lambda_0 \to \dots \tag{8.5}$$

Choose $I = \{i_0 < i_1 < \dots < i_\ell\} \subset \{0, \dots, d-1\}$. Then the periodic lattice $\mathcal{L}$ is isomorphic to the subchain of (8.5) where only Λ_i with $i \in I$ are kept, for suitable I. The points of the above functor with values in a $\mathcal{O}_E$-scheme S can now be interpreted as the isomorphism classes of commutative diagrams

$$\begin{array}{ccccccccc}
\Lambda_{i_0,S} & \longrightarrow & \Lambda_{i_1,S} & \longrightarrow & \dots & \longrightarrow & \Lambda_{i_\ell,S} & \longrightarrow & \pi^{-1}\Lambda_{i_0,S} \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
\mathcal{F}_0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \dots & \longrightarrow & \mathcal{F}_\ell & \longrightarrow & \pi^{-1}\mathcal{F}_0
\end{array}\quad ,$$

where $\Lambda_{i_j,S}$ is $\Lambda_{i_j} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$ and where the $\mathcal{F}_j$ are $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S$-submodules which locally on S are direct summands of $\Lambda_{i_j,S}$ as $\mathcal{O}_S$-modules and where we have an identity of polynomial functions on $\mathcal{O}_F$,

$$\det(a|\mathcal{F}_j) = \det{}_K(a|V_0)\quad ,\quad j = 0, \dots, \ell\quad . \tag{8.6}$$

Suppose that K is a sufficiently big Galois extension so that

$$F \otimes_{F_0} K = \bigoplus_{\varphi: F \to K} K\quad . \tag{8.7}$$

We obtain corresponding decompositions

$$V \otimes_{F_0} K = \bigoplus_{\varphi} V_\varphi\ ,\ V_0 \otimes_{F_0} K = \bigoplus_{\varphi} V_{0,\varphi}\ . \tag{8.8}$$

Then $\dim_K V_\varphi = d$ for all φ. Let $r_\varphi = \dim_K V_{0,\varphi}$. Then the determinant condition (8.6) can be rewritten as

$$\det(a|\mathcal{F}_j) = \prod_{\varphi} \varphi(a)^{r_\varphi}\quad . \tag{8.9}$$

In other words, each $\mathcal{F}_i$ is an S-valued point of the standard model for GL_d corresponding to $\mathbf{r} = (r_\varphi)$ (and Λ_i). Our functor is representable by a closed subscheme of the product over $\mathcal{O}_E$ of these standard models, one for each $i = 0, \dots, \ell$.

Now let us consider the general case. Let $\breve{F}_0$ be the completion of the maximal unramified extension of F_0 in F_0^{sep}. If F_1 is the maximal unramified extension of F_0 contained in F, we obtain a decomposition

$$F_1 \otimes_{F_0} \breve{F}_0 = \breve{F}_0 \oplus \dots \oplus \breve{F}_0\quad , \tag{8.10}$$

corresponding to the $f = [F_1 : F_0]$ different embeddings $\alpha : F_1 \to \breve{F}_0$ of F_1 into $\breve{F}_0$. For fixed α, the extension $\breve{F}_\alpha = F \otimes_{F_{1,\alpha}} \breve{F}_0$ is a totally ramified field extension of degree $e = [F : F_0]/f$ of $\breve{F}_0$. The simple central algebra $B \otimes_F \breve{F}_\alpha = B \otimes_{F_{1,\alpha}} \breve{F}_0$ splits, i.e. is isomorphic to a matrix algebra over $\breve{F}_\alpha$. Similarly, we obtain a decomposition

$$V \otimes_{F_0} \breve{F}_0 = \bigoplus_\alpha V_\alpha \quad , \tag{8.11}$$

where V_α is a $\breve{F}_\alpha$-vector space, all of the same dimension d.

Since F_1 is an unramified extension we obtain similar decompositions for the rings of integers, e.g.

$$\mathcal{O}_B \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_{\breve{F}_0} = \bigoplus_\alpha \mathcal{O}_B \otimes_{\mathcal{O}_{F_1},\alpha} \mathcal{O}_{\breve{F}_0} \quad , \tag{8.12}$$

where for each α the summand $\mathcal{O}_B \otimes_{\mathcal{O}_{F_1},\alpha} \mathcal{O}_{\breve{F}_0} = \mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}_\alpha}$ is a parahoric order in the matrix algebra $B \otimes_F \breve{F}_\alpha$. Similarly, the periodic lattice chain $\mathcal{L}$ corresponds to periodic $\mathcal{O}_{\breve{F}_\alpha}$-lattice chains $\mathcal{L}_\alpha$ in each $\breve{F}_\alpha$-vector space V_α in (8.11). Let K be a sufficiently big Galois extension of F_0. Then

$$V \otimes_{F_0} K = \bigoplus_\alpha V_\alpha \otimes_{\breve{F}_0} \breve{K} \quad , \tag{8.13}$$

where $\breve{K} = \breve{F}_0.K$, and for each α

$$V_\alpha \otimes_{\breve{F}_0} \breve{K} = V_{\alpha,0} \oplus V_{\alpha,1} \quad . \tag{8.14}$$

Let $\breve{E} = E.\breve{F}_0$. It now follows that our functor is representable by a projective scheme over $\mathcal{O}_E$ which after base change from $\mathcal{O}_E$ to $\mathcal{O}_{\breve{F}}$ becomes isomorphic to the product over all α of schemes considered before for the data $(\breve{F}_\alpha/\breve{F}_0, V_\alpha, \mu_\alpha, \mathcal{L}_\alpha)$.

Let us denote by M^{naive} the scheme over $\operatorname{Spec} \mathcal{O}_E$ associated in this way to the data fixed in the beginning of this section. This is what we call *a naive local model of EL-type.* We know from the special case of a standard model for GL_d that M^{naive} is rarely flat over Spec $\mathcal{O}_E$. We now define a closed subscheme M^{loc} of M^{naive} which stands a better chance of being flat over the base scheme.

Assume first that $B = F$ is a totally ramified extension. In the notation introduced after (8.5) let $\mathcal{L} = \mathcal{L}_I$. For every $i \in I$ we obtain a morphism

$$\pi_i : M^{\mathrm{naive}} \longrightarrow M^{\mathrm{naive}}(\Lambda_i) \quad i \in I \quad . \tag{8.15}$$

Here $M^{\mathrm{naive}}(\Lambda_i) = M(\Lambda_i, \mu)$ denotes the standard model associated to Λ_i (and (F, V, μ)). We then define in this case

$$M^{\mathrm{loc}} = \bigcap_{i \in I} \pi_i^{-1}(M^{\mathrm{loc}}(\Lambda_i)) \tag{8.16}$$

(scheme-theoretic intersection inside M^{naive}).

In the general case we have, with the notation used above,

$$M^{\mathrm{naive}} \otimes_{\mathcal{O}_E} \mathcal{O}_{\breve{E}} = \prod_{\alpha} M^{\mathrm{naive}}(\breve{F}_\alpha/\breve{F}_0, V_\alpha, \mu_\alpha, \mathcal{L}_\alpha) \quad . \tag{8.17}$$

Let

$$\breve{M}^{\mathrm{loc}} = \prod_{\alpha} M^{\mathrm{loc}}(\breve{F}_\alpha/\breve{F}_0, V_\alpha, \mu_\alpha, \mathcal{L}_\alpha) \quad . \tag{8.18}$$

The descent datum on $M^{\mathrm{naive}} \otimes_{\mathcal{O}_E} \mathcal{O}_{\breve{E}}$ respects the closed subscheme $\breve{M}^{\mathrm{loc}}$ and hence defines a closed subscheme M^{loc} of M^{naive}.

We conjecture that M^{loc} is flat over Spec $\mathcal{O}_E$. This would constitute the analogue of the result of Görtz [G] which confirms the conjecture in the case when F/F_0 is unramified.

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